Pion scattering from polarized ^{3}He

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The role of the three-body wave function in the scattering of pions from polarized ^{3}He is studied. It is found that for the case of the asymmetry from π^+ scattering, the inclusion of the D state can be very important, changing the asymmetry calculated by as much as a factor 2–3. The π^- scattering asymmetry is altered very little. [S0556-2813(99)04806-2]

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I. INTRODUCTION

Recent measurements of the asymmetry of scattered positive pions from polarized ^{3}He [1] indicate a considerable disagreement from the calculations using multiple scattering techniques [2] including only the S state of the three-body system. With only the S state present all of the nuclear spin is carried by the odd particle, the neutron in this case. Since the amplitude for elastic pion scattering from the neutron is (approximately) three times smaller at resonance than that for the proton and, since there are two protons, one might suspect that the asymmetry could well be sensitive to the smaller components of the three-body wave function which allow the proton to carry the spin. For the charge form factor there is no interference term between the S and D components but for the magnetic case there is an overlap. Thus asymmetry measurements provide an appropriate tool to observe the interference of the large and small components. For these reasons we study here the effect of the inclusion of the interference of the two principal small components, the S' and the D states.

Seminal work on this problem was done by Landau [3] some years ago. He used the same formalism as had been used for electron scattering, assuming that the same form factors would be valid for pion scattering, and found good agreement with the then-existing data. One of the interests in studying pion scattering from the three-nucleon system is to attempt to distinguish scattering from nucleons from scattering from extra-nucleonic components of the nucleus. Thus we treat the three-body system from first principles and ask what reasonable form factors are required to fit the data. It is not claimed that a final result will be presented here, it is only a first step. Other recent work has been done by Kamalov *et al.* [4].

While Faddeev calculations are capable of making predictions of the wave functions needed based on two- and three-body nucleon-based potentials, we proceed here with a phenomenological form based on general considerations. For the study of pion scattering from the three-nucleon system we employ the distorted wave impulse approximation (DWIA) which reduces the problem to the scattering from one of the nucleons with the ability to include the distortion of the incident and final waves by the other two nucleons.

The operator for the spin-dependent interaction of a π^+ with ^{3}He is given by

$$\mathcal{O}=\sum_{i=1,3}\sigma_i^z(g_0+g_1\tau_i^z)e^{i\mathbf{q}\cdot\mathbf{r}_i}$$

$$=\sum \sigma_i e^{i\mathbf{q}\cdot\mathbf{r}_i}\left[g_p\left(\frac{1+\tau_i^z}{2}\right)+g_n\left(\frac{1-\tau_i^z}{2}\right)\right], \tag{1}$$

where $g_0+g_1=g_p$ is the proton strength, $g_0-g_1=g_n$ is the neutron strength, and $\mathbf{q}$ is the difference between the incident and final wave vectors $\mathbf{q}=\mathbf{k}-\mathbf{k}'$. For the distorted wave version it is only necessary to replace the plane wave

$$e^{i\mathbf{q}\cdot\mathbf{r}_i}\rightarrow\psi^{(-)}(\mathbf{k}',\mathbf{r}_i)\psi^{(+)}(\mathbf{k},\mathbf{r}_i),$$

where the ψ are the pion waves distorted by the interaction with the other nucleons. Since bilinear products of the wave function are symmetric, we need take only one term of the operator and multiply by a factor of 3. To carry out this procedure it is necessary to have a characterization of the different states of this system.

We mention here a problem that exists in pion scattering from the three-nucleon system at higher momentum transfers. The large-angle measurements have revealed that the cross section rises at back angles. The shape looks very much like a form-factor effect but, while it is possible to find a density which will fit this shape, it does not seem to be reasonable physically. While this is an interesting problem and is under study, we consider it to be beyond the scope of the present work.

We will calculate the matrix element of the operator given in Eq. (1) between the initial and final ground states of ^{3}He using distorted waves for the pion. Section II reviews the group theoretic basis for the wave functions. In Sec. III the expressions are derived for the overlap of the dominant and small-component wave functions. Section IV presents the approximate forms for the controlling functions. Section V develops the formalism used for the distorted waves and Secs. VI and VII give the principal results and some conclusions.

II. CHARACTERIZATION OF THE PRINCIPAL STATES

The basic problem in the presentation of states of the three-nucleon system is that of satisfying certain fundamental principles in hadronic physics, namely, (1) conservation of total angular momentum, (2) conservation of isospin, and (3) the Pauli principle. This general procedure has been

TABLE I. Permutation table showing the action of the permutation operators on the symmetric, antisymmetric, and mixed states. Note that the subscript $\bar{s}$ (and not a) denotes antisymmetric.

	P_{23}	P_{12}	P_{13}
ψ_s	ψ_s	ψ_s	ψ_s
$\psi_{\bar{s}}$	$-\psi_{\bar{s}}$	$-\psi_{\bar{s}}$	$-\psi_{\bar{s}}$
ψ_a	ψ_a	$\frac{1}{2}(\sqrt{3}\psi_b-\psi_a)$	$-\frac{1}{2}(\sqrt{3}\psi_b+\psi_a)$
ψ_b	$-\psi_b$	$\frac{1}{2}(\psi_b+\sqrt{3}\psi_a)$	$\frac{1}{2}(\psi_b-\sqrt{3}\psi_a)$

known for some time [5–8]. In the present work we follow, to a large extent, the development of Ref. [9].

In order to treat the first requirement we must discuss the possible functions of angular momentum and spin. Since the spatial coordinates of the three-body system (center of mass motion removed) can be reduced to two vectors, the only total orbital angular momentum functions possible from a bilinear combination can be written as [10]

$$T_L^M(\mathbf{r},\mathbf{r}')=\sqrt{\frac{5}{6}}rr'\sum C_{1,1,L}^{m,m',M}Y_1^m(\mathbf{r})Y_1^{m'}(\mathbf{r}'). \quad (2)$$

These basis functions of the rotation group of representation L can then be combined with spin functions to obtain total angular momentum functions. Since L can take on the values of 0, 1, and 2 the states are classified as S, P, and D states. As has been known for a long time there is more than one possibility for each angular momentum. We shall attempt to choose only the most important among these states in order to be able to extract some information about the system.

For the three-nucleon system we must deal with states of the permutation group. While the overall state must be purely antisymmetric to satisfy the Pauli principle, it must be built out of states of mixed symmetry. For three bodies there exist symmetric states, antisymmetric states, and states of mixed symmetry. For spin and isospin there exist only the symmetric and mixed states. Since there are only two values of the variables, the antisymmetric state vanishes identically. Under the exchange of the coordinates the states follow the rule given in Table I.

From fundamental basis functions we can often build more complex functions with desired characteristics. Given two functions (say $\{\psi_a,\psi_b\}$ and $\{\eta_a,\eta_b\}$) which satisfy Table I we can always generate another set by the relations

$$\xi_s=\frac{1}{\sqrt{2}}(\psi_b\eta_b+\psi_a\eta_a), \quad \xi_{\bar{s}}=\frac{1}{\sqrt{2}}(\psi_a\eta_b-\psi_b\eta_a),$$

$$\xi_a=\frac{1}{\sqrt{2}}(\psi_b\eta_b-\psi_a\eta_a), \quad \xi_b=\frac{1}{\sqrt{2}}(\psi_a\eta_b+\psi_b\eta_a). \quad (3)$$

In this way we will generate the fully antisymmetric states needed from mixed symmetry states of spin, space, and isospin.

