# Watson's theorem for low-energy *p-d* radiative capture

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The use of Watson's theorem in the analysis of *p-d* radiative capture measurements at low energies is discussed. The principle of Watson's theorem is outlined, and a detailed description of how the theorem can be used in a matrix element analysis of radiative capture data is presented. It is shown that with Watson's theorem it is possible to reduce the number of unknown parameters in a matrix element analysis by essentially a factor of 2. This is done by employing a representation in which the capture matrix elements are all real. The phase information needed to construct the reaction amplitudes is then obtained from a separate phase shift analysis of elastic scattering data. Details concerning the extension of Watson's theorem to situations in which there is mixing between angular momentum states are given. The paper presents a consistent formulation which facilitates the simultaneous analysis of the elastic scattering and radiative capture channels. [S0556-2813(99)01304-7]

## I. INTRODUCTION

The radiative capture reaction $p+d\rightarrow\,^3\text{He}+\gamma$ is of interest for a number of reasons. It is known, for example, that the deuteron tensor analyzing powers are sensitive to *D*-state components in the $^3$He wave function [1], and measurements of these quantities have often been used in the past to extract *D*-state information [2–4]. There has also been a substantial amount of interest in obtaining information about the *M*1 contributions to the reaction process (see for example Refs. [5,6]). This interest arises in part from the desire to understand the role of meson exchange processes. Finally, the *p-d* capture reaction is of special importance since it is one of the basic processes that can be studied in the three-nucleon system. Recent theoretical and computational advances have made it possible to carry calculations in the $A=3$ system which are quantum mechanically exact, and as a result, *p-d* capture measurements have taken on added importance since they can now be used to test, at a rather fundamental level, our understanding of the nuclear three-body problem.

In radiative capture reactions, one can often learn much from measurements of cross sections or analyzing powers at only a few angles. However, it is well understood that information of a more fundamental nature can be obtained if one has a data set sufficiently complete to allow the extraction of individual reaction matrix elements. For capture reactions that have a simple spin-parity structure, the determination of the matrix elements can be relatively straightforward. However, for *p-d* capture this is not the case. For this reaction there are 5 *E*1 matrix elements, 5 *M*1 matrix elements and 6 *E*2 matrix elements. Since one knows that *E*1, *M*1 and *E*2 all play important roles at energies of a few MeV [7], a complete matrix element analysis in this energy range would involve 31 parameters—16 complex matrix elements with one irrelevant overall phase. Because the number of parameters is so large, complete matrix element analyses have not been possible in the past, although some restricted analyses have been reported [7].

The main point of the present paper is to demonstrate that the number of unknown parameters in a matrix analysis can be reduced by essentially a factor of 2 if the reaction is carried out at energies below the threshold for deuteron breakup. The essential idea is contained in a paper published by Watson in 1954 [8]. Watson's theorem, when applied to radiative capture reactions, essentially says that the complex phases of the matrix elements are related in a simple way to the phase shifts for elastic scattering. This result follows from the fact that the expectation values of the multipole operators are fundamentally real quantities (in the appropriate representation) provided that time reversal invariance holds. The matrix elements acquire a phase because the elastic scattering wave function for a given angular momentum state is required to satisfy the elastic scattering boundary conditions, and this fixes the phases of the scattering states.

Watson's theorem applies only to situations in which there are no open reaction channels other than radiative capture, and one of the central assumptions is that the capture amplitude is small enough to be calculated in perturbation theory. Equivalently, the assumption one makes is that the full wave function of the system is, to a good approximation, a pure elastic scattering wave function in the asymptotic region. Since the $\gamma+\,^3$He wave function amplitudes are very small compared to the elastic amplitudes, these conditions are well satisfied for *p-d* radiative capture at energies below the deuteron breakup threshold.

In principle then, Watson's theorem allows one to reduce the number of unknown parameters in a matrix analysis by essentially a factor of 2, provided that the phase shifts are known or can be determined from measurements of the elastic scattering. Since phenomenological phase shift analyses of *p-d* scattering have been successful in the past few years (see for example Ref. [9]) it seems possible that a complete matrix element analysis of *p-d* radiative capture might now be feasible with the aid of Watson's theorem.

The goal of the present paper is to establish the detailed formalism which one would use in such an analysis. In the simplest systems the implementation of Watson's theorem is relatively straightforward. However, the details become

more complicated when the spin structure is such that mixing between angular momentum states is allowed in the elastic scattering channel. Nevertheless, as we shall demonstrate, the basic principle of the Watson theorem still holds, provided that one defines the matrix elements appropriately.

The application of Watson's theorem to radiative capture is essentially equivalent to carrying out a simultaneous partial-wave analysis of the elastic scattering and reaction channels. As such, it is clear that one must be exceedingly careful about phase conventions, coupling schemes for angular momenta, and so on. For this reason, the discussion that follows is quite detailed, with clear and rigorous definitions presented for all relevant quantities.

We begin in Sec. II by defining the transition matrix for the radiative capture reaction, introducing the multipole expansion of the interaction Hamiltonian, and defining the phase conventions for the angular momentum states. In Sec. III we introduce the elastic scattering wave function and demonstrate Watson's theorem for the simple case of no coupling between angular momentum states. The generalization of Watson's theorem to situations in which angular momentum mixing is allowed is presented in Sec. IV. Section V presents a discussion of a few subtle points, and the final conclusions are given in Sec. VI.

## II. THE RADIATIVE CAPTURE REACTION

### A. The starting point

The radiative capture reaction is described as a process in which transitions between the initial state $\chi$, and the final state $\mu$, are induced by the electromagnetic interaction, $H_{\text{int}}$. The initial and final states are eigenstates of a Hamiltonian $H_0$, and the full Hamiltonian of the system is

$$H = H_0 + H_{\text{int}}. \qquad (2.1)$$

The interaction Hamiltonian is given by the usual expression

$$H_{\text{int}} = -\frac{1}{c} \int \mathbf{J}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) d^3 r, \qquad (2.2)$$

where $\mathbf{J}(\mathbf{r})$ is the current density and $\mathbf{A}(\mathbf{r})$ is the vector potential.

In a nuclear system, $H_{\text{int}}$ takes the form [10]

$$H_{\text{int}} = -\frac{e\hbar}{2mc} \sum_n [2g_{ln}\mathbf{p_n} \cdot \mathbf{A}(\mathbf{r_n}) + g_{sn}\mathbf{s_n} \cdot \mathbf{H}(\mathbf{r_n})], \quad (2.3)$$

where

$$\mathbf{H}(\mathbf{r}) \equiv \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}). \qquad (2.4)$$

In Eq. (2.3) the sum is over all nucleons, and the operators $\mathbf{p}$ and $\mathbf{s}$ have the definitions

$$\mathbf{p} = -i\boldsymbol{\nabla} \qquad (2.5)$$

and

$$\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}, \qquad (2.6)$$

where the components of $\boldsymbol{\sigma}$ are the Pauli matrices. The quantity $g_{ln}$ takes on the values 0 and 1 for neutrons and protons, respectively, while the corresponding values of $g_{sn}$ are $-3.826$ and $5.585$.

