Equivalence of the impulse approximation and the Gersch-Rodriguez-Smith series for structure functions

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For a nonrelativistic system we compare the Gersch-Rodriguez-Smith (GRS) and the impulse approximation (IA) approaches to the structure function. The first of these two approaches generates a series in $1/q$, whereas the second treats the interaction between the struck and core nucleons perturbatively. Instead of the IA series we derive a DWIA representation and prove that, up to and including terms of order $\mathcal{O}(1/q^2)$, it is contained in the GRS series of the same order. This clarifies the relation between the two approaches and suggests that the two approaches, when treated exactly, produce identical structure function to arbitrary order in $1/q$. [S0556-2813(99)02406-1]

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I. INTRODUCTION

Virtually all computations of structure functions of nuclei, as measured by inclusive scattering of high-energy electrons, use relativistic generalizations of either the nonrelativistic (NR), perturbative impulse approximation (IA) series [1,2], or of a nonperturbative theory [3], formulated by Gersch, Rodriguez, and Smith (GRS) [4].

The GRS approach produces an expansion of the response in inverse powers of the momentum transfer q with coefficient functions, depending on the inter-particle interaction V and the many-body density-matrices. The leading term in this series is the $q\to\infty$ limit while the correction terms contain the final state interactions (FSI).

To lowest order the IA is just the plane wave impulse approximation (PWIA) where the interaction $\overline{V}(\boldsymbol{r}_1)=\Sigma_{k\geqslant 2}V(\boldsymbol{r}_1-\boldsymbol{r}_k)$ between the struck nucleon and the core is neglected. The remaining terms in the IA series are then the contributions of increasing order in the initially neglected struck nucleon-core interaction, i.e., the FSI calculated perturbatively.

The GRS and the IA series are very dissimilar, yet each provides a representation of the structure function. Consequently an *exact* treatment of each must provide identical results. A frequently raised question is how the two are related, and which approach is better when treated *approximately*, i.e., truncated at finite order in some perturbation or small quantity. To our knowledge not even a criterion, to be followed in principle, has been previously formulated. The main purpose of the present article is just such a formulation, followed by a proof of equivalence.

The above quest is encumbered by the fact that we do not know of a manageable evaluation of FSI in the IA series as exists for the GRS theory. A prerequisite for a comparison is therefore a realistic model for FSI, replacing the IA series.

As to the nature of such a model, one is guided by the fact that the relative weight of FSI in the response diminishes with increasing q. It is therefore natural to consider on the one hand kinematic conditions, generally reached for scattering with high beam energies. On the other hand, high energies are by necessity accompanied by effects due to relativity, particle production and the like, whose treatment can never be exact. We therefore suggest as a starting point a well-defined nonrelativistic (NR) model, based on a Hamiltonian for point-particles which cannot be excited. Such a model can be treated exactly and provides insight which later can be incorporated in realistic situations.

We start in Sec. II with the GRS theory, recapitulate some formally exact expressions for the lowest order terms of the GRS series and cite results for partial summations of selected higher order terms. In Sec. III we consider the response of a semi-inclusive (SI) $A(e,e'p)X_{A-1}$ reaction in the PWIA, which features the one-hole spectral function. We then suggest a realistic form for FSI which in nuclear parlance is called the distorted wave impulse approximation (DWIA). Integration of the SI response over the momenta of the outgoing nucleon produces for that model the totally inclusive (TI) cross section. In Sec. IV we demonstrate that the GRS to $\mathcal{O}(1/q^2)$ contains the DWIA terms to the same order and attribute the absence of an extra term to the approximate nature of the chosen DWIA. In Sec. V we briefly discuss the embedding of the above in a relativistic theory.

II. THE GRS SERIES AND SOME RESUMMATIONS

We consider the TI structure function, or the response $S^{\rm TI}(q,\omega)$ for a NR many-body system to a scalar perturbation, defined as the ratio of cross sections for the inclusive scattering of a projectile from a composite target and from a constituent. The kinematic variables (q,ω) are the momentum and energy, transfered by the projectile to the target. The response per particle can be written as

$$S^{\rm TI}(q,\omega)=(2\pi A)^{-1}\langle\Phi_A^0|\rho_q^\dagger\delta(\omega+E_A^0-H_A)\rho_q|\Phi_A^0\rangle,$$
$$=(2\pi A)^{-1}\sum_m |\langle\Phi_A^0|\rho_q^\dagger|\Phi_A^m\rangle|^2\delta(\omega+E_A^0-E_A^m), \tag{1}$$