The possible (orthonormal) states for either spin or isospin projection $+\frac{1}{2}$ are given by

$$\left|\frac{1}{2}s\right\rangle=\frac{1}{\sqrt{3}}(|\downarrow\uparrow\uparrow\rangle+|\uparrow\downarrow\uparrow\rangle+|\uparrow\uparrow\downarrow\rangle), \quad (4)$$

$$\left|\frac{1}{2}a\right\rangle=\frac{1}{\sqrt{6}}(2|\downarrow\uparrow\uparrow\rangle-|\uparrow\downarrow\uparrow\rangle-|\uparrow\uparrow\downarrow\rangle),$$

$$\left|\frac{1}{2}b\right\rangle=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle-|\uparrow\uparrow\downarrow\rangle). \quad (5)$$

The symmetric state belongs to total spin $\frac{3}{2}$ and the mixed states to total spin $\frac{1}{2}$. Spin (isospin) projections of $-\frac{1}{2}$ are obtained by reversing the direction of all arrows.

The projection operators for proton and neutron (or spin-up and spin-down) are given by

$$P_p\equiv\frac{1}{2}(1+\tau_1^z), \quad P_n\equiv\frac{1}{2}(1-\tau_1^z). \quad (6)$$

The effect of the projection operators on these states is

$$P_p|a\rangle=\frac{1}{3}(|a\rangle-\sqrt{2}|s\rangle), \quad P_n|a\rangle=\frac{1}{3}(2|a\rangle+\sqrt{2}|s\rangle),$$

$$P_p|b\rangle=|b\rangle, \quad P_n|b\rangle=0,$$

$$P_p|s\rangle=\frac{1}{3}(-\sqrt{2}|a\rangle+2|s\rangle), \quad P_n|s\rangle=\frac{1}{3}(\sqrt{2}|a\rangle+|s\rangle). \quad (7)$$

We distinguish between the spin and isospin spinors with the subscript S or I. From these states we can construct a set of spin-isospin states

$$|s\rangle_{SI}=\frac{1}{\sqrt{2}}(|b\rangle_S|b\rangle_I+|a\rangle_S|a\rangle_I),$$

$$|\bar{s}\rangle_{SI}=\frac{1}{\sqrt{2}}(|a\rangle_S|b\rangle_I-|b\rangle_S|a\rangle_I),$$

$$|a\rangle_{SI}=\frac{1}{\sqrt{2}}(|b\rangle_S|b\rangle_I-|a\rangle_S|a\rangle_I),$$

$$|b\rangle_{SI}=\frac{1}{\sqrt{2}}(|a\rangle_S|b\rangle_I+|b\rangle_S|a\rangle_I). \quad (8)$$

Turning to the spatial functions, we take $\mathbf{r}_a$ and $\mathbf{r}_b$ to be the Jacoby variables defined by

$$\mathbf{r}_a=\frac{1}{\sqrt{3}}(2\mathbf{r}_1-\mathbf{r}_2-\mathbf{r}_3), \quad \mathbf{r}_b=\mathbf{r}_2-\mathbf{r}_3, \quad \mathbf{R}=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3. \quad (9)$$

Note that

$$r_a^2+r_b^2=\frac{2}{3}(r_{12}^2+r_{23}^2+r_{13}^2) \quad (10)$$

and

$$\mathbf{r}_1=\frac{\mathbf{R}}{3}+\frac{\mathbf{r}_a}{\sqrt{3}},\quad \mathbf{r}_2=\frac{\mathbf{R}}{3}-\frac{\mathbf{r}_a}{2\sqrt{3}}+\frac{\mathbf{r}_b}{2},\quad \mathbf{r}_3=\frac{\mathbf{R}}{3}-\frac{\mathbf{r}_a}{2\sqrt{3}}-\frac{\mathbf{r}_b}{2}. \tag{11}$$

With this definition the variables $\mathbf{r}_a$ and $\mathbf{r}_b$ themselves satisfy Table I.

A. *S* state wave function

From Eq. (2) with $L=0$ we find, setting, in turn, $\mathbf{r}=\mathbf{r}_a$, $\mathbf{r}'=\mathbf{r}_a$; $\mathbf{r}=\mathbf{r}_b$, $\mathbf{r}'=\mathbf{r}_b$; and $\mathbf{r}=\mathbf{r}_a$, $\mathbf{r}'=\mathbf{r}_b$, three functions which are scalars under rotation in space and which transform according to the permutation group

$$\chi_s=r_a^2+r_b^2,\quad \chi_a=r_b^2-r_a^2,\quad \chi_b=2\mathbf{r}_a\cdot\mathbf{r}_b. \tag{12}$$

Of course, any function of $r_a^2+r_b^2$ will be a symmetric function as well. The least complex spatial function usually provides the largest component of the ground state. We will choose the simplest functions in each category.

The minimal *S*-state wave function is given by the product of a symmetric function in space multiplied by an antisymmetric spin-isospin wave function

$$|S\rangle=\frac{1}{4\pi}S(t_{ab})|\overline{s}\rangle_{SI}, \tag{13}$$

where we have used the shorthand notation $t_{ab}=\sqrt{r_a^2+r_b^2}$. Since the s wave density is given by $S^2(t_{ab})/(4\pi)^2$ the normalization condition is

$$1=\langle S|S\rangle=\int r_a^2r_b^2dr_adr_bS^2(t_{ab}). \tag{14}$$

If we make the Irving transform [11]

$$r_a=t\cos\theta,\quad r_b=t\sin\theta$$

and use the result

$$\int_0^{\pi/2}\cos^m\theta\sin^n\theta d\theta=\frac{\Gamma[(m+1)/2]\Gamma[(n+1)/2]}{2[(m+n)/2]!}$$

for m and n even, the normalization condition becomes

$$1=\int t^5dt\ \cos^2\theta\sin^2\theta d\theta S^2(t)=\frac{\pi}{16}\int t^5dtS^2(t). \tag{15}$$

The density of any one particle will be given by

$$\rho_s(\mathbf{r})=\frac{\rho_s(r)}{4\pi}=\frac{1}{(4\pi)^2}\int d\mathbf{r}_ad\mathbf{r}_b\delta(\mathbf{r}-\mathbf{r}_a/\sqrt{3})S^2(t_{ab}), \tag{16}$$

$$\rho_s(r)=3\sqrt{3}\int_0^\infty r_b^2dr_bS^2(\sqrt{r_b^2+3r^2}), \tag{17}$$

where the normalization $\int_0^\infty r^2dr\rho_s(r)=1$ holds. The wave functions and densities for protons and neutrons with spin up and spin down are given by

$$p\uparrow\ \frac{1}{3}(|\overline{s}\rangle_{SI}+|b\rangle_S|s\rangle_I-|s\rangle_S|b\rangle_I)\frac{S(\mathbf{r}_a,\mathbf{r}_b)}{4\pi},\quad \frac{1}{3}\frac{S^2(\mathbf{r}_a,\mathbf{r}_b)}{(4\pi)^2},$$

$$p\downarrow\ \frac{1}{3}(|\overline{s}\rangle_{SI}+|b\rangle_{SI}+|s\rangle_S|b\rangle_I)\frac{S(\mathbf{r}_a,\mathbf{r}_b)}{4\pi},\quad \frac{1}{3}\frac{S^2(\mathbf{r}_a,\mathbf{r}_b)}{(4\pi)^2},$$

$$n\uparrow-\frac{1}{3}(|\overline{s}\rangle_{SI}-|b\rangle_{SI}+|b\rangle_S|s\rangle_I)\frac{S(\mathbf{r}_a,\mathbf{r}_b)}{4\pi},\quad \frac{1}{3}\frac{S^2(\mathbf{r}_a,\mathbf{r}_b)}{(4\pi)^2},$$

$$n\downarrow\ \ 0. \tag{18}$$

B. *S′* wave function

Since the wave function just described is the dominant one, it is necessary to go to one more level in complexity to include the next order. From the fundamental mixed representation in space we can construct more general spatial functions of mixed representation to be combined with the mixed spin-isospin representations:

$$S_a(\mathbf{r}_a,\mathbf{r}_b)\equiv\frac{U(t_{ab})}{4\pi}\chi_a(\mathbf{r}_a,\mathbf{r}_b),$$

$$S_b(\mathbf{r}_a,\mathbf{r}_b)\equiv\frac{U(t_{ab})}{4\pi}\chi_b(\mathbf{r}_a,\mathbf{r}_b). \tag{19}$$