Since the coupling between the initial and final states of the system is weak, the transition rates may be calculated in time-dependent perturbation theory. As seen in the following section, this allows us to find the connection between the matrix elements of $H_{\text{int}}$ and the radiative capture observables.

### B. The transition matrix

Our goal is to find the transition matrix $T_{fi}$ for the radiative capture process. Given $T_{fi}$, it is straightforward to obtain the reaction observables by employing the density matrix formalism. If the initial spin state is described by a density matrix $\rho_i$ then the final state density matrix $\rho_f$ is

$$\rho_f = T\rho_i T^{\dagger}. \qquad (2.7)$$

The reaction cross section is then simply the trace of the final state density matrix

$$\frac{d\sigma}{d\Omega} = \text{tr}\,\rho_f. \qquad (2.8)$$

To obtain $T_{fi}$ we use Fermi's golden rule, according to which the transition rate $w$ from state $i$ to state $f$ is given by

$$w = \frac{2\pi}{\hbar}\rho(E)|\langle f|H_{\text{int}}|i\rangle|^2. \qquad (2.9)$$

In the capture reaction the initial state, which we designate by $\chi$, is a full solution to the $p$-$d$ scattering problem for the Hamiltonian $H_0$ which satisfies the usual boundary conditions for elastic scattering. Thus, in the asymptotic region, $\chi$ consists of an incoming plane wave plus outgoing spherical waves. The final state, $f$, consists of a nuclear bound state $\mu$ plus a photon of momentum $\mathbf{k}$ and polarization $\hat{\boldsymbol{\epsilon}}$.

To obtain the reaction cross section we find the transition rate $\Delta w$ to states in which the photon momentum, $\mathbf{k}$, is contained in the solid angle $\Delta\Omega$. The result is

$$\Delta w = \frac{\Omega_N \omega^2}{4\pi^2 \hbar^2 c^3}|\langle \mu; (\mathbf{k}, \hat{\boldsymbol{\epsilon}})|H_{\text{int}}|\chi\rangle|^2 \Delta\Omega, \qquad (2.10)$$

where $\hbar\omega$ is the photon energy and $\Omega_N$ is the normalization volume.

We may now determine $T_{fi}$ by requiring that the reaction rates implied by Eqs. (2.8) and (2.10) must be equal. Assuming that $\chi$ is normalized for one projectile and one target in the volume $\Omega_N$, we obtain

$$T_{fi} = \frac{\Omega_N}{2\pi}\left[\frac{\mu_i \omega^2}{k_i(\hbar c)^3}\right]^{1/2} \langle \mu; (\mathbf{k}, \hat{\boldsymbol{\epsilon}})|H_{\text{int}}|\chi\rangle, \qquad (2.11)$$

where $k_i$ is the wave number for the initial $p$-$d$ state and $\mu_i$ is the $p$-$d$ reduced mass.

## C. Reduction of $H_{int}$

The first step in obtaining the multipole expansion of the matrix element in Eq. (2.11) is to simplify $H_{int}$. Strictly speaking, the quantities $\mathbf{A}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ in Eq. (2.3) should be treated as field operators. However, for present purposes it is preferable to adopt the semiclassical approach and refer the reader to Ref. [10] for a discussion of the more subtle aspects of the full quantum treatment.

In the semiclassical approach the electromagnetic wave is described by a vector potential of the form

$$\mathbf{A}(\mathbf{r},t) = A_0 \text{Re}[\hat{\boldsymbol{\epsilon}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]$$
$$= \frac{A_0}{2}[\hat{\boldsymbol{\epsilon}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \hat{\boldsymbol{\epsilon}}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]. \quad (2.12)$$

Following Ref. [10] we may then divide $H_{int}$ into two parts,

$$H_{int} = \frac{A_0}{2}[H_a(\boldsymbol{k},\hat{\boldsymbol{\epsilon}})e^{-i\omega t} + H_e(\boldsymbol{k},\hat{\boldsymbol{\epsilon}})e^{i\omega t}], \quad (2.13)$$

where

$$H_a = -\frac{e\hbar}{2mc}\sum_n [2g_l\mathbf{p}\cdot\hat{\boldsymbol{\epsilon}} e^{i\mathbf{k}\cdot\mathbf{r}} + g_s\mathbf{s}\cdot\boldsymbol{\nabla}\times\hat{\boldsymbol{\epsilon}} e^{i\mathbf{k}\cdot\mathbf{r}}], \quad (2.14)$$

and where $H_e$ is the Hermitian conjugate of $H_a$.

In time dependent perturbation theory, $H_a$ is responsible for the absorption of photons, while $H_e$ gives rise to the process of stimulated emission. Since we are interested here in radiative capture reactions, the process which concerns us is spontaneous emission. As is well known, this process is not described in the semiclassical treatment of electromagnetic radiation. However, as demonstrated in Ref. [10], the radiative capture amplitude can be determined by using the semiclassical theory to calculate the amplitudes for the inverse process, photon absorption on $^3$He. One may then use detailed balance to relate the two processes. The result, derived in Ref. [10], is

$$\langle\mu;(\boldsymbol{k},\hat{\boldsymbol{\epsilon}})|H_{int}|\chi\rangle = \left[\frac{2\pi\hbar c}{\Omega_N k}\right]^{1/2}\langle\chi|H_a(\boldsymbol{k},\hat{\boldsymbol{\epsilon}})|\mu\rangle^*. \quad (2.15)$$

Combining Eqs. (2.11) and (2.15) we then obtain our working equation,

$$T_{fi} = \left[\frac{\mu_i\omega\Omega_N}{2\pi\hbar^2 ck_i}\right]^{1/2}\langle\chi|H_a(\boldsymbol{k},\hat{\boldsymbol{\epsilon}})|\mu\rangle^*. \quad (2.16)$$

## D. The multipole expansion

One of the advantages of the approach outlined in the preceeding section is that it allows one to employ the standard multipole expansion of the operator $H_a$. In making the multipole expansion we follow the notation and conventions of Ref. [10].

For the unit vectors, $\hat{\boldsymbol{\epsilon}}$, that specify the photon polarization states we employ the usual spherical tensors

$$\hat{\boldsymbol{\epsilon}}_1 = -\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}) \quad (2.17a)$$

$$\hat{\boldsymbol{\epsilon}}_0 = \hat{\boldsymbol{z}} \quad (2.17b)$$

$$\hat{\boldsymbol{\epsilon}}_{-1} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}}-i\hat{\boldsymbol{y}}). \quad (2.17c)$$

At this point it is also useful to adopt a coordinate frame in which the $z$-axis is chosen to be along the photon momentum, $\boldsymbol{k}$. In this frame photons are emitted into only the $\lambda = 1$ and $\lambda = -1$ polarization states. In addition, the quantities $\hat{\boldsymbol{\epsilon}} e^{i\mathbf{k}\cdot\mathbf{r}}$ that appear in Eq. (2.14) take on a relatively simple form,

$$\hat{\boldsymbol{\epsilon}}_\lambda e^{ikz} = -\frac{1}{\sqrt{2}}\sum_L [\mathbf{A}_{L\lambda}^e + \lambda\mathbf{A}_{L\lambda}^m], \quad (2.18)$$

where $\mathbf{A}_{L\lambda}^e$ and $\mathbf{A}_{L\lambda}^m$ are appropriately defined vector spherical harmonics [10].

