Integral equation approach to relativistic Coulomb excitation

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(Received 8 October 1997; revised manuscript received 7 December 1998)

We develop a formalism describing Coulomb excitation in relativistic heavy-ion collisions as a general coupled-channel problem. The Schrödinger equation for the target nucleus is expressed as a set of coupled integral equations for a $T(\omega)$ matrix, whose on-shell matrix elements give the excitation amplitudes. The method of Kowalsky and Noyes is used to regularize the kernels of these integral equations, making the equations amenable to numerical solution. An application of the method is made to the study of the population of a model state representing an $L=2^+$ giant resonance in ^{40}Ca at an excitation energy of 20 MeV, due to the electromagnetic field of a ^{208}Pb projectile with kinetic energy per nucleon up to 100 GeV. The results of the numerical solution of the regularized coupled integral equations are compared to results obtained from the Born series, and from the long-wavelength approximation. These approximations are found to be adequate up to bombarding energies per nucleon of 3 GeV. We investigate the reasons for failure of these approximations at higher bombarding energy. Due to the finite travel time of the electromagnetic pulse across the target nucleus, the sudden approximation is found to be inapplicable to the excitation of the giant resonance. Thus the method of numerical solution of the regularized coupled integral equations seems to be the most suitable approach to the study of very high-energy Coulomb excitation of giant resonances. [S0556-2813(99)03404-4]

PACS number(s): 25.70.De, 13.40.−f, 24.10.Jv, 25.75.−q

I. INTRODUCTION

Most of the analyses of low-energy Coulomb excitation data have been based upon a classical time-dependent description of the relative motion of the colliding nuclei, with a quantum-mechanical description of the excitation of these nuclei as they proceed along their classical Coulomb-dominated orbits. A comprehensive discussion of this nonrelativistic theory is presented in the books of Alder and Winther [1], and of Biedenharn and Brussard [2]. The method is applicable, in principle, whenever the deBroglie wavelength associated with the relative motion is sufficiently small compared with the other important lengths in the problem.

A topic of major interest in current research in nuclear physics is Coulomb excitation in intermediate- and high-energy collisions between heavy nuclei (see, e.g., Refs. [3–8]). In this energy range, the high relative velocity of the colliding nuclei implies that the retardation of the electromagnetic interaction between them plays an important role. As a consequence of this retardation, the collision is more ''sudden,'' which enhances the direct and/or multistep excitation of high-energy states. Furthermore, at these energies, one benefits from the circumstance that the ratio between the resonance cross section and the background cross section is drastically improved [9].

The detailed quantitative analysis of the Coulomb excitation process can be performed in either the time domain or the frequency (ω domain). In the former, the time-dependent Schrödinger equation for the internal motion of the colliding nuclei is expressed as a set of coupled *differential* equations, which are numerically integrated. This approach requires a multipole expansion of the retarded interaction potential, expressed as a function of t. Expressions for this expansion are rather complicated, and until now it has been necessary to add a long-wavelength approximation in order to do actual calculations.

If we use the frequency domain, the amplitudes of the various channels are expressed as functions of ω. The Schrödinger equation is expressed as a set of coupled *integral* equations for these functions, with ω as the independent variable. In this approach, the retarded interaction potential enters the integral equations in terms of the Fourier transform of its time-dependent expression. A convenient multipole expansion of this Fourier transform was given by Alder and Winther [10]. It turns out that this multipole expansion has a simpler, more compact structure than the multipole expansion of the t-dependent interaction.

Until now, only the first few terms of the perturbation expansion of the excitation amplitude have been considered in the literature (see, e.g., Ref. [1] for the difficulties arising in developing further terms). This is equivalent to considering the first terms of the iterative solution of the integral equation in the ω variable. The first term is simply the on-shell interaction matrix element, i.e., the Fourier transform of the interaction, evaluated at the frequency corresponding to the energy difference between initial and final states. The second-order Born approximation has been used to account for the excitation of states that cannot be reached from the ground-state by a single-step process (see, e.g., Ref. [11]). It also provides an important correction to the first-order matrix element in the situation in which high-Z nuclei collide at low relative speeds [12]. The analyses available in the literature are mainly concerned with bombarding energy not exceeding 1 GeV per nucleon.

Analyses that use the first few terms of the perturbative series implicitly assume that this series converges. However, for a highly relativistic projectile, the convergence of the

Born series is made questionable by the strong retardation of the interaction. We will see below that as the projectile speed v approaches c, and $\gamma \equiv 1/\sqrt{1-v^2/c^2}$ becomes large, the magnitudes of some of the matrix elements of the retarded interaction potential diverge as $\ln(\gamma)$. Furthermore, the greater projectile speed implies a more sudden impulse, which broadens the ω range over which the Fourier transform of the interaction is non-negligible. Both these effects work against the convergence of the Born series at very high projectile velocity.

In order to test the reliability of various approximations to highly relativistic Coulomb excitation (bombarding energy up to 100 GeV per nucleon), we have attempted a nonperturbative solution of the coupled integral equations in ω. We first adapt the regularization procedure of Kowalski and Noyes [13] to remove the difficulties associated with the singular kernels of the integral equations. This makes the equations amenable to an approximate treatment, in which the continuous variable ω is replaced by a discrete index ω_i, and the coupled integral equations are replaced by a large set of linear simultaneous equations, which can be solved exactly. The results are then compared with solutions based on approximations such as truncation of the Born series, or the long-wavelength approximation, or the sudden approximation.

The semirealistic example we have chosen to illustrate the main features of our approach is the excitation of a 2^+ giant resonance in ^{40}Ca. This excited level is assumed to have a small permanent quadrupole distortion. The coupled-channel problem associated with this model provides an interesting study of reorientation effects. The small anharmonicities in the giant resonance in our simple model will affect the final population of the different M states. More generally, anharmonicities can have relevant effects on the values of multiple phonon cross sections [14,8], although the spacing of the levels is only slightly modified.

When nuclei collide at high relative kinetic energy, and with impact parameters greater the the sum of the nuclear radii, Coulomb excitation is not the only process that can occur. For example, the pion-production cross section can be appreciable at projectile energies of the order of 15–20 GeV per nucleon, depending upon the nuclei being considered. At somewhat higher energy ($\sim$50 GeV per nucleon), electron-positron pair creation can also be significant. Our goal is the more modest one of investigating the adequacy of various approximation methods for the study of pure high-energy Coulomb excitation. This seems to be a reasonable first step on the way to a complete description of high-energy collisions at large impact parameter. Several studies of Coulomb excitation at bombarding energies up to 100 GeV per nucleon have been developed without reference to pair creation and/or pion production (see, e.g., Refs. [3,15]).

In Sec. II, we first discuss the difficulties associated with the multipole expansion of the time-dependent Lienard-Wiechert retarded potential. Then the amplitude a_{fi}, for excitation of a final state f starting with initial state i, is expressed in terms of a $T(\omega)$ matrix, which is shown to satisfy a system of coupled integral equations [1].

In first-order treatments of the integral equations (Born approximation), the interaction responsible for Coulomb excitation enters only via its on-shell matrix elements, and these can be characterized completely by standard electric and magnetic multipole operators. However, a more complete analysis (Sec. III) of the integral equations requires the inclusion of off-shell matrix elements. We show that electric and magnetic multipole operators are not sufficient to calculate these off-shell matrix elements. An additional term is required, which vanishes on-shell.

In order to solve the coupled system of integral equations, we use the regularization method of Kowalski and Noyes (Sec. IV A) to remove the singularities in the kernels, and then we replace the regularized integral equations by an approximately equivalent set of simultaneous linear equations, which we solve exactly (Sec. IV B). We also develop an iterative solution of the integral equations (Born expansion, Sec. IV C). The sudden approximation is discussed in Sec. IV D.

The physical model is discussed in Sec. V A. The comparison between the exact excitation amplitudes and those calculated using various approximations is given in the following subsections.

In Sec. VI we show how the integral equation formulation may be modified if some of the excited states have finite widths. We also explore the connection between the integral equation approach to Coulomb excitation and the eikonal approximation. Some conclusions are drawn in Sec. VII.

II. THE EXCITATION AMPLITUDE

A. The differential form of the Schrödinger equation

Our aim is to determine the amplitudes for the population of excited states of projectile and target nuclei after they have been involved in a collision. We will use the standard semiclassical description, in which we take advantage of the short de Broglie wavelength associated with the relative motion to justify a classical description of the orbit of the projectile as it moves past the target. The projectile and target experience a mutual time-dependent interaction during the collision, which is added to the Hamiltonian describing their internal degrees of freedom. This Hamiltonian is now explicitly time dependent, and leads to a time-dependent Schrödinger equation, which governs the time evolution of the internal degrees of freedom. Quantum effects are fully taken into account in the description of the internal motion of the interacting projectile and target.

In order to simplify the following discussion, we will assume that the projectile remains in its ground state throughout the collision. We ignore the internal degrees of freedom of the projectile, and use ζ to represent all the internal degrees of freedom of the target. $H_0(\zeta)$ is the part of the interaction referring only to these internal target degrees of freedom, and does not depend explicitly on time. The time-independent eigenfunctions and eigenvalues of H_0 are ϕ_α and E_α, respectively. We assume that the relative motion has already been given in terms of a classical trajectory, from which we obtain $V(\zeta,t)$, the operator responsible for the projectile-target interaction. Thus the Schrödinger equation for the internal motion of the target can be written

$$\left(i\hbar\frac{\partial}{\partial t}-H_0\right)\psi(\zeta,t)=V(\zeta,t)\,\psi(\zeta,t). \tag{2.1}$$

In the standard method for solving this problem, the wave function is expanded in terms of the eigenstates of H_0,

$$\psi(\zeta,t)=\sum_{\alpha=0}^{N}\phi_\alpha a_\alpha(t)e^{-i\omega_\alpha t},$$

where $\omega_\alpha \equiv E_\alpha/\hbar$. If this expansion is substituted into the Schrödinger equation (2.1), it is found that the amplitudes $a_\alpha(t)$ satisfy the set of coupled differential equations

$$i\hbar\dot{a}_\alpha(t)=\sum_\beta \langle\phi_\alpha|V(t)|\phi_\beta\rangle e^{i(\omega_\alpha-\omega_\beta)t}a_\beta(t). \quad (2.2)$$

Since the target states ϕ_α are angular momentum eigenstates, the evaluation of the matrix elements in Eq. (2.2) requires a multipole expansion of $V(t)$, based on the target origin. The strong retardation effect associated with the high projectile speed makes this difficult. More specifically, suppose $\tau_{\rm pulse}$ is the duration of the electromagnetic pulse experienced at any point in the target and $\tau_{\rm sweep}$ is the time required for this pulse to sweep across the target. If $\tau_{\rm sweep} \gg \tau_{\rm pulse}$, different points in the target will feel the electromagnetic pulse at markedly different times. Conversely, at any given time, the spatial dependence of the projectile-target electromagnetic interaction will vary in a complicated way within the target, and will not be susceptible to a convenient multipole expansion.

If the projectile moves with speed v at impact parameter b, then

$$\tau_{\rm pulse}\sim\frac{b}{\gamma v}$$

and

$$\tau_{\rm sweep}\sim\frac{2R_{\rm T}}{v}.$$

Thus the condition $\tau_{\rm sweep}\gg\tau_{\rm pulse}$ occurs when

$$\frac{2R_{\rm T}}{b}\gamma\gg 1.$$

For a grazing collision, with $2R_{\rm T}\sim b$, this is equivalent to $\gamma\gg 1$. In this situation, the evaluation of $\langle\phi_\alpha|V(t)|\phi_\beta\rangle$ becomes difficult, and it has so far not been possible to develop a scheme for the exact evaluation of this matrix element.

In appropriate circumstances, one can use an approximate treatment of $\langle\phi_\alpha|V(t)|\phi_\beta\rangle$ based on the smallness of the ratio

$$\frac{\text{target dimensions}}{\text{dominant wavelength associated with the electromagnetic pulse}}.$$

This is analogous to the long-wavelength approximation, familiar from the theory of photon absorption and emission. However, in that situation, the smallness of the above ratio is guaranteed by the weakness of the electromagnetic coupling constant $e^2/\hbar c$. In the problem of relativistic Coulomb excitation, we do not know what to use for the denominator in this ratio until we have solved the Schrödinger equation. Thus the application of the long-wavelength approximation in this context may be dangerous. One of the goals of this work is to test the long-wavelength approximations in some cases in which we can solve the Schrödinger equation exactly. At relatively low energies (0.6–1 GeV per nucleon) good results have been obtained, e.g., in Refs. [16,17].

B. The integral form of the Schrödinger equation

Although retardation makes it difficult to work directly with $\langle\phi_\alpha|V(t)|\phi_\beta\rangle$, the Fourier transform of this time-dependent matrix element has a simpler, better convergent, structure. This was first demonstrated by Alder and Winther [10], who gave a full multipole expansion of the Fourier transform. In the quoted monograph [1], Alder and Winther had already demonstrated how the Coulomb excitation problem could be expressed entirely in terms of the Fourier transform of the interaction. This involves replacing the set of coupled differential equations in t by an equivalent set of coupled integral equations in the conjugate variable ω. The advantage of this approach is the simpler and more exact representation of the Fourier transform of the interaction. The disadvantage is that the solution of the coupled integral equations is computationally more demanding than the solution of the original set of coupled differential equations. However, we will see that the use of a regularization procedure makes the set of coupled integral equations readily amenable to numerical solution.

We begin by rewriting the differential equation (2.1) in the form

$$\frac{\partial}{\partial t}[e^{(i/\hbar)H_0t}\psi(\zeta,t)]=\frac{e^{(i/\hbar)H_0t}}{i\hbar}V(\zeta,t)\psi(\zeta,t). \quad (2.3)$$

This can be immediately integrated, to yield an integral equation

$$\psi_\alpha(\zeta,t)=e^{-(i/\hbar)E_\alpha t}\phi_\alpha + \int_{-\infty}^{t}\frac{dt'}{i\hbar}e^{-(i/\hbar)H_0(t-t')}V(\zeta,t')\psi_\alpha(\zeta,t') \quad (2.4)$$

whose solution satisfies the Schrödinger equation (2.1), corresponding to the initial internal wave function ϕ_α.