where Φ_A^m,E_A^m are states and energies of the exact A-body Hamiltonian H_A. For large q it is useful to introduce the reduced response $\phi(q,y)=(q/M)S(q,\omega)$, with M the mass of a particle and y a kinematic variable, replacing the energy loss ω [4,5]

$$y=\frac{M}{q}\left(\omega-\frac{q^2}{2M}\right). \tag{2}$$

Substitution of $\rho_q=\Sigma_j e^{i\mathbf{q}\cdot\mathbf{r}_j}$ into Eq. (1) produces incoherent and coherent components. When considering high-q responses, it suffices to consider the dominant incoherent part, where a single particle is tracked in its propagation through the medium [5]. We cite Ref. [4] for a derivation of the GRS series

$$\phi(q,y)=\sum_{n\geqslant 0}(1/v_q)^n F_n(y), \tag{3}$$

where $v_q=q/M$ is the recoil velocity, corresponding to a momentum transfer q. The coefficient functions $F_n(y)$ are functionals of the interparticle interaction V and density matrices $\rho_n(1'j;1j), j\geqslant 2$. Those are diagonal in all coordinates $j=\mathbf{r}_j$, except that of the struck nucleon, which is chosen to be ''1.'' All derive from $\rho_A(1',k;1,k), A\geqslant k\geqslant 2$ and satisfy in our convention the relations

$$\rho_n(1',2\ldots n;1,2\ldots n)$$
$$=\frac{1}{(A-n)!}\left(\Pi_{j=n+1}^{A}\int dj\right)\rho_A(1'j;1j),$$

$$\rho_A(1'k;1k)=A!\,\Phi_A^*(1',k)\Phi_A(1,k). \tag{4}$$

The appearance of exact many-body densities shows that from the onset the theory accounts for correlations of the target nucleons.

For our purposes it suffices to mention ρ_n for $n=1,2,3$ which enter expressions for the asymptotic limit F_0 and the two dominant FSI corrections F_1, F_2:

$$F_0(y)=\frac{1}{A!}\int\frac{ds}{2\pi}e^{iys}\int d1\left(\Pi_{k\geqslant 2}^{A}\int dk\right)\rho_A(1-s,k;1,k)$$
$$=\frac{1}{A}\int\frac{ds}{2\pi}e^{isy}\int d1\rho_1(1-s;1)=\frac{1}{4\pi^2}\int_{|y|}^{\infty}dp\,p\,n(p), \tag{5a}$$

$$\frac{1}{v_q}F_1(y)=\frac{i}{A!}\int\frac{ds}{2\pi}e^{iys}\int d1\left[\Pi_{k\geqslant 2}^{A}\int dk\right]$$
$$\times\rho_A(1-s,k;1k)\sum_{k\geqslant 2}\tilde{\chi}_q(1-k,s)$$
$$=\frac{i}{A}\int\frac{ds}{2\pi}e^{iys}\int\int d1d2\rho_2(1-s,2;12)$$
$$\times\tilde{\chi}_q(1-2,s), \tag{5b}$$

$$\frac{1}{v_q^2}F_2(y)=-\frac{1}{2A!}\int\frac{ds}{2\pi}e^{isy}\int d1\left[\Pi_{k\geqslant 2}^{A}\int dk\right]$$
$$\times\rho_A(1-s,k;1,k)\left[\int_0^s d\sigma\sum_{k\geqslant 2}\tilde{\chi}_q(1-k,s)\right]^2$$
$$+\frac{1}{v_q^2}F_2^{(r)}(y), \tag{5c}$$

$$\frac{1}{v_q^2}F_2^{(r)}(y)=-\frac{1}{A!}\int\frac{ds}{2\pi}e^{isy}\int d1\left[\Pi_{k\geqslant 2}^{A}\int dk\right]$$
$$\times\rho_A(1-s,k;1,k)\left[\frac{1}{2}\frac{\partial^2}{\partial s^2}\left(\sum_{k\geqslant 2}\int_0^s d\sigma\tilde{\chi}_q\right.\right.$$
$$\left.\left.\times(1-k,\sigma)\right)^2-\left(\sum_{k\geqslant 2}\tilde{\chi}_q(1-k,s)\right)^2\right] \tag{5d}$$

with $n(p)$ the single-particle momentum distribution. The expression for $F_2^{(r)}$ is easily derived from Eq. (14) in Ref. [6]. Above we also introduced

$$\tilde{\chi}_q(1,s)=-(1/v_q)\int_0^s d\sigma[V(1-\sigma)-V(1)]$$
$$\equiv-(1/v_q)\int_0^s d\sigma\mathcal{V}(1,\sigma)=-(1/v_q)$$
$$\times\int_0^s d\sigma[\mathcal{V}_\sigma^{(1)}(1)-\mathcal{V}^{(2)}(1)]. \tag{6}$$