Upon substituting this expression into Eq. (2.14) one can obtain the familiar multipole expansion formula

$$H_a(k;\hat{\boldsymbol{\epsilon}}_\lambda) = -\sum_L [\alpha_L^e(Q_{L\lambda}+Q'_{L\lambda}) + \lambda\alpha_L^m(M_{L\lambda}+M'_{L\lambda})]. \quad (2.19)$$

The definitions of operators $Q_{L\lambda}$, $Q'_{L\lambda}$, $M_{L\lambda}$ and $M'_{L\lambda}$ and of the complex coefficients $\alpha_L^e$ and $\alpha_L^m$ are given in Ref. [10].

The main focus of the present work is to demonstrate that with Watson's theorem one can obtain information about the phases of the matrix elements of the multipole operators. As it turns out, these phase relationships take on a relatively simple form if one replaces the multipole operators of Eq. (2.19) with quantities that Rose and Brink refer to as ''interaction multipole operators.'' These quantities are defined according to

$$T_{L\lambda}^e = \alpha_L^e(Q_{L\lambda}+Q'_{L\lambda}) \quad (2.20)$$

and

$$T_{L\lambda}^m = \alpha_L^m(M_{L\lambda}+M'_{L\lambda}), \quad (2.21)$$

and so we have

$$H_a(k;\hat{\boldsymbol{\epsilon}}_\lambda) = -\sum_L [T_{L\lambda}^e + \lambda T_{L\lambda}^m]. \quad (2.22)$$

## E. The time reversal operator

The phase relations that we shall obtain follow from the transformation properties of the multipole operators and of the angular momentum states under the operation of time reversal. We follow the commonly used convention and define the time reversal operator, $\theta_T$, as

$$\theta_T = e^{-i\pi S_y/\hbar}K_0, \quad (2.23)$$

where $K_0$ means complex conjugation, and where $S_y$ is the operator for the $y$-component of the total spin.

Since $\theta_T$ is an antiunitary operator, the matrix element of any operator $B$ must have the property [10]

$$\langle \phi | B | \psi \rangle = \langle \theta_T \phi | \theta_T B \theta_T^{-1} | \theta_T \psi \rangle^*. \qquad (2.24)$$

We now apply this formula to the matrix element of $T_{L\lambda}$ taken between two eigenstates of the total angular momentum. One can easily demonstrate that the electric and the magnetic multipole operators both transform according to the rule

$$\theta_T T_{L\lambda} \theta_T^{-1} = (-)^{L-\lambda} T_{L-\lambda}. \qquad (2.25)$$

Let us assume that for the angular momentum states we adopt phase conventions so that the time reversal transformation is

$$\theta_T | j, m \rangle = (-)^{j-m} | j, -m \rangle. \qquad (2.26)$$

With our definition of operator $\theta_T$ this equation is satisfied by the conventional spin angular momentum eigenstates. For eigenstates of orbital angular momentum Eq. (2.26) will be satisfied if we take

$$| l, m \rangle \equiv i^l Y_l^m, \qquad (2.27)$$

where $Y_l^m$ is a spherical harmonic defined with the usual phase convention [11]. Finally we note that if $| j_1, m_1 \rangle$ and $| j_2, m_2 \rangle$ satisfy Eq. (2.26), then

$$| j, m \rangle = \sum_{m_1, m_2} | j_1, m_1 \rangle | j_2, m_2 \rangle \langle j_1 m_1, j_2 m_2 | j m \rangle$$
$$(2.28)$$

does also.

It is now straightforward to demonstrate that matrix elements of the operators $T_{L\lambda}$ between angular momentum states that satisfy Eq. (2.26) must be real. To see this we substitute Eqs. (2.25) and (2.26) into Eq. (2.24) to obtain

$$\langle j_1, m_1 | T_{L\lambda} | j_2, m_2 \rangle = (-)^{j_1 - m_1 + j_2 - m_2 + L - \lambda}$$
$$\times \langle j_1, -m_1 | T_{L-\lambda} | j_2, -m_2 \rangle^*.$$
$$(2.29)$$

Since the operators $T_{L\lambda}$ are spherical tensors, we may use the Wigner-Eckart theorem to express the matrix elements in Eq. (2.29) in terms of Clebsch-Gordon coefficients and a reduced matrix element $\langle j_1 \| T_L \| j_2 \rangle$. The Clebsch-Gordon coefficients may then be eliminated by employing the appropriate symmetry relations and one obtains the result

$$\langle j_1 \| T_L \| j_2 \rangle = \langle j_1 \| T_L \| j_2 \rangle^*. \qquad (2.30)$$

We therefore conclude that the matrix elements of $T_{L\lambda}$ are real provided that the states $| j, m \rangle$ satisfy Eq. (2.26).

### III. THE ELASTIC SCATTERING STATES

#### A. Boundary conditions

We now focus on the properties of the elastic scattering state $\chi$. As indicated earlier, this wave function is the full solution to the $p$-$d$ scattering problem subject to the usual elastic scattering boundary conditions. The wave function depends on the magnetic quantum numbers $\nu_p$ and $\nu_d$ which specify the initial spin projections of the proton and deuteron respectively. Thus in the asymptotic region $(r \to \infty) \chi$ is of the form

$$\chi \to \frac{1}{\sqrt{\Omega_N}} [\psi_{\text{inc}} + \psi_{\text{scat}}], \qquad (3.1)$$

where

$$\psi_{\text{inc}} = e^{i \mathbf{k}_i \cdot \mathbf{r}} \phi_p^{\nu_p} \phi_d^{\nu_d}, \qquad (3.2)$$

and where $\psi_{\text{scat}}$ consists of outgoing spherical waves only. In Eq. (3.2) the quantity $\phi_p^{\nu_p}$ is the proton spin wave function while $\phi_d^{\nu_d}$ represents the internal wave function of the deuteron.