It is convenient to express the time-ordering implicit in Eq. (2.4) in terms of the step function $\Theta(t-t')$,

$$\psi_\alpha(\zeta,t)=e^{-(i/\hbar)E_\alpha t}\phi_\alpha + \int_{-\infty}^{\infty}\frac{dt'}{i\hbar}\Theta(t-t')e^{-(i/\hbar)H_0(t-t')} \times V(\zeta,t')\psi_\alpha(\zeta,t')$$

and then to use the Fourier transform of the step function

$$\Theta(t-t')=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\omega(t-t')}d\omega}{\omega-i\delta} \tag{2.5}$$

to rewrite the integral equation as

$$\psi_\alpha(\zeta,t)=e^{-(i/\hbar)E_\alpha t}\phi_\alpha-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{i(\omega-H_0/\hbar)t}}{\omega-i\delta} \times\int_{-\infty}^{\infty}\frac{dt'}{\hbar}e^{-i(\omega-H_0/\hbar)t'}V(\zeta,t')\psi_\alpha(\zeta,t'). \tag{2.6}$$

To find the amplitude for the system to end up in the H_0 eigenstate ϕ_β, we consider the quantity

$$T_{\beta\alpha}(\omega;\tau)\equiv\int_{-\tau}^{\tau}\frac{dt}{\hbar}e^{i(\omega+\omega_\alpha)t}\langle\phi_\beta|V|\psi_\alpha\rangle. \tag{2.7}$$

Here τ is a finite time, which we will allow to become infinite. Since ψ_α satisfies the Schrödinger equation (2.1), we can replace $V|\psi_\alpha\rangle$ in Eq. (2.7) by $[i\hbar(\partial/\partial t)-H_0]|\psi_\alpha\rangle$. Then an integration of t by parts yields

$$T_{\beta\alpha}(\omega;\tau)=(\omega+\omega_\alpha-\omega_\beta)\int_{-\tau}^{\tau}\frac{dt}{\hbar}e^{i(\omega+\omega_\alpha)t}\langle\phi_\beta|\psi_\alpha\rangle + i[\langle e^{-i(\omega+\omega_\alpha)\tau}\phi_\beta|\psi_\alpha(\tau)\rangle - \langle e^{i(\omega+\omega_\alpha)\tau}\phi_\beta|\psi_\alpha(-\tau)\rangle]. \tag{2.8}$$

For the particular ω value given by

$$\omega=\omega_\beta-\omega_\alpha=\frac{E_\beta-E_\alpha}{\hbar},$$

we have

$$T_{\beta\alpha}(\omega_\beta-\omega_\alpha;\tau)=-i[\langle e^{-i\omega_\beta\tau}\phi_\beta|\psi_\alpha(\tau)\rangle - \langle e^{i\omega_\beta\tau}\phi_\beta|\psi_\alpha(-\tau)\rangle]. \tag{2.9}$$

Now let $\tau\to\infty$, for which $\psi_\alpha(-\tau)\to e^{i\omega_\alpha\tau}\phi_\alpha$, and

$$a_{\beta\alpha}\equiv\lim_{\tau\to\infty}\langle e^{-i\omega_\beta\tau}\phi_\beta|\psi_\alpha(\tau)\rangle$$

approaches the asymptotic amplitude for populating the target state ϕ_β if the initial target state is ϕ_α. If we define $T_{\beta\alpha}(\omega)$ by

$$T_{\beta\alpha}(\omega)\equiv\lim_{\tau\to\infty}T_{\beta\alpha}(\omega;\tau),$$

then Eq. (2.9) can be rewritten as

$$T_{\beta\alpha}(\omega_\beta-\omega_\alpha)=i[a_{\beta\alpha}-\delta_{\beta\alpha}],$$
$$a_{\beta\alpha}=\delta_{\beta\alpha}-iT_{\beta\alpha}(\omega_\beta-\omega_\alpha), \tag{2.10}$$

and we see that the excitation amplitudes $a_{\beta\alpha}$ will be determined once we know the ''on-shell'' quantities $T_{\beta\alpha}(\omega_\beta-\omega_\alpha)$.

It is shown in Appendix B that $a_{\beta\alpha}=a_{\alpha\beta}$, and thus Eq. (2.10) implies that the on-shell T matrix is also symmetric. The unitarity of the a matrix is an expression of conservation of probability, which follows from the Hermiticity of the Hamiltonian.

The effect of a gauge transformation generated by a gauge function χ is to multiply every wave function for a system of charge q by the phase

$$e^{i(q\chi/\hbar c)}.$$

This operation will have no effect on any of the scalar products in Eq. (2.8). Thus we see that the functions $T_{\beta\alpha}(\omega)$, and the excitation amplitudes $a_{\beta\alpha}$ given by Eq. (2.10), are invariant with respect to a gauge transformation.

We can express the integral equation for the wave function as an integral equation for $T_{\beta\alpha}(\omega)$ by taking the scalar product of Eq. (2.6) with $e^{i(\omega+\omega_\alpha)t}\langle\phi_\beta|V$:

$$T_{\beta\alpha}(\omega)=V_{\beta\alpha}(\omega) - \frac{1}{2\pi}\sum_\lambda\int d\omega'\frac{V_{\beta,\lambda}(\omega+\omega')T_{\lambda\alpha}(-\omega')}{\omega'+\omega_\lambda-\omega_\alpha-i\delta}, \tag{2.11}$$

where we have defined the Fourier transform of the matrix elements of the interaction potential by

$$V_{\beta\alpha}(\omega)\equiv\int\frac{dt}{\hbar}e^{i\omega t}\langle\phi_\beta|V(t)|\phi_\alpha\rangle. \tag{2.12}$$

Equation (2.11) has a mathematical structure similar to the Lippmann-Schwinger equation of scattering theory. Note, however, that it is related to a response function, not to a cross section. Our goal is to solve this equation for the $T_{\beta\alpha}(\omega)$, and then to use the ''on-shell'' $T_{\beta\alpha}(\omega_\beta-\omega_\alpha)$ to determine the excitation amplitudes.

III. ON- AND OFF-SHELL MATRIX ELEMENTS OF THE ELECTROMAGNETIC INTERACTION

A. The retarded potentials

We now restrict our considerations to the excitation of a target nucleus as a result of its electromagnetic interaction with a spherically symmetric projectile of charge $Z_p e$. We know that the electromagnetic field of the projectile outside its charge distribution is the same as that of a point charge $Z_p e$ at its center. Since we are only interested in the projectile field at points outside its charge distribution, we will use this point charge as the source of the projectile electromagnetic field. We assume that the projectile center moves along a trajectory given by

$$\mathbf{r}=\mathbf{b}+vt\hat{\mathbf{z}}, \tag{3.1}$$

where $\mathbf{b}$ is the impact parameter vector perpendicular to $\hat{\mathbf{z}}$ and v is the constant projectile speed.

The scalar and vector potentials due to this projectile at the point $\mathbf{r}'$ of the target at time t are given by the Lienard-Wiechert expressions [18,10]

$$\varphi_C^{\rm ret}(\mathbf{r}',t)=\frac{Z_p e\gamma}{\sqrt{(x-x')^2+(y-y')^2+\gamma^2(vt-z')^2}}, \tag{3.2}$$

$$\mathbf{A}_C^{\rm ret}(\mathbf{r}',t)=\frac{v}{c}\varphi_C^{\rm ret}(\mathbf{r}',t)\hat{\mathbf{z}}. \tag{3.3}$$

Note that, because of the factor γ^2 which multiplies $(vt-z')^2$, this potential is not spherically symmetric.

B. The interaction

If the nonrelativistic target Hamiltonian is written

$$H_0(\zeta)=\sum_j \frac{1}{2m}\mathbf{p}_j\cdot\mathbf{p}_j+W(\zeta),$$

then we include the effect of the external electromagnetic potentials with

$$H(\zeta,t)=\sum_j \frac{1}{2m}\left(\mathbf{p}_j-\frac{e_j}{c}\mathbf{A}_C^{\rm ret}(\mathbf{r}'_j,t)\right)^2 + e_j\varphi_C^{\rm ret}(\mathbf{r}'_j,t)+W(\zeta),$$

so that the one-photon-exchange part of the perturbation is

$$V(\mathbf{r}',t)=-\sum_j \frac{e_j}{2mc}[\mathbf{p}_j\cdot\mathbf{A}_C^{\rm ret}(\mathbf{r}'_j,t) + \mathbf{A}_C^{\rm ret}(\mathbf{r}'_j,t)\cdot\mathbf{p}_j]+e_j\varphi_C^{\rm ret}(\mathbf{r}'_j,t). \tag{3.4}$$

Here $e_j=1$ or 0 depending upon whether the nucleon is a proton or a neutron, respectively. The matrix element of V between unperturbed eigenstates ϕ_α,ϕ_β is

$$\begin{aligned}\langle\phi_\beta|V(t)|\phi_\alpha\rangle&=\left\langle\phi_\beta\right|-\sum_j\frac{e_j}{2mc}[\mathbf{p}_j\cdot\mathbf{A}_C^{\rm ret}(\mathbf{r}'_j,t) + \mathbf{A}_C^{\rm ret}(\mathbf{r}'_j,t)\cdot\mathbf{p}_j]+e_j\varphi_C^{\rm ret}\left|\phi_\alpha\right\rangle\\ &=\int d^3\mathbf{r}'\left[\rho_{\beta\alpha}(\mathbf{r}')\varphi_C^{\rm ret}(\mathbf{r}',t) - \frac{1}{c}\mathbf{j}_{\beta\alpha}(\mathbf{r}')\cdot\mathbf{A}_C^{\rm ret}(\mathbf{r}',t)\right],\end{aligned} \tag{3.5}$$

where $\rho_{\beta\alpha}(\mathbf{r}')$ and $\mathbf{j}_{\beta\alpha}(\mathbf{r}')$ are matrix elements of the charge-density and current-density operators, respectively. These are single-particle operators, defined by their single-particle matrix elements as

$$\langle\chi_\beta|\rho(\mathbf{r}',t)|\chi_\alpha\rangle=e\chi_\beta^*(\mathbf{r}',t)\chi_\alpha(\mathbf{r}',t),$$

$$\langle\chi_\beta|\mathbf{j}(\mathbf{r}',t)|\chi_\alpha\rangle=\frac{e\hbar}{2mi}[\chi_\beta^*(\mathbf{r}',t)\boldsymbol{\nabla}\chi_\alpha(\mathbf{r}',t) - \chi_\alpha(\mathbf{r}',t)\boldsymbol{\nabla}\chi_\beta^*(\mathbf{r}',t)].$$

The conservation of target charge implies that

$$\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r}',t)+\frac{\partial\rho}{\partial t}(\mathbf{r}',t)=0,$$

whose matrix elements with respect to the unperturbed eigenstates are

$$\boldsymbol{\nabla}\cdot\mathbf{j}_{\beta\alpha}(\mathbf{r}')+i\frac{(E_\beta-E_\alpha)}{\hbar}\rho_{\beta\alpha}(\mathbf{r}')=\boldsymbol{\nabla}\cdot\mathbf{j}_{\beta\alpha}(\mathbf{r}')+i\omega_{\beta\alpha}\rho_{\beta\alpha}(\mathbf{r}') =0.$$

C. The Fourier transform of the interaction

The Fourier transform of $V_{\beta\alpha}(t)$ required in Eq. (2.11) is

$$\begin{aligned}V_{\beta\alpha}(\omega)&=\int\frac{dt}{\hbar}e^{i\omega t}V_{\beta\alpha}(t)\\ &=\int d^3\mathbf{r}'\frac{dt}{\hbar}e^{i\omega t}\left[\rho_{\beta\alpha}(\mathbf{r}')\varphi_C^{\rm ret}(\mathbf{r}',t) - \frac{1}{c}\mathbf{j}_{\beta\alpha}(\mathbf{r}')\cdot\mathbf{A}_C^{\rm ret}(\mathbf{r}',t)\right]\\ &=\int d^3\mathbf{r}'\left[\rho_{\beta\alpha}(\mathbf{r}')\varphi_C^{\rm ret}(\mathbf{r}',\omega) - \frac{1}{c}\mathbf{j}_{\beta\alpha}(\mathbf{r}')\cdot\mathbf{A}_C^{\rm ret}(\mathbf{r}',\omega)\right]\end{aligned} \tag{3.6}$$

with $\varphi_C^{\rm ret}(\mathbf{r}',\omega)$ and $\mathbf{A}_C^{\rm ret}(\mathbf{r}',\omega)$ the Fourier transforms of $\varphi_C^{\rm ret}(\mathbf{r}',t)$ and $\mathbf{A}_C^{\rm ret}(\mathbf{r}',t)$. The relation between the electric field and the potentials

$$\mathbf{E}_C^{\rm ret}(\mathbf{r}',t)=-\nabla\varphi_C^{\rm ret}(\mathbf{r}',t)-\frac{1}{c}\frac{\partial\mathbf{A}_C^{\rm ret}(\mathbf{r}',t)}{\partial t}$$

implies that the Fourier transform of $\mathbf{E}$ is related to the Fourier transforms of the potentials $(\phi,\mathbf{A})$ by

$$\mathbf{E}_C^{\rm ret}(\mathbf{r}',\omega)=-\nabla\varphi_C^{\rm ret}(\mathbf{r}',\omega)+\frac{i\omega}{c}\mathbf{A}_C^{\rm ret}(\mathbf{r}',\omega) \tag{3.7}$$

and Eq. (3.6) can be written

$$V_{\beta\alpha}(\omega)=\frac{i}{\omega}\int d^3\mathbf{r}'\mathbf{j}_{\beta\alpha}(\mathbf{r}')\cdot\mathbf{E}_C^{\rm ret}(\mathbf{r}',\omega) + \left(1-\frac{\omega_{\beta\alpha}}{\omega}\right)\int d^3\mathbf{r}'\rho_{\beta\alpha}(\mathbf{r}')\varphi_C^{\rm ret}(\mathbf{r}',\omega). \tag{3.8}$$

In particular, the on-shell matrix element $V_{\beta\alpha}(\omega_{\beta\alpha})$ is expressed entirely in terms of the electric field $\mathbf{E}_C^{\rm ret}$. Since $\mathbf{E}_C^{\rm ret}$ is invariant under a gauge transformation of the potentials, we see that on-shell matrix elements of the interaction are gauge-invariant, but off-shell matrix elements generally are not. Furthermore, the electric field that appears in Eq. (3.8) has the projectile as its source, so that its divergence is zero outside the projectile. If b is large enough so that the projectile and target do not overlap, then $\boldsymbol{\nabla}\cdot\mathbf{E}_C^{\rm ret}(\mathbf{r}',\omega)$ will be zero throughout the range of the $\mathbf{r}'$ integration. We will see below that this leads to a simplification in the expressions for the on-shell matrix elements.