Equation (6) defines the coordinate representation of the off-shell eikonal phase $\tilde{\chi}(1,s)$ corresponding to the total $\mathcal{V}$ and its components $\mathcal{V}^{(a)}, a=1,2$ which are characteristic of the GRS theory or of path integral methods for the response [7].

It is frequently useful to make resummations within the GRS series (3). We consider first a ladder sum of repeated interactions which results in the replacement $V\rightarrow t=V_{eff}$. This replacement is mandatory if the bare interaction V is singular. The corresponding change in the phase $\tilde{\chi}$, Eq. (6), is

$$i\tilde{\chi}\rightarrow\tilde{\Gamma}\equiv e^{i\tilde{\chi}}-1, \tag{7}$$

with $\tilde{\Gamma}$ the total off-shell profile function.

Next we consider a cumulant resummation which to lowest order reads [8]

$$\phi(q,y)=\frac{1}{A}\int\frac{ds}{2\pi}e^{isy}\int d1\rho_1(1-s;1)$$
$$\times\exp\left[\frac{\int d2\rho_2(1-s,2;1,2)\tilde{\Gamma}_q(1-2,s)}{\rho_1(1-s;1)}+\ldots\right]. \tag{8}$$

When expanded, it reproduces the lowest order terms in the GRS series, as well as selected higher order contributions (3). In the next section we shall compare with this partially summed results.

III. FSI CORRECTIONS TO THE PWIA RESPONSE

We approach the IA treatment of the TI response for a NR system of point-particles by considering first SI scattering. The corresponding response per nucleon is

$$S^{SI}(q,\omega;\boldsymbol{p})=\frac{1}{A}\sum_m |\langle\Phi_A^0|\rho_q^\dagger|\Psi^{(-)}_{(A-1)_m;\boldsymbol{p}+\boldsymbol{q}}\rangle|^2 \times\delta\left(\omega-\Delta_m-\frac{(\boldsymbol{p}+\boldsymbol{q})^2}{2M}\right), \tag{9}$$

where $\boldsymbol{p}$ is the momentum of the struck, and $\boldsymbol{p}+\boldsymbol{q}$ that of the detected outgoing nucleon after absorbing the momentum transfer $\boldsymbol{q}$. $\Psi^{(-)}_{(A-1)_m;\boldsymbol{p}+\boldsymbol{q}}$ is the state of that nucleon, scattered from a nucleus of $A-1$ particles in state m. Δ_m is the separation energy of a nucleon in the ground-state A-body system, with the daughter nucleus in the state Φ^m_{A-1}. We write the total Hamiltonian as

$$H_A(1;k)=H_{A-1}(k)+T(1)+\overline{V}(1) \tag{10}$$

with $\overline{V}(1)=\Sigma_{k\geqslant 2}V(1-k)$, the interaction of particle ''1'' with the core. Neglect of the latter defines the PWIA

$$[\Psi^{(-)}_{(A-1)_m;\boldsymbol{p}+\boldsymbol{q}}(\boldsymbol{r}_1;\boldsymbol{r}_k)]^{\rm PWIA}\rightarrow\Phi^m_{A-1}(\boldsymbol{r}_k)e^{-i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{r}_1}. \tag{11}$$