We now expand $\psi_{\text{inc}}$ in partial waves. Since the relative phases of the various partial wave states depend on the coupling scheme, it is important to adopt a clear and consistent convention for the coupling order. We shall use the conventions of Seyler and Weller [1], in which the channel-spin $\mathbf{S}$ is given by

$$\mathbf{S} = \mathbf{S}_p + \mathbf{S}_d, \qquad (3.3)$$

while the coupling order for the total angular momentum $\mathbf{J}$ is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}. \qquad (3.4)$$

Our generalized spin-angle functions $\mathcal{Y}$ are therefore defined as

$$\mathcal{Y}_{Jls}^M = \sum_{\nu_p, \nu_d, \nu, m} \langle s_p \nu_p, s_d \nu_d | s \nu \rangle \langle l m, s \nu | J M \rangle$$
$$\times i^l Y_l^m(\hat{r}) \phi_p^{\nu_p} \phi_d^{\nu_d}. \qquad (3.5)$$

It follows from Eqs. (2.27) and (2.28) that these functions obey the time reversal relation Eq. (2.26).

The expansion of $\psi_{\text{inc}}$ in terms of the spin-angle functions is readily obtained by employing the symmetry properties of the Clebsch-Gordon coefficients. The result is

$$\psi_{\text{inc}} = 4\pi \sum_{\substack{J, l, s \\ M, \nu, m}} \langle s_p \nu_p, s_d \nu_d | s \nu \rangle \langle l m, s \nu | J M \rangle$$
$$\times j_l(k_i r) \mathcal{Y}_{Jls}^M Y_l^{m*}(\hat{k}_i). \qquad (3.6)$$

The asymptotic form of the incident wave may then be seen by substituting into Eq. (3.6) the familiar expression for $j_l$ at large $r$:

$$j_l(k_i r) \to \frac{1}{2 i k_i r} [e^{i[k_i r - l(\pi/2)]} - e^{-i[k_i r - l(\pi/2)]}]. \qquad (3.7)$$

Thus, if there were no scattering, the wave function $\chi$ at large $r$ would be of the form prescribed by Eqs. (3.6) and (3.7). In general, however, the scattering modifies $\chi$ giving rise to additional outgoing spherical waves.

#### B. The simple case—no angular momentum mixing

To incorporate the effects of scattering it is helpful to begin with the simple case in which there is no mixing between angular momentum states. We will return shortly to the more general case, but for now we assume that the solu-

tions to the Schrödinger equation are eigenfunctions of $L^2$ and $S^2$, so that $l$ and $s$ are good quantum numbers. In addition we assume throughout that there are no open reaction channels. It follows that the most general solution of the Schrödinger equation in a given angular momentum state will consist of ingoing and outgoing spherical waves in the asymptotic region. In other words the solutions at large $r$ will be characterized by radial wave functions that can be written in the form

$$g_l(r) \rightarrow \frac{1}{2ik_ir}[be^{i[k_ir-l(\pi/2)]} - ae^{-i[k_ir-l(\pi/2)]}], \quad (3.8)$$

where $a$ and $b$ are constants. The notation we adopt in Eq. (3.8) and in the following discussion is that of Blatt and Biedenharn [12].

Since the outgoing wave evolves from the ingoing wave, it follows that the outgoing wave amplitude $b$ must be a function of the ingoing wave amplitude $a$. Then, since the Schrödinger equation is linear, we conclude that $a$ and $b$ must be related by an equation of the form

$$b = Sa, \quad (3.9)$$

where the proportionality constant $S$ is the $S$-matrix element for the angular momentum state in question. If there are no open reaction channels then flux conservation requires $|a| = |b|$ and we may write

$$S = e^{2i\delta}, \quad (3.10)$$

where the phase shift $\delta$ is real.

Now to obtain the full elastic scattering wave function, $\psi^{\nu_p,\nu_d}$, we need to satisfy the boundary conditions implied by Eq. (3.1). Thus the constants $a$ must be chosen so that the full wave function has ingoing waves that match those of Eq. (3.6). Defining

$$\psi^{\nu_p,\nu_d} = \sqrt{\Omega_N}\chi \quad (3.11)$$

we conclude that $\psi^{\nu_p,\nu_d}$ must have the asymptotic form

$$\psi^{\nu_p,\nu_d} \rightarrow 4\pi \sum_{\substack{J,l,s \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d|s\nu\rangle\langle lm,s\nu|JM\rangle\left(\frac{1}{2ik_ir}\right)$$
$$\times[e^{2i\delta_J^{ls}}e^{i[k_ir-l(\pi/2)]} - e^{-i[k_ir-l(\pi/2)]}]\mathcal{Y}_{Jls}^M Y_l^{m*}(\hat{\mathbf{k}}_i). \quad (3.12)$$

At this point we wish to express the scattering wave function in a form that allows us to readily make use of the result given in Eq. (2.30). For this purpose one needs to write $\psi^{\nu_p,\nu_d}$ as an expansion over angular momentum eigenstates that satisfy Eq. (2.26). To accomplish this we first rewrite Eq. (3.12) in the form

$$\psi^{\nu_p,\nu_d} \rightarrow 4\pi \sum_{\substack{J,l,s \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d|s\nu\rangle\langle lm,s\nu|JM\rangle$$
$$\times e^{i\delta_J^{ls}}\left(\frac{1}{k_ir}\right)\sin\left(k_ir - l\frac{\pi}{2} + \delta_J^{ls}\right)\mathcal{Y}_{Jls}^M Y_l^{m*}(\hat{\mathbf{k}}_i). \quad (3.13)$$

This result suggests that it is useful to introduce a new set of scattering eigenstates, $|ls;JM\rangle$. These wave functions are defined to be full solutions to the Schrödinger equation in a given angular momentum state having the asymptotic form

$$|ls;JM\rangle \rightarrow \left(\frac{1}{k_ir}\right)\sin\left(k_ir - l\frac{\pi}{2} + \delta_J^{ls}\right)\mathcal{Y}_{Jls}^M. \quad (3.14)$$

The full wave function is then given in terms of the scattering eigenstates by the expansion

$$\psi^{\nu_p,\nu_d} = 4\pi \sum_{\substack{J,l,s \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d|s\nu\rangle\langle lm,s\nu|JM\rangle$$
$$\times e^{i\delta_J^{ls}}|ls;JM\rangle Y_l^{m*}(\hat{\mathbf{k}}_i). \quad (3.15)$$

Since the states $|ls;JM\rangle$ are full solutions to the Schrödinger equation, this last result is valid in all space.