D. Multipole expansion of the interaction matrix element

The transition charge and current densities in Eq. (3.6) will be determined by the model we use to describe the target states. To evaluate the integrals in Eq. (3.6), it is useful to have expansions of $\varphi_C^{\rm ret}(\mathbf{r}',\omega)$ and $\mathbf{A}_C^{\rm ret}(\mathbf{r}',\omega)$ in terms of multipoles defined relative to the target center. Such expansions were first given by Alder and Winther [10]. Their result can be expressed as

$$V_{\beta\alpha}(\omega)=\frac{2Z_Pe}{\hbar v}\sum_\mu\ e^{-i\mu\phi_b}K_\mu\left(\frac{|\omega|b}{v\gamma}\right)\times\sum_{\lambda=|\mu|}^{\infty}\mathcal{G}_{\lambda\mu}\int d^3r'.\left[\rho_{\beta\alpha}(\mathbf{r}')-\frac{v}{c^2}\hat{\mathbf{z}}\cdot\mathbf{j}_{\beta\alpha}(\mathbf{r}')\right]j_\lambda\left(\frac{|\omega|}{c}\,r'\right)Y_\mu^\lambda(\theta',\phi'). \tag{3.9}$$

Here r', θ', and ϕ' are the spherical polar coordinates of $\mathbf{r}'$ defined relative to a target-centered origin, and ϕ_b is the angle between the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane and the reaction plane. The coefficient $\mathcal{G}_{\lambda\mu}$ has the explicit expression

$$\mathcal{G}_{\lambda,\mu}=\frac{i^{\lambda+\mu}}{(2\gamma)^\mu}\left(\frac{\omega}{|\omega|}\right)^{\lambda-\mu}\left(\frac{c}{v}\right)^\lambda\sqrt{4\pi(2\lambda+1)(\lambda-\mu)!(\lambda+\mu)!}\times\sum_n\frac{1}{(2\gamma)^{2n}(n+\mu)!n!(\lambda-\mu-2n)!}. \tag{3.10}$$

It can also be expressed in terms of solid harmonics or Gegenbauer polynomials

$$\mathcal{G}_{\lambda,\mu}=i^\lambda 4\pi\left(\frac{\omega}{|\omega|}\right)^{\lambda-\mu}\left(\frac{c}{v}\right)^l\mathcal{Y}_\mu^\lambda\left(\frac{1}{i\gamma},0,1\right)=i^{\lambda-\mu}\left(\frac{\omega}{|\omega|}\right)^{\lambda-\mu}\left(\frac{c}{\gamma v}\right)^{|\mu|}\sqrt{4\pi(2\lambda+1)\frac{(\lambda-|\mu|)!}{(\lambda+|\mu|)!}}\times(2|\mu|-1)!!C_{\lambda-|\mu|}^{|\mu|+1/2}\left(\frac{c}{v}\right). \tag{3.11}$$

In Appendix A we give an alternative derivation of the expansion (3.9), which we believe is somewhat more direct than the original derivation of Alder and Winther.

Since the nuclear states used in the calculation of the transition current density have well-defined parity and angular momentum, it is convenient to express the vector fields appearing in Eq. (3.9) in terms of vector spherical harmonics. For a given radial dependence $j_\lambda(\kappa r')$, there are three linearly independent vector fields with rotational transformation properties (λ,μ). The choices usually made are

$$\mathbf{L}j_\lambda(\kappa r')Y_\mu^\lambda(\hat{\mathbf{r}}')=\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla})j_\lambda(\kappa r')Y_\mu^\lambda(\hat{\mathbf{r}}'), \tag{3.12}$$

$$(\boldsymbol{\nabla}\times\mathbf{L})j_\lambda(\kappa r')Y_\mu^\lambda(\hat{\mathbf{r}}'), \tag{3.13}$$

$$\boldsymbol{\nabla}j_\lambda(\kappa r')Y_\mu^\lambda(\hat{\mathbf{r}}'). \tag{3.14}$$

The fields (3.12) and (3.13) are solenoidal, i.e., they have zero divergence. Field (3.14) is irrotational, i.e., it has zero curl.

If $\hat{\mathbf{z}}j_\lambda(\kappa r')Y_\mu^\lambda(\hat{\mathbf{r}}')$ is expressed in terms of the multipole fields (3.12)–(3.14), the multipole expansion of $V_{\beta\alpha}$ can be written

$$V_{\beta\alpha}(\omega)=\frac{2Z_Pe}{\hbar v}\sum_\mu\ e^{-i\mu\phi_b}K_\mu\left(\frac{|\omega|b}{v\gamma}\right)\times\sum_{\lambda=|\mu|}^{\infty}[X_\mu^\lambda(E)+X_\mu^\lambda(M)+X_\mu^\lambda(G)], \tag{3.15}$$

where

$$X_\mu^\lambda(E)=\frac{iv}{c\hbar\omega}\left[\frac{\mathcal{G}_{\lambda-1,\mu}}{\lambda}\sqrt{\frac{\lambda^2-\mu^2}{(2\lambda+1)(2\lambda-1)}}+\frac{\mathcal{G}_{\lambda+1,\mu}}{\lambda+1}\sqrt{\frac{(\lambda+1)^2-\mu^2}{(2\lambda+1)(2\lambda+3)}}\right]\times\int\mathbf{j}_{\beta\alpha}\cdot(\boldsymbol{\nabla}\times\mathbf{L})j_\lambda\left(\frac{\omega}{c}r'\right)Y_\mu^\lambda(\hat{\mathbf{r}}')d^3r', \tag{3.16}$$

$$X_\mu^\lambda(M)=-\frac{v\mu}{c^2\hbar}\frac{\mathcal{G}_{\lambda,\mu}}{\lambda(\lambda+1)}\int\mathbf{j}_{\beta\alpha}\cdot\mathbf{L}j_\lambda\left(\frac{\omega}{c}r'\right)Y_\mu^\lambda(\hat{\mathbf{r}}')d^3r', \tag{3.17}$$

$$X_\mu^\lambda(G)=\left[1-\frac{\omega_{\beta\alpha}}{\omega}\right]\mathcal{G}_{\lambda\mu}\int\rho_{\beta\alpha}(\mathbf{r}')j_\lambda\left(\frac{\omega}{c}r'\right)Y_\mu^\lambda(\hat{\mathbf{r}}')d^3r'. \tag{3.18}$$

In deriving Eqs. (3.16)–(3.18), we have used the relation

$$\mathcal{G}_{\lambda\mu}=\frac{iv}{c}\left[\sqrt{\frac{\lambda^2-\mu^2}{(2\lambda+1)(2\lambda-1)}}\mathcal{G}_{\lambda-1,\mu}-\sqrt{\frac{(\lambda+1)^2-\mu^2}{(2\lambda+1)(2\lambda+3)}}\mathcal{G}_{\lambda+1,\mu}\right],$$

which follows when Eq. (3.11) is applied to the familiar spherical harmonic relation

$$Y_0^1(\hat{r})Y_\mu^\lambda(\hat{r})=\sum_l (1\lambda 0\mu|l\mu) \times(1\lambda 00|l0)\sqrt{\frac{3(2\lambda+1)}{4\pi(2l+1)}}Y_\mu^l(\hat{r}).$$

It was pointed out above that for on-shell matrix elements, when $\omega=\omega_{\beta\alpha}$, the electromagnetic field seen by the target is solenoidal. This is confirmed by Eqs. (3.16)–(3.18), where we see that the on-shell matrix elements involve only integrals of $\mathbf{j}_{\beta\alpha}$ with $\mathbf{L}j_\lambda Y_\mu^\lambda$ and $(\nabla\times\mathbf{L})j_\lambda Y_\mu^\lambda$. These integrals can be expressed in terms of the usual multipole matrix elements $\mathcal{M}(M\lambda,\mu)$ and $\mathcal{M}(E\lambda,\mu)$, using the definitions [19]

$$\int d^3r'\mathbf{j}_{\beta\alpha}(\mathbf{r}')\cdot\mathbf{L}j_\lambda\left(\frac{\omega}{c}r'\right)Y_\mu^\lambda(\hat{\mathbf{r}}') =i\left(\frac{\omega}{c}\right)^\lambda\frac{\hbar c(\lambda+1)}{(2\lambda+1)!!}\mathcal{M}(M\lambda,\mu), \tag{3.19}$$

$$\int d^3r'\mathbf{j}_{\beta\alpha}(\mathbf{r}')\cdot\nabla\times\mathbf{L}j_\lambda\left(\frac{\omega}{c}r\right)Y_\mu^\lambda(\hat{\mathbf{r}}') =\left(\frac{\omega}{c}\right)^{\lambda+1}\frac{\hbar c(\lambda+1)}{(2\lambda+1)!!}\mathcal{M}(E\lambda,\mu). \tag{3.20}$$

Thus we see that although $\mathcal{M}(M\lambda,\mu)$ and $\mathcal{M}(E\lambda,\mu)$ are sufficient to determine the interaction matrix elements in first-order perturbation theory, where only on-shell matrix elements are needed, in a more complete calculation we will also need the integral of $\rho_{\beta\alpha}j_\lambda Y_\mu^\lambda$. If the long-wavelength approximation is applicable, in which

$$\frac{\omega}{c}R_{\text{target}}\ll 1, \tag{3.21}$$

then we may use

$$\int \rho_{\beta\alpha}(\mathbf{r}')j_\lambda\left(\frac{\omega}{c}r'\right)Y_\mu^\lambda(\hat{\mathbf{r}}')d^3r'\sim\frac{(\omega/c)^\lambda}{(2\lambda+1)!!}\mathcal{M}(E\lambda,\mu). \tag{3.22}$$

It is shown in Appendix B that $V_{\beta\alpha}(\omega)$ is real if the trajectory lies in the $\hat{y}-\hat{z}$ plane, and if the phases of the target states are chosen so that the effect of complex conjugation on ϕ_M^J is to change it into $(-1)^{(J-M)}\phi_{-M}^J$. It is also shown that the hermiticity of $V(\zeta,t)$ implies that

$$V_{\beta\alpha}(\omega)=[V_{\alpha\beta}(-\omega)]^*=V_{\alpha\beta}(-\omega).$$

IV. METHODS OF SOLUTION OF THE COUPLED INTEGRAL EQUATIONS

A. Removing singularities of the kernel

If we attempt a direct numerical solution of the integral Eq. (2.11), we will be faced with difficulties associated with its singular kernel. Kowalski and Noyes (see, e.g., Ref. [13]) have devised a very effective method for removing singularities in scattering problems. In this section, we will adapt the Kowalski-Noyes approach to our equation for $T_{\beta\alpha}(\omega)$.

The general method will be clearer if we first consider a one-channel example of Eq. (2.11):

$$T(\omega)=V(\omega)-\frac{1}{2\pi}\int d\omega'\frac{V(\omega+\omega')T(-\omega')}{\omega'-i\delta}. \tag{4.1}$$

In particular, the $\omega=0$ version of this equation is

$$T(0)=V(0)-\frac{1}{2\pi}\int d\omega'\frac{V(\omega')T(-\omega')}{\omega'-i\delta}. \tag{4.2}$$

If we multiply Eq. (4.2) by $V(\omega)/V(0)$ and subtract it from Eq. (4.1), we get

$$\begin{aligned}T(\omega)&=T(0)\frac{V(\omega)}{V(0)}-\frac{1}{2\pi}\int d\omega' \times\frac{V(\omega+\omega')-V(\omega')[V(\omega)/V(0)]}{\omega'-i\delta}T(-\omega') \\ &=T(0)\frac{V(\omega)}{V(0)}-\frac{1}{2\pi}\int d\omega' \times\frac{V(\omega+\omega')-V(\omega')[V(\omega)/V(0)]}{\omega'}T(-\omega').\end{aligned} \tag{4.3}$$

We can omit the $-i\delta$ in the last denominator of Eq. (4.3) because the corresponding numerator vanishes when $\omega'=0$. Equivalently,

$$\Gamma(\omega)=U(\omega)-\frac{1}{2\pi} \times\int d\omega'\frac{V(\omega+\omega')-V(\omega')U(\omega)}{\omega'}\Gamma(-\omega'), \tag{4.4}$$

where we have introduced the ratios

$$U(\omega)\equiv\frac{V(\omega)}{V(0)},\ \Gamma(\omega)\equiv\frac{T(\omega)}{T(0)}.$$

Once we have solved the nonsingular Eq. (4.4), we can substitute its solution into the integral in Eq. (4.2) to get

$$T(0)=V(0)-\frac{1}{2\pi}\int d\omega'\frac{V(\omega')\Gamma(-\omega')}{\omega'-i\delta}T(0),$$

$$T(0)=\frac{V(0)}{1+(1/2\pi)\int d\omega'[V(\omega')\Gamma(-\omega')/(\omega'-i\delta)]},$$

$$T(\omega)=\frac{\Gamma(\omega)}{1+(1/2\pi)\int d\omega'[V(\omega')\Gamma(-\omega')/(\omega'-i\delta)]}V(0)$$

$$=\frac{\Gamma(\omega)}{1+(1/2\pi)\mathcal{P}\int(d\omega'/\omega')\,V(\omega')\Gamma(-\omega')+(i/2)V(0)}\times V(0). \tag{4.5}$$

Although we still have a principal-value integral to perform in the final step of Eq. (4.5), the singularity has been removed from the integral equation (4.4).

We now apply an analogous procedure to the many-channel coupled integral equations (2.11). We use the symbol $V_{\alpha\beta}$, without the argument ω, to represent the on-shell interaction matrix

$$V_{\alpha\beta}\equiv V_{\alpha\beta}(\omega_\alpha-\omega_\beta).$$

Similarly, $T_{\alpha\beta}$, without the argument ω, is used to represent the on-shell T matrix

$$T_{\alpha\beta}\equiv T_{\alpha\beta}(\omega_\alpha-\omega_\beta).$$

Then the matrix $U(\omega)$ is defined by

$$U_{\beta\gamma}(\omega)\equiv\sum_\lambda V_{\beta\lambda}(\omega-\omega_\lambda)[V^{-1}]_{\lambda\gamma}, \tag{4.6}$$

from which we see that

$$U_{\beta\gamma}(\omega_\beta)=\sum_\lambda V_{\beta\lambda}(\omega_\beta-\omega_\lambda)[V^{-1}]_{\lambda\gamma}=\sum_\lambda V_{\beta\lambda}[V^{-1}]_{\lambda\gamma}$$

$$=\delta_{\beta\gamma}.$$

Similarly, the matrix $\Gamma(\omega)$ is defined by

$$\Gamma_{\beta\gamma}(\omega)\equiv\sum_\lambda T_{\beta\lambda}(\omega-\omega_\lambda)[T^{-1}]_{\lambda\gamma}, \tag{4.7}$$

from which it follows that

$$\Gamma_{\beta\gamma}(\omega_\beta)=\delta_{\beta\gamma}.$$

Then by taking a linear combination of Eq. (2.11) and the version of Eq. (2.11) with $\beta=\gamma$ and $\omega=\omega_\gamma-\omega_\alpha$, we find that

$$T_{\beta\alpha}(\omega)=\sum_\gamma U_{\beta\gamma}(\omega+\omega_\alpha)T_{\gamma\alpha}-\frac{1}{2\pi}\sum_\lambda\int\frac{d\omega'}{\omega'+\omega_\lambda-\omega_\alpha}$$

$$\times\left[V_{\beta\lambda}(\omega+\omega')-\sum_\gamma U_{\beta\gamma}(\omega+\omega_\alpha)\right.$$

$$\left.\times V_{\gamma\lambda}(\omega_\gamma-\omega_\alpha+\omega')\right]T_{\lambda\alpha}(-\omega'). \tag{4.8}$$

