Approach towards *N*-nucleon effective generators of the Poincaré group derived from a field theory

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It is shown that the ten Hermitian generators of the Poincaré group derived from standard Hermitian Lagrangians which describe interacting fields can be block diagonalized by one and the same unitary transformation such that the space of a fixed number of nucleons is separated from the rest of the space. The existence proof is carried through using a formal power series expansion in the coupling constant to all orders. In this manner one arrives at effective Hermitian generators of the Poincaré group which act in the two subspaces separately. [S0556-2813(99)00304-0]

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I. INTRODUCTION

Low-energy nuclear physics below the meson threshold is naturally formulated in terms of a fixed number of nucleonic degrees of freedom. In the overwhelming number of applications a nonrelativistic framework is used. This, however, is not sufficient if one investigates for instance electron scattering with high three-momentum transfers as one encounters in typical experiments performed nowadays. Also it is still an open question whether relativistic effects play a significant role when calculating the binding energy of nuclei. In three-nucleon scattering it has been found recently [1] that the total nd cross section evaluated with most modern NN forces and based on rigorous solutions of the $3N$ Faddeev equations deviate from the data above ≈ 100 MeV nucleon laboratory energy. That discrepancy reaches about 10% at 300 MeV and is very likely caused by the neglect of relativity. On all these grounds a relativistic generalization of the usual Schrödinger equation for N interacting nucleons is highly desirable.

In Ref. [2] Dirac proposed three forms of relativistic quantum mechanics for a given number of interacting particles. A realization thereof in the instant form was given by Bakamjian and Thomas [3]. That scheme, however, violates cluster separability [4]. Being less ambitious and searching just for relativistic correction terms to the generators of the Galilean group in leading orders Foldy and Krajcik have discussed [5] a $1/c^2$ expansion of the ten generators of the Poincaré group. This scheme has been applied recently in a realistic context in the $3N$ system [6]. A way to treat the defect in the Bakamjian and Thomas scheme with respect to the cluster separability has been found by Sokolov [7] and also worked out by Coester and Polyzou [8]. An extensive overview over the whole subject is given in Ref. [9].

There is, however, also another approach to the generators of the Poincaré group for a fixed number of particles. Relativistic field theory provides generators which act in the full space with an infinite number of particles. Thinking of applications for nuclear physics one considers interacting fields of nucleons and mesons. To arrive at generators which act in the space of a fixed number of N nucleons one has to eliminate the mesonic degrees of freedom as well as the ones for antiparticles. A way to do this has been proposed in Ref. [10] and worked out in lowest order in the coupling constant for a field theory of ''scalar nucleons'' interacting with a scalar meson field. While this has been formulated in the instant form a corresponding derivation can also be performed in the light front form [11]. Numerical investigations based on those effective generators determined in leading order in the coupling constant have been carried through in Refs. [12], [13], and [14].

In this article we want to show that the derivation proposed in Ref. [10] can be carried through to arbitrary order in the coupling constant. Thus the effective generators of the Poincaré group in an N nucleon subspace do exist at least in the sense of a formal series expansion. It will be interesting to investigate whether those generators are automatically also cluster separable. This is left to a future study.

In Sec. II we formulate our way to derive the effective generators in the N nucleon subspace out of a field theoretical model of interacting nucleon and meson fields. The existence proof is carried through in Sec. III. We discuss the properties of the new generators and outline possible applications in Sec. IV before we briefly summarize in Sec. V.

