Pairing in low-density Fermi gases

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We consider pairing in a dilute system of fermions with a short-range interaction. While the theory is ill-defined for a contact interaction, the BCS equations can be solved in the leading order of low-energy effective field theory. The integrals are evaluated with the dimensional regularization technique, giving analytic formulas relating the pairing gap, the density, and the energy density to the two-particle scattering length. [S0556-2813(99)04804-9]

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In the theory of fermionic matter, the expansion about the low-density limit has been invaluable for understanding the structure of the theory and the role of the interaction. At low densities, the interaction needs only be characterized by its scattering length to get expansions for the energy density, excitation spectrum, etc. [1]. However, to our knowledge the pairing singularity has never been incorporated into this framework. We have for example only the qualitative statement in Ref. [1] that the pairing singularity is logarithmic and unimportant for integrated quantities. A more quantitative statement is needed to have complete understanding of low-density fermionic matter.

Another motivation for our study is the general reexamination of nuclear physics with effective field theory which is now taking place [2–9]. In the effective field theory approach, the interaction is systematically expanded in a power series in momentum with the object of getting relationships between observables such that the details of the short-distance interaction need not be parameterized. We shall show here that the BCS theory of pairing is amenable to this approach, and the low-energy theory gives finite and analytic results. Within effective field theory many results can be obtained analytically opposed to the numerical treatment of potential models. In this sense our approach complements the large body of literature of pairing in nuclear and neutron matter that is based on potential models [10–16].

We consider a Fermi gas with two-fold degeneracy interacting with a short-range attractive interaction. Examples are neutron matter or gaseous ^{3}He. The Hamiltonian is idealized to be of the form

$$H=V\int \frac{d^3k}{(2\pi)^3}\epsilon_k(a^\dagger_{k,\uparrow}a_{k,\uparrow}+a^\dagger_{-k,\downarrow}a_{-k,\downarrow})+gV^2\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}a^\dagger_{k,\uparrow}a^\dagger_{-k,\downarrow}a_{-k',\downarrow}a_{k',\uparrow}, \quad (1)$$

where $\epsilon_k=k^2/2m$ is the kinetic energy and V the volume. In effective field theory the contact interaction is the leading term in a derivative expansion of the many-body system. This limits the validity of the Hamiltonian (1) to the regime of long wave lengths or small densities. However, corrections can systematically be implemented. We have only retained terms in the contact interaction that are needed in the wave function. The BCS wave function has the form $|\Psi\rangle=\Pi_k(U_k+V_k a^\dagger_{k,\uparrow}a^\dagger_{-k,\downarrow})|0\rangle$; the energy is minimized with respect to U_k,V_k to get the BCS equations [17]. The equation for the pairing gap Δ is

$$1=-\frac{gV}{2(2\pi)^3}\int\frac{d^3k}{\sqrt{(\epsilon_k-\lambda)^2+\Delta^2}}, \quad (2)$$

where λ is the chemical potential. The density is given in terms of these parameters by

$$\frac{N}{V}=\int\frac{d^3k}{(2\pi)^3}\left[1-\frac{\epsilon_k-\lambda}{\sqrt{(\epsilon_k-\lambda)^2+\Delta^2}}\right]. \quad (3)$$

Finally, the energy density of the paired state is given by

$$\frac{E}{V}=\int\frac{d^3k}{(2\pi)^3}\left[\epsilon_k-\frac{\epsilon_k(\epsilon_k-\lambda)}{\sqrt{(\epsilon_k-\lambda)^2+\Delta^2}}-\frac{1}{2}\frac{\Delta^2}{\sqrt{(\epsilon_k-\lambda)^2+\Delta^2}}\right]. \quad (4)$$

Note that the last two integrals are finite, although each integrand is a sum of terms that are individually divergent.