We are again able to omit the $-i\delta$ from the denominator in the integrand, because the linear combination within square brackets has been constructed so that it vanishes when $\omega'=\omega_\alpha-\omega_\lambda$. If we replace ω in this equation by $\omega-\omega_\alpha$, and multiply from the right by T^{-1}, we get the nonsingular set

$$\Gamma_{\beta\alpha}(\omega)=U_{\beta\alpha}(\omega)-\frac{1}{2\pi}\sum_\lambda\int\frac{d\omega'}{\omega'+\omega_\lambda}$$

$$\times\left[V_{\beta\lambda}(\omega+\omega')-\sum_\gamma U_{\beta\gamma}(\omega)\right.$$

$$\left.\times V_{\gamma\lambda}(\omega_\gamma+\omega')\right]\Gamma_{\lambda\alpha}(-\omega'), \tag{4.9}$$

which generalizes Eq. (4.4). Finally, the solution of Eq. (4.9) is used, together with Eq. (4.7) in the on-shell version of Eq. (2.11) to obtain the desired on-shell T-matrix elements

$$T_{\beta\alpha}=\sum_\gamma[\mathcal{M}^{-1}]_{\beta\gamma}V_{\gamma\alpha}, \tag{4.10}$$

$$\mathcal{M}_{\beta\gamma}=\delta_{\beta\gamma}+\frac{1}{2\pi}\sum_\lambda\int\frac{V_{\beta\lambda}(\omega_\beta+\omega')\Gamma_{\lambda\gamma}(-\omega')}{\omega'+\omega_\lambda-i\delta}d\omega'=\delta_{\beta\gamma}$$

$$+\frac{1}{2\pi}\sum_\lambda\mathcal{P}\int\frac{d\omega'}{\omega'+\omega_\lambda}V_{\beta\lambda}(\omega_\beta+\omega')\Gamma_{\lambda\gamma}(-\omega')$$

$$+\frac{i}{2}\sum_\lambda V_{\beta\lambda}\Gamma_{\lambda\gamma}(\omega_\lambda)=\delta_{\beta\gamma}$$

$$+\frac{1}{2\pi}\sum_\lambda\mathcal{P}\int\frac{d\omega'}{\omega'+\omega_\lambda}V_{\beta\lambda}(\omega_\beta+\omega')\Gamma_{\lambda\gamma}(-\omega')$$

$$+\frac{i}{2}V_{\beta\gamma}. \tag{4.11}$$

This generalizes Eq. (4.5).

We see from Eq. (4.11) that the imaginary part of $\mathcal{M}$ is completely defined by the on-shell interaction matrix. The solution $\Gamma(\omega)$ of the integral equation (4.9) only contributes to $\mathcal{M}$ by its presence in the integrand of the principal-value integral.

The method described above is only applicable in situations in which the on-shell T matrix is nonsingular. If it *were* singular, it follows from Eq. (2.10) that the a matrix would have an eigenvalue of unity. In this situation, there would exist a set of initial amplitudes of the target states which are such that they are also the final amplitudes remaining after the interaction with the projectile has occurred. While this is a conceivable situation, we imagine that it must be very unlikely in a system of reasonable complexity, and thus we believe that the nonsingularity of the T matrix is a safe assumption. Similarly, a singular on-shell V matrix would imply that there exists a set of initial target-state amplitudes that are unaffected by the interaction in first order, which is also unlikely in a reasonably complex system.

B. Solution of the integral equation by matrix inversion

The most direct approach to numerical solution of a set of coupled linear integral equations is to approximate them by a set of coupled linear algebraic equations. The continuous variable ω is replaced by a discrete variable ω_i ($i=1,2,\ldots$), and the ω' integral in Eq. (4.9) is approximated by a sum over i. Equation (4.9) is thereby approximated by

the following set of simultaneous linear algebraic equations for the unknown numbers $\Gamma_{\beta\alpha}(\omega_i)$:

$$\sum_{\lambda j} A_{\beta i,\lambda j}\Gamma_{\lambda\alpha}(\omega_j)=U_{\beta\alpha}(\omega_i),$$

with

$$A_{\beta i,\lambda j}\equiv\delta_{\beta\lambda}\delta_{\omega_i-\omega_j}+\frac{\Delta}{2\pi(\omega_\lambda-\omega_j)}\left[V_{\beta\lambda}(\omega_i-\omega_j)-\sum_\gamma U_{\beta\gamma}(\omega_i)V_{\gamma\lambda}(\omega_\gamma-\omega_j)\right], \tag{4.12}$$

whose solution is

$$\Gamma_{\beta\alpha}(\omega_i)=\sum_{\lambda j}[A^{-1}]_{\beta i,\lambda j}U_{\lambda\alpha}(\omega_j). \tag{4.13}$$

Here Δ is the grid spacing, $\omega_{i+1}-\omega_i$. The numbers $\Gamma_{\beta\alpha}(\omega_i)$ determined from Eq. (4.13) are used for the calculation of the principal-value integrals in Eq. (4.11), and then the on-shell T matrix is calculated using Eq. (4.10).

Note that the matrix of coefficients $[A]$ defined in Eq. (4.12) is independent of α. $\Gamma_{\beta\alpha}(\omega_i)$, for different values of α, are obtained by changing the value of α on the right-hand side of Eq. (4.13), using the same matrix $[A]$. If our grid size and range determine that we have N values of ω_i, and we choose to couple n target states, then the dimension of the matrix $[A]$ will be Nn.

The grid spacing Δ should be small compared to the ω interval over which the $V_{\beta\alpha}(\omega)$ change appreciably, and also small compared to the differences between the on-shell values ω_λ. If the electromagnetic pulse lasts for time $\tau_{\rm sweep}$, then the $V_{\beta\alpha}(\omega)$ can be expected to vary significantly when ω changes by an amount $1/\tau_{\rm sweep}$. For relativistic projectiles and typical nuclear dimensions,

$$\frac{1}{\tau_{\rm sweep}}\sim\frac{1}{(2R)/c}\sim\frac{\hbar c}{2R\hbar}\sim\frac{100\,\text{MeV fm}}{R\hbar}.$$

For excitation of nuclear giant resonances, the ω_λ differences are of the order of 10–20 MeV/$\hbar$. These considerations indicate that a choice of Δ of a few MeV/$\hbar$ should be an adequate approximation to the continuous variable ω, and we have used this choice in our numerical calculations.

If $V_{\beta\alpha}(t)$ varies appreciably over a time interval δT, then the $V_{\beta\alpha}(\omega)$ will be significant over an $|\omega|$ range up to about $1/\delta T$. For the sharp electromagnetic pulse associated with a relativistic projectile, δT will be of the order of magnitude of the time it takes for the pulse to move across significant features of the nuclear density, such as the nuclear skin thickness. This leads to an estimate of a total ω width of

$$\frac{1}{0.5\,\text{fm}/c}\sim\frac{\hbar c}{0.5\,\text{fm}\,\hbar}\sim400\frac{\text{MeV}}{\hbar}.$$

It will be seen in our numerical examples that $V_{\beta\alpha}(\omega)$ is very small outside the ω range of -400 MeV/$\hbar<\omega<400$ MeV/$\hbar$. Of course, we can check whether this choice is adequate by seeing if the calculated $\Gamma_{\beta\alpha}(\omega_i)$ are negligibly small when ω_i approaches the limits of the chosen range or by testing the sensitivity of the calculated excitation probabilities to changes in this range.

C. The Born series for the on-shell T matrix

The Born series is an expansion of the on-shell T matrix in powers of the interaction V

$$T=t_1+t_2+t_3+\cdots. \tag{4.14}$$

Here t_n represents all the terms containing n products of V. A corresponding expansion of $\mathcal{M}$ of Eq. (4.11) in powers of V is

$$\mathcal{M}=m_0+m_1+m_2+\cdots, \tag{4.15}$$

with $m_0=\mathbf{1}$. Then Eq. (4.10) implies that

$$\mathcal{M}\cdot T=V,$$

$$(\mathbf{1}+m_1+m_2+m_3+\cdots)\cdot(t_1+t_2+t_3+\cdots)=V. \tag{4.16}$$

Identifying terms on the left-hand side of Eq. (4.17) corresponding to n powers of V, we get

$$t_1=V,$$

$$t_n=-\sum_{r=1}^{n-1}m_{n-r}t_r\ (n=2,3,4,\ldots). \tag{4.17}$$

The first few terms of Eq. (4.17) imply that

$$t_2=-m_1t_1=-m_1V,$$

$$t_3=-m_2t_1-m_1t_2=(-m_2+m_1^2)V,$$

and so forth. We can obtain the coefficients m_{n-r} we need in Eq. (4.17) by combining the definition (4.11) for $\mathcal{M}$ with an expansion of $\Gamma(-\omega')$ in powers of V:

$$\Gamma(-\omega')=\Gamma^{(0)}(-\omega')+\Gamma^{(1)}(-\omega')+\Gamma^{(2)}(-\omega')+\cdots. \tag{4.18}$$

If this expansion is substituted into the integral equation for Γ [Eq. (4.9)], and then terms of n powers of V are identified, one finds that

$$\Gamma^{(n+1)}_{\beta\alpha}(\omega)=-\frac{1}{2\pi}\sum_\lambda\int\frac{d\omega'}{\omega'+\omega_\lambda}\times\left[V_{\beta\lambda}(\omega+\omega')-\sum_\gamma U_{\beta\gamma}(\omega)V_{\gamma\lambda}(\omega_\gamma+\omega')\right]\times\Gamma^{(n)}_{\lambda\alpha}(-\omega').$$

This recursion relation can be solved for the $\Gamma^{(n)}(\omega)$, starting with $\Gamma^{(0)}(\omega)=U(\omega)$, and these $\Gamma^{(n)}(\omega)$ can be used together with Eqs. (4.18) and (4.11) to obtain the coefficients m_{n-r} needed in Eq. (4.17). Thus we have a well-defined procedure for determining successive terms of the Born ex-

pansion of the on-shell T matrix. The relationship between this expansion and the perturbative expansion of Alder and Winther [1] is discussed in Appendix C.

We illustrate this procedure by calculating t_2, the second-order contribution to the on-shell T matrix. This requires m_1, the first-order contribution to $\mathcal{M}$ of Eq. (4.11),

$$\begin{aligned}(m_1)_{\beta\gamma}&=\frac{i}{2}V_{\beta\gamma}+\frac{1}{2\pi}\sum_\lambda \mathcal{P}\int\frac{d\omega'}{\omega'+\omega_\lambda}V_{\beta\lambda}(\omega_\beta+\omega')\\&\quad\times\Gamma^{(0)}_{\lambda\gamma}(-\omega')\\&=\frac{i}{2}V_{\beta\gamma}+\frac{1}{2\pi}\sum_\lambda \mathcal{P}\int\frac{d\omega'}{\omega'+\omega_\lambda}V_{\beta\lambda}(\omega_\beta+\omega')\\&\quad\times U_{\lambda\gamma}(-\omega').\end{aligned}$$

Then

$$\begin{aligned}(t_2)_{\beta\alpha}&=(-m_1t_1)_{\beta\alpha}=(-m_1V)_{\beta\alpha}\\&=-\frac{i}{2}\sum_\lambda V_{\beta\lambda}V_{\lambda\alpha}-\frac{1}{2\pi}\sum_\lambda \mathcal{P}\\&\quad\times\int\frac{d\omega'}{\omega'+\omega_\lambda}V_{\beta\lambda}(\omega_\beta+\omega')V_{\lambda\alpha}(-\omega'-\omega_\alpha)\\&=-\frac{1}{2\pi}\int d\omega'\sum_\lambda\frac{V_{\beta\lambda}(\omega_\beta+\omega')V_{\lambda\alpha}(-\omega'-\omega_\alpha)}{\omega'+\omega_\lambda-i\delta}\\&=-\frac{1}{2\pi}\int d\omega'\sum_\lambda\frac{V_{\beta\lambda}(\omega_\beta-\omega_\alpha+\omega')V_{\lambda\alpha}(-\omega')}{\omega'+\omega_\lambda-\omega_\alpha-i\delta}.\end{aligned}$$

This agrees with the second-order term one can obtain from the Born expansion of the integral equation (2.11) for $T_{\beta\alpha}(\omega_\beta-\omega_\alpha)$.

D. Sudden approximation

Suppose that the external impulse experienced by the target is of much shorter duration than the characteristic oscillation modes of the target. In this case, the interaction matrix elements $V_{\beta\lambda}(\omega+\omega')$ in Eq. (2.11) vary by small fractions of themselves in the ω range defined by the differences $\omega_\lambda-\omega_\alpha$ in the denominator of Eq. (2.11). Then we can get a useful approximation to Eq. (2.11) by ignoring these differences $\omega_\lambda-\omega_\alpha$, so that Eq. (2.11) is replaced by

$$T_{\beta\alpha}(\omega)=V_{\beta\alpha}(\omega)-\frac{1}{2\pi}\sum_\lambda\int d\omega'\frac{V_{\beta,\lambda}(\omega+\omega')T_{\lambda\alpha}(-\omega')}{\omega'-i\delta}. \tag{4.19}$$

This is equivalent to ignoring the presence of $e^{(i/\hbar)H_0t}$ in the time-dependent Schrödinger equation (2.3), leading to

$$\frac{\partial}{\partial t}\psi(\zeta,t)=\frac{1}{i\hbar}V(\zeta,t)\psi(\zeta,t).$$

The solution of this equation, corresponding to initial condition $\psi(\zeta,-\infty)=\phi_\alpha(\zeta)$, is

$$\psi_\alpha(t)=e^{\int_{-\infty}^t(dt'/i\hbar)V(t')}\phi_\alpha.$$

If this solution is substituted into the definition (2.7) of $T_{\beta\alpha}(\omega)$, the result is

$$T_{\beta\alpha}(\omega)=\int_{-\infty}^{\infty}dte^{i\omega t}\langle\phi_\beta|V(t)|e^{\int_{-\infty}^t(dt'/i\hbar)V(t')}\phi_\alpha\rangle. \tag{4.20}$$

It can be verified that Eq. (4.20) satisfies the sudden-approximation integral equation (4.19).

The excitation amplitudes are obtained, according to Eq. (2.10), from the on-shell values of $T_{\beta\alpha}(\omega)$. In the sudden-approximation limit, on-shell implies $\omega=0$. Thus we need

$$\begin{aligned}T_{\beta\alpha}(0)&=\int_{-\infty}^{\infty}dt\langle\phi_\beta|V(t)|e^{\int_{-\infty}^t(dt'/i\hbar)\,V(t')}\phi_\alpha\rangle\\&=\int_{-\infty}^{\infty}dt\frac{d}{dt}\langle\phi_\beta|e^{\int_{-\infty}^t(dt'/i\hbar)\,V(t')}|\phi_\alpha\rangle\\&=\langle\phi_\beta|e^{\int_{-\infty}^{\infty}(dt'/i\hbar)\,V(t')}|\phi_\alpha\rangle-\delta_{\beta\alpha}.\end{aligned}$$

If this result is used in Eq. (2.10), we get the sudden-approximation excitation amplitudes

$$\begin{aligned}a_{\beta\alpha}&=\delta_{\beta\alpha}-iT_{\beta\alpha}(0),\\&=\langle\phi_\beta|e^{\int_{-\infty}^{\infty}(dt'/i\hbar)\,V(t')}|\phi_\alpha\rangle.\end{aligned}$$

This formula has a strong similarity with a corresponding expression for the multichannel eikonal approximation. This is not surprising since, as will be demonstrated in Sec. VI, there is a close correspondence between Coulomb excitation theory and the multichannel eikonal approximation.