When substituted into Eq. (9), it produces the standard PWIA approximation for the SI response

$$S^{\rm SI;PWIA}(q,\omega;\boldsymbol{p})=\int dEP(\boldsymbol{p},E)\,\delta\left(E-\omega-\frac{(\boldsymbol{p}+\boldsymbol{q})^2}{2M}\right), \tag{12a}$$

$$\phi^{\rm SI;PWIA}(q,y_0;\boldsymbol{p})\approx\delta(y_0-p_z)n(p), \tag{12b}$$

$$n(p)=\int dEP(\boldsymbol{p},E). \tag{12c}$$

Here $P(\boldsymbol{p},E)$ is the single-hole spectral function, dependent on the separation-energies of each of the daughter states m. Equation (12b) results from the approximation $\Delta_m\rightarrow\langle\Delta\rangle$ with $\langle\Delta\rangle$ an average separation energy. One may then replace the energy loss ω by the IA scaling variable, also in terms of $\langle\Delta\rangle$

$$y_0=-q+\sqrt{2M(\omega-\langle\Delta\rangle)}. \tag{13}$$

FSI corrections to the PWBA result Eq. (12b) are, by definition, contributions due to the residual interaction V, treated perturbatively. With no practical way to do so systematically, we proceed in an approximative manner.

Whereas the core particles k have momenta of the order of the Fermi momentum p_F, for particle 1 after absorption of the high-mass virtual photon, $|\boldsymbol{p}+\boldsymbol{q}|\approx q\gg p_F$, i.e., its momentum exceeds by far the average momentum of a core particle. Whenever a state contains both 1 and k, one may to lowest order neglect $|\boldsymbol{p}|$ altogether, or equivalently, freeze in a standard fashion the core coordinates k. For the final scattering states in Eq. (8) we now suggest [9–11]

$$\Psi^{(-)}_{(A-1)_m;\boldsymbol{p}+\boldsymbol{q}}(1;k)\approx\Phi^m_{A-1}(k)\,\psi^{(-)}_{\boldsymbol{p}+\boldsymbol{q}}(1;\langle k\rangle). \tag{14}$$

One notes that, contrary to the perturbative nature of the actual IA series, the approximation (14) is nonperturbative.

The eikonal approximation for a state, describing scattering of the high-momentum, knocked-out particle 1 from a static, noncentral field $\Sigma_{k\geqslant 2}V(1-\langle k\rangle)$ reads [12]

$$\psi^{\pm}_{\boldsymbol{\kappa}}(1;\langle k\rangle)=e^{i\kappa z_1}\xi^{\pm}_{\boldsymbol{\kappa}}(1;\langle k\rangle) \tag{15}$$

with distortion function ξ

$$\xi^{(-)}_q(1;\langle k\rangle)=\exp\left[-\frac{i}{v_q}\sum_k\int_{z_1}^{\infty}d\zeta V(1-\langle k\rangle-\zeta)\right]. \tag{16}$$

Substituting Eq. (14) into Eq. (9) and replacing again state-dependent separation energies by an average, one performs closure over states of the daughter nucleus and obtains

$$\begin{aligned}\phi^{\rm SI}(q,y_0;\boldsymbol{p})&\approx\delta(y_0-p_z)\langle\Phi_A^0(1';k)|e^{-i\boldsymbol{q}\cdot\boldsymbol{r}'_1}|\psi^{(-)}_{\boldsymbol{p}+\boldsymbol{q}}(1';k)\rangle\\&\quad\times\langle\psi^{(-)}_{\boldsymbol{p}+\boldsymbol{q}}(1;k)|e^{i\boldsymbol{q}\cdot\boldsymbol{r}_1}|\Phi_A^0(1;k)\rangle^*\\&\approx\delta(y_0-p_z)\langle\Phi_A^0(1';k)|e^{-i\boldsymbol{p}\cdot\boldsymbol{r}'_1}|\xi^{(-)}_{\boldsymbol{p}+\boldsymbol{q}}(1';\langle k\rangle)\rangle\\&\quad\times\langle\xi^{(-)}_{\boldsymbol{p}+\boldsymbol{q}}(1;\langle k\rangle|e^{i\boldsymbol{p}\cdot\boldsymbol{r}_1}|\Phi_A^0(1;k)\rangle^*.\end{aligned} \tag{17}$$

The distorted wave, Eq. (15), for the outgoing particle 1 depends implicitly on all other coordinates $\langle k\rangle$. Having performed closure, we treat those again as dynamical coordinates and obtain for real V the following expression for the SI response in the DWIA ($v_{\boldsymbol{p}+\boldsymbol{q}}\approx v_q$)

$$\begin{aligned}\phi^{\rm SI,DWIA}(q,y_0;\boldsymbol{p})&=\frac{1}{A!}\,\delta(y_0-p_z)\int d\boldsymbol{s}e^{i\boldsymbol{p}.\boldsymbol{s}}\\&\quad\times\int d1\left[\Pi_{k\geqslant 2}\int dk\right]\rho_A(1-s,k;1,k)\\&\quad\times\exp\left[-\frac{i}{v_q}\sum_k\int_{z'_1}^{z_1}d\zeta V(1-k-\zeta)\right].\end{aligned} \tag{18}$$