### C. Watson's theorem

The value of this particular form of the scattering wave function is that it allows us to express the capture amplitude in terms of a set of real matrix elememts. This is because the scattering eigenstates $|ls;JM\rangle$ satisfy Eq. (2.26). To see this is the case, recall that these eigenstates are solutions to the Schrödinger equation $H_0|ls;JM\rangle = E|ls;JM\rangle$. We assume that $H_0$ is invariant under time reversal, and it follows that $\theta_T|ls;JM\rangle$ must also solve the Schrödinger equation. The properties of this solution are apparent from Eq. (3.14). According to this formula, $|ls;JM\rangle$ in the asymptotic region is the product of a real function and a quantity $\mathcal{Y}_{Jls}^M$ that satisfies Eq. (2.26), and it follows that $\theta_T|ls;JM\rangle$ has the asymptotic form

$$\theta_T|ls;JM\rangle \rightarrow (-)^{J-M}\left(\frac{1}{k_ir}\right)\sin\left(k_ir - l\frac{\pi}{2} + \delta_J^{ls}\right)\mathcal{Y}_{Jls}^{-M}. \quad (3.16)$$

Except for an overall sign, we see that $\theta_T|ls;JM\rangle$ matches the function $|ls;J-M\rangle$ in the asymptotic region, and since the Schrödinger equation allows only one solution with this particular asymptotic form, we conclude that

$$\theta_T|ls;JM\rangle = (-)^{J-M}|ls;J-M\rangle. \quad (3.17)$$

At this point it is useful to combine results to obtain an expression for the transition amplitude for radiative capture. We want the transition amplitude for a reaction proceeding from an initial state characterized by quantum numbers $\nu_p$ and $\nu_d$ to a final state in which the photon has circular polarization $\lambda$ and the residual ${}^3$He nucleus has spin projection $\sigma$. This quantity, which we now designate as $T_{\lambda\sigma,\nu_p\nu_d}$, is obtained by substituting Eqs. (2.22) and (3.15) into Eq. (2.16). With the help of Eq. (3.11) we have

$$T_{\lambda\sigma,\nu_p\nu_d} = -4\pi\left[\frac{\mu_i\omega}{2\pi\hbar^2 ck_i}\right]^{1/2}\sum_{\substack{J,l,s,L \\ M,\nu,m}}\langle s_p\nu_p,s_d\nu_d|s\nu\rangle$$

$$\times\langle lm,s\nu|JM\rangle e^{i\delta_J^{ls}}$$

$$\times\langle ls;JM|T_{L\lambda}^e+\lambda T_{L\lambda}^m|\phi_{\mathrm{He}}^\sigma\rangle^* Y_l^{m*}(\hat{\boldsymbol{k}}_i), \quad (3.18)$$

where $\phi_{\mathrm{He}}$ is the $^3$He bound state wave function, which we assume to be defined in accordance with the phase convention Eq. (2.26).

We now further simplify this result by employing the Wigner-Eckart theorem. Writing

$$\langle ls;JM|T_{L\lambda}|\phi_{\mathrm{He}}^\sigma\rangle = \langle L\lambda,s_c\sigma|JM\rangle\langle ls;J||T_L||\phi_{\mathrm{He}}\rangle, \quad (3.19)$$

where $s_c$ is the $^3$He spin, we obtain

$$T_{\lambda\sigma,\nu_p\nu_d} = -4\pi\left[\frac{\mu_i\omega}{2\pi\hbar^2 ck_i}\right]^{1/2}\sum_{\substack{J,l,s,L \\ M,\nu,m}}\langle s_p\nu_p,s_d\nu_d|s\nu\rangle$$

$$\times\langle lm,s\nu|JM\rangle\langle L\lambda,s_c\sigma|JM\rangle$$

$$\times e^{i\delta_J^{ls}}[\langle ls;J||T_L^e||\phi_{\mathrm{He}}\rangle^*$$

$$+\lambda\langle ls;J||T_L^m||\phi_{\mathrm{He}}\rangle^*]Y_l^{m*}(\hat{\boldsymbol{k}}_i). \quad (3.20)$$

The reader will note at this point that the complex conjugation symbols associated with the various matrix elements have been carried along through Eqs. (3.18) and (3.20). However, these complex conjugations are superfluous since the matrix elements are necessarily real by virtue of the results of Sec. II E. Equation (3.20) can thus be viewed as the statement of Watson's theorem for the $p$-$d$ radiative capture reaction for the case of no angular momentum mixing. In this special situation, the phase associated with each matrix element is simply the corresponding elastic scattering phase shift.

## IV. THE GENERALIZED WATSON THEOREM

### A. The elastic $S$-matrix

We now proceed to the general case in which mixing between angular momentum states is permitted. In this case the solutions to the Schrödinger equation are eigenfunctions of $J^2$ and $J_z$, but $l$ and $s$ are no longer good quantum numbers. If we focus on a particular $J$ and parity, the most general solutions to the Schrödinger equation, $\phi_J^M$, will involve sums of ingoing and outgoing waves:

$$\phi_J^M\to\sum_\alpha\frac{1}{2ik_ir}[b_\alpha e^{i[k_ir-l_\alpha(\pi/2)]}-a_\alpha e^{-i[k_ir-l_\alpha(\pi/2)]}]\mathcal{Y}_{Jl_\alpha s_\alpha}^M, \quad (4.1)$$

where the sum now runs over all $l$-$s$ combinations that are consistent with the specified $J^\pi$ value. For $p$-$d$ scattering this sum involves 2 terms when $J=\frac{1}{2}$ and 3 terms for all higher $J$.

Following Blatt and Biedenharn [12], we argue that since all the states are coupled, each $b_\alpha$ coefficient must now be a function of all the $a$'s. Thus for the case of 3-state mixing one would have

$$b_1=S_{11}a_1+S_{12}a_2+S_{13}a_3,$$

$$b_2=S_{21}a_1+S_{22}a_2+S_{23}a_3,$$

$$b_3=S_{31}a_1+S_{32}a_2+S_{33}a_3, \quad (4.2)$$

or in matrix form

$$b=Sa. \quad (4.3)$$

Now for elastic scattering with no open reaction channels, the $S$-matrix must be unitary and symmetric. It follows (see for example Ref. [13]) that $S$ can be diagonalized by a matrix transformation that has the form of a generalized Blatt-Biedenharn ''rotation.'' Equivalently, the $S$-matrix can be written in the form

$$S=u^\dagger S_0 u, \quad (4.4)$$

where $S_0$ is a diagonal, unitary matrix

$$S_0=\begin{bmatrix}e^{2i\delta_1} & 0 & 0 \\ 0 & e^{2i\delta_2} & 0 \\ 0 & 0 & e^{2i\delta_3}\end{bmatrix}. \quad (4.5)$$

The phase shifts, $\delta_\alpha$ that appear in Eq. (4.5) are real parameters, commonly referred to as the eigenphaseshifts.

The mixing matrix $u$ that appears in Eq. (4.4) is both real and orthonormal. For two-state mixing this matrix is usually parametrized in terms of a single mixing parameter $\epsilon$,

$$u=\begin{bmatrix}\cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon\end{bmatrix}. \quad (4.6)$$

For the case of three-state mixing, three ''mixing parameters'' are required. One possible choice for the matrix $u$ in this case is given in Ref. [13]; however, for our purposes adoption of a specific parametrization of the matrix $u$ is not required.