II. CONDITIONS FOR THE EFFECTIVE GENERATORS

We consider a field theory of interacting scalar ''nucleons'' and ''antinucleons,'' and mesons given by a Hermitian Lagrangian of the form

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I, \tag{1}$$

where $\mathcal{L}_0$ is the free part and the interacting part $\mathcal{L}_I$ is linear in the coupling constant g. We also assume that $\mathcal{L}_I$ is linear in creation and absorption operators for mesons as is the case for often used field theories [see, for instance, Eq. (100)]. In a standard manner [15] one arrives at the ten Hermitian generators of the Poincaré group for constant time slices (instant form). The Hamiltonian and the three boost operators carry interactions, whereas the total momentum and angular momentum operators are the free ones. The latter two leave the plane $t=$const invariant. Thus one has in obvious notation

$$H=H_0+H_I, \tag{2}$$

$$K_i = K_{0i} + K_{Ii}, \tag{3}$$

$$P_i = P_{0i}, \tag{4}$$

$$J_i = J_{0i}, \tag{5}$$

where due to Eq. (1)

$$H_I \sim g,$$

$$K_{Ii} \sim g. \tag{6}$$

These ten operators fulfil the Lie algebra of the Poincaré group

$$[P_i, H] = 0, \tag{7}$$

$$[J_i, H] = 0, \tag{8}$$

$$[P_i, P_j] = 0, \tag{9}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \tag{10}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k, \tag{11}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k, \tag{12}$$

$$[H, K_i] = -iP_i, \tag{13}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k, \tag{14}$$

$$[P_i, K_j] = -i\delta_{ij} H. \tag{15}$$

Formally one can verify that using the equal time commutation relations of the underlying fields. Following the derivation scheme for the generators starting from a field theoretical Lagrangian density we note that the operators in Eqs. (2)–(5) can be expressed as sums and integrals over particle creation and absorption operators which fulfil the standard commutation relations equivalent to the equal time commutation relations. As a consequence the set of commutation relations is fulfilled by matrices of those generators with respect to Fock space states. This matrix form underlies our algebra presented below and in Sec. III.

Because $\mathcal{L}_I$ is assumed to be linear in the creation and annihilation operators for mesons, H_I and K_{Ii} will be linear in these operators too. Hence the eigenstates of H will necessarily contain an infinite number of mesons in addition to the nucleons (and antinucleons). The behavior of such an eigenstate under Lorentz transformation, however, is transparent. We regard the operator of four momentum

$$P^\mu \equiv (H, P_1, P_2, P_3) \tag{16}$$

and consider a Lorentz transformation $T(\Gamma, a)$ defined by

$$x^\mu \xrightarrow{T} x'^\mu = \Gamma^\mu{}_\nu x^\nu + a^\mu. \tag{17}$$

Related to T is a unitary operator $U(\Gamma, a)$ acting in the Hilbert space spanned by the eigenstates of H. A consequence of the commutation relations (7)–(15) are the transformation properties of P^μ:

$$P^\mu \xrightarrow{T} P'^\mu = UP^\mu U^\dagger = \Gamma_\nu{}^\mu P^\nu. \tag{18}$$

Because of Eqs. (7) and (9) there exist simultaneous eigenstates related to the four components of the four-momentum operator, which fulfil

$$P^\mu |\Psi_p\rangle = p^\mu |\Psi_p\rangle. \tag{19}$$

Applying U and using Eq. (18) one gets

$$\Gamma_\nu{}^\mu P^\nu U |\Psi_p\rangle = p^\mu U |\Psi_p\rangle. \tag{20}$$

This can be rewritten into

$$P^\nu U |\Psi_p\rangle = \Gamma^\nu{}_\mu p^\mu U |\Psi_p\rangle. \tag{21}$$

Thus up to a phase factor we get

$$U |\Psi_p\rangle = |\Psi_{\Gamma p}\rangle \tag{22}$$

and the ''four-dimensional Schrödinger equation'' (19) reads in the new frame of reference

$$P^\mu |\Psi_{\Gamma p}\rangle = (\Gamma p)^\mu |\Psi_{\Gamma p}\rangle. \tag{23}$$

Therefore the simultaneous eigenstates of P^μ in the new frame are eigenstates in the old frame with Lorentz transformed eigenvalues of the overall four momentum.