Unfortunately, the sudden approximation is of little use in the investigation of reactions in which states in the giant resonance region are excited, for which we encounter $\Delta_{\lambda,\alpha}=\omega_\lambda-\omega_\alpha\sim10-20$ MeV/$\hbar$. At relativistic projectile speeds, the duration of the excitation pulse is of the order of $2R/c$, so the applicability of the sudden approximation would require that

$$\frac{2R}{c}\ll\frac{1}{\Delta_{\lambda,\alpha}\text{MeV}/\hbar},$$

$$2R\ll\frac{\hbar c}{\Delta_{\lambda,\alpha}\,\text{MeV}}\sim\frac{200\text{ MeV fm}}{\Delta_{\lambda,\alpha}\,\text{MeV}}\sim10-20\,\text{fm}.$$

This criterion would be satisfied only for the lightest target nuclei, and would not be satisfied for a medium-weight target such as calcium. However, if we were interested in the population of low-energy rotational levels, for which $\omega_\lambda-\omega_\alpha\sim0.1$ MeV/$\hbar$, the criterion for the validity of the sudden approximation in a peripheral relativistic collision would be satisfied by all nuclei.

V. EXAMPLES OF NUMERICAL RESULTS AND EXPLORATORY STUDIES

A. The physical model

To illustrate the methods described in the previous sections, we will analyze a simple semirealistic example of relativistic Coulomb excitation. We consider the excitation of a 2^+ level of ^{40}Ca at 20 MeV, exhausting 100% of the EWSR, which amounts to a reduced transition probability $B(E2,0\rightarrow 2)=450\ e^2\,\mathrm{fm}^4$. To this 2^+ level, we ascribe a static quadrupole moment Q_0 of 15 e fm^2. The occurrence of a static quadrupole moment implies the presence of anharmonic terms in the nuclear Hamiltonian. According to Eq. (6-240) of Ref. [19], anharmonic effects that can yield this value of Q_0 can also yield ''reorientation reduced transition probabilities'' of about 75 $e^2\,\mathrm{fm}^4$ between one-phonon states. These are about 1/6 of the reduced transition probability for the allowed ($\Delta n=1, 0^+\rightarrow 2^+$) transition. The excitation of these states is due to the electromagnetic field of a relativistic ^{208}Pb nucleus, moving parallel to the $\hat{z}$ axis in the $\hat{z}$-$\hat{y}$ plane.

We approximate the transition current and charge densities needed in Eqs. (3.16)–(3.18) by using the Tassie model [20], which describes the 2^+ resonance as a one-quantum vibrational oscillation of an incompressible irrotational fluid. This leads to expressions of the form

$$\mathbf{j}(\mathbf{r})=\rho_0(r)\sum_{\lambda,\mu} K_\mu^\lambda(\zeta) r^{\lambda-1}([Y^{\lambda-1}(\hat{r})\hat{\xi}^1]_\mu^\lambda)^*, \tag{5.1}$$

$$\begin{aligned}\rho(\mathbf{r})&=\frac{i}{\omega_{\beta\alpha}}\nabla\cdot\mathbf{j}(\mathbf{r})\\ &=\frac{i}{\omega_{\beta\alpha}}\sqrt{\frac{\lambda}{2\lambda+1}}r^{\lambda-1}\rho_0'(r)\sum_\mu K_\mu^\lambda(\zeta)Y_\mu^\lambda(\hat{r}),\end{aligned} \tag{5.2}$$

where ρ_0 is the equilibrium proton density, and $K_\mu^\lambda(\zeta)$ are operators that act on the target degrees of freedom ζ. The $\hat{\xi}_\mu^1$ are unit vectors defined by

$$\hat{\xi}_{\pm 1}^1=(\mp 1)\frac{1}{\sqrt{2}}[\hat{x}\pm\hat{y}],$$

$$\hat{\xi}_0^1=\hat{z}.$$

In the present application we have $\lambda=2$ and $\omega_{\beta\alpha}=20$ MeV/$\hbar$. The operators K_μ^2 are normalized in terms of the value of

$$B(E2,J_\alpha\rightarrow J_\beta)\equiv\sum_{M_\beta,\mu}|\langle\phi_{M_\beta}^{J_\beta}|\mathcal{M}(E2,\mu)|\phi_{M_\alpha}^{J_\alpha}\rangle|^2,$$

calculated using Eq. (5.1). We follow Suzuki and Rowe [21] in using

$$\rho_0(r)=\frac{1}{4\pi^{3/2}b^3}\left(40+32\frac{r^4}{b^4}\right)e^{-(r^2/b^2)}\ (b=1.887\ \mathrm{fm}).$$

Since the operators K_μ^2 involve the creation or annihilation of a vibrational quantum, they yield zero matrix elements between the different magnetic substates of the 2^+ level. Since these substates are degenerate,

$$\omega_{\beta\alpha}=0=\nabla\cdot\mathbf{j}(\mathbf{r}').$$

For simplicity, we calculate these ''reorientation'' matrix elements by making the further assumption that $\mathbf{j}_{\beta\alpha}=0$, so that only the contribution (3.18) survives, which we calculate using

$$\rho(\mathbf{r})=\rho_0(r)\sum_\mu Q_\mu^2(\zeta)r^2[Y_\mu^2(\hat{r})]^* \tag{5.3}$$

in Eq. (3.18), and normalize the reduced matrix element of $Q_\mu^2(\zeta)$ in terms of the static quadrupole moment of the excited level. The ω-grid spacing was chosen to allow the significant ω range to be divided in 801 points. This yielded $\Delta\omega=1$ MeV/$\hbar$ at $E_P/A=1$ GeV, and $\Delta\omega=2.5$ MeV/$\hbar$ at $E_P/A=100$ GeV.

Since the trajectory of the projectile is in the $\hat{z}$-$\hat{y}$ plane, the only target states that are connected to the ground state by the interaction are those that are symmetric with respect to a reflection across the $\hat{z}-\hat{y}$ plane. Thus we only need to include the four states

$$\begin{aligned}\phi_0&\equiv\phi_0^{0^+}(\text{ground state}),\\ \phi_1&\equiv\phi_0^{2^+},\\ \phi_2&\equiv\frac{1}{\sqrt{2}}(\phi_1^{2^+}-\phi_{-1}^{2^+}),\\ \phi_3&\equiv\frac{1}{\sqrt{2}}(\phi_2^{2^+}+\phi_{-2}^{2^+}).\end{aligned} \tag{5.4}$$

This requires the calculation and inversion of a 3204×3204 real matrix, which can be done without difficulty on a modern computer workstation.

B. General characteristics of the interaction matrix elements

Many of the features of our numerical results for excitation probabilities depend upon the general characteristics of the interaction matrix elements $V_{0\alpha}(\omega)$ ($\alpha=1,2,3$) connecting the ground state and the states of the 2^+ level. Figures 1(a)–1(c) show plots of $V_{0\alpha}(\omega)$ as a function of ω for various bombarding energies. It is immediately apparent that for very high projectile energies ($\gamma\gg 1$), the $\Delta M=\pm 1,\pm 2$ matrix elements are very nearly independent of projectile energy, whereas the $\Delta M=0$ matrix element continues to increase. It is shown in Appendix A that these behaviors are a consequence of the high-energy asymptotic form of the electromagnetic pulse. The $\Delta M=0$ matrix element also exhibits interesting behavior at the on-shell value of ω (20 MeV/$\hbar$ in the present example). There is some indication in Fig. 1(a) that this matrix element is very small on-shell, and we now show that this is indeed the case.

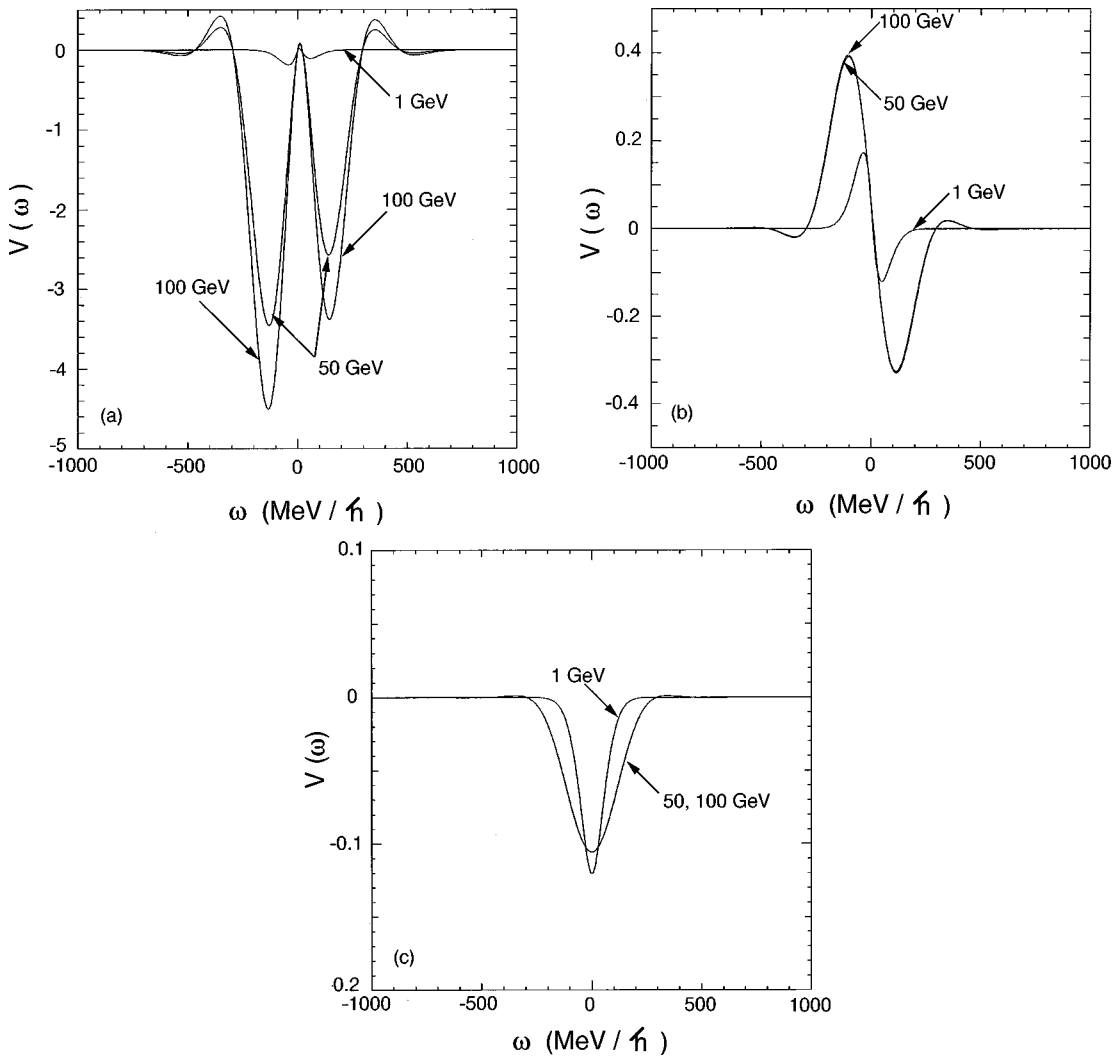

FIG. 1. (a) The Fourier transform of the matrix element of the electromagnetic interaction between the ground state in ^{40}Ca and the component $L=2^+$, $M=0$ of our model state at 20 MeV. The interaction is with a ^{208}Pb projectile, at an impact parameter of 10 fm, and with kinetic energy per nucleon of 1, 50, or 100 GeV. (b) The same as for (a), except that the excited state of ^{40}Ca is the reflection-symmetric linear combination $(\phi_1^2-\phi_{-1}^2)/\sqrt{2}$. (c) The same as for (a), except that the excited state of ^{40}Ca is the reflection-symmetric linear combination $(\phi_2^2+\phi_{-2}^2)/\sqrt{2}$.

According to Eq. (3.8), the on-shell value of $V_{\beta\alpha}(\omega)$ is given by

$$V_{\beta\alpha}(\omega_{\beta\alpha})=\frac{i}{\omega}\int d^3\mathbf{r}'\mathbf{j}_{\beta\alpha}(\mathbf{r}')\cdot\mathbf{E}_C^{\rm ret}(\mathbf{r}',\omega_{\beta\alpha}). \qquad (5.5)$$

Using Eqs. (3.7) and (3.3), we have

$$\mathbf{E}_C^{\rm ret}(\mathbf{r}',\omega)=-\nabla\varphi_C^{\rm ret}(\mathbf{r}',\omega)+\frac{i\omega}{c}\mathbf{A}_C^{\rm ret}(\mathbf{r}',\omega)$$

$$=-\nabla\varphi_C^{\rm ret}(\mathbf{r}',\omega)+\frac{i\omega v}{c^2}\varphi_C^{\rm ret}(\mathbf{r}',\omega)\hat{z}.$$

Now let us use for the $M=0$ part of $\varphi_C^{\rm ret}(\mathbf{r}',\omega)$ the simple approximate expression (A17), valid when $\gamma v/\omega \gg b$,

$$\varphi_C^{\rm ret}(\mathbf{r}',\omega)=\frac{2Z_Pe}{\hbar v}\sqrt{4\pi}e^{i(\omega z'/v)}\times\left[-\ln\left(\frac{b|\omega|}{v\gamma}\right)Y_0^0(\theta',\phi')\right].$$

Since the only spatial dependence is in the factor $e^{i(\omega z'/v)}$, it follows that

$$\nabla\varphi_C^{\rm ret}(\mathbf{r}',\omega)=i\frac{\omega}{v}\times\varphi_C^{\rm ret}(\mathbf{r}',\omega)\hat{z}$$

so that the $M=0$ part of $\mathbf{E}_C^{\rm ret}(\mathbf{r}',\omega)$ is approximately given by

$$\left(-i\frac{\omega}{v}+\frac{i\omega v}{c^2}\right)\varphi_C^{\rm ret}(\mathbf{r}',\omega)\hat{z}=-i\frac{\omega}{v\gamma^2}\varphi_C^{\rm ret}(\mathbf{r}',\omega)\hat{z}$$

$$=i\frac{\omega}{v\gamma^2}\frac{2Z_Pe}{\hbar v}$$

$$\times\ln\left(\frac{b|\omega|}{v\gamma}\right)e^{i(\omega z'/v)}.$$

TABLE I. Excitation probabilities of the states of the $L=2^+$ level. Comparison between the Born approximation and the exact result for values of bombarding energies of the projectile ranging from 1 to 100 GeV per nucleon. In this and in the following tables we use the notation -1, $+3$, e.g., for the multiplicative factors 10^{-1}, 10^{+3}.

b	E_P/A	$M=0$		$M=\pm1$		$M=\pm2$	
(fm)	(GeV)	Born	Exact	Born	Exact	Born	Exact
	1	0.722-3	0.744-3	0.461-2	0.470-2	0.122-1	0.119-1
	5	0.219-4	0.104-3	0.287-2	0.314-2	0.109-1	0.101-1
	10	0.310-5	0.122-2	0.279-2	0.341-2	0.108-1	0.896-2
10	15	0.894-6	0.269-2	0.278-2	0.364-2	0.108-1	0.803-2
	20	0.357-6	0.358-2	0.277-2	0.364-2	0.108-1	0.730-2
	50	0.165-7	0.251-3	0.277-2	0.132-2	0.108-1	0.481-2
	100	0.146-8	0.445-1	0.277-2	0.241-2	0.108-1	0.293-2
20	100	0.106-8	0.896-3	0.692-3	0.313-3	0.675-3	0.324-3

This vanishes in the $\gamma\to\infty$ limit, which implies that the on-shell $\Delta M=0$ interaction matrix element will become very small at sufficiently high bombarding energy. It is shown in Appendix A that the off-shell $\Delta M=0$ matrix element diverges at high bombarding energy in proportion to $\ln(\gamma)$.