### B. The eigenstates of $S$

To obtain the generalized version of Watson's theorem, we would like to write the scattering wave function in terms of a set of scattering states that satisfy Eq. (2.26). As we shall see below, the functions we want are the so-called eigenstates of scattering matrix [12]. To define these states we introduce a shorthand notation for the ingoing and outgoing waves in a given partial wave state,

$$\chi_\alpha^{in}=-\left(\frac{1}{2ik_ir}\right)e^{-i[k_ir-l_\alpha(\pi/2)]}\mathcal{Y}_{Jl_\alpha s_\alpha}^M \quad (4.7)$$

and

$$\chi_\alpha^{out}=\left(\frac{1}{2ik_ir}\right)e^{i[k_ir-l_\alpha(\pi/2)]}\mathcal{Y}_{Jl_\alpha s_\alpha}^M. \quad (4.8)$$

The general wave function of Eq. (4.1) may then be written as

$$\phi\to\sum_\beta\left[a_\beta\chi_\beta^{in}+\sum_\gamma S_{\beta\gamma}a_\gamma\chi_\beta^{out}\right]. \quad (4.9)$$

Let us now define a set of three functions, $\phi_\alpha$ in which the ingoing-wave amplitudes $a_\beta$ are defined according to the rule

$$a_\beta = u_{\alpha\beta} \qquad (4.10)$$

so that

$$\phi_\alpha \rightarrow \sum_\beta \left[ u_{\alpha\beta} \chi_\beta^{\text{in}} + \sum_\gamma S_{\beta\gamma} u_{\alpha\gamma} \chi_\beta^{\text{out}} \right]. \qquad (4.11)$$

Then by making use of Eq. (4.4) and by recalling that $u$ is real and unitary and that $S_0$ is diagonal, one may easily re-write these functions in the form

$$\phi_\alpha \rightarrow \sum_\beta u_{\alpha\beta} \chi_\beta^{\text{in}} + e^{2i\delta_\alpha} \sum_\beta u_{\alpha\beta} \chi_\beta^{\text{out}}. \qquad (4.12)$$

From this last result one can see why these particular functions are referred to as the eigenstates of the scattering matrix. If we prepare an ingoing-wave state made up of the particular linear combination of partial wave amplitudes specified in Eq. (4.10), then the resulting outgoing wave is just an overall phase, $e^{2i\delta_\alpha}$, times the same linear combination of outgoing partial wave amplitudes.

For our purposes, the importance of this result can be seen by inserting the definitions of $\chi^{in}$ and $\chi^{out}$ into Eq. (4.12). After some manipulation one obtains

$$\phi_\alpha \rightarrow e^{i\delta_\alpha} \sum_\beta u_{\alpha\beta} \left( \frac{1}{k_i r} \right) \sin\left( k_i r + \delta_\alpha - l_\beta \frac{\pi}{2} \right) \mathcal{Y}_{Jl_\beta s_\beta}^M. \qquad (4.13)$$

Recalling once again that the elements of the matrix $u$ are real, we see that the wave functions $\phi_\alpha$ have a well defined behavior under the operation of time reversal. We introduce states $|\alpha;JM\rangle$ defined by the equation

$$|\alpha;JM\rangle = e^{-i\delta_\alpha} \phi_\alpha, \qquad (4.14)$$

where $\delta_\alpha$ is the appropriate eigenphaseshift. Then, according to the arguments of Sec. III C, these states must obey the time-reversal transformation

$$\theta_T |\alpha;JM\rangle = (-)^{J-M} |\alpha;J-M\rangle. \qquad (4.15)$$

### C. The generalized theorem

Since we have now succeeded in identifying states that transform according to the rule (2.26), we can readily obtain the Watson theorem formula for the general case. To do so, we simply need to express the full scattering wave function in terms of the scattering eigenstates.

From Eqs. (3.6) and (3.7) the incident plane wave can be written in the form

$$\psi_{\text{inc}} \rightarrow 4\pi \sum_{\substack{J,\pi,\beta \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d | s_\beta\nu\rangle \langle l_\beta m, s_\beta\nu|JM\rangle$$

$$\times (\chi_\beta^{\text{in}} + \chi_\beta^{\text{out}}) Y_{l_\beta}^{m*}(\hat{\mathbf{k}}_i), \qquad (4.16)$$

where summation over the $\beta$ is meant to imply summation over all $l$-$s$ values associated with each particular $J^\pi$. To proceed we now construct the full scattering wave function by making a superposition of scattering eigenstates in which the ingoing waves match those of $\psi_{\text{inc}}$. Starting from Eq. (4.12) one readily obtains

$$\sum_\alpha u_{\alpha\beta} \phi_\alpha \rightarrow \chi_\beta^{\text{in}} + \sum_\alpha S_{\alpha\beta} \chi_\alpha^{\text{out}} \qquad (4.17)$$

and so the full elastic scattering wave function can be written in the form

$$\psi^{\nu_p,\nu_d} = 4\pi \sum_{\substack{J,\pi,\beta \\ M,\nu,m}} \langle s_p\nu_p, s_d\nu_d | s_\beta\nu\rangle \langle l_\beta m, s_\beta\nu|JM\rangle$$

$$\times \left( \sum_\alpha u_{\alpha\beta} \phi_\alpha \right) Y_{l_\beta}^{m*}(\hat{\mathbf{k}}_i). \qquad (4.18)$$

The generalized form of Watson's theorem is then obtained by substituting this result into Eq. (2.16). We replace the scattering eigenstates $\phi_\alpha$ with quantities $|\alpha;JM\rangle$ defined in Eq. (4.14) and thus our expression for the transition amplitude involves matrix elements of the form $\langle\alpha;JM|T_{L\lambda}|\phi_{\text{He}}^\sigma\rangle$. We then introduce reduced matrix elements defined by the formula

$$\langle\alpha;JM|T_{L\lambda}|\phi_{\text{He}}^\sigma\rangle = \langle L\lambda, s_c\sigma|JM\rangle\langle\alpha;J||T_L||\phi_{\text{He}}\rangle. \qquad (4.19)$$

Combining the various formulas we then obtain the final result

$$T_{\lambda\sigma,\nu_p\nu_d} = -4\pi \left[ \frac{\mu_i\omega}{2\pi\hbar^2 c k_i} \right]^{1/2} \sum_{\substack{J,\pi,\alpha,\beta \\ L,M,\nu,m}} \langle s_p\nu_p, s_d\nu_d | s_\beta\nu\rangle$$

$$\times \langle l_\beta m, s_\beta\nu|JM\rangle\langle L\lambda, s_c\sigma|JM\rangle$$

$$\times u_{\alpha\beta} e^{i\delta_\alpha} [\langle\alpha;J||T_L^e||\phi_{\text{He}}\rangle^*$$

$$+ \lambda\langle\alpha;J||T_L^m||\phi_{\text{He}}\rangle^*] Y_{l_\beta}^{m*}(\hat{\mathbf{k}}_i), \qquad (4.20)$$

where, by virtue of Eq. (4.15), it is seen that the reduced matrix elements must once again be real.

### V. DISCUSSION

We have now reached the goal of obtaining a single formula which can be used as the basis of a matrix-element analysis of $p$-$d$ radiative capture data obtained at energies below the deuteron breakup threshold. As we have outlined previously, the approach one uses in such an analysis is to treat the reduced matrix elements as parameters and determine these quantities by fitting measurements.