The problem with Eq. (2) as derived is that the contact interaction is singular in three dimensions. One often introduces a cutoff to make the integrals converge. However, in effective field theory cutoffs are not explicitly introduced. Rather, the computed observables are expressed directly in terms of other physical quantities. To leading order in a low-energy expansion of the interaction, the physical quantity is the scattering length. With the same Hamiltonian, the scattering length a is given by a similar divergent integral,

$$-\frac{mgV}{4\pi a}+1=-\frac{gV}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\epsilon_k}. \quad (5)$$

Let us now subtract Eqs. (2) and (5) to obtain

$$\frac{mg}{4\pi a}=-\frac{g}{2(2\pi)^3}\int d^3k\left[\frac{1}{\sqrt{(\epsilon_k-\lambda)^2+\Delta^2}}-\frac{1}{\epsilon_k}\right]. \quad (6)$$

Notice that the integral is now convergent and so any cutoff can be taken to infinity. Furthermore, the strength of the contact interaction g, which is also an unphysical quantity, can be divided out. It is convenient to evaluate both terms of the integral (6) separately by dimensional regularization

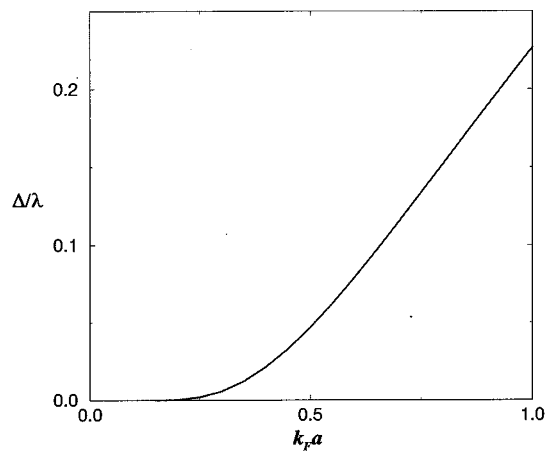

FIG. 1. Energy gap Δ/λ as a function of $k_F a$.

(DR) [18]. In DR, integrals of powers are zero so the second term in the integrand in Eq. (6) can be dropped. The first term can be evaluated using [19] (3.252.11),

$$\int_0^\infty dz \frac{z^\alpha}{[(z-1)^2+x^2]^{1/2}} = -\frac{\pi}{\sin \pi\alpha}(1+x^2)^{\alpha/2}P_\alpha(-1/\sqrt{1+x^2}), \quad (7)$$

where P_α denotes the Legendre function.

We write the final result in the form

$$\frac{1}{k_F a} = (1+x^2)^{1/4}P_{1/2}(-1/\sqrt{1+x^2}), \quad (8)$$

where $k_F = \sqrt{2m\lambda}$ is the Fermi momentum and $x = \Delta/\lambda$. This is our main result. Equation (8) is graphed in Fig. 1. For small values of $k_F a$ the gap is exponentially small as in the usual BCS theory,

$$\Delta = \frac{8}{e^2}\lambda \exp\left(\frac{-\pi}{2k_F|a|}\right). \quad (9)$$

This comes about by the behavior of $P_{1/2}(z)$, which has a logarithmic singularity at $z=-1$ [20]. Equation (9) agrees with the result derived in Ref. [14]. For large values of $k_F a$, the gap is proportional to λ, approaching $\Delta \approx 1.16\lambda$.

For neutron matter the solution of Eq. (8) agrees with numerical results from potential models only for small values of the Fermi momentum. The large value of the scattering length ($a=-18.8$ fm) clearly limits the domain of validity of the Hamiltonian (1). In the Appendix we consider pairing in the effective range approximation. This improves on the precision of the calculation in the low-density regime but does not enlarge the domain of validity.

