β- and double-β-decay transitions in a schematic model

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(Received 18 November 1998)

Single- and double-β decay transitions of Fermi type are discussed within a schematic model. The amplitudes associated with these transitions as well as some basic properties of the model like ground state and excitation energies are reproduced quite well in terms of a multistep variational procedure. The procedure, fully developed in a boson space, searches for the best ground state wave function through a series of minimizations. Excited states are constructed in terms of a phonon operator acting on the ground state so derived and their structure is also determined via a variational mechanism. Results obtained within the quasiparticle random phase approximation (QRPA) are shown for reference and also the comparison with other approaches like the renormalized QRPA is discussed. A considerable improvement is obtained within the present approach. [S0556-2813(99)05104-3]

PACS number(s): 21.60.Jz, 23.40.−s

I. INTRODUCTION

In recent years the quasiparticle random phase approximation (QRPA) has represented the most widely used nuclear structure method for the study of double-β ($\beta\beta$) decay physics. As well known [1–6], however, the evaluation of the results within this approach has proved to be particularly difficult due to the fact that the matrix elements associated with $\beta\beta$ processes are highly sensitive to the particle-particle component of the residual interaction and that the physical value of this force is usually close to a point in which the QRPA ''collapses.'' These difficulties have encouraged every effort aimed at improving this theory.

Several methods have been proposed in this context [7–12] but the one which has raised more interest so far is the so-called renormalized QRPA (RQRPA). The leading principle in the elaboration of this method has been the removal of the basic inconsistency of the RPA approach, namely, the quasiboson approximation (QBA) [13], which consists in approximating the correlated ground state with the uncorrelated one when solving the equations of motion. The method can be traced back to some work of Hara [14], Rowe [15] and da Providencia [16]; it has been the object of recent elaborations [17,18], and its first application to $\beta\beta$ decay physics is due to Toivanen and Suhonen [19] followed, then, by several other works [20–24].

The RQRPA is an approximation scheme developed in fermion space. In a recent paper [25], still with reference to $\beta\beta$ decay physics, we have instead examined an alternative method having among its features that of being fully developed in a boson space. Working in a boson space has the advantage that the QBA problem can be faced in a very natural way. On the other hand, other important problems appear which are just related to the use of bosons: in particular, a reliable mapping procedure is now needed in order to transform any fermion operator onto its boson image.

In order to test this boson approach, in Ref. [25] we have considered a schematic model which has been recently formulated in connection with $\beta\beta$ decay physics and used as a testing ground for RQRPA calculations [26]. Some important improvements with respect to the QRPA results have been indeed observed, in particular for what concerns the stability of the solutions. It is also true, however, that some deficiencies have remained especially at the level of β transition amplitudes for values of the interaction strength larger than the QRPA critical point.

The purpose of the present paper is that of showing that a much better description of the properties of the model of Ref. [26] is possible in terms of a different approach. This approach still makes use of a boson formalism and draws inspiration from a previous work [27]. In short, the approach is based on a multistep variational mechanism in which a derivation of the best ground state wave function is searched for through a series of minimizations. A variational mechanism also allows one to determine a phonon operator which is used for the definition of the excited states. We will show that, in comparison with the standard QRPA results, energies and β and $\beta\beta$ transition amplitudes are reproduced much more accurately and for values of the interaction strength far beyond the QRPA critical point. Similar conclusions will emerge also from a comparison with the RQRPA calculations [26] as well as with those of the previous boson approach [25].

The paper is organized as follows. In Sec. II, we will discuss the model. In Sec. III, we will resume the main points of the mapping procedure which is used to construct boson images of the fermion operators. In Sec. IV, we will describe the procedure for constructing ground and excited states. In Sec. V, we will discuss the results. In Sec. VI, we will search for a more accurate analysis of these results by making a comparison with previous calculations. Finally, in Sec. VII, we will provide a summary of the paper and draw some conclusions.

II. MODEL

We consider a system of protons and neutrons occupying both a single j shell and interacting via the Hamiltonian

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$$H=H_p+H_n+H_{\rm res}, \tag{1}$$

where

$$H_p=e_p\sum_m\ a^\dagger_{pm}a_{pm}-G_pS^\dagger_pS_p, \tag{2}$$

$$H_n=e_n\sum_m\ a^\dagger_{nm}a_{nm}-G_nS^\dagger_nS_n, \tag{3}$$

$$H_{\rm res}=2\chi\beta^-\beta^+-2kP^-P^+, \tag{4}$$

$$S^\dagger_p=\frac{1}{2}\sum_m\ a^\dagger_{pm}\tilde{a}^\dagger_{pm}, \quad S^\dagger_n=\frac{1}{2}\sum_m\ a^\dagger_{nm}\tilde{a}^\dagger_{nm}, \tag{5}$$

$$\beta^-=\sum_m\ a^\dagger_{pm}a_{nm}, \quad \beta^+=(\beta^-)^\dagger, \tag{6}$$

$$P^-=\sum_m\ a^\dagger_{pm}\tilde{a}^\dagger_{nm}, \quad P^+=(P^-)^\dagger. \tag{7}$$

In this expression $a^\dagger_{pm}$ is the operator which creates a proton with angular momentum j_p and projection m, $\tilde{a}^\dagger_{pm}=(-1)^{j-m}a^\dagger_{p-m}$ is its time reversal, and $a^\dagger_{nm}$, $\tilde{a}^\dagger_{nm}$ are the equivalent operators for neutrons. By performing a standard Bogolyubov transformation [13] for protons and neutrons, i.e., introducing the operators

$$\alpha^\dagger_{im}=u_ia^\dagger_{im}-v_i\tilde{a}_{im} \tag{8}$$

(where $u_i^2+v_i^2=1$, $i=p,n$), under the hypothesis that $j_p=j_n\equiv j$, $G_p=G_n\equiv G$, and by neglecting the so-called scattering terms $B^\dagger=[\alpha^\dagger_p\tilde{\alpha}_n]^{J=0}$ and B, it has been shown by Hirsch *et al.* [26] that the above Hamiltonian can be reduced to the simple form