We pose now the question if one can find a matrix representation of the Poincaré algebra being restricted to a subspace with a fixed number of particles. We want to call such matrices ''effective.'' In other words those effective matrices are block diagonal with respect to the subspace with a fixed number of particles and the rest of the Fock space. If it is possible to find an effective representation of the Poincaré group one is able to formulate an effective Schrödinger equation in the subspace of a given number of particles, say N nucleons and no mesons, as in Eq. (19). The interesting point about that is that this equation would be easier to solve than Eq. (19) since the number of degrees of freedom is finite now. In addition, since we assume the Poincaré algebra to be fulfilled in that subspace, this effective Schrödinger equation inherits the nice transformation properties of Eq. (23).

A way to find effective generators is to unitarily transform the generators (2)–(5) by an operator $\mathcal{U}$ such that all ten generators are put into a block diagonal shape at the same time. One block would refer to the N nucleon subspace, the other block to the rest and the two blocks would not be coupled. Under a unitary transformation the commutation relations remain valid, of course. Let us denote the projection on the subspace of N nucleons by η and the projection on the rest by $\Lambda \equiv 1 - \eta$. Then what we are looking for is a unitary transformation $\mathcal{U}$, the existence of which is needed to be proved and which is of the form

$$H \xrightarrow{\mathcal{U}} \tilde{H} = \eta \tilde{H} \eta + \Lambda \tilde{H} \Lambda, \tag{24}$$

$$K_i \xrightarrow{\mathcal{U}} \tilde{K}_i = \eta \tilde{K}_i \eta + \Lambda \tilde{K}_i \Lambda, \tag{25}$$

$$P_i \xrightarrow{\mathcal{U}} \tilde{P}_i = \eta \tilde{P}_i \eta + \Lambda \tilde{P}_i \Lambda, \tag{26}$$

$$J_i \overset{\mathcal{U}}{\rightarrow} \tilde{J}_i = \eta \tilde{J}_i \eta + \Lambda \tilde{J}_i \Lambda. \tag{27}$$

While H and K_i (in the instant form) couple the η and Λ spaces, this is by assumption no longer the case for $\tilde{H}$ and $\tilde{K}_i$ and the operators $\eta \tilde{H} \eta, \eta \tilde{K}_i \eta, \eta \tilde{P}_i \eta$, and $\eta \tilde{J}_i \eta$ are effective generators of the Poincaré group. Now one can look for eigenstates of $\tilde{P}^\mu$ whose Λ components are zero. Lorentz transformations on those states are generated by the effective operators and we may write down the effective Schrödinger equation

$$\eta \tilde{P}^\mu \eta |\tilde{\psi}\rangle = p^\mu \eta |\tilde{\psi}\rangle. \tag{28}$$

In Ref. [10] such a path has been initiated and will be worked out more stringently now. In Ref. [16] Okubo proposed a way to transform an arbitrary Hermitian operator

$$O = \begin{pmatrix} \eta O \eta & \eta O \Lambda \\ \Lambda O \eta & \Lambda O \Lambda \end{pmatrix} \tag{29}$$

into a block diagonal form by means of a unitary transformation $\mathcal{U}$:

$$O \rightarrow \tilde{O} = \mathcal{U} O \mathcal{U}^\dagger = \eta \tilde{O} \eta + \Lambda \tilde{O} \Lambda. \tag{30}$$

We follow Okubo for the choice of the unitary operator

$$\mathcal{U} = \begin{pmatrix} \eta \mathcal{U} \eta & \eta \mathcal{U} \Lambda \\ \Lambda \mathcal{U} \eta & \Lambda \mathcal{U} \Lambda \end{pmatrix} = \begin{pmatrix} (1+A^\dagger A)^{-1/2} \eta & (1+A^\dagger A)^{-1/2} A^\dagger \\ -(1+AA^\dagger)^{-1/2} A & (1+AA^\dagger)^{-1/2} \Lambda \end{pmatrix}, \tag{31}$$

where A has the form

$$A = \Lambda A \eta. \tag{32}$$

Unitary transformations within the subspaces η and Λ are put to 1. Using the forms (29) and (31) the requirement for block diagonalization is