We complete this discussion by computing the energy density (4) and the density (3). The finite integrals involved are very similar to the previous one and can be evaluated using the same DR integral, Eq. (7). The density of the BCS state is given by

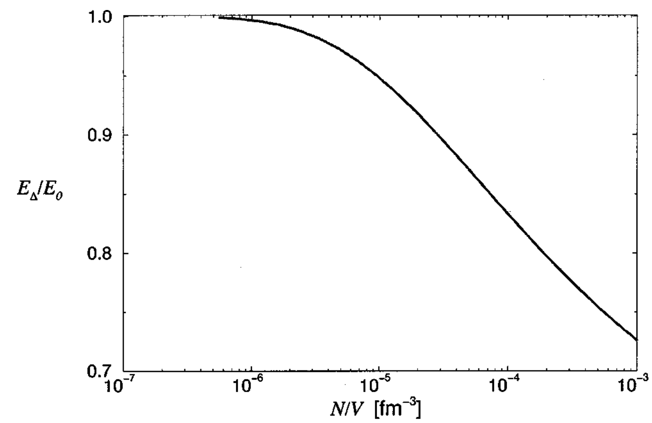

FIG. 2. Energy E_Δ of interacting neutron matter normalized to the energy E_0 of noninteracting neutrons as a function of the density.

$$\frac{N}{V} = -\frac{k_F^3}{4\pi}(1+x^2)^{1/4}[P_{1/2}(-1/\sqrt{1+x^2}) + \sqrt{1+x^2}\,P_{3/2}(-1/\sqrt{1+x^2})], \quad (10)$$

and the energy density by

$$\frac{E}{V} = -\frac{3}{20\pi}k_F^3\lambda(1+x^2)^{1/4}[(1+x^2/6)P_{1/2}(-1/\sqrt{1+x^2}) + \sqrt{1+x^2}P_{3/2}(-1/\sqrt{1+x^2})]. \quad (11)$$

For fixed density and scattering length Eqs. (8) and (10) can be solved for the pairing gap and the Fermi energy. Put into Eq. (11) this yields the energy of the interacting system at fixed density. Figure 2 shows a comparison with the energy of noninteracting neutrons. For $|k_F a| \approx 1$ (i.e., $N/V \approx 5 \times 10^{-6}\,\mathrm{fm}^{-3}$) pairing lowers the energy about 3% confirming the qualitative statement that the effects of pairing on the binding are mild.

Discussion. We now discuss the domain of validity of this low-density expansion. As pointed out above, the applicability of Hamiltonian (1) is limited to the regime of long wave length $|k_F a| \ll 1$ or small densities. The description of neutron or nuclear matter at nuclear densities requires the inclusion of the effective range and pions. Comparing with more microscopic calculations involving phenomenological potentials it appears that deviations from the low-density behavior are set by the scattering length. Similar considerations can be made for ^{3}He where the scattering length of the Aziz potential [21] is large on an atomic scale. Many-body correlation effects will become important when $k_F a \approx 1$. It might be possible to treat them by modifying the strength of the pairing and the density of states in Eq. (1). The sign would be to increase the pairing, but we have not attempted to calculate these effects.

Another consideration is whether the low-density phase exists for fermionic systems with attractive scattering lengths. In the case of ^{3}He, a low-density phase could only be metastable at zero temperature, because there is a finite binding of the liquid phase. However, the metastability could be quite significant, because the minimum size for a bound drop is thought to be of the order of fifty particles. Another indication of the metastability of a low-density phase is the sound velocity in the scattering length expansion. Taking the first three terms, the sound velocity is positive at all densities, and thus small deviations from uniformity are energetically unfavorable. In the case of neutron matter, it is thought that pressure is always positive as a function of density, so the low-density state would be stable.

In summary, we have considered the pairing in low-density Fermi systems within effective field theory. This model independent approach yields analytical expressions which relate the pairing gap, the density and the ground state energy to the scattering length. The analytical derivation of these results is quite interesting.