The advantage of the present formulation over the conventional one is that the matrix elements are necessarily real. Information on the relative phases of the terms that appear in the sum over multipoles and angular momentum states is derived from the elastic scattering channel. The assumption we make is that the eigenphaseshifts and the mixing matrix

elements are known from a separate phase shift analysis of elastic scattering.

In a conventional analysis, the matrix elements appearing in the expansion of $T_{fi}$ would be matrix elements of some multipole operator taken between the bound state and a partial-wave scattering state, $\phi_{ls,J}^M$, defined with the boundary condition that the ingoing asymptotic wave should be a pure angular momentum state involving only a single $l$-$s$ combination. For these states the outgoing waves have a relatively complicated asymptotic form that, in general, involves all three eigenphaseshifts. As a result the phase of the resulting matrix element is not apparent. In contrast, the matrix elements used in the present formulation are defined in terms of the eigenstates of the $S$-matrix, which have a simpler asymptotic form that involves only a single eigenphase. Since we use eigenchannel wave functions, we refer to the matrix elements of Eq. (4.19) as eigenchannel matrix elements.

The connection between the eigenchannel matrix elements and the conventional ones is fairly simple. We define a set of matrix element parameters $P_\alpha^{\mathrm{EL}}$ and $P_\alpha^{\mathrm{ML}}$ by the equations

$$P_\alpha^{\mathrm{EL}} = -\left[\frac{8\mu_i k_i \omega}{\hbar^2 c}\right]^{1/2} \frac{1}{\sqrt{2L+1}} \langle \alpha;J||T_L^e||\phi_{\mathrm{He}}\rangle \quad (5.1)$$

and

$$P_\alpha^{\mathrm{ML}} = -\left[\frac{8\mu_i k_i \omega}{\hbar^2 c}\right]^{1/2} \frac{1}{\sqrt{2L+1}} \langle \alpha;J||T_L^m||\phi_{\mathrm{He}}\rangle, \quad (5.2)$$

and then introduce a set of transformed matrix elements,

$$R_\beta = \sum_\alpha u_{\alpha\beta} e^{i\delta_\alpha} P_\alpha. \quad (5.3)$$

Except for a possible overall phase, these quantities appear to be identical to the $R$ parameters used by Seyler and Weller [1]. In terms of these transformed matrix elements, the transition amplitude may be written as

$$T_{\lambda\sigma,\nu_p\nu_d} = \frac{2\pi}{k_i} \sum_{\substack{J,\pi,\beta,L \\ M,\nu,m}} \left[\frac{2L+1}{4\pi}\right]^{1/2} \langle s_p\nu_p, s_d\nu_d|s_\beta\nu\rangle$$

$$\times \langle l_\beta m, s_\beta\nu|JM\rangle \langle L\lambda, s_c\sigma|JM\rangle$$

$$\times [R_\beta^{\mathrm{EL}} + \lambda R_\beta^{\mathrm{ML}}] Y_{l_\beta}^{m*}(\hat{\boldsymbol{k}}_i). \quad (5.4)$$

While the equations we have obtained for the transition amplitude may appear to be quite straightforward, there are still a few subtle points. The first concerns the choice of coordinate frame. Recall that in Sec. II D we adopted a coordinate frame with the $z$-axis along the direction of the photon's momentum, $\boldsymbol{k}$. We shall now refer to this as the ''photon frame.'' Let us choose the $y$-axis of this coordinate frame to be along the direction $\boldsymbol{k}_i \times \boldsymbol{k}$, where $\hat{\boldsymbol{k}}_i$ is the proton's momentum direction in the c.m. frame. The reaction angle $\theta_{\mathrm{c.m.}}$ is then taken to be the angle between the incident proton and the outgoing $\gamma$-ray (in the c.m. frame) and it follows that

$$Y_l^{m*}(\hat{\boldsymbol{k}}_i) = Y_l^{m*}(\theta_{\mathrm{c.m.}},\pi) = Y_l^{-m}(\theta_{\mathrm{c.m.}},0). \quad (5.5)$$

The next point concerns the calculation of the deuteron analyzing powers. These quantities are normally defined in accordance with the Madison Convention [14], in which the spin operators and observables are referenced to a coordinate frame which does not coincide with the frame we are using here. If we adopt the spherical tensor notation [14], the Madison Convention analyzing powers, $T_{kq}$, may be obtained by first calculating the analyzing powers in the photon frame, and then transforming to the Madison frame. In the photon frame we have

$$\hat{T}_{kq} = \mathrm{tr}[T\tau_{kq}T^\dagger]/\mathrm{tr}[TT^\dagger], \quad (5.6)$$

where the $\tau_{kq}$'s are the usual spherical tensor spin operators. Since the analyzing powers are spherical tensors, we may use the rotation matrices to transform to the Madison frame. The appropriate transformation is

$$T_{kq} = \sum_{q'} d_{q'q}^k(\pi-\theta_{\mathrm{c.m.}})\hat{T}_{kq'}, \quad (5.7)$$

where the $d$-functions are defined, for example, in Ref. [15].

The next point concerns the phase conventions. If one wishes to carry out a matrix element analysis of the kind we are proposing, care must be taken to ensure that the mixing matrices are used correctly. This requires careful adherence to the phase conventions and coupling orders for the angular momenta. In particular, it is important to use the same conventions in the elastic scattering analysis as in the reaction analysis. It should be noted that many of the published phase shift analyses (see for example Ref. [9]) employ the coupling scheme of Ref. [13] which differs from that used in the present formulation. The effect of changing the coupling order is that some of the off-diagonal $S$-matrix elements change sign, and the result is that the matrix $u$ is altered. For purposes of completeness, we have recorded in the Appendix the relevant formulas for the elastic scattering amplitudes in the present coupling scheme.

Finally we need to make a few comments concerning the treatment of Coulomb effects in the elastic scattering channel. It will be noted that in the treatment presented above, there is no explicit reference to the Coulomb phase shifts. The understanding, then, is that the $S$-matrix being referred to in Eqs. (4.3) and (4.4) is the complete elastic scattering $S$-matrix.

Since the Coulomb phases do not converge to zero, the normal practice in a phase shift analysis is to separate off the Coulomb amplitude and work with the ''nuclear'' part of $S$ (see Ref. [13]). This quantity is then ordinarily parametrized with an equation of the form (4.4). Given the nuclear $S$-matrix, $\widetilde{S}$, the complete scattering matrix, $S$, is obtained by a matrix transformation (see the Appendix)

$$S = W\widetilde{S}W, \quad (5.8)$$

where $W$ is a diagonal matrix whose elements are the Coulomb phase factors $e^{i\sigma_l}$ associated with the individual angular momentum states that make up $S$. The phase parameters, $\sigma_l$, are obtained in the usual way,

$$\sigma_l = \arg \Gamma(l+1+i\eta). \quad (5.9)$$

It may be noted, however, that one could use modified Coulomb phases

$$\omega_l = \sigma_l - \sigma_0, \quad (5.10)$$

in place of the $\sigma_l$'s in making the transformation of Eq. (5.8). The only consequence would be that the elements of $S$ would all be rotated by a common phase, $2\sigma_0$, which means that the extracted eigenphases would change by $\sigma_0$, and the resulting transition amplitude would thus simply acquire an additional overall phase.