$$\Lambda([O,A] + O - AOA)\eta = 0. \tag{33}$$

Since it is *a priori* not obvious that it will be possible to block diagonalize each generator using the same $\mathcal{U}$ we label A with the generator to be block diagonalized. Noting Eqs. (4) and (5) telling that P_i and J_i do not connect the η and Λ spaces the conditions for the ten operators A_H, A_{K_i}, A_{P_i}, and A_{J_i} turn out to be

$$\Lambda([H_0, A_H] + H_I A_H + H_I - A_H H_I A_H)\eta = 0, \tag{34}$$

$$\Lambda([K_{0_i}, A_{K_i}] + K_{Ii} A_{K_i} + K_{I_i} - A_{K_i} K_{Ii} A_{K_i})\eta = 0, \tag{35}$$

$$\Lambda[P_i, A_{P_i}]\eta = 0, \tag{36}$$

$$\Lambda[J_i, A_{J_i}]\eta = 0. \tag{37}$$

Here we made use of the assumption that $\mathcal{L}_I$ and hence H_I and K_{Ii} are linear in the meson operators such that $\eta H_I \eta = 0 = \eta K_{Ii} \eta$. If one and the same A can be found that fulfils the set of conditions (34)–(37) the existence of ten effective generators of the Poincaré group in the separate subspaces η and Λ is proven. In that case we find the following form of the effective generators:

$$\begin{aligned} \eta \tilde{O} \eta &= \eta \mathcal{U}(\Lambda + \eta) O (\Lambda + \eta) \mathcal{U}^\dagger \eta \\ &= (1+A^\dagger A)^{-1/2} (\eta + A^\dagger) O (\eta + A)(1+A^\dagger A)^{-1/2}. \end{aligned} \tag{38}$$

Because these effective generators are derived by a unitary transformation they have to fulfil the Poincaré algebra. Nevertheless we want to show by explicit calculation that this is indeed the case. As we will see now we only require the properties (32) and (33).

With the help of the decoupling equation (33) we add zeros in two ways

$$\begin{aligned} \eta \tilde{O} \eta &= (1+A^\dagger A)^{-1/2} (\eta + A^\dagger) O (\eta + A)(1+A^\dagger A)^{-1/2} \\ &= [(1+A^\dagger A)^{-1/2}(\eta + A^\dagger) + (1+AA^\dagger)^{-1/2} \\ &\quad \times (-A + \Lambda)] O (\eta + A)(1+A^\dagger A)^{-1/2} \\ &= (1+A^\dagger A)^{-1/2}(\eta + A^\dagger) O [(\eta + A)(1+A^\dagger A)^{-1/2} \\ &\quad + (\Lambda - A^\dagger)(1+AA^\dagger)^{-1/2}]. \end{aligned} \tag{39}$$

Now we regard

$$\begin{aligned} [\eta \tilde{O}_1 \eta, \eta \tilde{O}_2 \eta] &= (1+A^\dagger A)^{-1/2}(\eta + A^\dagger)\{O_1[(\eta + A)(1+A^\dagger A)^{-1/2} + (\Lambda - A^\dagger)(1+AA^\dagger)^{-1/2}] \\ &\quad \times [(1+A^\dagger A)^{-1/2}(\eta + A^\dagger) + (1+AA^\dagger)^{-1/2}(-A + \Lambda)] O_2 - O_2 \\ &\quad \times [(\eta + A)(1+A^\dagger A)^{-1/2} + (\Lambda - A^\dagger)(1+AA^\dagger)^{-1/2}][(1+A^\dagger A)^{-1/2}(\eta + A^\dagger) \\ &\quad + (1+AA^\dagger)^{-1/2}(-A + \Lambda)] O_1\} (\eta + A)(1+A^\dagger A)^{-1/2}. \end{aligned} \tag{40}$$