Of course more complicated functions can be constructed using Eqs. (3)

$$\chi_a^{(1)}=\frac{1}{\sqrt{2}}(\chi_b^2-\chi_a^2),\quad \chi_b^{(1)}=\sqrt{2}\chi_a\chi_b,$$

$$\chi_a^{(2)}=\frac{1}{\sqrt{2}}(\chi_b\chi_b^{(1)}-\chi_a\chi_a^{(1)}),\quad \chi_b^{(2)}=\frac{1}{\sqrt{2}}(\chi_a\chi_b^{(1)}+\chi_b\chi_a^{(1)}),$$

$$\chi_a^{(3)}=\frac{1}{\sqrt{2}}(\chi_b\chi_b^{(2)}-\chi_a\chi_a^{(2)}),\quad \chi_b^{(3)}=\frac{1}{\sqrt{2}}(\chi_a\chi_b^{(2)}+\chi_b\chi_a^{(2)}),$$

$$\text{etc.} \tag{20}$$

Using these mixed representation of the spatial wave functions we can write

$$|S'\rangle=\frac{1}{4\pi}[S_a(\mathbf{r}_a,\mathbf{r}_b)|b\rangle_{SI}-S_b(\mathbf{r}_a,\mathbf{r}_b)|a\rangle_{SI}] \tag{21}$$

with the normalization

$$\begin{aligned}1&=\langle S'|S'\rangle\\&=\frac{1}{(4\pi)^2}\int U(t_{ab})^2[(r_b^2-r_a^2)^2+4(\mathbf{r}_a\cdot\mathbf{r}_b)^2]d\mathbf{r}_ad\mathbf{r}_b\\&=\int U^2(t_{ab})\left[(r_b^2-r_a^2)^2+\frac{4}{3}r_a^2r_b^2\right]r_a^2r_b^2dr_adr_b.\end{aligned} \tag{22}$$

Using the Irving transform again we find

$$1=\frac{\pi}{32}\int t^9 dt S'^2(t). \tag{23}$$

Similar to the S state, the density of one particle is given by

$$\rho_{s'}(r)=3\sqrt{3}\int_0^\infty r_b^2 dr_b U^2(\sqrt{r_b^2+3r^2})[(r_b^2-3r^2)^2+4r^2r_b^2]. \tag{24}$$

The wave functions and densities of the particles are given by

$$p\uparrow \frac{1}{9\sqrt{2}}\{3S_a[(|a\rangle_I-\sqrt{2}|s\rangle_I)|b\rangle_S+(|a\rangle_S-\sqrt{2}|s\rangle_S)|b\rangle_I] \\ -S_b[9|b\rangle_S|b\rangle_I-(|a\rangle_S-\sqrt{2}|s\rangle_S)(|a\rangle_I-\sqrt{2}|s\rangle_I)]\}, \\ \frac{1}{3}S_a^2+\frac{5}{9}S_b^2,$$

$$p\downarrow \frac{1}{9\sqrt{2}}[3S_a(2|a\rangle_S+\sqrt{2}|s\rangle_S)|b\rangle_I \\ +S_b(|a\rangle_I-\sqrt{2}|s\rangle_I)(2|a\rangle_S+\sqrt{2}|s\rangle_S)], \quad \frac{1}{3}S_a^2+\frac{1}{9}S_b^2,$$

$$n\uparrow \frac{1}{9\sqrt{2}}[3S_a(2|a\rangle_I+\sqrt{2}|s\rangle_I)|b\rangle_S \\ +S_b(2|a\rangle_I+\sqrt{2}|s\rangle_I)(|a\rangle_S-\sqrt{2}|s\rangle_S)], \quad \frac{1}{3}S_a^2+\frac{1}{9}S_b^2,$$

$$n\downarrow \frac{1}{9\sqrt{2}}S_b(2|a\rangle_I+\sqrt{2}|s\rangle_I)(2|a\rangle_S+\sqrt{2}|s\rangle_S) \quad \frac{2}{9}S_b^2. \tag{25}$$

C. *P*-state wave function

For the P state, the coupling to unity in Eq. (2) yields only one (antisymmetric) state,

$$|P\rangle=\sum C^{M,\mu,1/2}_{1,1/2,1/2}T_1^M(\mathbf{r}_a,\mathbf{r}_b)|s\mu\rangle_{SI}, \tag{26}$$

where μ represents the spin projection, the isospin projection being held fixed. A quartet P state can be constructed by using the $\frac{3}{2}$ spin function, which is always symmetric, and the symmetric isospin wave function. The probability of this state is thought to be very small. In any case we will not consider the P states further since their overlap with the S state with the operator σ_z is zero so that there is no interference term.

D. *D*-state wave function

Since the D state must be coupled to spin $\frac{3}{2}$ to construct the spin $\frac{1}{2}$ ^{3}He-T system, and since the spin $\frac{3}{2}$ wave function is purely symmetric, we must use a spatial mixed representation to form an antisymmetric wave function. For the D state Eq. (2) becomes

$$T_2^M(\mathbf{r}_a,\mathbf{r}_b)\equiv\sqrt{\frac{5}{6}}r_a r_b\sum C^{m,m',M}_{1,1,2}Y_1^m(\hat{\mathbf{r}}_a)Y_1^{m'}(\hat{\mathbf{r}}_b). \tag{27}$$

Note that

$$T_2^M(\mathbf{r},\mathbf{r})=\frac{r^2}{\sqrt{4\pi}}Y_2^M(\hat{\mathbf{r}}). \tag{28}$$

We can now define functions of the correct symmetry for the spatial part of the three-body wave function

$$\phi_s^M(\mathbf{r}_a,\mathbf{r}_b)=\frac{1}{2}[T_2^M(\mathbf{r}_b,\mathbf{r}_b)+T_2^M(\mathbf{r}_a,\mathbf{r}_a)], \tag{29}$$

$$\phi_a^M(\mathbf{r}_a,\mathbf{r}_b)=\frac{1}{2}[T_2^M(\mathbf{r}_b,\mathbf{r}_b)-T_2^M(\mathbf{r}_a,\mathbf{r}_a)];$$

$$\phi_b^M(\mathbf{r}_a,\mathbf{r}_b)=T_2^M(\mathbf{r}_a,\mathbf{r}_b). \tag{30}$$

Their angular integrals are given by

$$\int d\Omega_a d\Omega_b \phi_s^{*M'}(\mathbf{r}_a,\mathbf{r}_b)\phi_s^M(\mathbf{r}_a,\mathbf{r}_b) \\ =\int d\Omega_a d\Omega_b \phi_a^{*M'}(\mathbf{r}_a,\mathbf{r}_b)\phi_a^M(\mathbf{r}_a,\mathbf{r}_b)=\delta_{MM'}\frac{r_a^4+r_b^4}{4} \tag{31}$$

and

$$\int d\Omega_a d\Omega_b \phi_b^{*M'}(\mathbf{r}_a,\mathbf{r}_b)\phi_b^M(\mathbf{r}_a,\mathbf{r}_b)=\delta_{MM'}\frac{5}{6}r_a^2r_b^2. \tag{32}$$

Since $\mathbf{r}_a$ and $\mathbf{r}_b$ are variables which satisfy the permutation group (Table I) the reader can easily verify that the ϕ's do as well.