$$H_F=\epsilon C+\lambda_1A^\dagger A+\lambda_2(A^\dagger A^\dagger+AA), \tag{9}$$

where a constant term has been neglected and where

$$A^\dagger=[\alpha^\dagger_p\alpha^\dagger_n]^{J=0}, \tag{10}$$

$$C=\sum_m\ \alpha^\dagger_{pm}\alpha_{pm}+\sum_m\ \alpha^\dagger_{nm}\alpha_{nm}, \tag{11}$$

with

$$\epsilon=\frac{\Omega}{2}G, \tag{12}$$

$$\lambda_1=4\Omega[\chi(u_p^2v_n^2+v_p^2u_n^2)-k(u_p^2u_n^2+v_p^2v_n^2)], \tag{13}$$

$$\lambda_2=4\Omega(\chi+k)u_pv_pu_nv_n. \tag{14}$$

In the previous expressions, $\Omega=j+\frac{1}{2}$ and

$$v_i=\sqrt{\frac{N_i}{2\Omega}}, \quad i=p,n, \tag{15}$$

$$u_i=\sqrt{1-\frac{N_i}{2\Omega}}, \quad i=p,n. \tag{16}$$

The operators $\{A,A^\dagger,C\}$ satisfy the SU(2) Lie algebra

$$[A,A^\dagger]=1-\frac{C}{2\Omega}, \quad [C,A^\dagger]=2A^\dagger, \tag{17}$$

and so the Hamiltonian (9) resembles that of the standard Lipkin model [28] with, in addition, the λ_1 term.

In this paper, we will concentrate on the solutions of the Hamiltonian (9), keeping for the coefficients $\epsilon,\lambda_1,\lambda_2$ the dependence on the parameters k,χ which results from Eqs. (12)–(14). In particular, calculations will be performed for a fixed value of χ and for k ranging in a given interval, as specified in Sec. V. The reason for this choice is that, as already observed in [26], calculated quantities like excitation energies and β and $\beta\beta$ transition amplitudes will exhibit a dependence on these parameters similar to that observed for realistic quantities in terms of the particle-particle (g_{pp}) and particle-hole (g_{ph}) strenghts [1,2,4]. In this way, the model Hamiltonian (9), although not meant to reproduce actual nuclear properties, will provide at least some qualitative features of a realistic pn-QRPA calculation.

In the following, then, whenever talking about exact energies and eigenstates, we will always mean those obtained from the diagonalization of Eq. (9) in the space

$$F\equiv\{|n\rangle=(A^\dagger)^n|0\rangle, \ \ 0\leqslant n\leqslant 2\Omega\}, \tag{18}$$

where $|0\rangle$ is the vacuum of the quasiparticle operators. These eigenvalues are, of course, not those of the Hamiltonian (1) since H_F, Eq. (9), only provides an approximation of Eq. (1). However, aiming in this work at making a relative comparison among different approximate schemes and having the Hamiltonian (9) already provide a basis for similar calculations [25,26], the use of this Hamiltonian appears quite suitable for such a comparative analysis.

III. BOSON IMAGES OF FERMION OPERATORS

As anticipated in the Introduction, the approach discussed in this paper is developed in a boson formalism. Therefore, as a preliminary operation, it is necessary to construct the boson images of all the fermion operators of interest. In this work we will not discuss the details of the mapping procedure adopted, for which we refer to Ref. [25] or, more in general, to Ref. [29], but we simply recall its main points.

Let $b^\dagger,b$ be creation, annihilation $J=0$ boson operators and $|0)$ the corresponding vacuum, and let us define the space

$$B\equiv\left\{|n)=\frac{1}{\sqrt{n!}}(b^\dagger)^n|0), \ \ 0\leqslant n\leqslant 2\Omega\right\}. \tag{19}$$

We can establish a one-to-one correspondence between the states of F and B. We define boson image of the fermion operator O_F in B a boson operator O_B such that all the eigenvalues of O_F in F are also eigenvalues of O_B in B. As a

result of the orthonormality of the states (19), this requirement simply amounts to a search for a boson operator O_B such that

$$\frac{1}{\sqrt{N_n}}\frac{1}{\sqrt{N_{n'}}}\langle n|O_F|n'\rangle=(n|O_B|n'),\quad 0\leqslant n,\ n'\leqslant 2\Omega, \tag{20}$$

where $N_n=\langle n|n\rangle$. This condition, which preserves matrix elements between corresponding states and, therefore, is of Marumori type [13], defines the operator O_B.

The boson operator is constructed by requiring the fulfillment of the condition (20) for increasingly larger fermion and boson subspaces. The larger these spaces are, in general, the more complicated the structure of the resulting boson operator will be. In this paper we will consider boson images having at most four-boson terms. This requires involving at most the states of the subspaces

$$F^{(3)}=\{|0\rangle,A^\dagger|0\rangle,(A^\dagger)^2|0\rangle,(A^\dagger)^3|0\rangle\} \tag{21}$$

and

$$B^{(3)}=\left\{|0),b^\dagger|0),\frac{1}{\sqrt{2}}(b^\dagger)^2|0),\frac{1}{\sqrt{6}}(b^\dagger)^3|0)\right\} \tag{22}$$

in the fulfillment of the condition (20).

The boson image of the Hamiltonian (9) which is construted with this procedure has the form

$$H_B=\alpha b^\dagger b+\beta(b^\dagger b^\dagger+bb)+\gamma b^\dagger b^\dagger bb + \delta(b^\dagger b^\dagger b^\dagger b+b^\dagger bbb), \tag{23}$$

with

$$\alpha=2\epsilon+\lambda_1, \tag{24}$$

$$\beta=\lambda_2\sqrt{1-\frac{1}{2\Omega}}, \tag{25}$$

$$\gamma=-\frac{\lambda_1}{2\Omega}, \tag{26}$$

$$\delta=\lambda_2\left[\sqrt{\left(1-\frac{1}{3\Omega}\right)\left(1-\frac{1}{2\Omega}\right)}-\sqrt{1-\frac{1}{2\Omega}}\right]. \tag{27}$$

This Hamiltonian has been seen to provide an excellent image of Eq. (9) [25].