The need to transform from the nuclear $S$-matrix to the complete $S$-matrix introduces some complications. In particular one finds that there is no simple relationship between the mixing matrix $\tilde{u}$ that diagonalizes $\tilde{S}$ and the matrix $u$ that diagonalizes $S$. Similarly the eigenphases of $S$ are not related in any simple way to the eigenphases of $\tilde{S}$. The procedure one might follow to find the needed parameters would be to use the elastic phase shift parameters to construct $\tilde{S}$, find $S$ by the transformation (5.8), and then rediagonalize $S$ to obtain the new eigenphases and mixing matrix elements.

## VI. CONCLUSIONS

The use of Watson's theorem for the analysis low-energy radiative capture reactions has been discussed in detail. Particular emphasis has been given to the question of how Watson's theorem can be used in situations in which there is mixing between angular momentum states. Although the present work focuses on the $p$-$d$ capture reaction, the results obtained here can be applied to other systems as well. The main limitation is that Watson's theorem is valid only in situations in which there are no open reaction channels that significantly reduce the flux in the elastic scattering channel.

It is anticipated that the formalism introduced here will be employed in the near future to carry out a matrix element analysis of a set of $p$-$d$ radiative capture measurements at $E_{\text{c.m.}} = 2\,\text{MeV}$ obtained recently in experiments at the University of Wisconsin [16]. The results of this analysis will be presented in a future publication.

## APPENDIX

For completeness we record here the relevant formulas for the elastic scattering amplitudes. These amplitudes are easily extracted from the scattering wave function given in Eq. (4.18). Using the definition of $\phi_\alpha$ from Eq. (4.11) one obtains after some algebra

$$\psi^{\nu_p,\nu_d} \to 4\pi \sum_{\substack{J,l,l',s,s' \\ M,m,\nu}} \langle s_p\nu_p, s_d\nu_d | s\nu \rangle \langle lm, s\nu | JM \rangle$$
$$\times [\delta_{l,l'}\delta_{s,s'}\chi_{Jls}^{\text{in}} + S_{l's',ls}^J \chi_{Jl's'}^{\text{out}}] Y_l^{m*}(\hat{k}_i). \quad (A1)$$

To find the outgoing wave amplitudes we subtract the incident plane wave, Eq. (3.6). Using the definitions of $\chi^{\text{in}}$ and $\chi^{\text{out}}$ from Eqs. (4.7) and (4.8), and inserting the formula for spin-angle functions from Eq. (3.5) one obtains the result

$$\psi_{\text{scat}}^{\nu_p,\nu_d} \to 4\pi \sum_{\substack{J,l,l',s,s' \\ M,m,m',\nu,\nu'}} \sum_{\nu_p',\nu_d'} \langle s_p\nu_p, s_d\nu_d | s\nu \rangle \langle lm, s\nu | JM \rangle$$
$$\times \langle s_p\nu_p', s_d\nu_d' | s'\nu' \rangle \langle l'm', s'\nu' | JM \rangle$$
$$\times \frac{1}{2i}[S_{l's',ls}^J - \delta_{l,l'}\delta_{s,s'}]$$
$$\times Y_l^{m*}(\hat{k}_i) Y_{l'}^{m'}(\hat{r}) \frac{e^{ik_i r}}{k_i r} \phi_p^{\nu_p'} \phi_d^{\nu_d'}. \quad (A2)$$

One may now read off the outgoing wave amplitude in spin state $\nu_p', \nu_d'$. Adopting the standard coordinate frame for elastic scattering in which the $z$-axis is along $k_i$ and the $y$-axis is along $k_i \times k_f$ we have

$$M_{\nu_p'\nu_d';\nu_p\nu_d} = \frac{i\sqrt{\pi}}{k_i} \sum_{\substack{J,l,l',s,s' \\ M,m',\nu,\nu'}} [2l+1]^{1/2} \langle s_p\nu_p, s_d\nu_d | s\nu \rangle$$
$$\times \langle l0, s\nu | JM \rangle \langle s_p\nu_p', s_d\nu_d' | s'\nu' \rangle$$
$$\times \langle l'm', s'\nu' | JM \rangle$$
$$\times [\delta_{l,l'}\delta_{s,s'} - S_{l's',ls}^J] Y_{l'}^{m'}(\theta,0). \quad (A3)$$

The formulas needed to incorporate the Coulomb scattering in an explicit way are found, for example, in Ref. [17]. By using the analytic expression for the Coulomb amplitude together with the partial wave expansion of the same quantity one obtains the result

$$M_{\nu_p'\nu_d';\nu_p\nu_d} = e^{2i\sigma_0} \frac{\sqrt{\pi}}{k_i} \Bigg\{ -C(\theta)\delta_{\nu_p,\nu_p'}\delta_{\nu_p,\nu_p'}$$
$$+ i \sum_{\substack{J,l,l',s,s' \\ M,m',\nu,\nu'}} [2l+1]^{1/2} \langle s_p\nu_p, s_d\nu_d | s\nu \rangle$$
$$\times \langle l0, s\nu | JM \rangle \langle s_p\nu_p', s_d\nu_d' | s'\nu' \rangle$$

$$\times\langle l'm',s'\nu'|JM\rangle e^{i(\omega_l+\omega_l')}[\delta_{l,l'}\delta_{s,s'}$$

$$-e^{-i(\sigma_l+\sigma_l')}S^J_{l's',ls}]Y^{m'}_{l'}(\theta,0)\Bigg\}, \qquad \text{(A4)}$$

where $\sigma_l$ and $\omega_l$ are defined in Eqs. (5.9) and (5.10), respectively, and where $C(\theta)$ is the Coulomb amplitude

$$C(\theta)=\frac{1}{\sqrt{4\pi}}\frac{\eta}{\sin^2\frac{\theta}{2}}e^{-i\eta\,\ln[\sin^2(\theta/2)]}. \qquad \text{(A5)}$$

We define the nuclear $S$-matrix elements to be

$$\tilde{S}^J_{l's',ls}=e^{-i(\sigma_l+\sigma_l')}S^J_{l's',ls}, \qquad \text{(A6)}$$

and in a phase shift analysis the submatrix of $\tilde{S}$ for each $J^\pi$ value would be parametrized in the form of Eq. (4.4),

$$\tilde{S}=\tilde{u}^\dagger\tilde{S}_0\tilde{u}. \qquad \text{(A7)}$$

In the matrix element analysis of the capture reaction, one then uses Eq. (5.8) to reconstruct the full $S$-matrix from $\tilde{S}$.

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