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APPENDIX

To include the effective range we add the effective range potential

$$g_2V^2\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}(k-k')^2a^\dagger_{k,\uparrow}a^\dagger_{-k,\downarrow}a_{-k',\downarrow}a_{k',\uparrow} \tag{12}$$

to the Hamiltonian (1). The gap equation then becomes

$$\Delta_p=-\frac{V}{2}\int\frac{d^3k}{(2\pi)^3}\frac{g+g_2(p-k)^2}{\sqrt{(\epsilon_k-\lambda)^2+\Delta_k^2}}\Delta_k \tag{13}$$

and is explicitly momentum dependent. We make the quadratic ansatz $\Delta_p=\Delta+p^2\delta$ for the momentum dependence and obtain two coupled equations that express Δ and δ in terms of (divergent) integrals. To deal with the divergencies we observe that the integrals' dependence on the Fermi momentum is given by

$$\int\frac{d^3k}{(2\pi)^3}\frac{k^{2n}}{\sqrt{(\epsilon_k-\lambda)^2+(\Delta+k^2\delta)^2}}=\frac{k_F^{2n+1}m}{2\pi^2\lambda}J_{n+1/2}(x,y), \tag{14}$$

where $x=\Delta/\lambda$, $y=\delta k_F^2/\lambda$ and $J_{n+1/2}(x,y)$ is the dimensionless function

$$J_\alpha(x,y)\equiv\int_0^\infty dt\,\frac{t^\alpha}{\sqrt{(t-1)^2+(x+yt)^2}}. \tag{15}$$

In effective field theory an expansion in momenta is quite useful [3,8]. In what follows we truncate each of the gap equations to its leading order in the Fermi momentum and obtain

$$1=-\frac{Vgmk_F}{4\pi^2}J_{1/2}(x,y), \tag{16}$$

$$\delta=-\frac{Vg_2\Delta mk_F}{4\pi^2}J_{1/2}(x,y).$$

Obviously we have $\delta/\Delta=g_2/g$. To make contact with low energy scattering data we expand the scattering amplitude

$$\mathcal{A}(p)=Vg\left[1+VgI(p)+\frac{g_2}{g}p^2+(VgI(p))^2\right] \tag{17}$$

up to quadratic order in momenta. The loop integral is

$$I(p)\equiv\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\epsilon_p-\epsilon_k+i\eta}=\frac{mp}{4\pi^2}\int_0^\infty dt\,\frac{t^{1/2}}{1-t+i\eta}. \tag{18}$$

At low energies the scattering amplitude is given in terms of the scattering length a and the effective range r_0

$$\mathcal{A}(p)=\frac{4\pi a}{m}[1-iap+(ar_0/2-a^2)p^2]. \tag{19}$$

Note that the divergence of the integral $J_{1/2}(x,y)$ appearing in the gap equations (16) is similar to that of the loop integral $I(p)$ appearing in the expression (17) for the scattering amplitude. Thus, both divergencies may be taken care off by a renormalization of the coupling constants g and g_2. We use dimensional regularization to compute the divergent integrals. One obtains $I(p)=-i(m/4\pi)p$ and a comparison of Eqs. (17) and (19) yields $g_2/g=ar_0/2$ and $g=4\pi a/m$. Finally we have

$$J_\alpha(x,y)=-\frac{\pi}{\sin\pi\alpha}(1+y^2)^{-1/2}\left(\frac{1+x^2}{1+y^2}\right)^{\alpha/2}P_\alpha(-z), \tag{20}$$

where $z=(1-xy)/\sqrt{(1+x^2)(1+y^2)}$. This yields the final results

$$\frac{1}{k_Fa}=(1+y^2)^{-1/2}\left(\frac{1+x^2}{1+y^2}\right)^{1/4}P_{1/2}(-z),\quad y=\frac{ar_0}{2}k_F^2x. \tag{21}$$

Note that these equations add corrections of the order $\sim k_F^2ar_0$ to the gap equation (8). These corrections are small only in the low-density regime $k_Fa\ll1$. For a description of neutron matter ($a=-18.8$ fm, $r_0=2.75$ fm) at larger densities, at least the inclusion of pions seems to be necessary. Note also, that the gap equations (21) become singular for $k_F^2\to-2/ar_0$ (i.e., $y\to-x$) due to the logarithmic singularity of the Legendre function for $z\to-1$. This behavior results from the quadratic approximation for the interaction potential and the truncations in the gap equation. It is related to the change in sign of the truncated potential at $k_F=\sqrt{-2/ar_0}$ [14]. Again, the introduction of pions or higher potential terms seem to be necessary to alter this behavior.

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