For what concerns the Fermi β transition operators (6), in order to make clear the role played by different truncations, we will consider two different images. The first one is

$$(\beta^-)_B^{(1)}=\sqrt{2\Omega}(u_pv_nb^\dagger+v_pu_nb) \tag{28}$$

and is obtained by truncating the boson expansion at the lowest order. The second image is

$$(\beta^-)_B^{(2)}=(\beta^-)_B^{(1)}+c_1b^\dagger b^\dagger b+c_2b^\dagger bb, \tag{29}$$

with

$$c_1=\sqrt{2\Omega}u_pv_n\left(\sqrt{1-\frac{1}{2\Omega}}-1\right), \tag{30}$$

$$c_2=\sqrt{2\Omega}v_pu_n\left(\sqrt{1-\frac{1}{2\Omega}}-1\right), \tag{31}$$

and contains up to three boson terms. It is also

$$(\beta^+)_B^{(i)}=((\beta^-)_B^{(i)})^\dagger,\quad i=1,2. \tag{32}$$

IV. PROCEDURE

In order to keep the notation as clear as possible, we will refer directly to the model under discussion. Starting point for our procedure is the introduction of a RPA-like phonon operator

$$q_1^\dagger=X_1b^\dagger-Y_1b. \tag{33}$$

We define the state $|gs_1)$ as the vacuum of the q_1 operator, i.e.,

$$q_1|gs_1)=0. \tag{34}$$

As is well known [13], this condition implies

$$|gs_1)\propto e^{S_1^\dagger}|0), \tag{35}$$

where

$$S_1^\dagger=\frac{Y_1}{2X_1}b^\dagger b^\dagger. \tag{36}$$

The state (35) has components carrying any number of bosons but only those components which have up to 2Ω bosons can belong to the space B and so to have a physical counterpart in F. We get rid of the spurious components of Eq. (35) by acting on this state with the identity operator of B,

$$I_B=\sum_{n=0}^{2\Omega}|n)(n|, \tag{37}$$

and obtain

$$|\widetilde{gs}_1)\equiv I_B|gs_1)=\mathcal{N}_1\sum_{n=0}^{2\Omega,2}\frac{c_n^{(1)}}{\sqrt{n!}}|n), \tag{38}$$

with

$$c_n^{(1)}=\frac{n!}{(n/2)!}\left(\frac{Y_1}{2X_1}\right)^{n/2} \tag{39}$$

(where we have also introduced the normalization factor $\mathcal{N}_1$).

In order to fix the amplitudes X_1 and Y_1 of Eq. (33) and so to fix the structure of $|\widetilde{gs}_1)$, we minimize the expectation value of the boson Hamiltonian (23) in this state, i.e., the energy

$$E_{gs}^{(1)}=(\widetilde{gs}_1|H_B|\widetilde{gs}_1). \tag{40}$$

Actually, it is clear from Eq. (39) that the state $|\widetilde{gs}_1)$ only depends on the ratio Y_1/X_1. The absolute values of these amplitudes can be fixed by requiring the normalization of the one-phonon state

$$|q_1) \equiv I_B q_1^\dagger |\widetilde{gs}_1) = \mathcal{N}_1 \sum_{n=1}^{2\Omega-1,2} d_n^{(1)} |n), \tag{41}$$

where

$$d_n^{(1)} = \left(\frac{c_{n-1}^{(1)}}{(n-1)!} X_1 - \frac{c_{n+1}^{(1)}}{n!} Y_1 \right) \sqrt{n!}. \tag{42}$$

This normalization would reduce to the standard condition $(X_1)^2 - (Y_1)^2 = 1$ if we neglected the finite dimensionality of the space B. We define the state which results from the minimization of E_1 as the ''first-order'' approximation of the ground state of our system.

The state (38) has a very ''rigid'' structure which results from the condition (34). In other words, all the $c_n^{(1)}$ coefficients are strictly bound one to the other and fixing $c_2^{(1)}$ [or, equivalently, the structure of the operator $S_1^\dagger$, Eq. (36)] implies fixing them all at once. To search for a better ground state necessarily means to free these coefficients from this severe constraint and this is the basic idea which has inspired the following steps of the procedure.

As a next step, then, we introduce the state $|\widetilde{gs}_2)$ which is defined by the following *Ansatz*:

$$|\widetilde{gs}_2) = \mathcal{N}_2 (1 + S_2) |\widetilde{gs}_1), \tag{43}$$

where $S_2 = Z_2 bb$ and where $|\widetilde{gs}_1)$ is the state which has already been fixed in the first minimization. This state has still the form (38), i.e.,

$$|\widetilde{gs}_2) = \mathcal{N}_2 \sum_{n=0}^{2\Omega,2} \frac{c_n^{(2)}}{\sqrt{n!}} |n), \tag{44}$$

but with coefficients

$$c_n^{(2)} = \mathcal{N}_1 [c_n^{(1)} + Z_2 (1 - \delta_{n,2\Omega}) c_{n+2}^{(1)}]. \tag{45}$$

These coefficients can be fixed by minimizing the energy

$$E_{\text{g.s.}}^{(2)} = (\widetilde{gs}_2 | H_B | \widetilde{gs}_2) \tag{46}$$

with respect to the variable Z_2. The advantage of $|\widetilde{gs}_2)$ with respect to $|\widetilde{gs}_1)$ is that this new state is no longer bound to an exponential form of the type (35). Since $|\widetilde{gs}_2)$ is chosen such to correspond to a minimum in the energy, $E_{\text{g.s.}}^{(2)}$ can only provide an approximate ground state energy better than (or, at worst, equal to) $E_{\text{g.s.}}^{(1)}$. We define $|\widetilde{gs}_2)$ as the ''second-order'' approximation of the ground state of our system.