Since in the model, degrees of freedom other than point-particles are absent, the TI response is obtained by integrating $\phi^{\rm SI}$ over missing momenta $\boldsymbol{p}$ leading to

$$\begin{aligned}\phi^{\rm TI,DWIA}(q,y_0)&=\frac{1}{A!}\int\frac{ds}{2\pi}e^{iy_0s}\int d1\Pi_{k\geqslant 2}\int dk\rho_A\\&\quad\times(1-s,k;1,k)\exp\left[i\sum_{k\geqslant 2}\tilde{\chi}^{(1)}_q(1-k,s)\right]\end{aligned} \tag{19}$$

with $\boldsymbol{s}=\boldsymbol{r}_1-\boldsymbol{r}'_1=s\hat{\boldsymbol{q}}$ lying in the direction of $\boldsymbol{q}$. The above result still has the full complexity of a many-body problem, present in the A-body density matrix. That complexity is considerably reduced in a Kirkwood independent-pair approximation

$$\rho_A(1k;1'k)\approx\frac{(A-1)!}{(A-1)^{A-1}}\frac{\Pi_{k\geqslant 2}^A\rho_2(1k;1'k)}{[\rho_1(1;1')]^{A-2}},\qquad(20)$$

which respects the sum rules (4). Substitution in Eq. (19) produces for the reduced TI response per nucleon in the DWIA

$$\begin{aligned}\phi^{\text{TI,DWIA}}&(q,y_0)\\ &\approx\frac{1}{A(A-1)^{A-1}}\int\frac{ds}{2\pi}e^{iy_0 s}\frac{\int d1}{[\rho_1(1-s;1)]^{A-2}}\\ &\quad\times\left[\Pi_{k\geqslant 2}\int dk\rho_A(1-s,k;1,k)\tilde{\Gamma}_q^{(1)}(1-k,s)\right]\\ &\approx\frac{1}{A}\int\int\frac{ds}{2\pi}e^{iy_0 s}d1\rho_1(1-s;1)\\ &\quad\times\exp\left[\frac{\int d2\rho_2(1-s,2;1,2)\tilde{\Gamma}_q^{(1)}(1-2,s)}{\rho_1(1-s;1)}\right]\qquad(21a)\\ &\approx\frac{1}{A}\int dy_0'F_0(y_0-y_0')\mathcal{R}_q(y_0'),\qquad(21b)\end{aligned}$$

$$\mathcal{R}_q(y_0)\approx\int\frac{ds}{2\pi}e^{iy_0 s}\int d1\times\exp\left[\frac{\int d2\rho_2(1-s,2;1,2)\tilde{\Gamma}_q^{(1)}(1-2,s)}{\rho_1(1-s;1)}\right].\qquad(21c)$$

For later use we expressed the response (21a) as a convolution of the asymptotic limit and a generalized FSI factor [cf. Eq. (5a), of the last article of Ref. [3]].

There clearly is a formal similarity in the expressions (8) and (21a) for the TI response in, respectively, the first cumulant expression in the GRS theory, and the approximate IA series. The apparent differences amount to (i) the appearance of y_0 instead of $y=y_w$ and (ii) the presence in the DWIA of a profile function $\tilde{\Gamma}^{(1)}$, related to the first potential in Eq. (6) and not to both, as is the case in the GRS theory. In the following section we shall investigate whether, and to what extent, these apparently similar expressions coincide.

IV. MEASURE OF EQUIVALENCE OF GRS AND APPROXIMATE IA SERIES

There are two, in principle, equivalent ways to compare the exact IA and GRS series for the response, namely by isolating and counting powers in either the residual interaction $\overline{V}(1)$ or in $1/q$. However, in view of the fact that the IA series is treated approximately, the exact GRS series becomes the natural standard. Both approaches shall be traced to terms up to, and including $\mathcal{O}(1/q^2)$.