Three minimal D-state wave functions can be constructed from the symmetry functions given above as

$$|D_\alpha\rangle=\frac{1}{\sqrt{2}}D_\alpha(t_{ab})\sum[\phi_a^M(\mathbf{r}_a,\mathbf{r}_b)|b\rangle_I \\ -\phi_b^M(\mathbf{r}_a,\mathbf{r}_b)|a\rangle_I]C^{M,\mu,1/2}_{2,3/2,1/2}\left|\frac{3}{2}\mu\right\rangle, \tag{33}$$

$$|D_\beta\rangle=\frac{1}{\sqrt{2}}D_\beta(t_{ab})\sum\{[\chi_b\phi_a^M(\mathbf{r}_a,\mathbf{r}_b) \\ +\chi_a\phi_b^M(\mathbf{r}_a,\mathbf{r}_b)]|a\rangle_I-[\chi_b\phi_b^M(\mathbf{r}_a,\mathbf{r}_b) \\ -\chi_a\phi_a^M(\mathbf{r}_a,\mathbf{r}_b)]|b\rangle_I\}C^{M,\mu,1/2}_{2,3/2,1/2}\left|\frac{3}{2}\mu\right\rangle, \tag{34}$$

and

$$|D_\gamma\rangle=\frac{1}{\sqrt{2}}D_\gamma(t_{ab})(\chi_b|a\rangle_I-\chi_a|b\rangle_I)\times\sum C^{M,\mu,1/2}_{2,3/2,1/2}\phi^M_s(\mathbf{r}_a,\mathbf{r}_b)\left|\frac{3}{2}\mu\right\rangle. \tag{35}$$

The normalization of the three symmetric functions is taken so that

$$\int d\mathbf{r}_a d\mathbf{r}_b D^2_\alpha(t_{ab})|\phi^M_a|^2=\int d\mathbf{r}_a d\mathbf{r}_b D^2_\alpha(t_{ab})|\phi^M_b|^2=1, \tag{36}$$

$$\int d\mathbf{r}_a d\mathbf{r}_b D^2_\beta(t_{ab})|\chi_b\phi^M_a+\chi_a\phi^M_b|^2=\int d\mathbf{r}_a d\mathbf{r}_b D^2_\beta(t_{ab})|\chi_b\phi^M_b-\chi_a\phi^M_a|^2=1, \tag{37}$$

$$\int d\mathbf{r}_a d\mathbf{r}_b D^2_\gamma(t_{ab})|\chi_b\phi^M_s|^2=\int d\mathbf{r}_a d\mathbf{r}_b D^2_\gamma(t_{ab})|\chi_a\phi^M_s|^2=1. \tag{38}$$

The complete D state can be written as

$$|D\rangle=\frac{1}{\sqrt{2}}\sum\{[-\alpha D_\alpha\phi^M_b+\beta D_\beta(t_{ab})(\chi_b\phi^M_a+\chi_a\phi^M_b)+\gamma D_\gamma\chi_b\phi^M_s]|a\rangle_I,+[\alpha D_\alpha\phi^M_a-\beta D_\beta(\chi_b\phi^M_b-\chi_a\phi^M_a)-\gamma D_\gamma\chi_a\phi^M_s]|b\rangle_I\}C^{M,\mu,1/2}_{2,3/2,1/2}\left|\frac{3}{2}\mu\right\rangle. \tag{39}$$

The β D state is orthogonal to the other two but the α and γ states are not orthogonal to each other. We can construct a set of orthogonal states by defining the combinations

$$A^M_\pm=-(D_\alpha\phi^M_b\pm D_\gamma\chi_b\phi^M_s)N_\pm,$$

$$A^M_0=\frac{1}{\sqrt{2}}D_\beta(\chi_b\phi^M_a+\chi_a\phi^M_b), \tag{40}$$

$$B^M_\pm=(D_\alpha\phi^M_a\pm D_\gamma\chi_a\phi^M_s)N_\pm,$$

$$B^M_0=-\frac{1}{\sqrt{2}}D_\beta(\chi_b\phi^M_b-\chi_a\phi^M_a), \tag{41}$$

with the normalization factors chosen so that

$$\int d\mathbf{r}_a d\mathbf{r}_b|A_\pm|^2=\int d\mathbf{r}_a d\mathbf{r}_b|A_0|^2=\int d\mathbf{r}_a d\mathbf{r}_b|B_\pm|^2=\int d\mathbf{r}_a d\mathbf{r}_b|B_0|^2=1. \tag{42}$$

Then, with

$$C_+=\frac{\alpha+\gamma}{2N_+},\quad C_-=\frac{\alpha-\gamma}{2N_-},\quad C^0=\beta, \tag{43}$$

the wave function can be written as

$$|D\rangle=\frac{1}{\sqrt{2}}\sum[(C_+A^M_++C_-A^M_-+C_0A^M_0)|a\rangle_I+(C_+B^M_++C_-B^M_-+C_0B^M_0)|b\rangle_I]C^{M,\mu,1/2}_{2,3/2,1/2}\left|\frac{3}{2}\mu\right\rangle. \tag{44}$$

The β and γ D states have a more complicated internal structure than the D_α state presented above. The S' state is believed to have a probability of less than 2% and the P state less than 1% so it is reasonable to believe that the D_β and D_γ states are also smaller than the D_α state. Gibson found that their q dependence for small values started with a higher power [8].

In the present work we restrict ourselves to the α state alone. This choice gives a rough approximation to the Faddeev results. We now give the formulas for the normalization and densities for this state. The density of the D_α wave will be given by

$$D^2_\alpha(t_{ab})\sum(C^{M,\mu,1/2}_{2,3/2,1/2})^2[|\phi^M_a(\mathbf{r}_a,\mathbf{r}_b)|^2+|\phi^M_b(\mathbf{r}_a,\mathbf{r}_b)|^2] \tag{45}$$

and the normalization

$$1=\langle D_\alpha|D_\alpha\rangle=\int r_a^2r_b^2dr_adr_bD^2_\alpha(t_{ab})\left[\frac{r_a^4+r_b^4}{4}+\frac{5}{6}r_a^2r_b^2\right] \tag{46}$$

$$=\int r_a^2r_b^2dr_adr_bD^2_\alpha(t_{ab})\left[\frac{r_a^4}{2}+\frac{5}{6}r_a^2r_b^2\right]=\frac{1}{2}\int r_a^6r_b^2dr_adr_bD^2_\alpha(t_{ab})+\frac{5}{6}\int r_a^4r_b^4dr_adr_bD^2_\alpha(t_{ab}). \tag{47}$$

Using the Irving transform we can rewrite Eq. (47) as

$$1=\frac{1}{2}\int t^9\cos^6\theta\sin^2\theta dtd\theta D^2(t)+\frac{5}{6}\int t^9\cos^4\theta\sin^4\theta dtd\theta D^2_\alpha(t) \tag{48}$$

$$=\frac{1}{2}\frac{5\pi}{256}\int_0^\infty t^9D^2_\alpha(t)dt+\frac{3\pi}{256}\frac{5}{6}\int_0^\infty t^9D^2_\alpha(t)dt \tag{49}$$

$$=\frac{1}{2}\frac{5\pi}{256}\int_0^\infty t^9D^2_\alpha(t)dt+\frac{5\pi}{256}\frac{1}{2}\int_0^\infty t^9D^2_\alpha(t)dt=\frac{5\pi}{256}\int_0^\infty t^9D^2_\alpha(t)dt. \tag{50}$$

The square of the a term of the density integrated on angle gives

$$\rho_a(r_a,r_b)=\frac{5}{6}D_\alpha^2(t_{ab})r_a^2r_b^2 \tag{51}$$

while the b term gives

$$\rho_b(r_a,r_b)=\frac{1}{4}D_\alpha^2(t_{ab})(r_a^4+r_b^4). \tag{52}$$

A more complete treatment of the D states can be found in Appendix B.

III. OVERLAP BETWEEN THE LARGE AND SMALL COMPONENTS

We will write the wave function as

$$|^3\mathrm{He}\rangle=C_S|S\rangle+C_{S'}|S'\rangle+C_D|D\rangle. \tag{53}$$

In the present work we consider only the α form of the D state.