Having fixed $|\widetilde{gs}_2)$, the procedure could go on by iterating the step just described. So we could introduce a state $|\widetilde{gs}_3)$ of the type (43), namely in which an operator $(1+S_3)$ acts on $|\widetilde{gs}_2)$, we could minimize the ground state energy with respect to the new variable Z_3, and so on. As a matter of fact we have verified a more effective way of proceeding that works as follows.

The *Ansatz* (43) is not the only one that can be thought to add some ''flexibility'' to the c_n coefficients. A similar result can be obtained, for instance, with a state $|\widetilde{gs}_3)$ defined by the following *Ansatz*:

$$|\widetilde{gs}_3) = \mathcal{N}_3 I_B (1 + S_3^\dagger) |\widetilde{gs}_2), \tag{47}$$

where $S_3^\dagger = Z_3 b^\dagger b^\dagger$ [we have in this case introduced the projection operator I_B which was unnecessary in Eq. (43)]. This state has the form

$$|\widetilde{gs}_3) = \mathcal{N}_3 \sum_{n=0}^{2\Omega,2} \frac{c_n^{(3)}}{\sqrt{n!}} |n), \tag{48}$$

with coefficients

$$c_n^{(3)} = \mathcal{N}_2 [c_n^{(2)} + Z_3 n(n-1) c_{n-2}^{(2)}]. \tag{49}$$

These coefficients can be fixed as before by minimizing the energy

$$E_{\text{g.s.}}^{(3)} = (\widetilde{gs}_3 | H_B | \widetilde{gs}_3) \tag{50}$$

with respect to the variable Z_3. The same considerations already made for $|\widetilde{gs}_2)$ also hold for $|\widetilde{gs}_3)$ and so this state is expected to represent an improvement with respect to $|\widetilde{gs}_2)$.

The best way of proceeding that we have experienced consists in making an alternate use of the *Ansätze* (43) and (47). Qualitatively, such a result can be understood by noticing that the operators $(1+S)$ and $(1+S^\dagger)$ contain nothing but the first two terms of the power expansion of the operators e^S and $e^{S^\dagger}$, respectively. Had we used such operators, only their alternate use would have been meaningful since the sequential action of two exponential operators of the same form in the minimization procedure would be exactly equivalent to the use of just one of them. We could have indeed used these full exponential forms and in the special model under discussion this would have given no special problems. However, having in mind possible applications of the procedure under discussion to realistic cases we have preferred to test an approach as simple as possible.

In conclusion, we define the state $|\widetilde{gs}_3)$ (47) as the ''third-order'' approximation of the ground state of our system and we proceed by alternatively minimizing forms of the types (43) and (47). As already noticed above, one can expect an improvement of the approximate ground state energy at each step of this procedure. However, one should also not forget that, as far as the boson Hamiltonian that one is using is not an exact image of the fermion one (and this is generally the case), this procedure cannot converge to the exact solution. The quality of the final result will, of course, strictly depend on the quality of the Hamiltonian itself.

For what concerns excited states, we have already considered in Eq. (41) the one-phonon state $|q_1)$. An important feature of this state is that, in the spirit of RPA, the ground state $|\widetilde{gs}_1)$ entering the definition of $|q_1)$ is the vacuum (a part from a projection) of the phonon operator q_1. Whenever dealing with a ground state $|\widetilde{gs}_n)$ with $n>1$, instead, a definition of this kind becomes impossible since this state is no longer the vacuum of any phonon operator. A quite natural

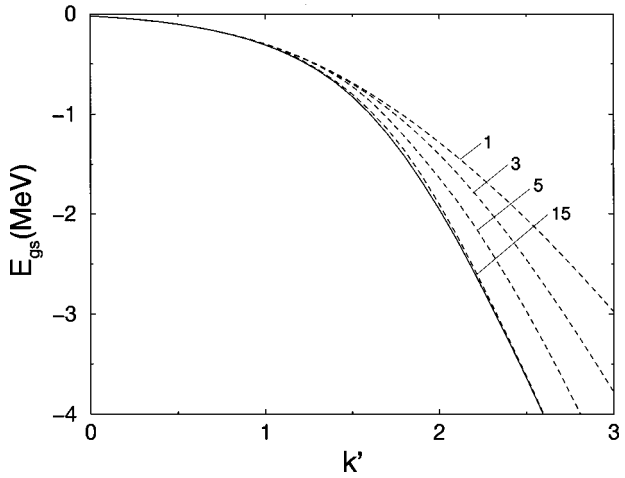

FIG. 1. Comparison of the ground state energies resulting from the diagonalization of H_F, Eq. (9), in the basis (18) (solid line) with those obtained in our approach (dashed lines). Dashed lines are labeled by numbers which indicate the order of the approximation of the calculation (see text).

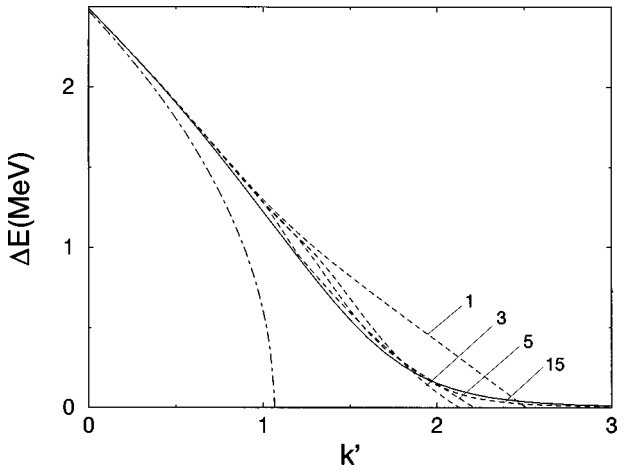

FIG. 2. Energies of the first excited state referred to the ground state. Solid line shows the exact result, dash-dotted line the QRPA one, and dashed lines refer to our results at different orders of approximation.

way of constructing an excited one-phonon state based on $|\widetilde{gs}_n)$ is, then, that of defining

$$|q_n)=I_B q^\dagger|\widetilde{gs}_n), \tag{51}$$

where $q^\dagger=Xb^\dagger-Yb$ is a phonon operator to be determined. The X and Y variables of this operator can be fixed by the minimization of

$$E_q^{(n)}=(q_n|H_B|q_n) \tag{52}$$

under the constraint $(q_n|q_n)=1$. The difference between this state $|q_n)$ $(n>1)$ and the state $|q_1)$, Eq. (41), is crucial. While, in fact, an evident interdependence exists between the ground state $|\widetilde{gs}_1)$ and the excited state $|q_1)$ due to the fact that the same phonon operator enters in the definition of both these states, this is no longer true in the case of $|\widetilde{gs}_n)$ and $|q_n)$ for $n>1$. This leaves more freedom in the choice of the best excited state. The analysis of this interdependence has been the object of a recent Letter [30] and will be further discussed in Sec. VI. To conclude this section, we remark that, hereafter, whenever referring to an ''nth-order'' approximation, we will always mean a calculation involving the states $|\widetilde{gs}_n)$ and $|q_n)$.