We start with the GRS series (3)

$$\phi(q,y;\mathcal{V})=\sum_{n\geqslant 0}(1/v_q)^nF_n(y;\mathcal{V}).\qquad(22)$$

Using (6) we make explicit the two components of the "full" interaction action in Eq. (6)

$$F_0(y)=\frac{1}{4\pi^2}\int_{|y|}^{\infty}dppn(p),\qquad(23a)$$

$$\begin{aligned}F_1(y)&=-\frac{i}{A!}\int\frac{ds}{2\pi}e^{iys}\int d1\left[\Pi_{k\geqslant 2}^A\int dk\right]\rho_A(1-s,k;1k)\\ &\quad\times\sum_{k\geqslant 2}\int_0^s d\sigma[\mathcal{V}_\sigma^{(1)}(1-k)-\mathcal{V}^{(2)}(1-k)]\\ &=-\frac{1}{A}\int\frac{ds}{2\pi}e^{iys}\int\int d1d2\rho_2(1-s,2;12)\\ &\quad\times\int_0^s d\sigma[\mathcal{V}_\sigma^{(1)}(1-2)-\mathcal{V}^{(2)}(1-2)],\qquad(23b)\end{aligned}$$

$$\begin{aligned}F_2(y)&=\frac{i^2}{A!}\int\frac{ds}{2\pi}e^{isy}\int d1\left[\Pi_{k\geqslant 2}^A\int dk\right]\rho_A(1-s,k;1k)\\ &\quad\times\frac{1}{2}\left[\int_0^s d\sigma\sum_{k\geqslant 2}\int_0^s d\sigma[\mathcal{V}_\sigma^{(1)}(1-k)-\mathcal{V}^{(2)}(1-k)]\right]^2\\ &\quad+F_2^{(r)}(y).\qquad(23c)\end{aligned}$$

Next we expand in parallel the approximate DWIA expression (21a)

$$\begin{aligned}\phi^{\text{TI,DWIA}}&(q,y_0;\mathcal{V}^{(1)})\\ &=\frac{1}{A}\int\frac{ds}{2\pi}e^{iy_0 s}\int d1\left[\rho_1(1-s;1)\right.\\ &\quad-\frac{i}{v_q}\int d2\rho_2(1-s,2;1,2)\int_0^s d\sigma\mathcal{V}_\sigma^{(1)}(1-2)\\ &\quad-\frac{1}{2v_q^2}\int d2\left(\int_0^s d\sigma\mathcal{V}_\sigma^{(1)}(1-2)\right)^2\\ &\quad-\frac{1}{2v_q^2}\int d2\int d3\rho_3(1-s,2,3;1,2,3)\\ &\quad\left.\times\int_0^s d\sigma\mathcal{V}_\sigma^{(1)}(1-2)\int_0^s d\sigma'\mathcal{V}_{\sigma'}^{(1)}(1-3)\right]+\mathcal{O}(1/v_q^3).\end{aligned}\qquad(24)$$

It is then our quest to investigate whether, and to what extent, the terms (23a)–(23c) of the GRS series contain the DWIA counterparts, Eq. (24). We do so as follows:

(i) Separate in Eqs. (23) terms that depend exclusively on $\mathcal{V}^{(1)}$. The former we expect to meet in Eq. (24).

(ii) Track in the remainder of Eqs. (23) parts where $\mathcal{V}^{(2)}$ acts on the ground state in ρ_A. Using Eq. (8) one has

$$\left[\sum_{l\geqslant 2} V(1-k)\right]\Phi(1,k)=[H_A-H_{A-1}-T(1)]\Phi(1,k)$$

$$\approx-\int\frac{d\boldsymbol{p}}{(2\pi)^3}e^{i\boldsymbol{p}\cdot\boldsymbol{r}}\left(\langle\Delta\rangle+\frac{p^2}{2M}\right)\Phi(\boldsymbol{p},k), \tag{25}$$

where, in line with the assumption made above, separation energies are again replaced by an average.

(iii) Collect terms, which enable the replacement of the GRS-West scaling variable by the IA one, making use of

$$y(=y_w)=y_0+\frac{1}{v_q}\left(\frac{y_0^2}{2M}+\langle\Delta\rangle\right). \tag{26}$$