A. SS' overlap

From Eq. (18), we have

$$\langle S'|p\uparrow|S\rangle=0,\quad \langle S'|p\downarrow|S\rangle=\frac{1}{3}(r_b^2-r_a^2)\frac{S(\mathbf{r}_a,\mathbf{r}_b)U(\mathbf{r}_a,\mathbf{r}_b)}{(4\pi)^2},$$

$$\langle S'|n\downarrow|S\rangle=0,$$

$$\langle S'|n\uparrow|S\rangle=-\frac{1}{3}(r_b^2-r_a^2)\frac{S(\mathbf{r}_a,\mathbf{r}_b)U(\mathbf{r}_a,\mathbf{r}_b)}{(4\pi)^2}. \tag{54}$$

While the density looks the same the sign difference causes the interference with the dominant S state to provide the major contribution to the radius difference between the even and odd nucleon radii in ^{3}He and the triton. The full integral over all coordinates gives orthogonality of the S and S' states but there is a contribution to the rms radius from this interference term.

For the expectation value of the spin operator we have

$$\langle S'|\sigma_1^z|S\rangle=-\frac{2}{3}(r_b^2-r_a^2)\frac{S(\mathbf{r}_a,\mathbf{r}_b)U(\mathbf{r}_a,\mathbf{r}_b)}{(4\pi)^2},$$

while

$$\langle S'|\sigma_1^z\tau_1^z|S\rangle=0. \tag{55}$$

Hence the plane-wave form factor of the overlap is

$$\begin{aligned}2\langle S'|\mathcal{O}|S\rangle&=\frac{-4g_0C_SC_{S'}}{(4\pi)^2}\\&\quad\times\int S(t_{ab})U(t_{ab})(r_b^2-r_a^2)e^{i\mathbf{q}\cdot\mathbf{r}_1}d\mathbf{r}_a d\mathbf{r}_b\\&=-12\sqrt{3}g_0C_SC_{S'}\int S(\sqrt{3r^2+r_b^2})\\&\quad\times U(\sqrt{3r^2+r_b^2})(r_b^2-3r^2)j_0(qr)r^2r_b^2drdr_b.\end{aligned} \tag{56}$$

We now define

$$\rho_{ss'}(r)=3\sqrt{3}\int r_b^2dr_bS(\sqrt{3r^2+r_b^2})U(\sqrt{3r^2+r_b^2})(r_b^2-3r^2), \tag{57}$$

so that

$$\begin{aligned}2\langle S'|\mathcal{O}|S\rangle&=-4g_0C_SC_{S'}\int r^2dr\rho_{ss'}(r)j_0(qr)\\&=-2(g_p+g_n)C_SC_{S'}\int r^2dr\rho_{ss'}(r)j_0(qr).\end{aligned} \tag{58}$$

B. SD overlap

The overlap of the charge form factor for the S and D waves is zero since the quartet and doublet spin states are orthogonal. Including the spin operator

$$\begin{aligned}\langle D_\alpha|p\uparrow|S\rangle&=-\frac{S(t_{ab})D_\alpha(t_{ab})}{4\pi}\phi_a(\mathbf{r}_a,\mathbf{r}_b)C^{0,1/2,1/2}_{2,3/2,1/2}\langle D_\alpha|n\uparrow|S\rangle\\&=0,\\\langle D_\alpha|p\downarrow|S\rangle&=\frac{S(t_{ab})D_\alpha(t_{ab})}{4\pi}\phi_a(\mathbf{r}_a,\mathbf{r}_b)C^{0,1/2,1/2}_{2,3/2,1/2}\langle D_\alpha|n\downarrow|S\rangle\\&=0.\end{aligned} \tag{59}$$

For the SD overlap we find

$$\begin{aligned}\langle S|\sigma_1^z|D\rangle&=\langle S|\tau_1^z\sigma_1^z|D\rangle\\&=-\frac{1}{4\pi}S(t_{ab})D_\alpha(t_{ab})C^{0,1/2,1/2}_{2,3/2,1/2}\frac{2}{3}\phi_a^0(\mathbf{r}_a,\mathbf{r}_b).\end{aligned} \tag{60}$$

Since the two results are equal, only the protons (the like particles) will contribute.

Including the factor of 3 for the three terms in the operator, we have for the contribution from the cross term for the plane-wave result

$$\begin{aligned}2\langle S|\mathcal{O}|D_\alpha\rangle&=-\frac{C_SC_Dg_p}{\pi\sqrt{5}}\int d\mathbf{r}_ad\mathbf{r}_bS(t_{ab})D_\alpha(t_{ab})\\&\quad\times e^{i\mathbf{q}\cdot\mathbf{r}_1}\phi_a^0(\mathbf{r}_a,\mathbf{r}_b).\end{aligned} \tag{61}$$

Using the relation between $\mathbf{r}_1$ and $\mathbf{r}_a$ we see that only one term of $\phi_a^0(\mathbf{r}_a,\mathbf{r}_b)$ contributes

$$2\langle S|\mathcal{O}|D_\alpha\rangle = C_S C_D g_p \frac{2}{4\pi\sqrt{5}}\frac{1}{\sqrt{4\pi}}\int d\mathbf{r}_a d\mathbf{r}_b r_a^2 S(t_{ab}) \times D_\alpha(t_{ab}) e^{i\mathbf{q}\cdot\mathbf{r}_1} Y_2^0(\mathbf{r}_a) \quad (62)$$

$$= C_S C_D g_p \frac{9\sqrt{3}}{4\pi\sqrt{5}}\frac{2}{\sqrt{4\pi}}\int d\mathbf{r} d\mathbf{r}_b r^2 \times S(\sqrt{3r^2+r_b^2}) D_\alpha(\sqrt{3r^2+r_b^2}) \times e^{i\mathbf{q}\cdot\mathbf{r}} Y_2^0(\mathbf{r}). \quad (63)$$

Defining

$$\rho_{SD}(r) \equiv \frac{3\sqrt{3}}{4\pi} r^2 \int d\mathbf{r}_b S(\sqrt{3r^2+r_b^2}) D_\alpha(\sqrt{3r^2+r_b^2}) \quad (64)$$

$$= 3\sqrt{3} r^2 \int_0^\infty r_b^2 dr_b S(\sqrt{3r^2+r_b^2}) D_\alpha(\sqrt{3r^2+r_b^2}), \quad (65)$$

we can write

$$2\langle S|\mathcal{O}|D\rangle = C_S C_D g_p \frac{6}{\sqrt{20\pi}}\int d\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} Y_2^0(\hat{\mathbf{r}}) \rho_{SD}(r)$$

$$= -C_S C_D g_p 6\sqrt{\frac{4\pi}{5}} Y_2^0(\mathbf{q}) \int r^2 dr j_2(qr) \rho_{SD}(r). \quad (66)$$

Thus

$$2\langle S|\mathcal{O}|D\rangle = 3 C_S C_D g_p \int r^2 dr j_2(qr) \rho_{SD}(r). \quad (67)$$

IV. APPROXIMATE FORMS

To obtain an estimate for the overall scalar functions we fit forms to the neutron and proton Faddeev densities. It can be expected that these functions will tend rapidly to zero for small values of the argument because of the hard core repulsion of the nucleon-nucleon interaction. Since small values of the argument t correspond to all three nucleons being close together the suppression can be expected to be very strong. This suppression is supplied by the exponential in the denominator of the expressions below.