V. RESULTS

The parameters that we have chosen for our model are $j=9/2$, $Z=4$, $N=6$, and $\epsilon=1$ MeV. Moreover, in order to avoid dealing with small numbers, we have also redefined the two parameters k and χ as follows: $k\rightarrow k'\equiv 2\Omega k$ and $\chi\rightarrow\chi'\equiv 2\Omega\chi$. Calculations have been performed for $\chi'=0.5$ while k' has been varied in the interval (0,3).

A. Energies

In Fig. 1, we show the ground state energies for different orders of approximations and compare them with the exact values. In Fig. 2, a similar comparison is made for the energies of the first excited state (relative to the ground state). In this case, we also plot the energies calculated within the QRPA (for these as well as for the other QRPA results we refer to Ref. [26]). The latter figure well illustrates the extent of the interval chosen for the variable k' [(0,3)] when one compares this with the range of applicability of the QRPA which is approximately (0,1). It emerges that, within the QRPA range, already the first-order approximation of our calculations provides results which are quite well in agreement with the exact ones. Outside this range, instead, the larger k' is, the more these $n=1$ values start deviating from the exact ones. In this region, however, a visible improvement in the results is observed when resorting to higher order approximations and, for $n=15$, both ground and excitation energies are very well reproduced in the whole range of k'. These figures, therefore, underline the high degree of convergence which is reached by the procedure under discussion.

B. Expectation values of the quasiparticle number operator

As a simple test on the quality of the wave functions, we have performed a comparison between exact and approximate expectation values of half the quasiparticle number operator (11), $C/2$, in the ground state and in the first excited state. The interest in such a comparison is due to the fact that the boson image of $C/2$ which is constructed with the procedure of Sec. III, namely, the operator $N_B=b^\dagger b$, is quite simple and *exact*. It follows that any difference between exact and approximate values in the above quantities cannot be ascribed to a bad quality of the boson operator but rather reflects the quality of the wave functions of the states involved.

We define

$$N_{\text{g.s.}}=\left\langle \text{g.s.}\left|\frac{C}{2}\right|\text{g.s.}\right\rangle \tag{53}$$

and

$$\Delta N=\left\langle \text{exc}\left|\frac{C}{2}\right|\text{exc}\right\rangle-\left\langle \text{g.s.}\left|\frac{C}{2}\right|\text{g.s.}\right\rangle, \tag{54}$$

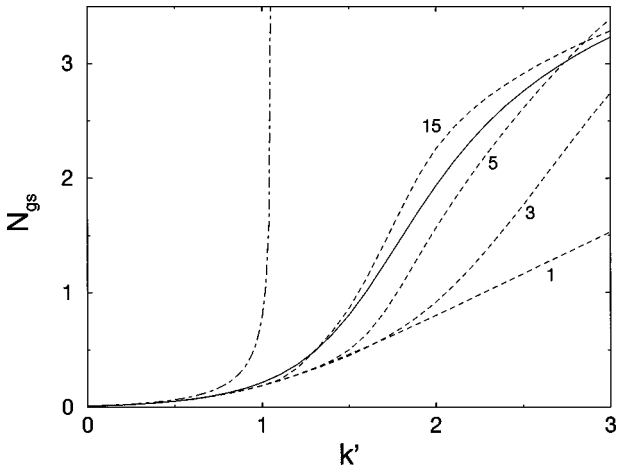

FIG. 3. Expectation values of half the quasiparticle number operator (11), $C/2$, in the ground state. For symbols see Fig. 2.

where $|\text{g.s.}\rangle$ is the ground state and $|\text{exc}\rangle$ is the first excited state of the system. In Figs. 3 and 4, we show the exact values, the QRPA ones, and those obtained within the present procedure at different orders of approximation. These results confirm qualitatively those already seen for the energies. Indeed, already for $n=1$, both $N_{\text{g.s.}}$ and ΔN are approximated reasonably well within the range of applicability of the QRPA while for larger values of k' a clear disagreement is visible for this value of n. However, also in this case, higher order approximations in our calculations cancel to a very good extent this disagreement. The convergence towards the exact results provides evidence of the improved quality of the wave functions.

C. Fermi β and $\beta\beta$ transition amplitudes

A comparison between matrix elements of the β operators (6) presents some extra difficulties with respect to the previous cases. Indeed, one has now to deal with boson operators which are not exact images and so one has also to take into account the effect of this further approximation when evaluating the results. To test such an effect, we have performed calculations using two different β transition operators: operators truncated at one-boson terms $(\beta^{\pm})_B^{(1)}$ [Eq. (28] and at three-boson terms $(\beta^{\pm})_B^{(2)}$ [Eq. (29]. Matrix elements of these operators between the ground state and the first excited state are compared with the corresponding fermion quantities,

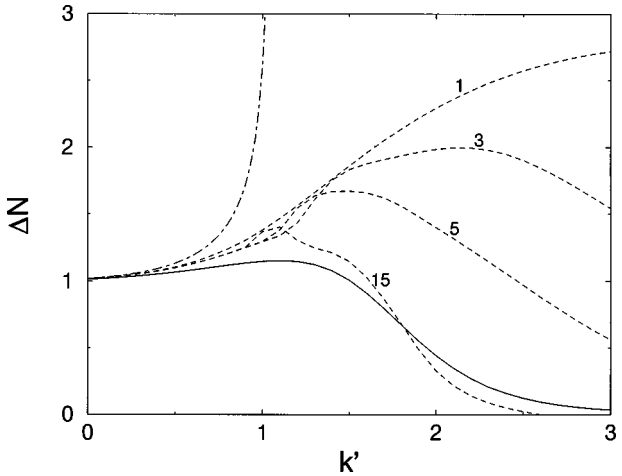

FIG. 4. Differences between the expectation values of half the quasiparticle number operator (11), $C/2$, in the first excited state and in the ground state. For symbols see Fig. 2.