C. Comparison of Born approximation with exact excitation probabilities

Table I shows excitation probabilities for the $M=0$ state of the $L=2$ level, and for the reflection-symmetric $M=\pm1$ and $M=\pm2$ combinations of states, at various bombarding energies per nucleon between 1 and 100 GeV. Comparison is made between the exact results and those obtained by means of the first Born approximation, in which the on-shell $T_{\beta\alpha}(\omega_{\beta\alpha})$ is approximated by the on-shell interaction matrix element $V_{\beta\alpha}(\omega_{\beta\alpha})$.

It is seen that when $E_P/A=1$ GeV, the Born approximation gives a very good representation of the excitation probabilities. However, as the bombarding energy increases, the fit between the Born approximation and the exact values deteriorates. This is especially true in the case of the excitation of the $L=2$, $M=0$ state. As explained in the previous section, the on-shell interaction matrix element is expected to become very small at high bombarding energy, leading to a very small Born approximation prediction for the $L=2$, $M=0$ excitation probability. However, the exact calculation of this excitation probability involves off-shell matrix elements of the interaction potential, and indirect transitions to the $L=2$, $M=0$ state via the $L=2$, $M=\pm1$ and $L=2$, $M=\pm2$ states. Therefore the weakness of the direct $L=0$, $M=0\to L=2$, $M=0$ on-shell matrix element does not prevent the exact calculation from yielding a relatively large excitation probability. The constancy, with increasing bombarding energy, of the Born predictions for the $M=\pm1$ and $M=\pm2$ excitation probabilities is a consequence of the constancy of the corresponding on-shell matrix elements $V_{\beta\alpha}(\omega_{\beta\alpha})$, as explained in Appendix A. This constancy is *not* exhibited by the exact values of the $M=\pm1,\pm2$ excitation probabilities, because the latter are affected by the strong variation with bombarding energy of the $M=0$ matrix element. It is seen from Table I that whereas the $M=\pm2$ state is excited most strongly at $E_P/A=1$ GeV, the $M=0$ state is populated most strongly at 100 GeV/nucleon.

The last line of Table I shows excitation probabilities at $E_P/A=100$ GeV at an impact parameter of 20 fm. Although the excitation probabilities are weaker than they were at the grazing impact parameter of 10 fm, the ratios of the different matrix elements are comparable. In particular, the Born approximation to the population of the $L=2$, $M=0$ state is many orders of magnitude smaller than the exact value.

The differences between the Born and exact excitation probabilities shown in Table I are due to the inclusion in the exact calculation of the effect of the off-shell values of ω, brought in by the integral equation (2.11), and to the inclusion of reorientation couplings between the different $L=2$ states. To disentangle these two effects, we did a series of calculations in which the $L=0$, $M=0$ ground state was separately coupled to each of the $L=2$ states, with no reorientation couplings. These results are shown in Table II. Now the differences between the Born and exact values are due only to the inclusion of off-shell ω values in the latter. Not surprisingly, these differences are not as great as when reorientation couplings are included. Nevertheless, the exact $M=0$ excitation probability is 5 orders of magnitude greater than the Born value at $E_P/A=100$ GeV, showing the importance of off-shell ω values in this case.

D. The convergence of the Born series

In Sec. IV C it was shown how the full Born series could be calculated from an iterative solution of the Γ-matrix equation. Table III summarizes some results for the convergence of the Born series at bombarding energies per nucleon of 1, 3, 10, and 100 GeV. It is seen that at $E_P/A=1$ and 3 GeV, the convergence to the exact values is smooth and rapid. At $E_P/A=10$ GeV, the series approximation improves as we proceed from the first Born approximation to the inclusion of the first five terms of the Born series. At this stage we are within a few percent of the exact values. However, as further terms of the Born series are included, the approximation to the exact values deteriorates, and by the 12th iteration the series is wildly divergent. Similar behavior is observed at bombarding energy per nucleon from 10 to 15 GeV. At

TABLE II. Excitation probabilities of the states of the $L=2+$ level, without any re-orientation transitions among the M states. The impact parameter is 10 fm. The Born approximation and the exact solution are compared.

	$M=0$		$M=\pm1$		$M=\pm2$	
E_P/A	Born	Exact	Born	Exact	Born	Exact
1	0.722-3	0.727-3	0.461-2	0.462-2	0.122-1	0.122-1
5	0.219-4	0.594-4	0.287-2	0.293-2	0.109-1	0.109-1
20	0.357-6	0.443-2	0.277-2	0.288-2	0.108-1	0.108-1
50	0.165-7	0.408-2	0.277-2	0.288-2	0.108-1	0.108-1
70	0.513-8	0.153-2	0.277-2	0.289-2	0.108-1	0.108-1
100	0.146-8	0.184-3	0.277-2	0.289-2	0.108-1	0.107-1

higher bombarding energies, there is no stage of iteration at which the Born series gives a good approximation to the exact values.

The tentative convergence, and then divergence, at $E_P/A=10$ GeV can be interpreted in terms of the Schmidt-Weinberg method for the solution of integral equations [22]. In this approach, the solution is separated into a convergent series, plus a finite term which is associated with one or more eigenvalues of the resolvent which are greater than unity. If the coefficient of this finite term is small, it plays a very minor role at the early stages of the iteration process. However, eventually it grows exponentially and overwhelms the convergent part of the solution. It is evident from Table III that even in a situation in which the Born series eventually diverges, it is possible to get a very good approximation to the exact solution by stopping the iteration at the appropriate place. In our situation we can see where that place is, since we know the exact result. Unfortunately, in the absence of an exact solution, it is difficult to extract reliable information from a divergent series.

The divergence of the Born series at high bombarding energies is associated with the divergence of the $M=0$ interaction matrix element. This is shown by comparing Table III with Table IV where we include only separate couplings of the ground state with each $L=2$ state, with no reorientation couplings. It is seen that the Born series converges in the $(L=0, M=0\leftrightarrow L=2, M=\pm1)$ and $(L=0, M=0\leftrightarrow L=2, M=\pm2)$ systems at all bombarding energies. In the $(L=0, M=0\leftrightarrow L=2, M=0)$ system we have convergence up to $E_P=5$ GeV, but divergence at $E_P=7$ GeV (and above).

E. The long-wavelength approximation

The long-wavelength approximation (LWL) is based upon the substitution

$$j_\lambda\left(\frac{\omega}{c}r'\right)\sim\frac{[(\omega/c)r']^\lambda}{2(\lambda+1)!!}$$

in the evaluation of the matrix elements (3.16)–(3.18). This substitution is valid when

$$\frac{\omega}{c}r'<\lambda,$$

TABLE III. Excitation probabilities of the states of the $L=2^+$ level, calculated using different orders of the Born series. The impact parameter is 10 fm. (See the text for more details.)

E_P/A	M	Born	3rd iter.	5th iter.	10th iter.	15th iter.	Exact
1	0	0.722-3	0.766-3	0.744-3	0.744-3	0.744-3	0.744-3
GeV	±1	0.461-2	0.471-2	0.470-2	0.470-2	0.470-2	0.470-2
	±2	0.122-1	0.121-1	0.119-1	0.119-1	0.119-1	0.119-1
E_P/A	M	Born	3rd iter.	5th iter.	10th iter.	15th iter.	Exact
3	0	0.785-4	0.267-4	0.269-4	0.269-4	0.269-4	0.270-4
GeV	±1	0.306-2	0.321-2	0.319-2	0.319-2	0.319-2	0.319-2
	±2	0.111-1	0.107-1	0.106-1	0.106-1	0.106-1	0.106-1
E_P/A	M	Born	3rd iter.	5th iter.	8th iter.	12th iter.	Exact
10	0	0.310-5	0.179-2	0.124-2	0.936-3	0.658+1	0.122-2
GeV	±1	0.279-2	0.367-2	0.337-2	0.379-2	0.117+3	0.341-2
	±2	0.108-1	0.912-2	0.892-2	0.746-2	0.189+3	0.896-2
E_P/A	M	Born	2nd iter.	3rd iter.	5th iter.	7th iter.	Exact
100	0	0.146-8	0.171+0	0.267+0	0.378+0	0.226+7	0.445-1
GeV	±1	0.277-2	0.303-2	0.153-1	0.210+0	0.161+7	0.241-2
	±2	0.108-1	0.117-1	0.488-2	0.851-3	0.597+4	0.293-2

TABLE IV. Excitation probabilities of the states of the $L=2^+$, without any reorientation transition among the M states, calculated using different orders of the Born series. The impact parameter is 10 fm. More details can be found in the text.

M	E_P/A	Born	3rd iter.	5th iter.	10th iter.	15th iter.	Exact
0	1	0.722-3	0.727-3	0.727-3	0.727-3	0.727-3	0.727-3
	5	0.219-4	0.583-4	0.594-4	0.594-4	0.594-4	0.594-4
	7	0.875-5	0.909-4	0.969-4	0.964-4	0.241+3	0.931-4
±1	1	0.461-2	0.462-2	0.462-2	0.462-2	0.462-2	0.462-2
	100	0.277-2	0.289-2	0.289-2	0.289-2	0.289-2	0.289-2
±2	1	0.122-1	0.122-1	0.122-1	0.122-1	0.122-1	0.122-1
	100	0.108-1	0.108-1	0.108-1	0.108-1	0.108-1	0.107-1

$$\hbar\omega < \frac{\hbar c}{r'}\lambda = \frac{197.3\ \text{MeV fm}}{r'}\lambda.$$

For our off-diagonal matrix elements, $\lambda=2$, and r' extends out to about 5 fm. This implies that we can use the LWL to evaluate our matrix elements as long as

$$\hbar\omega < \sim 80\ \text{MeV}.$$

Thus we can use the LWL for evaluating the on-shell matrix elements (for which $\hbar\omega=20$ MeV). However, if the solution of Eq. (4.9) depends upon $V_{\beta\alpha}(\omega')$ with ω' far off-shell, then the LWL may not be adequate. It is difficult to assess the importance of far-off-shell ω' in the integral equation, since the ω' integration can involve cancellation from different regions of the ω' range. Thus we have calculated our matrix elements using the exact expression for the $j_\lambda[(\omega/c)r']$, and the LWL, and used both sets of matrix elements in the exact solution of the integral equation. Some examples of this comparison are shown in Table V. It is seen that the LWL does well at $E_P/A=1$ GeV. Above $E_P/A=3$ GeV, the LWL appears to give an inadequate representation of the excitation amplitudes.

VI. FURTHER DEVELOPMENTS

Until now, the giant resonance states included in our calculations have been treated as if they were normalizable bound states, although they are really continuum states. In this section, we generalize our treatment so as to include both bound and continuum target states, the latter being used to describe the actual giant resonance states.

TABLE V. Test of the use of the long-wavelength approximation, Eq. (5.6), in the evaluation of the interaction matrix elements. The impact parameter is 10 fm.

E_P/A	L,M	Exact	LWL
1	2,0	0.744-3	0.730-3
GeV	2,±1	0.470-2	0.461-2
	2,±2	0.119-1	0.118-1
3	2,0	0.269-4	0.366-4
GeV	2,±1	0.319-2	0.326-2
	2,±2	0.106-1	0.100-1
5	2,0	0.104-3	0.202-2
GeV	2,±1	0.314-2	0.300-2
	2,±2	0.101-1	0.729-2

Our derivation of the T-matrix equation (2.11) began by premultiplying the integral form of the Schrödinger equation (2.4) by $V(t)$. Then the target-state completeness relation was used. We proceed now in a similar manner, but we separate the completeness relation for the target states into discrete and continuum parts:

$$1=\sum_\lambda |\phi_\lambda\rangle\langle\phi_\lambda| + \int d\epsilon |\phi_\epsilon\rangle\langle\phi_\epsilon|.$$

This leads to

$$\begin{aligned} V(t)\psi_\alpha(\zeta,t) = {} & e^{-(i/\hbar)E_\alpha t}V\phi_\alpha \\ & + \sum_\lambda \int_{-\infty}^{t}\frac{dt'}{i\hbar}e^{-(i/\hbar)E_\lambda(t-t')}V(t')\phi_\lambda\rangle \\ & \times\langle\phi_\lambda V(t')\psi_\alpha(\zeta,t') \\ & + \int d\epsilon\int_{-\infty}^{t}\frac{dt'}{i\hbar}e^{-(i/\hbar)\epsilon(t-t')}V(t')\phi_\epsilon\rangle \\ & \times\langle\phi_\epsilon V(t')\psi_\alpha(\zeta,t'). \end{aligned} \tag{6.1}$$

We now assume that the continuum integration is dominated by a series of ''resonance'' states $|\Phi_\mu\rangle$ centered at energies E_μ. In the vicinity of E_μ, we further assume that

$$\langle\phi_{E_\mu+\epsilon}|V \approx \langle\phi_{E_\mu+\epsilon}|\Phi_\mu\rangle\langle\Phi_\mu|V$$

$$V|\phi_{E_\mu+\epsilon}\rangle \approx V|\Phi_\mu\rangle\langle\Phi_\mu|\phi_{E_\mu+\epsilon}\rangle,$$

i.e., we assume that $\phi_{E_\mu+\epsilon}$ is affected by the interaction only to the extent to which $\phi_{E_\mu+\epsilon}$ overlaps with Φ_μ. Moreover it is assumed that the ''strength function'' $s_\mu(\epsilon)$, defined by

$$s_\mu(\epsilon)\equiv\langle\Phi_\mu|\phi_{E_\mu+\epsilon}\rangle\langle\phi_{E_\mu+\epsilon}|\Phi_\mu\rangle,$$

can be approximated by a Breit-Wigner form

$$s_\mu(\epsilon)=\frac{1}{2\pi}\frac{\Gamma_\mu}{\epsilon^2+\Gamma_\mu^2/4}.$$

Then the integral

$$\frac{\Gamma_\mu}{2\pi}\int d\epsilon \frac{e^{-(i/\hbar)\,\epsilon(t-t')}}{\epsilon^2+(\Gamma_\mu/2)^2} = e^{-(\Gamma_\mu/2\hbar)\,(t-t')}\ (t>t')$$

allows us to approximate the part of the continuum integral in Eq. (6.1) near E_μ by

$$\int_{-\infty}^{t} \frac{dt'}{i\hbar} e^{-(i/\hbar)[E_\mu - i(\Gamma_\mu/2)](t-t')} V(t)|\Phi_\mu\rangle\langle\Phi_\mu|V(t)\psi_\alpha. \tag{6.2}$$

Comparison of Eq. (6.2) with the terms in the discrete sum in Eq. (6.1) shows that, if the above approximations are valid, the continuum region in the vicinity of E_μ may be treated as a single discrete state Φ_μ, with a complex energy $E_\mu - i(\Gamma_\mu/2)$. If this is the case, we can incorporate the integral over ϵ in Eq. (6.1) into the sum over λ, making the replacement

$$E_\mu \Rightarrow E_\mu - i\frac{\Gamma_\mu}{2} \text{ and } \omega_\mu \Rightarrow \omega_\mu - i\frac{\Gamma_\mu}{2\hbar}$$

where necessary. This implies that the T-matrix equation (2.11) is generalized to

$$T_{\beta\alpha}(\omega) = V_{\beta\alpha}(\omega) - \frac{1}{2\pi}\sum_\lambda \int d\omega' \frac{V_{\beta\lambda}(\omega+\omega')T_{\lambda\alpha}(-\omega')}{\omega'+\omega_\lambda - i(\Gamma_\lambda/2\hbar) - \omega_\alpha - i\delta}. \tag{6.3}$$

The fact that the state Φ_λ is a continuum resonance state rather than a discrete state is reflected in the appearance of a width Γ_λ in the corresponding energy denominator. A description of the strength function by means of a Lorentz form, rather than a Breit-Wigner form, gives similar results.