We start with

$$F_1(\mathcal{V})=F_1^{(1)}(\mathcal{V}^{(1)})+F_1^{(2)}(\mathcal{V}^{(2)}), \tag{27}$$

where the superscripts indicate dependence on $\mathcal{V}^{(1)},\mathcal{V}^{(2)}$. Following (i) we consider the part

$$\begin{aligned}\frac{1}{v_q}F_1^{(2)}(y)&=-\frac{i}{A!}\frac{\partial}{\partial y}\int\frac{ds}{2\pi}e^{iys}\int d1\left[\Pi_{k\geqslant 2}\int dk\right]\rho_A\\&\quad\times(1-s,k;1,k)\frac{1}{v_q}\sum_{l\geqslant 2}V(l)\\&=\frac{\partial}{\partial y}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\delta(p_z-y)\frac{1}{v_q}\left(\langle\Delta\rangle+\frac{p^2}{2M}\right)n(p)\\&=(y-y_0)\frac{dF_0(y)}{dy}.\end{aligned} \tag{28}$$

Continuing with F_2 we write [cf. Eqs. (5c), (23c)]

$$F_2(\mathcal{V})=F_2^{(1)}+F_2^{(2)}+F_2^{(1,2)}+F_2^{(r)}, \tag{29}$$

with $F_2^{(1,2)}$ containing mixed $\mathcal{V}^{(1)},\mathcal{V}^{(2)}$ terms. The reasoning which leads to Eq. (28) produces

$$\begin{aligned}\frac{1}{v_q^2}F_2^{(2)}(y)&=\frac{1}{2A!}\frac{\partial^2}{\partial y^2}\int\frac{ds}{2\pi}e^{iys}\int d1\left[\Pi_{l\geqslant 2}\int dl\right]\\&\quad\times\rho_A(1-s,k;1,k)\frac{1}{v_q}\sum_{k\geqslant 2}V(1-k)\frac{1}{v_q}\\&\quad\times\sum_{l\geqslant 2}V(1-l)\\&=\frac{1}{2}\left[\frac{1}{v_q}\left(\langle\Delta\rangle+\frac{y^2}{2M}\right)^2\right]\frac{d^2F_0(y)}{dy^2}\\&=\frac{1}{2}(y-y_0)^2\frac{d^2F_0(y)}{dy^2}.\end{aligned} \tag{30}$$

Finally for the mixed term

$$\begin{aligned}\frac{1}{v_q}F_2^{(1,2)}(y)&=-\frac{1}{A!}\frac{\partial}{\partial y}\int\frac{ds}{2\pi}e^{iys}\int d1\left[\Pi_{l\geqslant 2}\int dl\right]\\&\quad\times\rho_A(1-s,k;1,k)\frac{1}{v_q}\sum_{k\geqslant 2}V(1-k)\frac{1}{v_q}\\&\quad\times\sum_{l\geqslant 2}\int_0^s d\sigma V(1-l-\sigma)]\\&=\frac{1}{v_q}\left(\frac{y^2}{2M}+\langle\Delta\rangle\right)\frac{dF_1^{(1)}(y)}{dy}=(y-y_0)\frac{dF_1^{(1)}(y)}{dy}.\end{aligned} \tag{31}$$

Assembling the last three results and using Eq. (26) one finds

$$\phi(q,y;\mathcal{V})=F_0(y)+\frac{1}{v_q}F_1(y;\mathcal{V})+\frac{1}{v_q^2}F_2(y;\mathcal{V})+\mathcal{O}(1/v_q^3) \tag{32a}$$

$$\begin{aligned}&=\left[F_0(y)+(y-y_0)\frac{dF_0(y)}{dy}+\frac{1}{2}(y-y_0)^2\frac{d^2F_0(y)}{dy^2}\right]\\&\quad+\frac{1}{v_q}\left[F_1^{(1)}(y)+(y-y_0)\frac{dF_1^{(1)}(y)}{dy}\right]+\frac{1}{v_q^2}[F_2^{(1)}(y)]\\&\quad+\frac{1}{v_q^2}F_2^{(r)}(y)+\mathcal{O}(1/v_q^3)\end{aligned} \tag{32b}$$

$$\begin{aligned}&=\left[F_0(y_0)+\frac{1}{v_q}F_1^{(1)}(y_0)+\frac{1}{v_q^2}F_2^{(1)}(y_0)\right]+\frac{1}{v_q^2}F_2^{(r)}(y)\\&\quad+\mathcal{O}(1/v_q^3),\end{aligned} \tag{32c}$$

and thus

$$\phi(q,y;\mathcal{V})=\phi(q,y_0;\mathcal{V}^{(1)})+\frac{1}{v_q^2}F_2^{(r)}(y;\mathcal{V})+\mathcal{O}(1/v_q^3), \tag{33}$$

where, as in Eq. (32a), we reinstated for greater clarity the dependence on $\mathcal{V}$ and its component $\mathcal{V}^{(1)}$.