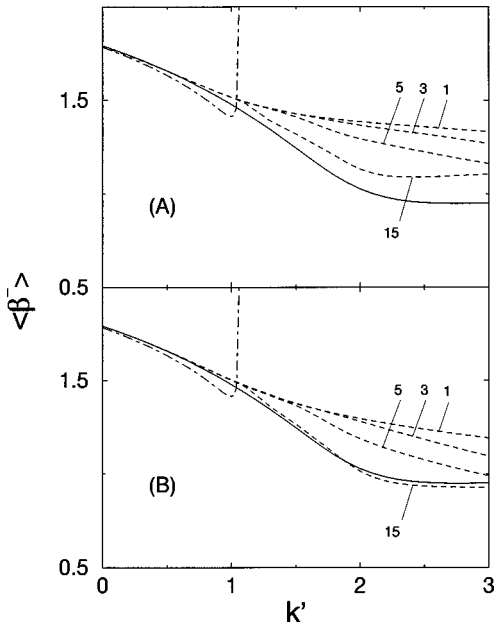

FIG. 5. Transition amplitudes of the operator β^- (6) between ground and first excited states. (A) refers to the use of the boson operator $(\beta^-)_B^{(1)}$, Eq. (28), while (B) to $(\beta^-)_B^{(2)}$, Eq. (29). For symbols see Fig. 2.

$$\langle\beta^{\pm}\rangle \equiv \langle \text{exc}|\beta^{\pm}|\text{g.s.}\rangle, \tag{55}$$

in Figs. 5 and 6. These results confirm qualitatively those

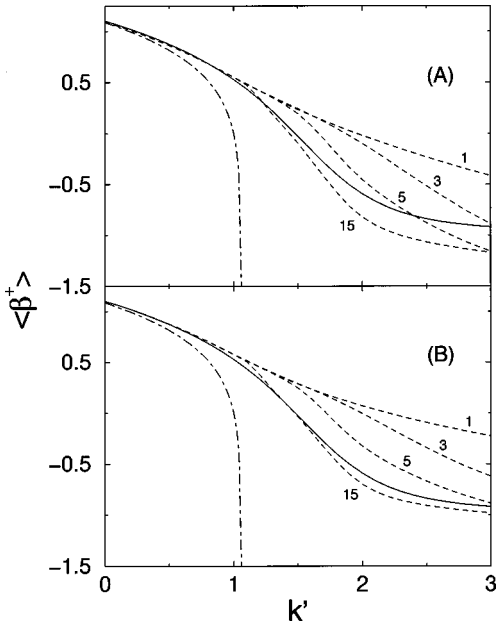

FIG. 6. Transition amplitudes of the operator β^+, Eq. (6), between ground and first excited states. (A) refers to the use of the boson operator $(\beta^+)_B^{(1)}$, while (B) to $(\beta^+)_B^{(2)}$, Eq. (32). For symbols see Fig. 2.

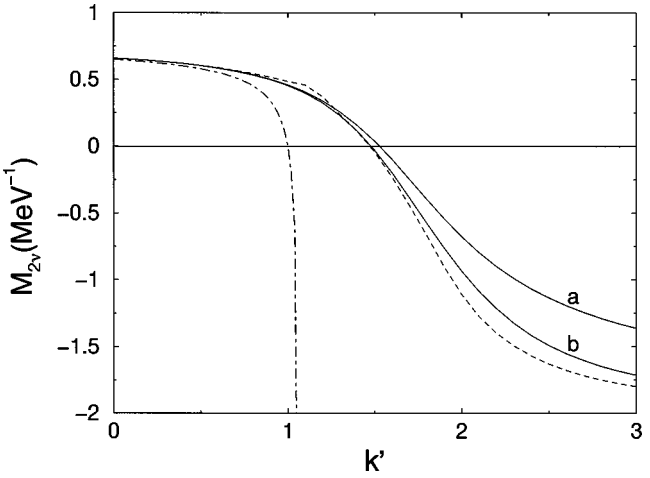

FIG. 7. Two-neutrino decay amplitudes (56) for $\Delta=0.5$ MeV. The solid line (*a*) refers to the exact values obtained summing over all the excited states while for line *b* the summation has been restricted to the first excited state only. The dash-dotted line refers to the QRPA result while the dashed line to our $n=15$ calculation.

already found for the energies as well as for $N_{\text{g.s.}}$ and ΔN. In addition, one can now examine the role of the two different truncations in the boson operators employed. As expected, the use of the $(\beta^{\pm})_B^{(2)}$ in place of $(\beta^{\pm})_B^{(1)}$ results in a better agreement with the exact values. However, the difference between the two cases becomes really appreciable only for rather large values of the strength k'.

In order to examine two-neutrino $\beta\beta$ decay amplitudes in this model, we follow the same lines of Ref. [26] and make the approximation $\langle(\text{g.s.})_{\text{final}}|\beta^-|\lambda\rangle\approx\langle(\text{g.s.})_{\text{initial}}|\beta^-|\lambda\rangle=\langle\lambda|\beta^+|\text{g.s.}\rangle$. In this way, we define

$$M_{2\nu}=\sum_{\lambda}\frac{\langle\lambda|\beta^+|\text{g.s.}\rangle\langle\lambda|\beta^-|\text{g.s.}\rangle}{E_\lambda+\Delta}, \tag{56}$$

where the sum runs over all the excited states of the Hamiltonian (9) and the E_λ's are the excitation energies of these states.