We simplify the terms in the brackets in Eq. (40):

$$[\cdots]\times[\cdots]=(\eta+A)\frac{1}{1+A^\dagger A}(\eta+A^\dagger)+(\eta+A)(1+A^\dagger A)^{-1/2}\times(1+AA^\dagger)^{-1/2}(-A+\Lambda)(\Lambda-A^\dagger)\times(1+AA^\dagger)^{-1/2}(1+A^\dagger A)^{-1/2}(\eta+A^\dagger)+(\Lambda-A^\dagger)\frac{1}{1+AA^\dagger}(-A+\Lambda). \qquad (41)$$

Due to Eq. (32) this reduces to

$$[\cdots]\times[\cdots]=(\eta+A)\frac{1}{1+A^\dagger A}(\eta+A^\dagger)+(\Lambda-A^\dagger)\frac{1}{1+AA^\dagger}(-A+\Lambda)=(\eta+A)\frac{1}{1+A^\dagger A}(1-\Lambda+A^\dagger)+(\Lambda-A^\dagger)\frac{1}{1+AA^\dagger}(-A+1-\eta). \qquad (42)$$

We use Eq. (32) again to give

$$[\cdots]\times[\cdots]=(\eta+A)\frac{1}{1+A^\dagger A}+(\eta+A)\frac{1}{1+A^\dagger A}A^\dagger+(\Lambda-A^\dagger)\frac{1}{1+AA^\dagger}-(\Lambda-A^\dagger)\frac{1}{1+AA^\dagger}A=(\eta+A)\frac{1}{1+A^\dagger A}+(\eta+A)A^\dagger\frac{1}{1+AA^\dagger}+(\Lambda-A^\dagger)\frac{1}{1+AA^\dagger}-(\Lambda-A^\dagger)A\frac{1}{1+A^\dagger A}=\eta\frac{1}{1+A^\dagger A}+AA^\dagger\frac{1}{1+AA^\dagger}+\Lambda\frac{1}{1+AA^\dagger}+A^\dagger A\frac{1}{1+A^\dagger A}. \qquad (43)$$

In the last step we used the following identity:

$$\frac{1}{1+AA^\dagger}A=A\frac{1}{1+A^\dagger A} \qquad (44)$$

which can be addressed by applying an operator expansion of the fraction. Next we rearrange the projection operators and get

$$[\cdots]\times[\cdots]=(1-\Lambda)\frac{1}{1+A^\dagger A}+AA^\dagger\frac{1}{1+AA^\dagger}+(1-\eta)\frac{1}{1+AA^\dagger}+A^\dagger A\frac{1}{1+A^\dagger A}=\frac{1}{1+A^\dagger A}-\Lambda+AA^\dagger\frac{1}{1+AA^\dagger}+\frac{1}{1+AA^\dagger}-\eta+A^\dagger A\frac{1}{1+A^\dagger A}=(1+AA^\dagger)\frac{1}{1+AA^\dagger}+(1+A^\dagger A)\frac{1}{1+A^\dagger A}-\Lambda-\eta=1. \qquad (45)$$

We conclude that

$$[\eta\tilde{O}_1\eta,\eta\tilde{O}_2\eta]=(1+A^\dagger A)^{-1/2}(\eta+A^\dagger)\times[O_1,O_2](\eta+A)(1+A^\dagger A)^{-1/2}. \qquad (46)$$

Presuming

$$[O_1,O_2]=O_3 \qquad (47)$$

we get

$$[\eta\tilde{O}_1\eta,\eta\tilde{O}_2\eta]=(1+A^\dagger A)^{-1/2}(\eta+A^\dagger)\times O_3(\eta+A)(1+A^\dagger A)^{-1/2}=\eta\tilde{O}_3\eta. \qquad (48)$$

Thus we arrive indeed at the desired result that the effective operators in the η space fulfil the same algebra as the original ones

$$[O_1,O_2]=O_3\Rightarrow[\eta\tilde{O}_1\eta,\eta\tilde{O}_2\eta]=\eta\tilde{O}_3\eta, \qquad (49)$$

whenever we transform three operators O_1,O_2,O_3 by a transformation given by Eqs. (31), (32), and (33).