The forms used were

$$S(t) = N_S \frac{e^{-a_S t}}{(t^{7/4}+\eta_S)(1+e^{(c_S-t)/b_S})}, \quad (68)$$

$$S'(t) = N_{S'} \frac{e^{-a_{S'} t}}{(t^{15/4}+\eta_{S'})(1+e^{(c_{S'}-t)/b_{S'}})}, \quad (69)$$

and

TABLE II. Parameters for the scalar functions.

	a	b	c	η	N
S	0.55	0.2855	1.7200	2.0	4.036265
S'	1.00	0.4000	3.3378	1.0	52.54929
D	1.2	0.2000	2.26900	2.0	53.4300

$$D(t) = N_D \frac{e^{-a_D t}}{(t^{15/4}+\eta_D)(1+e^{(c_D-t)/b_D})}. \quad (70)$$

From $S(t)$, $S'(t)$, and $D(t)$, the densities of ^{3}He and ^{3}H can be reconstructed and from these densities, the proton and neutron distributions of ^{3}He and ^{3}H and the charge form factors of ^{3}He and ^{3}H can be obtained. The parameters in the $S(t)$, $S'(t)$, and $D(t)$ (shown in Table II) were obtained from fitting the charge form factors of ^{3}He and ^{3}H to Faddeev densities with the additional constraint that the rms radii of ^{3}He and ^{3}H nuclei were required to be consistent with the experimental results. Figure 1 shows plots of these functions.

V. DISTORTED WAVE CALCULATIONS

The pion-^{3}He amplitude can be written as

$$A(\mathbf{k},\mathbf{k}') = F(\mathbf{k},\mathbf{k}') + G(\mathbf{k},\mathbf{k}')\sigma\cdot\hat{\mathbf{k}}\times\hat{\mathbf{k}}'. \quad (71)$$

The spin-independent part, $F(\mathbf{k},\mathbf{k}')$, is calculated by means of a finite range optical model. Although the calculation is done in r space it is finite range and includes the same physics as momentum space models. The techniques used in this case are the same as have been described elsewhere [2,19].

The spin-dependent part, $G(\mathbf{k},\mathbf{k}')$, is calculated using a distorted wave impulse approximation. For the case of the S state the calculation is the same as in Ref. [2]. For the S-S' interference the expressions for the DWIA are the same as for the pure S state, only the distorted waves are different.

For the S-D interference term the expressions are more complicated and will be given here. The distorted waves are

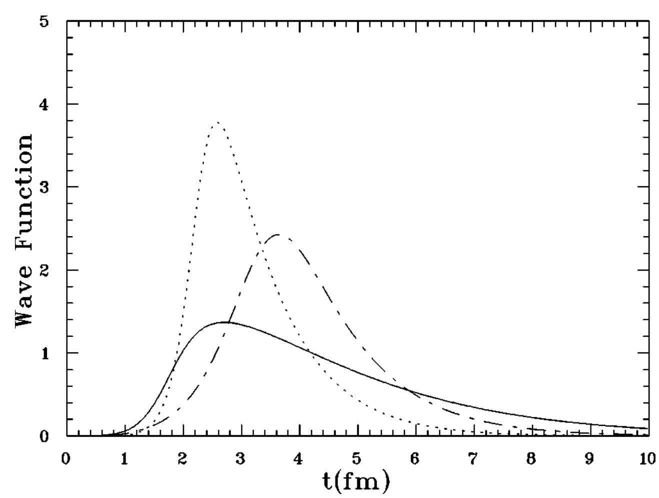

FIG. 1. Overall functions controlling the wave functions. Solid line: $t^{5/2}S(t)$, dotted line: $t^{9/2}D(t)$, dot-dashed line: $t^{9/2}S'(t)$.

calculated as due to the scattering from a neutron-proton pair since the spin is carried by one of the protons.

The full operator (with its off-shell extension) is taken to be proportional to

$$O_G(\mathbf{q},\mathbf{q}')=v(\mathbf{q})\sigma\cdot\mathbf{q}\times\mathbf{q}'v(\mathbf{q}'), \tag{72}$$

where

$$v(\mathbf{q})\equiv\frac{\Lambda^2+k^2}{\Lambda^2+q^2}. \tag{73}$$

Here $\mathbf{q}$ is to be interpreted as $-i\nabla$. The unprimed quantities act on the incoming pion wave function and the primed ones act on the final pion wave function.

The action of $v(\mathbf{q})$ can be calculated on a function $f(r)$ by

$$\tilde{f}(\mathbf{r})\equiv v(-i\nabla)f(\mathbf{r})=\int d\mathbf{r}'d\mathbf{q}e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}f(\mathbf{r}')\frac{\Lambda^2+k^2}{\Lambda^2+q^2}. \tag{74}$$

Using a partial-wave expansion for $f(\mathbf{r})$ and the exponential, the components of $f(\mathbf{r})$ can be transformed each partial wave at a time. With the wave functions from the scattering from the other nucleons denoted by $\psi(\mathbf{k},\mathbf{r})$ the transformed wave functions are $\tilde{\psi}(\mathbf{k},\mathbf{r})$. For the *S*-state contribution the *G* amplitude is generated from scattering from the neutron so the distortion of the incoming and outgoing waves is by the protons. For the *D* state the action is on one of the protons so the appropriate distortion is from a neutron-proton pair. For the S' state there are two cases since the interaction is with a one proton and one neutron.

The *G* amplitude from the *SD* interference will be given by

$$G(\mathbf{k},\mathbf{k}')=3g_pC_SC_D\int d\mathbf{r}\psi(\mathbf{k},\mathbf{r})O_G(\mathbf{q},\mathbf{q}')\psi(\mathbf{k}',\mathbf{r}). \tag{75}$$

Hence it is necessary to evaluate the integral

$$G_{SD}=3g_pC_SC_D\int d\mathbf{r}[\nabla\tilde{\psi}(\mathbf{k},\mathbf{r})]\times[\nabla\tilde{\psi}(\mathbf{k}',\mathbf{r})]Y_2^0(\mathbf{r})\rho_{SD}(r). \tag{76}$$

Using the result

$$\nabla f\times\nabla g=\frac{f'\hat{L}g-g'\hat{L}f}{r}+\frac{\hat{L}f\times\hat{L}g}{r^2}, \tag{77}$$

where

$$\hat{L}\equiv\mathbf{r}\times\nabla \tag{78}$$

and

$$\hat{L}^\mu Y_l^m(\hat{\mathbf{r}})=i\sqrt{l(l+1)}C_{l,1,l}^{m,\mu,m+\mu}Y_l^{m+\mu}(\hat{\mathbf{r}}), \tag{79}$$

we can express the result as a sum of radial integrals:

$$g_{SD}=-\frac{3}{\sqrt{5}}g_pC_SC_D(2F_1+F_2), \tag{80}$$

$$F_1(\theta)=\sum_{l,m}m(-1)^m\sin m\theta\{f_1^{00}(l)C_{l,l,2}^{0,0,0}C_{l,l,2}^{m,-m,0}Q_{l,m}^2-[f_1^{02}(l)+f_1^{20}(l)]C_{l,l+2,2}^{0,0,0}C_{l,l+2,2}^{m,-m,0}Q_{l,m}Q_{l+2,m}\}, \tag{81}$$

$$\begin{aligned}F_2(\theta)=\sum_{l,m}(-1)^m\sin m\theta\{&f_2^{00}(l)C_{l,l,2}^{0,0,0}\{[l(l+1)-m(m-1)]C_{l,l,2}^{m-1,-m+1,0}-[l(l+1)-m(m+1)]C_{l,l,2}^{m+1,-m-1,0}\}Q_{l,m}^2\\&+f_2^{02}(l)C_{l,l+2,2}^{0,0,0}\{\sqrt{[l(l+1)-m(m-1)][(l+2)(l+3)-m(m-1)]}C_{l,l,2}^{m-1,-m+1,0}\\&-\sqrt{[l(l+1)-m(m+1)][(l+2)(l+3)-m(m+1)]}C_{l,l+2,2}^{m+1,-m-1,0}\}Q_{l,m}Q_{l+2,m}\},\end{aligned} \tag{82}$$

where

$$f_1^{\lambda,\lambda'}\equiv\int_0^\infty\left[U'_{l+\lambda}(r)-\frac{U_{l+\lambda}(r)}{r}\right]U_{l+\lambda'}(r)\rho_{SD}(r)dr, \tag{83}$$

$$f_2^{\lambda,\lambda'}\equiv\int_0^\infty\frac{U_{l+\lambda}(r)U_{l+\lambda'}(r)}{r}\rho_{SD}(r)dr, \tag{84}$$

and

$$Q_{l,m}\equiv(2l+1)P_l^m(0)\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}. \tag{85}$$

$U_l(r)$ is the pion partial wave function, multiplied by r, coming from the solution of the Schrödinger equation modified to correct for the finite-range effects as discussed above.