In the simple case of a two-channel problem in which only the ground state and a resonance with energy E_μ are coupled, the second-order T-matrix element for the elastic channel is given by

$$T_{00}(\omega) = -\frac{1}{2\pi}\int d\omega' \frac{V_{0\mu}(\omega+\omega')V_{\mu 0}(-\omega')}{\omega'+\omega_\mu - i(\Gamma_\mu/2\hbar) - \omega_0}.$$

This result is directly comparable with Eq. (60) of Ref. [16], when that formula is particularized to the this order of approximation.

This formalism can be easily extended to the description of problems in which there is an escape width, in addition to the spreading width considered above. For a recent discussion of this problem, see, e.g., Ref. [23].

Another approach to nuclear excitation, valid when the de Broglie wavelength associated with the relative motion is small, is the multichannel eikonal approximation [24–27]. Here one seeks an approximate expression for the full time-independent Schrödinger wave function of the system $\psi(\mathbf{r};\zeta) = \psi(\mathbf{b},z;\zeta)$, with $\mathbf{r} = \mathbf{b} + z\hat{\mathbf{z}}$ the relative target-projectile vector. It is shown in Refs. [25–27] that the multichannel eikonal approximation implies that $\psi(\mathbf{b},z;\zeta)$ satisfies

$$i\hbar\frac{\partial}{\partial(z/v)}[e^{-i\mathcal{K}z}\psi(\mathbf{b},z;\zeta)] = e^{-i\mathcal{K}z}V(\mathbf{b},z;\zeta)e^{i\mathcal{K}z}[e^{-i\mathcal{K}z}\psi(\mathbf{b},z;\zeta)], \tag{6.4}$$

with the linear momentum operator $\mathcal{K}$ defined by

$$\mathcal{K} \equiv \frac{1}{\hbar}\sqrt{2m(E-H_0)},$$

so that

$$\mathcal{K}\phi_\alpha(\zeta) = k_\alpha\phi_\alpha(\zeta).$$

Comparison of Eqs. (6.4) and (2.3) shows that there is a very close correspondence between the semiclassical and multichannel eikonal approaches, with t and ω_α in the former replaced by z/v and $-k_\alpha$ in the latter. Thus all our work so far, which has been formulated in terms of the semiclassical approach, can be immediately applied to the multichannel eikonal approach.

The final result of each calculation should be a set of numbers which can be compared with experimental data. In the semiclassical method, one calculates

$$\text{Excitation probability of target state } \phi_\beta = \int d^2b|a_{\beta\alpha}(\Delta\omega_{\beta\alpha}\mathbf{b})|^2, \tag{6.5}$$

if the intial target state is ϕ_α. But the multichannel eikonal approach yields not only excitation probabilities, but also differential cross sections:

$$\frac{d\sigma_{\beta\alpha}}{d\Omega}(\mathbf{q}) = \left|\frac{k_\beta}{2\pi}\int d^2b e^{i\mathbf{q}\cdot\mathbf{b}} a_{\beta\alpha}(\Delta k_{\beta\alpha}\mathbf{b})\right|^2. \tag{6.6}$$

Here $\mathbf{q}$ is the tranverse momentum transferred to the projectile, and is assumed to be much smaller than the longitudinal momenta k_α, k_β. Thus the multichannel eikonal approach brings us closer to useful comparison with real experimental data. At high energy one has

$$\Delta k \simeq \frac{\Delta\omega}{v}$$

so that one can calculate angular distributions using the same matrix elements as those evaluated in the time-dependent approach.

VII. SUMMARY AND CONCLUSIONS

Our goal has been the study of the various components of a calculation of excitation of a target nucleus due to the electromagnetic field of a relativistic projectile. The strongly retarded interaction points to the description of the process in terms of frequency ω, rather than time t. We have seen that whereas on-shell matrix elements can be expressed purely in terms of solenoidal electromagnetic fields [and so in terms of multipole matrix elements $\mathcal{M}(E\lambda,\mu)$ and $\mathcal{M}(M\lambda,\mu)$], the off-shell matrix elements also involve nonsolenoidal fields. The natural expression of the Schrödinger equation in terms of ω-dependent matrix elements is as a set of coupled inte-

gral equations, reminiscent of the Lippmann-Schwinger equation. We have used the Kowalski-Noyes regularization procedure to remove the singularities from the kernels of these integral equations, which makes them more amenable to numerical solution. By replacing the integrals by sums, we constructed an approximately equivalent set of simultaneous linear equations, which could be solved exactly. This yielded a set of ''exact'' transition amplitudes. In a semirealistic application of these methods, we found that the first Born approximation yielded a very good approximation to the exact amplitudes for bombarding energies per nucleon up to about 1 GeV. At higher bombarding energies, the first Born amplitude becomes very small for $\Delta M=0$ transitions, because the on-shell $\Delta M=0$ matrix element becomes small. However, reorientation processes included in the exact calculation lead to appreciable indirect population of the excited $M=0$ state. Furthermore, for bombarding energies per nucleon up to about 3 GeV, the Born series was found to converge rapidly to yield the exact transition amplitudes. However, at bombarding energies per nucleon above 10 GeV, the first few terms of Born series seemed to be converging towards the exact values, but addition of further terms led to divergence. This behavior could be understood in terms of the Schmidt-Weinberg analysis of the kernel of the integral equations.

Even at ultrarelativistic bombarding energies, the sudden approximation is not applicable to the study of giant resonances with excitation energies in the 10 to 20 MeV region. Moreover, the long-wavelength approximation is unreliable at bombarding energies per nucleon above a few GeV. Thus at very high projectile energy, the discretization of the coupled integral equations appears to be the most promising approach.

We have seen that the integral equation approach can readily be generalized to include the excitation of giant resonances in the continuum part of the target spectrum. Moreover, the entire formalism can be put in correspondence with the multichannel-eikonal approximation. This suggests that the formalism can be generalized to include nuclear interactions between the projectile and target, using a version of the multichannel Glauber model [24,27]. At relativistic projectile energies, nuclear interactions are likely to manifest themselves mainly through a smooth cutoff due to absorption.

According to the harmonic model, the multiphonon cross-section saturates as the bombarding energy per nucleon increases to 10–20 GeV, and does not increases as the bombarding energy increases further. It would be interesting to see whether this prediction would be borne out by a more exact treatment, such as we have presented here, in which anharmonic and retardation effects are taken into account.

ACKNOWLEDGMENTS

We are grateful to M. A. Nagarajan and A. Vitturi for important suggestions and comments. One of us (B.F.B.) acknowledges financial support from the INFN.

APPENDIX A: MULTIPOLE EXPANSION OF $V_{\beta\alpha}(\omega)$

The source of the potential $\varphi_C^{\rm ret}$ can be taken to be a point charge $Z_P e$ located at $\mathbf{r}=\mathbf{b}+v t\hat{\mathbf{z}}$. Thus the charge density associated with this source is $Z_P e\,\delta(\boldsymbol{\rho}'-\mathbf{b})\,\delta(z'-vt)$, with $\boldsymbol{\rho}'=x'\hat{\mathbf{x}}+y'\hat{\mathbf{y}}$. Then $\varphi_C^{\rm ret}$ satisfies the d'Alembert equation

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi_C^{\rm ret}(\mathbf{r}',t,\mathbf{b})=-4\pi Z_P e\,\delta(\boldsymbol{\rho}'-\mathbf{b})\,\delta(z'-vt). \tag{A1}$$

We have included $\mathbf{b}$ as an argument of $\varphi_C^{\rm ret}$ to emphasize the dependence of the potential on the projectile orbit. We seek the particular solution of Eq. (A1) in which z' and t enter in the combination $z'-vt$, since that is the way they appear in the source term.

We need the Fourier transform of $\varphi_C^{\rm ret}$,

$$\begin{aligned}\varphi_C^{\rm ret}(\mathbf{r}',t,\mathbf{b})&=\frac{\hbar}{2\pi}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\varphi_C^{\rm ret}(\mathbf{r}',\omega,\mathbf{b})\\&=\frac{\hbar}{2\pi}\int_{-\infty}^{\infty}d\omega e^{i\omega(z'/v-t)}\varphi_C^{\rm ret}(\boldsymbol{\rho}',\omega,\mathbf{b}).\end{aligned} \tag{A2}$$

If this is substituted into Eq. (A1), we find that $\varphi_C^{\rm ret}(\boldsymbol{\rho}',\omega,\mathbf{b})$ satisfies

$$\left(\nabla_{\boldsymbol{\rho}'}^2-\frac{\omega^2}{v^2}\left[1-\frac{v^2}{c^2}\right]\right)\varphi_C^{\rm ret}(\boldsymbol{\rho}',\omega,\mathbf{b})$$

$$=\left(\nabla_{\boldsymbol{\rho}'}^2-\frac{\omega^2}{v^2\gamma^2}\right)\varphi_C^{\rm ret}(\boldsymbol{\rho}',\omega,\mathbf{b}) \tag{A3}$$

$$=-\frac{4\pi}{\hbar v}Z_P e\,\delta(\boldsymbol{\rho}'-\mathbf{b}). \tag{A4}$$

In order to reduce Eq. (A4) to a set of ordinary differential equations in ρ', we expand both sides in Fourier series in the azimuthal angle difference $\phi'-\phi_b$:

$$\varphi_C^{\rm ret}(\boldsymbol{\rho}',\omega,\mathbf{b})=\sum_\mu\ \varphi_{C,\mu}^{\rm ret}(\rho',\omega,b)e^{i\mu(\phi'-\phi_b)}, \tag{A5}$$

$$\begin{aligned}\delta(\boldsymbol{\rho}'-\mathbf{b})&=\frac{\delta(\rho'-b)}{\rho'}\,\delta(\phi'-\phi_b)\\&=\frac{\delta(\rho'-b)}{2\pi\rho'}\sum_\mu\ e^{i\mu(\phi'-\phi_b)}.\end{aligned} \tag{A6}$$

Then Eq. (A4) implies that

$$\left(\frac{d^2}{d\rho'^2}+\frac{1}{\rho'}\frac{d}{d\rho'}-\frac{\mu^2}{\rho'^2}-\frac{\omega^2}{v^2\gamma^2}\right)\varphi_{C,\mu}^{\rm ret}(\rho',\omega,b)$$

$$=-\frac{2Z_P e}{\hbar v}\,\frac{\delta(\rho'-b)}{\rho'}. \tag{A7}$$

If $\rho'\neq b$, the right-hand side of Eq. (A7) is zero and its solution is a linear combination of the modified Bessel functions $I_\mu(\omega\rho'/v\gamma)$ and $K_\mu(\omega\rho'/v\gamma)$. If Eq. (A7) is integrated across an infinitesimal ρ' interval containing b, it is

seen that the derivative of $\varphi^{\rm ret}_{C,\mu}(\rho',\omega,b)$ with respect to ρ' has a discontinuity at $\rho'=b$ equal to $-(2Z_Pe/\hbar v b)$. Both these conditions are satisfied by

$$\varphi^{\rm ret}_{C,\mu}(\rho',\omega,b)=\frac{2Z_Pe}{\hbar v}I_\mu\left(\frac{|\omega|\rho'}{v\gamma}\right)K_\mu\left(\frac{|\omega|b}{v\gamma}\right)\ \text{when}\ \rho'<b$$
$$=\frac{2Z_Pe}{\hbar v}I_\mu\left(\frac{|\omega|b}{v\gamma}\right)K_\mu\left(\frac{|\omega|\rho'}{v\gamma}\right)\ \text{when}\ \rho'>b. \tag{A8}$$

Moreover, Eq. (A8) is continuous at $\rho'=b$, is regular at $\rho'=0$, and vanishes as $\rho'\to\infty$, the boundary conditions appropriate to this problem. Thus we use Eq. (A8) in Eqs. (A5) and (A2). The result is

$$\varphi^{\rm ret}_C(\mathbf{r}',\omega,\mathbf{b})=\frac{2Z_Pe}{\hbar v}\sum_\mu\ e^{i\mu(\phi'-\phi_b)}e^{i(\omega/v)z'}$$
$$\times I_\mu\left(\frac{|\omega|\rho'}{v\gamma}\right)K_\mu\left(\frac{|\omega|b}{v\gamma}\right), \tag{A9}$$

$$A^{\rm ret}_C(\mathbf{r}',\omega,\mathbf{b})=\frac{v}{c}\hat{\mathbf{z}}\varphi^{\rm ret}_C(\mathbf{r}',\omega,\mathbf{b}) \tag{A10}$$

when $\rho'<b$.