III. PROOF OF THE EXISTENCE OF A

We now want to show that in the case of the ten generators H, K_i, P_i, and J_i we can find one operator A that satisfies Eqs. (32) and (34)–(37). We note that the conditions (34)–(37) are nonlinear in the A's. One can linearize them by searching for A in the form of a Taylor expansion in the coupling constant

$$A=\sum_{\nu=1}^{\infty}A_\nu g^\nu. \qquad (50)$$

The term of order g^0 is absent since for a free theory ($g\rightarrow 0$) the generators are already block diagonal and $\mathcal{U}=1$ is achieved with $A=0$. It is easy to rewrite the set under the assumption (50) by equating equal powers of g. Keeping in mind that we assumed $\eta H_I\eta=0=\eta K_{Ii}\eta$ we get the result

$$[H_0,A_{H_1}]=-H_I\eta, \tag{51}$$

$$[H_0,A_{H_2}]=-\Lambda H_I A_{H_1}, \tag{52}$$

$$[H_0,A_{H_{n+1}}]=-\Lambda H_I A_{H_n}+\sum_{\nu=1}^{n-1} A_{H_\nu}H_I A_{H_{n-\nu}},\quad n\geqslant 2, \tag{53}$$

$$[K_{0_i},A_{K_{i_1}}]=-K_{I_i}\eta, \tag{54}$$

$$[K_{0_i},A_{K_{i_2}}]=-\Lambda K_{I_i}A_{K_{i_1}}, \tag{55}$$

$$[K_{0_i},A_{K_{i_{n+1}}}]=-\Lambda K_{I_i}A_{K_{i_n}}+\sum_{\nu=1}^{n-1} A_{K_{i_\nu}}K_{I_i}A_{K_{i_{n-\nu}}},\quad n\geqslant 2, \tag{56}$$

$$[P_i,A_{P_{i_n}}]=0,\quad n\geqslant 1, \tag{57}$$

$$[J_i,A_{J_{i_n}}]=0,\quad n\geqslant 1. \tag{58}$$

Let us introduce a shorthand notation. An arbitrary eigenstate of H_0 in the Λ space describing a certain number of noninteracting particles with momentum $\boldsymbol{p}$ is simply denoted by $|\Lambda\rangle$. Its energy, the eigenvalue to H_0, is denoted by E_Λ. The projection operator into the Λ space is then given by

$$\Lambda\equiv\int_\Lambda d^3p_\Lambda|\Lambda\rangle\langle\Lambda|, \tag{59}$$

where d^3p_Λ stands for all momentum integrations. Similarly we denote an arbitrary state in the η space by $|\eta\rangle$, its energy by E_η and the projection operator into the η space by

$$\eta\equiv\int_\eta d^3p_\eta|\eta\rangle\langle\eta|. \tag{60}$$

Since, as stated above, the generators (2)–(5) can be expressed in terms of particle creation and annihilation operators, it follows from the basic commutation relations that we can write:

$$(\Lambda+\eta)G=G(\Lambda+\eta)=G, \tag{61}$$

G being any of the ten generators. Then it is very convenient to use the following notation:

$$\int_{\eta,\Lambda} d^3p_\Lambda d^3p_\eta\frac{1}{E_\Lambda-E_\eta}|\Lambda\rangle\langle\Lambda|B|\eta\rangle\langle\eta|\equiv\frac{1}{E_\Lambda-E_\eta}\Lambda B\eta, \tag{62}$$

where B is an arbitrary operator.