VI. RESULTS

Figures 2 and 3 show the cross sections obtained and Figs. 4 and 5 the asymmetries for π^+ and π^- scattering from ^{3}He. There is reasonable agreement with the data except at

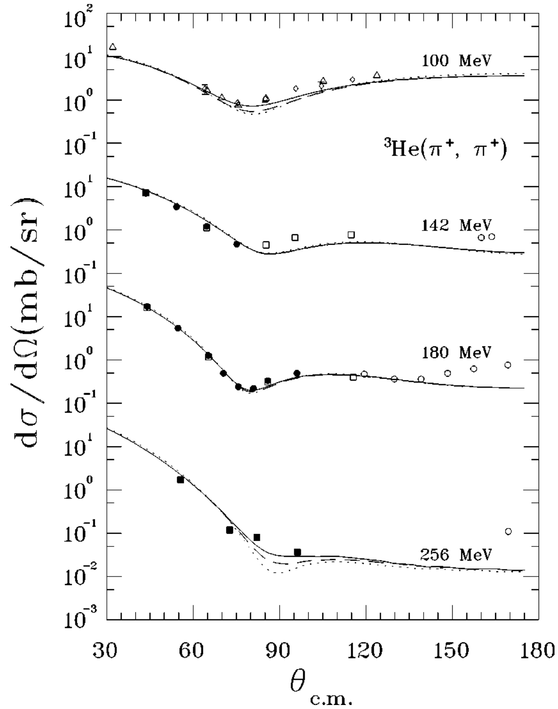

FIG. 2. Cross sections for π^{+3}He scattering. The data are compared to DWIA calculations with our fitted densities. Dotted line: only *S*-state, dotted-dash line: *S* state and *SD* overlap, solid line: *S* state plus *SS′* overlap and *SD* overlap. (Full circles: Ref. [12], open circles: Ref. [13], open squares: Ref. [14], full squares: Ref. [16], diamonds: Ref. [15], and open triangles: Ref. [17]).

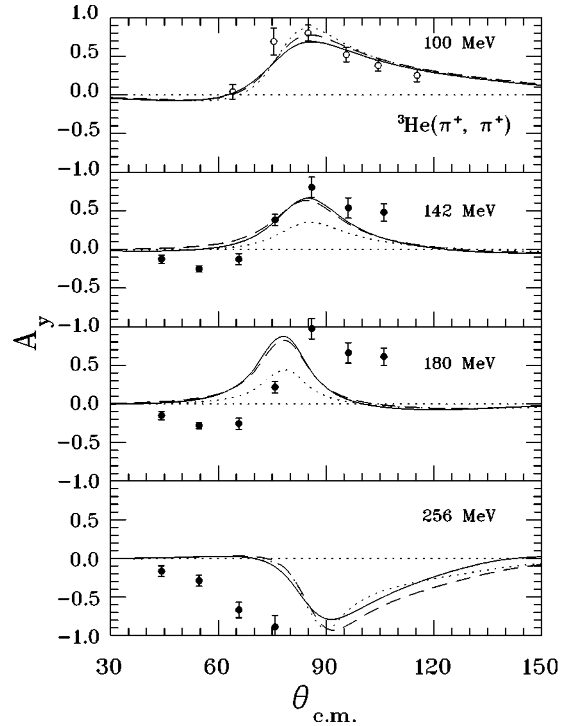

FIG. 4. Asymmetry (A_y) angular distribution for π^{+}-^{3}He elastic scattering. The curve shows the DWIA calculation with our fitted densities. dotted line: only *S*-state, dotted-dash line: *S* state and *SD* overlap, Solid line: *S* state plus *SS′* overlap and *SD* overlap. (Full circles: Ref. [1], open circles: Ref. [15].)

the back angles where a rise is seen in the experimental values which is not reproduced by the theory.

The asymmetries at 100 MeV for $\pi^{+}$and 100 and 180 MeV for $\pi^{-}$are in good agreement with the data. At 142 and 180 MeV a strong dependence is seen on the inclusion of the *S*-*D* interference term. It increases the asymmetry by about a factor of 2 at 142 MeV and about a factor of 3 at 180 MeV. The *S*-*S′* interference leads to a small contribution.

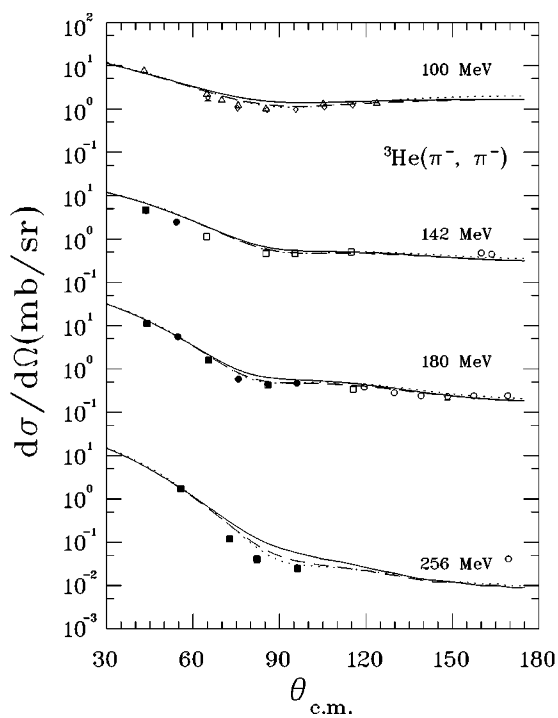

FIG. 3. Cross sections for π^{-3}He scattering. The symbols and lines have the same meaning as in Fig. 2.

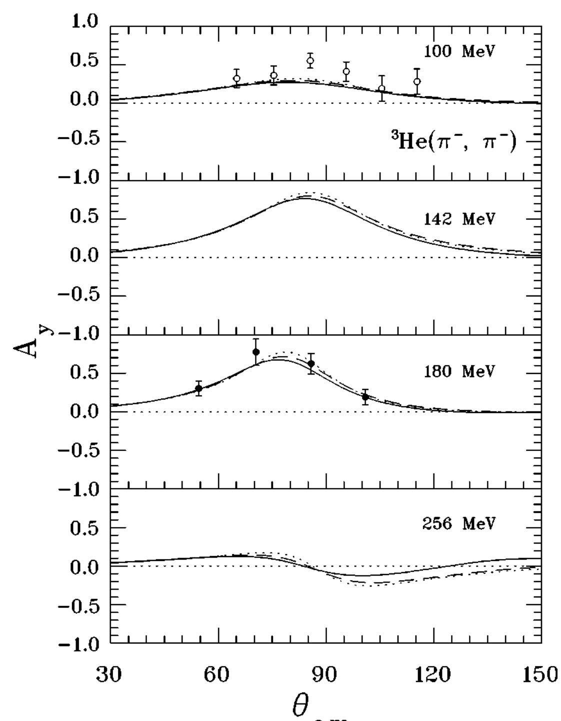

FIG. 5. Same as Fig. 4, but for π^{-}-^{3}He elastic scattering. (Full circles: Ref. [18], open circles: Ref. [15].)

The peak of the asymmetry occurs at the minimum in the cross section, as is very common in many calculations. It is interesting to note that the experimental maximum of the asymmetry does not occur at the cross section minimum. The minimum moves to smaller angles at the resonance where the nucleus becomes ''blacker'' and then to larger angles as the absorption decreases in both the theory and the experiment.