To express the $\mathbf{r}'$ dependence of Eq. (A9) in terms of spherical harmonics, we write the partial-wave expansion of a plane wave

$$e^{i\boldsymbol{\kappa}\cdot\mathbf{r}'}=\sum_\lambda\ i^\lambda(2\lambda+1)j_\lambda(|\kappa r'|)P_\lambda(\hat{\boldsymbol{\kappa}}\cdot\hat{\mathbf{r}}')$$
$$=4\pi\sum_{\lambda,\mu}\ (-1)^\mu i^\lambda j_\lambda(|\kappa r'|)Y^\lambda_\mu(\hat{\mathbf{r}}')Y^\lambda_{-\mu}(\hat{\kappa})$$
$$=4\pi\sum_{\lambda,\mu}\ (-1)^\mu i^\lambda\frac{j_\lambda(|\kappa r'|)}{|\kappa|^\lambda}Y^\lambda_\mu(\hat{\mathbf{r}}')\mathcal{Y}^\lambda_{-\mu}(\boldsymbol{\kappa}).$$

Here $\mathcal{Y}^\lambda_{-\mu}(\boldsymbol{\kappa})$ is a ''solid harmonic,'' a homogeneous polynomial of degree λ in $\kappa_x,\kappa_y,\kappa_z$:

$$\mathcal{Y}^l_m(\kappa_x,\kappa_y,\kappa_z)=\sqrt{\frac{2l+1}{4\pi}(l+m)!(l-m)!}$$
$$\times\sum_n\ (-1)^{m+n}$$
$$\times\frac{(\kappa_x+i\kappa_y)^m(\kappa_x^2+\kappa_y^2)^n\kappa_z^{l-m-2n}}{(2n)!!(2n+2m)!!(l-m-2n)!}.$$

Also

$$\frac{j_\lambda(\kappa r')}{\kappa^\lambda}$$

is a polynomial in κ_x, κ_y, κ_z. In these polynomials, substitute

$$\kappa_x=\frac{\omega}{iv\gamma},\ \kappa_y=0,\ \kappa_z=\frac{\omega}{v}.$$

Then

$$i\boldsymbol{\kappa}\cdot\mathbf{r}'=\frac{\omega}{v\gamma}x'+i\frac{\omega}{v}z',$$
$$\kappa^2\equiv\kappa_x^2+\kappa_y^2+\kappa_z^2=\frac{\omega^2}{v^2}\left[1-\frac{1}{\gamma^2}\right]=\frac{\omega^2}{v^2}\left[\frac{v^2}{c^2}\right]=\frac{\omega^2}{c^2}$$

so that

$$e^{(\omega/v\gamma)x'+i(\omega/v)z'}=\sum_{\lambda,\mu}\mathcal{F}_{\lambda,\mu}j_\lambda\left(\frac{|\omega|}{c}r'\right)Y^\lambda_\mu(\hat{\mathbf{r}}') \tag{A11}$$

with

$$\mathcal{F}_{\lambda,\mu}=(-1)^\mu i^\lambda\frac{4\pi}{|\kappa|^\lambda}\mathcal{Y}^\lambda_{-\mu}\left(\frac{\omega}{iv\gamma},0,\frac{\omega}{v}\right)$$
$$=(-1)^\mu i^\lambda 4\pi\left(\frac{c}{v}\frac{\omega}{|\omega|}\right)^\lambda\mathcal{Y}^\lambda_{-\mu}\left(\frac{1}{i\gamma},0,1\right)$$
$$=i^\lambda 4\pi\left(\frac{c}{v}\frac{\omega}{|\omega|}\right)^\lambda\mathcal{Y}^\lambda_{\mu}\left(\frac{1}{i\gamma},0,1\right). \tag{A12}$$

But we can also expand the left-hand side of Eq. (A11) in cylindrical coordinates

$$e^{(\omega/v\gamma)x'+i(\omega/v)z'}=e^{(\omega/v\gamma)\rho'\cos\phi'+i(\omega/v)z'}$$
$$=\sum_\mu\ e^{i\mu\phi'}I_\mu\left(\frac{\omega\rho'}{v\gamma}\right)e^{i(\omega/v)z'}$$
$$=\sum_\mu\ e^{i\mu\phi'}\left(\frac{\omega}{|\omega|}\right)^\mu I_\mu\left(\frac{|\omega|\rho'}{v\gamma}\right)e^{i(\omega/v)z'}. \tag{A13}$$

If we identify the coefficients of $e^{i\mu\phi'}$ in Eqs. (A11) and (A13), we get

$$e^{i\mu\phi'}I_\mu\left(\frac{|\omega|\rho'}{v\gamma}\right)e^{i(\omega/v)z'}=\sum_{\lambda=|\mu|}^\infty\left(\frac{\omega}{|\omega|}\right)^{-\mu}$$
$$\times\mathcal{F}_{\lambda,\mu}j_\lambda\left(\frac{|\omega|r'}{c}\right)Y^\lambda_\mu(\hat{\mathbf{r}}')$$
$$=\sum_{\lambda=|\mu|}^\infty\mathcal{G}_{\lambda,\mu}j_\lambda\left(\frac{|\omega|r'}{c}\right)Y^\lambda_\mu(\hat{\mathbf{r}}'), \tag{A14}$$

with

$$\mathcal{G}_{\lambda,\mu}=\left(\frac{\omega}{|\omega|}\right)^{-\mu}\mathcal{F}_{\lambda,\mu}=i^\lambda 4\pi\left(\frac{\omega}{|\omega|}\right)^{\lambda-\mu}\left(\frac{c}{v}\right)^\lambda\mathcal{Y}^\lambda_\mu\left(\frac{1}{i\gamma},0,1\right). \tag{A15}$$

If Eq. (A14) is substituted into Eq. (A9), we obtain the multipole expansion of the Fourier transform of the scalar potential

$$\varphi_C^{\rm ret}(\mathbf{r}',\omega,\mathbf{b})=\frac{2Z_Pe}{\hbar v}\sum_{\mu}\ e^{-i\mu\phi_b}K_\mu\left(\frac{|\omega|b}{v\gamma}\right)\sum_{\lambda=|\mu|}^{\infty}\ \mathcal{G}_{\lambda,\mu}\times j_\lambda\left(\frac{|\omega|}{c}r'\right)Y_\mu^\lambda(\hat{\mathbf{r}}'). \tag{A16}$$

If this expansion and Eq. (A10) are used in Eq. (3.6) the result is the multipole expansion (3.9) of the Fourier transform of the interaction matrix element.

It is of some interest to focus on the distance parameter that determines the rate of variation of the retarded potential as we move perpendicular to the projectile orbit. It is clear from Eq. (A7) that this distance parameter is $\gamma v/\omega$. When this is large compared to the impact parameter and nuclear radius, we can replace $I_\mu(|\omega|\rho'/v\gamma)$ and $K_\mu(|\omega|b/v\gamma)$ by their limiting values for small arguments. If this is done, Eq. (A9) can be approximated by

$$\varphi_C^{\rm ret}(\mathbf{r}',\omega,\mathbf{b})=\frac{2Z_Pe}{\hbar v}\sqrt{4\pi}e^{i(\omega z'/v)}\times\left[-\ln\left(\frac{b|\omega|}{v\gamma}\right)Y_0^0(\theta',\phi')+\sum_{\mu=1,2,3,\ldots}\frac{1}{2\mu}\left(\frac{r'}{b}\right)^\mu\sqrt{\frac{(2\mu)!!}{(2\mu+1)!!}}\times[Y_{-\mu}^\mu(\theta',\phi')e^{i\mu\phi_b}+(-1)^\mu Y_\mu^\mu(\theta',\phi')e^{-i\mu\phi_b}]\right]. \tag{A17}$$

We will make use of this limiting form at high bombarding energy, when $\gamma\gg1$ and $v\sim c$. It is seen that the $\Delta M=0$ matrix elements of $\varphi_C^{\rm ret}$, which arise from the $\mu=0$ term of Eq. (A17), will diverge logarithmically as projectile energy (and hence γ) increases. On the other hand, the only projectile-energy dependence of the matrix elements connecting states with $\Delta M\neq0$ comes from the factor $e^{i(\omega/v)z'}$, which changes very little with increasing projectile energy when v is close to c.

APPENDIX B: SYMMETRIES

We will consider situations in which the projectile and target are both in 0^+ states before the collision. We first show that $V_{\beta\alpha}(\omega)$ defined in Eq. (2.12) is real if the coordinate system is chosen such that the trajectory of the projectile (which is parallel to the $\hat{z}$ axis) lies in the $\hat{y}$-$\hat{z}$ plane. In this situation, we use Eq. (3.2) with $x=0$. It will be convenient to use the symbol $\mathcal{I}_z$ to represent reflection across the $\hat{x}$-$\hat{y}$ plane, $\mathcal{I}_x$ to represent reflection across the $\hat{y}$-$\hat{z}$ plane, and R_y to represent a rotation through π about the $\hat{y}$ axis.

We have

$$V_{\beta\alpha}(\omega)=\int\frac{dt}{\hbar}e^{i\omega t}\times\left\langle\phi_\beta\left|\frac{Z_Pe\gamma}{\sqrt{x'^2+(y-y')^2+\gamma^2(vt-z')^2}}\right|\phi_\alpha\right\rangle,$$

$$V_{\beta\alpha}(\omega)^*=\int\frac{dt}{\hbar}e^{-i\omega t}\times\left\langle\phi_\beta^*\left|\frac{Z_Pe\gamma}{\sqrt{x'^2+(y-y')^2+\gamma^2(vt-z')^2}}\right|\phi_\alpha^*\right\rangle.$$

We use ''time-reversal'' phases, for which the effect of complex conjugation on angular momentum eigenstates is given by

$$(\psi_M^J)^*=(-1)^{J-M}\psi_{-M}^J=R_y\psi_M^J. \tag{B1}$$

Then

$$\begin{aligned}\langle\phi_{M_\beta}^{J_\beta}|V(\omega)|\phi_{M_\alpha}^{J_\alpha}\rangle^*&=\int\frac{dt}{\hbar}e^{-i\omega t}\left\langle R_y\phi_{M_\beta}^{J_\beta}\left|\frac{Z_Pe\gamma}{\sqrt{x'^2+(y-y')^2+\gamma^2(vt-z')^2}}\right|R_y\phi_{M_\alpha}^{J_\alpha}\right\rangle\\&=\int\frac{dt}{\hbar}e^{-i\omega t}\left\langle\phi_{M_\beta}^{J_\beta}\left|R_y^{-1}\frac{Z_Pe\gamma}{\sqrt{x'^2+(y-y')^2+\gamma^2(vt-z')^2}}R_y\right|\phi_{M_\alpha}^{J_\alpha}\right\rangle\\&=\int\frac{dt}{\hbar}e^{-i\omega t}\left\langle\phi_{M_\beta}^{J_\beta}\left|\frac{Z_Pe\gamma}{\sqrt{x'^2+(y-y')^2+\gamma^2(vt+z')^2}}\right|\phi_{M_\alpha}^{J_\alpha}\right\rangle\\&=\int\frac{dt}{\hbar}e^{i\omega t}\left\langle\phi_{M_\beta}^{J_\beta}\left|\frac{Z_Pe\gamma}{\sqrt{x'^2+(y-y')^2+\gamma^2(-vt+z')^2}}\right|\phi_{M_\alpha}^{J_\alpha}\right\rangle=\langle\phi_{M_\beta}^{J_\beta}|V(\omega)|\phi_{M_\alpha}^{J_\alpha}\rangle.\end{aligned} \tag{B2}$$

Thus $V_{\beta\alpha}$ is real if ''time-reversal'' phases are used for the target states, and if the coordinate system is chosen so that the trajectory lies in the $\hat{y}$-$\hat{z}$ plane. A similar argument can be used to show that

$$V_{\beta\alpha}(\omega)=V_{\alpha\beta}(-\omega). \tag{B3}$$

To prove the symmetry of the a and T matrices, we introduce the notation $a_{\beta,\alpha}(+)$ to represent the transition amplitude from initial state ϕ_α to final state ϕ_β when the projectile moves in the $+\hat{z}$ direction. This is the quantity that we have previously called $a_{\beta\alpha}$. Let $a_{\beta,\alpha}(-)$ represent the transition amplitude from initial state ϕ_α to final state ϕ_β when the projectile moves in the $-\hat{z}$ direction. Then

$$a_{\beta,\alpha}(+)=a_{\mathcal{I}_z\alpha,\mathcal{I}_z\beta}(-)=a_{R_y\mathcal{I}_z\alpha,R_y\mathcal{I}_z\beta}(+). \tag{B4}$$

But since $R_y\mathcal{I}_z$ has the same effect as $\mathcal{I}_x$, a reflection across the $\hat{y}$-$\hat{z}$ plane, Eq. (B4) implies that

$$a_{\beta,\alpha}(+)=a_{\mathcal{I}_x\alpha,\mathcal{I}_x\beta}(+). \tag{B5}$$

If the projectile trajectory is in the $\hat{y}$-$\hat{z}$ plane, the Hamiltonian is invariant under the reflection $\mathcal{I}_x$. Since the initial target state (0^+) is invariant under $\mathcal{I}_x$, the target state will remain invariant throughout the reaction. Thus it is sufficient to restrict our attention to target states ϕ_α and ϕ_β that satisfy

$$\mathcal{I}_x\phi_\alpha=\phi_\alpha,\ \mathcal{I}_x\phi_\beta=\phi_\beta,$$

and Eq. (B5) implies that

$$a_{\beta,\alpha}(+)=a_{\alpha,\beta}(+),$$

$$a_{\beta,\alpha}=a_{\alpha,\beta}, \tag{B6}$$

and the a matrix is symmetric. From this it follows that the on-shell T matrix

$$T_{\beta\alpha}=i[a_{\beta\alpha}-\delta_{\beta\alpha}]$$

is also symmetric.

APPENDIX C: CONNECTION TO THE K MATRIX

Let us use the symbol $\mathcal{P}$ to represent the sum of principal-value integrals in Eq. (3.21), and define the matrix K by

$$K\equiv-\frac{1}{2}(1+\mathcal{P})^{-1}V=-\frac{1}{2}(1-\mathcal{P}+\mathcal{P}^2-\mathcal{P}^3+\cdots)V. \tag{C1}$$

As defined in Sec. IV A, the symbol V, without its argument ω, represents the on-shell interaction matrix. The excitation amplitude matrix a can be expressed in terms of K by using

$$\begin{aligned}a&=1-iT=1-i\mathcal{M}^{-1}V=\mathcal{M}^{-1}(\mathcal{M}-iV)\\&=\left(1+\mathcal{P}+\frac{i}{2}V\right)^{-1}\left(1+\mathcal{P}-\frac{i}{2}V\right)\\&=(1-iK)^{-1}(1+iK).\end{aligned}$$

The expansion of $\mathcal{P}$ in powers of the interaction is, apart from the linear term, the same as the expansion of $\mathcal{M}$, Eq. (4.15),

$$\mathcal{P}=\left(m_1-i\frac{V}{2}\right)+m_2+m_3+m_4+\cdots.$$

If this expansion is used in Eq. (C1), we recover the expansion of K in powers of the interaction given in Eq. (II 3.26) of Ref. [1].

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