Equation (33) is our final result. It makes explicit that all terms of the IA series up to and including $\mathcal{O}(1/v_q^2)$ are contained in the GRS series of the same order, which however, has one additional term, not reproduced in the DWIA. This can be traced to the approximation (14). Higher order eikonal terms [13] to the distortion function ξ in Eq. (15) are at least of order $1/v_q^2$ and are expected to account for the above difference to that order.

Equations (28), (30), and (31) are truly remarkable. Those show that the $\mathcal{V}^{(2)}$ dependence of coefficient functions $(1/v_q)^nF_n(\mathcal{V}^{(1)}+\mathcal{V}^{(2)})$ of order n can be expressed in terms of mth order derivatives of $(1/v_q)^{n-m}F_{n-m}(\mathcal{V}^{(1)})$ of lower order $n-m$, and which are free of the component $\mathcal{V}^{(2)}$. Properly grouped terms ultimately produce the replacement $y\to y_0$ and bring about significant cancellations in the GRS series.

The above completes the equivalence proof of the two expressions of the structure function for any NR many-body system. Were it not for the use of average separation energies, Eq. (33) would be exact [14]. In a way this approximation is unavoidable, because the appearance of an essentially kinematic IA scaling variable y_0 as in Eq. (15) requires an average separation energy. An earlier attempt to keep actual separation energies invites other approximations (cf. Eqs. (2.23) and following in Ref. [14]), but we shall not pursue that extension here.

We conclude this section by mentioning previous incorporations of FSI interactions in the IA series. In particular Benhar and co-workers [2,15] have advocated a convolution of the PWIA spectral function and an expression for the FSI in the energy loss variable ω, and not in y. Otherwise their FSI correctly features only the component $\mathcal{V}^{(1)}$ as in Eq. (21a).

An expression for a structure function as a convolution of the lowest order asymptotic limit and a FSI factor has been *proved* for the GRS theory and the appropriate variable is the GRS-West scaling variable y and not ω [8]. A presumed generalization, valid for the IA series, certainly requires a parallel proof, which to our knowledge has not been provided. Such a proof would select the convolution variable.

Let us put aside the *ad hoc* convolution in Refs. [2,15] and attempt to replace ω by a scaling variable. That is possible for any candidate, built from purely kinematic variables as is y_w, Eq. (2). The result is clearly neither Eq. (8) nor Eq. (21a): The latter manifestly requires the IA variable Eq. (26), which in principle is ruled out, because it assumes the existence of an average separation energy. This is foreign to the IA approach in terms of the exact spectral function with a state-dependent Δ_m. The latter is just demonstrated by Eq. (21a), which may be written as a convolution (21b) in the ''natural'' IA variable y_0.

Finally we mention a conference report (currently not published) [16] which also uses the above folding procedure. A discussion should await the publication of a complete account.

V. SUMMARY AND CONCLUSION

The major goal in of this paper was a comparison of the GRS and IA approaches to calculating the structure function of a NR system of point-particles. Whereas for the former there exists a formally exact expression, we do not know of a manageable expression for the IA series. Any comparison therefore requires an approximation for this second series. Such an approximation was developed in Sec. III and the comparison done in Sec. IV.

Our demonstration starts with of the GRS series up to and including $\mathcal{O}(1/v_q^2)$, with coefficient functions of the GRS-West scaling variable y. We then proved striking cancelations, producing the same lowest order terms from the DWIA expressed in the parallel IA scaling variable y_0. The unretrieved term in the GRS series is undoubtedly due to the DWIA we chose to approximate the FSI within the actual IA series. Thus there is a very close connection between the GRS and IA approaches.

The above success naturally elicits the question of a relativistic extension. There clearly is no hope to derive results with comparable rigor. It is nevertheless of interest to recall here some models where nuclear and nucleon structure functions are related by a generalized convolution [17]

$$F^A = f^{PN} * F_N,$$

with $f^{PN} \propto \phi$. Here ϕ is in principle the structure function of a nucleus, composed of point-particles, where internucleon potentials as in Eq. (7) are replaced by scattering amplitudes which also have meaning in a relativistic theory. We refer to Ref. [3] where a generalization of the effectively 2-component interaction in the above spirit is discussed.

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