This quantity is shown in Fig. 7 (line *a*) for $\Delta=0.5$ MeV and is compared with the QRPA result and with the values which are obtained making use of the β matrix elements illustrated in Figs. 5 and 6 in the best approximation examined so far, namely, $n=15$. A quite good agreement is found between our results and the exact ones up to the cancellation point of $M_{2\nu}$. After this point an increasing difference appears between the two results. To better understand the origin of this difference, in Fig. 7 we have also plotted the quantity $M_{2\nu}$ calculated by restricting the summation in Eq. (56) to the first excited state only (line *b*). It should be remembered, in fact, that both in the QRPA and in the present approach only this state is involved in the summation. This line (*b*) closely approaches the one obtained in the present approach in the whole range of variation of k'. The difference between our results and the exact ones (line *a*) is therefore essentially ascribable to the contribution of all the excited states which are not included in our calculations. This contribution, however, becomes appreciable only for large values of k'.

VI. COMPARISON WITH PREVIOUS CALCULATIONS

In the calculations just described we have resorted to $|\widetilde{gs}_{15})$ and $|q_{15})$ in order to have good ground and first excited states in the whole range $0\leqslant k'\leqslant 3$. The mechanism of these calculations is such, however, that the order of the approximation needed to reach the ''convergence'' is strictly related to the initial ground state trial function. So far, this has been taken of the form (38), namely, as the vacuum (properly projected) of the standard phonon (33). In this section, instead, we believe it interesting to consider also the case of a quite more elaborate phonon operator, that is,

$$Q^\dagger=Xb^\dagger-Yb+Zb^\dagger b^\dagger b-Wb^\dagger bb. \tag{57}$$

The vacuum of the Q operator, namely, the state $|GS)$ such that

$$Q|GS)=0, \tag{58}$$

has still the form (38) but with c_n coefficients which are now quite more complicated than Eq. (39). They are defined by the recursion formula

$$c_{n+1}=\frac{nY+n(n-1)W}{X+nZ}c_{n-1}, \tag{59}$$

with $c_0=1$, $c_1=0$ [as for the state (38), we consider this state $|GS)$ projected on B]. This state has no longer an exponential form of the type (35).

Similarly to Eq. (41), we construct the one-phonon state

$$|Q)=I_BQ^\dagger|GS), \tag{60}$$

whose coefficients d_n are now given by

$$d_n=\left(\frac{c_{n-1}}{(n-1)!}X+\frac{c_{n-1}}{(n-2)!}Z-\frac{c_{n+1}}{n!}Y-\frac{c_{n+1}}{(n-1)!}W\right)\sqrt{n!}. \tag{61}$$

In Ref. [30] it was shown that, minimizing the energy $E_{GS}=(GS|H_B|GS)$ with respect to the variables X, Y, W, and Z, one obtained values quite close to the exact ones in the whole range of k'. These values are shown in the column $E_{\text{g.s.}}^{([30])}$ of Table I. In the same table, column $E_{\text{g.s.}}^{(n=15)}$, we show the best results obtained in this work. The similarity between the two column is apparent. It therefore emerges clearly that the state $|\widetilde{gs}_{15})$ of our procedure is to a good extent equivalent to the vacuum of the more elaborate Q phonon operator (57). In other words, either taking the ''simple'' phonon operator (33) and applying the multistep variational procedure described in this work (up to the order $n=15$) or taking the ''sophisticated'' phonon operator (57) and just minimizing the energy of its vacuum (i.e., remaining at the order $n=1$) gives rise to results which are quite similar. The first way, however, is the one which looks more accessible in a realistic case.

Using the X, Y, W, Z amplitudes resulting from this minimization, it was also seen in Ref. [30] that the energies $E_Q=(Q|H_B|Q)$ reproduced reasonably well the exact ones only up to $k'\sim 0.8$. After this value, they started deviating completely. It was also realized, however, that, introducing a new one-phonon operator $\overline{Q}^\dagger$, constructing the one-phonon state

TABLE I. Comparison between approximate and exact energies as a function of the variable k'. $E_{\text{g.s.}}^{(n=15)}$, $E_{\text{g.s.}}^{([30])}$, and $E_{\text{g.s.}}^{(\text{exact})}$ are the ground state energies obtained, respectively, in the best approximation tested in this work, in Ref. [30] (see discussion in Sec. VI) and in the exact calculations. A similar notation is used for the excitation energies ΔE.

k'	$E_{\text{g.s.}}^{(n=15)}$	$E_{\text{g.s.}}^{([30])}$	$E_{gs}^{(\text{exact})}$	$\Delta E^{(n=15)}$	$\Delta E^{([30])}$	$\Delta E^{(\text{exact})}$
0.0	−0.02108	−0.02109	−0.02109	2.48877	2.48845	2.48845
0.2	−0.04462	−0.04466	−0.04466	2.26446	2.26330	2.26330
0.4	−0.08025	−0.08047	−0.08047	2.02776	2.02426	2.02426
0.6	−0.13154	−0.13239	−0.13239	1.78001	1.77032	1.77033
0.8	−0.20328	−0.20635	−0.20635	1.52691	1.50171	1.50173
1.0	−0.30166	−0.31153	−0.31152	1.28364	1.22149	1.22153
1.2	−0.45384	−0.46237	−0.46235	0.95286	0.93820	0.93826
1.4	−0.66647	−0.68030	−0.68023	0.70580	0.66871	0.66876
1.6	−0.95734	−0.99157	−0.99141	0.48213	0.43722	0.43713
1.8	−1.35853	−1.41569	−1.41539	0.28889	0.26426	0.26383
2.0	−1.89819	−1.95121	−1.95072	0.13938	0.15215	0.15125
2.2	−2.54238	−2.57819	−2.57747	0.06398	0.08650	0.08514
2.4	−3.25098	−3.27290	−3.27191	0.03055	0.04984	0.04816
2.6	−4.00324	−4.01658	−4.01527	0.01506	0.02956	0.02771
2.8	−4.78748	−4.79625	−4.79457	0.00793	0.01819	0.01630
3.0	−5.59641	−5.60318	−5.60106	0.00525	0.01167	0.00981