VII. CONCLUSIONS

We have included the contribution of the interference between the dominant S state and the smaller D and S' states to the spin-dependent amplitude for pion ^{3}He scattering in a distorted wave calculation. We have presented scalar functions representing the three-body system in a ''minimal'' approximation. At the 33 resonance we find these effects to be important (at least for the π^+ case). Any theory attempting to fit this data should include these smaller components.

ACKNOWLEDGMENTS

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APPENDIX A: EXPLICIT EXPRESSIONS

With $N=\sqrt{15/2(1/8\pi)}$ we can write

$$T_2^2(\mathbf{r}_a,\mathbf{r}_b)=N[x_ax_b-y_ay_b+i(x_ay_b+y_ax_b)], \quad (A1)$$

$$T_2^1(\mathbf{r}_a,\mathbf{r}_b)=-N[x_az_b+z_ax_b+i(y_az_b+z_ay_b)],$$

$$T_2^0(\mathbf{r}_a,\mathbf{r}_b)=N'[3z_az_b-\mathbf{r}_a\cdot\mathbf{r}_b], \quad (A2)$$

where $N'=N\sqrt{2/3}$. Explicit forms are

$$\phi_a^0(\mathbf{r}_a,\mathbf{r}_b)=\frac{1}{2}N'(3z_b^2-3z_a^2-r_b^2+r_a^2),$$

$$\phi_a^1(\mathbf{r}_a,\mathbf{r}_b)=N[x_az_a+x_bz_b+i(y_az_a+y_bz_b)], \quad (A3)$$

$$\phi_a^2(\mathbf{r}_a,\mathbf{r}_b)=\frac{1}{2}N[x_b^2-y_b^2-x_a^2+y_a^2+2i(x_by_b-x_ay_a)]. \quad (A4)$$

Under the replacement

$$\mathbf{R}_a=\frac{\mathbf{r}_b+\mathbf{r}_a}{\sqrt{2}}, \quad \mathbf{R}_b=\frac{\mathbf{r}_b-\mathbf{r}_a}{\sqrt{2}}, \quad (A5)$$

$$\phi_s^M(\mathbf{R}_a,\mathbf{R}_b)=\phi_s^M(\mathbf{r}_a,\mathbf{r}_b),$$

$$\phi_a^M(\mathbf{R}_a,\mathbf{R}_b)=-\phi_b^M(\mathbf{r}_a,\mathbf{r}_b), \quad \phi_b^M(\mathbf{R}_a,\mathbf{R}_b)=\phi_a^M(\mathbf{r}_a,\mathbf{r}_b). \quad (A6)$$

APPENDIX B: COMPLETE D STATES

The square of the a term of the density integrated on angle gives

$$\rho_a(r_a,r_b)=\alpha^2D_\alpha^2P_{bb}+\frac{1}{2}\beta^2D_\beta^2(P_{bbaa}+2P_{abab}+P_{aabb}) -\sqrt{2}\alpha\beta D_\alpha D_\beta(P_{bab}+P_{abb})-\alpha\gamma D_\alpha D_\gamma P_{bbs} +\sqrt{2}\beta\gamma D_\beta D_\gamma(P_{bbas}+P_{abbs})+\gamma^2D_\gamma^2P_{bbss}, \quad (B1)$$

while the b term gives

$$\rho_b(r_a,r_b)=\alpha^2D_\alpha^2P_{aa}+\frac{1}{2}\beta^2D_\beta^2(P_{bbbb}-2P_{abab}+P_{aaaa}) -\sqrt{2}\alpha\beta D_\alpha D_\beta(P_{bab}-P_{aaa}), -\alpha\gamma D_\alpha D_\gamma P_{aas}+\sqrt{2}\beta\gamma D_\beta D_\gamma(P_{abbss}-P_{aaas}) +\gamma^2D_\gamma^2P_{aass}, \quad (B2)$$

where the P functions are defined as

$$P_{ij}=\int d\Omega_a d\Omega_b \phi_i^{M*}(\mathbf{r}_a,\mathbf{r}_b)\phi_j^M(\mathbf{r}_a,\mathbf{r}_b), \quad (B3)$$

$$P_{ijk}=\int d\Omega_a d\Omega_b \chi_i(\mathbf{r}_a,\mathbf{r}_b)\phi_j^{M*}(\mathbf{r}_a,\mathbf{r}_b)\phi_k^M(\mathbf{r}_a,\mathbf{r}_b), \quad (B4)$$

$$P_{ijkl}=\int d\Omega_a d\Omega_b \chi_i(\mathbf{r}_a,\mathbf{r}_b)\chi_j(\mathbf{r}_a,\mathbf{r}_b) \times\phi_k^{M*}(\mathbf{r}_a,\mathbf{r}_b)\phi_l^M(\mathbf{r}_a,\mathbf{r}_b), \quad (B5)$$

and are given in Table III. Of interest also (for the calculation of the normalization and the rms radius) are the integrals

TABLE III. Functions which make up the d-state densities. The column labeled N_n contains a common factor of $\pi/3\times 2^{10}$ and that labeled R_n $\pi/3\times 2^{10}$.

Integral	Function	N_n	R_n
P_{aa}	$\frac{1}{4}(r_a^4+r_b^4)$	30	30
P_{bb}	$\frac{5}{6}r_a^2r_b^2$	30	30
P_{aaa}	$\frac{1}{4}(r_b^2-r_a^2)(r_a^4+r_b^4)$	0	−15
P_{abb}	$\frac{5}{6}(r_b^2-r_a^2)r_a^2r_b^2$	0	−5
P_{bab}	$\frac{1}{3}(r_b^2-r_a^2)r_a^2r_b^2$	0	−2
P_{aas}	$\frac{1}{4}(r_b^2-r_a^2)(r_b^4-r_a^4)$	12	12
P_{bbs}	$\frac{1}{3}r_a^2r_b^2(r_b^2+r_a^2)$	12	12
P_{aaaa}	$\frac{1}{4}(r_b^2-r_a^2)^2(r_a^4+r_b^4)$	9	9
P_{aabb}	$\frac{5}{6}(r_b^2-r_a^2)^2r_a^2r_b^2$	5	5
P_{bbaa}	$\frac{1}{3}r_a^2r_b^2(r_a^4+r_b^4-\frac{4}{5}r_a^2r_b^2)$	5	5
P_{abab}	$\frac{1}{3}(r_b^2-r_a^2)^2r_a^2r_b^2$	2	2
P_{bbbb}	$\frac{6}{5}r_a^4r_b^4$	9	9
P_{aaas}	$\frac{1}{4}(r_b^2-r_a^2)^2(r_b^4-r_a^4)$	0	−6
P_{bbas}	$\frac{1}{3}r_a^2r_b^2(r_b^4-r_a^4)$	0	−2
P_{abbs}	$\frac{1}{3}r_a^2r_b^2(r_b^4-r_a^4)$	0	−2
P_{aass}	$\frac{1}{4}(r_b^2-r_a^2)^2(r_a^4+r_b^4)$	9	9
P_{bbss}	$\frac{1}{3}r_a^2r_b^2(r_a^4+r_b^4+\frac{4}{5}r_a^2r_b^2)$	9	9

$$\int r_a^2 dr_a r_b^2 dr_b D^2(t_{ab}) P(r_a, r_b) = N_n \int t^n D^2(t) dt, \tag{B6}$$

and

$$\int r_a^4 dr_a r_b^2 dr_b D^2(t_{ab}) P(r_a, r_b) = R_n \int t^{n+2} D^2(t) dt, \tag{B7}$$

where n is the dimension of the P function plus 5. The constants in front of the integral on the right hand side can be obtained with the Irving transformation and are given in Table III:

$$P_{ij} = P_{ji}, \quad P_{ijk} = P_{ikj}, \quad P_{ijkl} = P_{jikl} = P_{ijlk}.$$

All P's with an odd number of indices b vanish under the angular integration. All P's with an odd number of indices a vanish under the Irving integration.

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