$|\overline{Q}) = I_B \overline{Q}^\dagger |GS)$ [where $|GS)$ is the ground state already fixed in the previous minimization], and minimizing the energy of this state with respect to the new variables $\overline{X}, \overline{Y}, \overline{W}, \overline{Z}$ which definined this new phonon, one obtained results very close to the exact ones. These are shown in Table I. Once again one notices the similarity between these results and those of the column ''$n=15$.'' A common feature to both these results is that they have been obtained by employing an independent phonon operator in the definition of the excited state, i.e., a phonon by no means bound to the ground state. Differently from what is seen for the ground state energies, however, the effects of using the $Q^\dagger$ operator (57) rather than $q^\dagger$, Eq. (33), are now much less evident.

VII. SUMMARY AND CONCLUSIONS

In this paper β and $\beta\beta$ decay transitions of Fermi type have been analyzed within a schematic model. The amplitudes associated with these transitions as well as some basic properties of the model like ground state and excitation energies have been reproduced quite well in terms of a multistep variational procedure.

The procedure has been fully developed in a boson space and so, as a preliminary step, we have constructed the boson images of all the fermion operators of interest. A RPA-like phonon operator depending on the standard X, Y amplitudes has been introduced and we have assumed the vacuum of this operator (properly projected onto the ''physical'' space) as an initial approximation of the ground state of the system. The X and Y amplitudes have been fixed by minimizing the energy of this state.

The purpose of the remaining part of the procedure has been that of drawing up a scheme able to improve step by step the starting approximation of the ground state. For that, new *Ansätze* for the ground state have been introduced which depended on new variables. These variables have been fixed by minimizing the energy of the corresponding state at each step. As a result of these minimizations we have verified a continous improvement of the ground state energies and their final convergence to values quite close to the exact ones.

It is worth stressing that, with the only exception of the starting approximation ($n=1$), the ground state so constructed is not the vacuum of any phonon operator. Even for $n=1$, however, this feature is not strictly necessary and, in principle, one could assume any trial function as a starting approximation. It remains understood, of course, as also evidenced in Sec. VI, that the better the starting approximation is, the sooner the procedure will have a chance to converge.

Excited states have been constructed in terms of a new phonon operator acting on the ground state so derived and fixing the related X and Y variables by means of a further minimization. Also the excitation energies associated with these states have exhibited a convergence to the exact values quite analogous to the ground state energies.

In this paper, as a reference for our calculations, we have shown the results obtained within the QRPA. An analysis of previous works also allows a comparison with other approaches like the RQRPA [26] and a different boson procedure [25]. In all the cases examined so far, the present approach has provided considerable improved results.

In conclusion, the picture which emerges of the multistep variational procedure discussed in this paper is highly positive and encourages an extension of this technique to realistic cases. Of course, such an extension will imply facing some problems that have not been met within the present, rather simple, model and we cannot even exclude the possibility that the solution of these problems might require some variations in the procedure just illustrated. These final notes are meant to clarify this point.

All the calculations performed so far have been greatly facilitated by the fact that, as a result of the particular model

employed, only one boson operator $b^\dagger$ and, consequently, one phonon operator $q^\dagger$ have been introduced. This has implied that, at each step of the variational procedure for the determination of the best ground state wave function, the minimization has been performed with respect to one variable only. Clearly, this situation is bound to become more complicated in a realistic case where more than one $b^\dagger$ will appear, labeled by some indices, and a set of phonon operators of the form

$$q_\nu^\dagger = \sum_{\lambda=1}^{N} (X_\lambda^{(\nu)} b_\lambda^\dagger - Y_\lambda^{(\nu)} b_\lambda) \quad (\nu = 1,2,\ldots,N) \tag{62}$$

will replace the single phonon (33).

Following exactly the indications of Sec. IV, in a realistic case one should assume the vacuum of the operators q_ν as the starting approximation for the ground state wave function and this state has the form [13]

$$|gs) = N_0 e^{S^\dagger} |0), \tag{63}$$

where

$$S^\dagger = \frac{1}{2} \sum_{\lambda,\lambda'} Z_{\lambda\lambda'} b_\lambda^\dagger b_{\lambda'}^\dagger, \tag{64}$$

N_0 is a normalization constant, and the (symmetric) matrix Z is related to the matrices X and Y through the expression

$$Z = YX^{-1} \tag{65}$$

(in the hypothesis of real quantities). In this case, then, the first step of the procedure would require performing a minimization with respect to $N(N+1)/2$ variables $Z_{\lambda\lambda'}$. Depending on N, there could be situations in which the number of these variables might become rather large and so cause some difficulties in the calculations. In such a circumstance, however, one has to remember that, as already pointed out, the choice (63),(64) is not at all an obliged one and, being only meant to provide a starting approximation for the ground state wave function, one can choose alternative and more tractable forms for it. One possibility could be, for instance, that of assuming a state still of the exponential form (63) but with the operator $S^\dagger$ expressed as

$$S^\dagger = \frac{1}{2} \left(\sum_\lambda z_\lambda b_\lambda^\dagger \right)^2. \tag{66}$$

This choice, which amounts to assume $Z_{\lambda\lambda'} \equiv z_\lambda z_{\lambda'}$ in Eq. (64), would considerably simplify the calculations since the number of variables involved in the minimization would be reduced to N and this number is expected not to cause particular problems in realistic cases. Of course, this choice also leaves less freedom in the initial determination of the wave function and so it appears worse than Eq. (64). Thanks to the multistep mechanism, however, this fact should only give rise to a slower convergence torward the exact solution (in Sec. VI, we have just seen the opposite situation where a better choice of the initial wave function has caused a faster convergence).

The choices (64) and (66) are undistinguishable within the model discussed in this paper and so nothing more can be said at present on this point. More elaborate calculations are under way to explore this subject in more detail and we plan to illustrate them in a future publication.

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