The set (51)–(53) can now be solved recursively. Note that according to Eq. (32) A connects the Λ and the η spaces. Using now the notation (62) one finds the following expressions for A_{H_n}:

$$A_{H_1}=-\frac{1}{E_\Lambda-E_\eta}\Lambda H_I\eta, \tag{63}$$

$$A_{H_2}=-\frac{1}{E_\Lambda-E_\eta}\Lambda H_I A_{H_1}\eta, \tag{64}$$

$$A_{H_{n+1}}=-\frac{1}{E_\Lambda-E_\eta}\Lambda H_I A_{H_n}\eta+\sum_{\nu=1}^{n-1}\frac{1}{E_\Lambda-E_\eta}\Lambda A_{H_\nu}H_I A_{H_{n-\nu}}\eta,\quad n\geqslant 2. \tag{65}$$

These expressions are formal solutions to Eqs. (51)–(53) and are used to carry out our existence proof. If one wants to calculate A_{H_n} explicitly special care is needed when performing the integrations over the energy denominators in Eqs. (63)–(65) as they may vanish. As an example we refer to Sec. IV where this is briefly addressed in the case of particle production. In this sense the above expressions are a symbolic notation for A_{H_n}.

With the help of the Lie algebra (7)–(15) we will now show by induction that A_{H_n} given by Eqs. (63)–(65) satisfies also Eqs. (54)–(58) and therefore one and the same A block diagonalizes all ten generators.

First we look at Eq. (57). Because of

$$[P_i,\Lambda]=[P_i,\eta]=0 \tag{66}$$

one has the following identity for any operator $B(H)$:

$$\left[P_i,\frac{1}{E_\Lambda-E_\eta}\Lambda B(H)\eta\right]=\frac{1}{E_\Lambda-E_\eta}\Lambda[P_i,B(H)]\eta. \tag{67}$$

Further, since P_i commutes with H_0 and H, it also commutes with $H_I\equiv H-H_0$. Consequently we get

$$[P_i,A_{H_1}]=-\left[P_i,\frac{1}{E_\Lambda-E_\eta}\Lambda H_I\eta\right]=-\frac{1}{E_\Lambda-E_\eta}\Lambda[P_i,H_I]\eta=0. \tag{68}$$

By induction this carries over to A_{H_2} and A_{H_n} with $n\geqslant 3$ and we can write

$$A_{P_i}=A_H. \tag{69}$$

The proof of Eq. (58) using A_H is very similar. Equations (66)–(68) remain valid replacing P_i by J_i and J_i commutes with H and H_0. Consequently we can also put

$$A_{J_i}=A_H. \tag{70}$$

The proof that A_{H_n} solves the set (54)–(56) is the difficult one. We need the following relations. From

$$[H,K_i]=-iP_i \tag{71}$$

and the linear dependence of H and K_i on g [see Eqs. (2), (3), and (6)] we find easily by equating operators related to equal powers in g

$$[H_0,K_{0_i}]=-iP_i, \tag{72}$$

$$[H_0,K_{I_i}]=[K_{0_i},H_I], \tag{73}$$

$$[H_I,K_{I_i}]=0. \tag{74}$$

Further one has

$$[K_{0_i},C(H_0)]=iP_i\frac{\partial}{\partial H_0}C(H_0), \tag{75}$$

where C depends on H_0 only. Because of the free state kinematics it is also easily seen that

$$[K_{0_i},\Lambda]=[K_{0_i},\eta]=0. \tag{76}$$

Using all that one verifies that

$$\begin{aligned}\left[K_{0_i},\frac{1}{E_\Lambda-E_\eta}\Lambda B(H)\eta\right]&=\left[K_{0_i},\frac{1}{H_0-E_\eta}\Lambda B(H)\eta\right]\\&=\left[K_{0_i},\frac{1}{H_0-E_\eta}\right]\Lambda B\eta\\&\quad+\frac{1}{E_\Lambda-E_\eta}\Lambda[K_{0i},B]\eta+\Lambda B\left[K_{0_i},\frac{1}{E_\Lambda-H_0}\right]\eta \qquad (77)\\&=\frac{1}{E_\Lambda-E_\eta}\Lambda[K_{0_i},B]\eta. \qquad (78)\end{aligned}$$

This enables us now to show that A_{H_1} solves Eq. (54):

$$\begin{aligned}[K_{0_i},A_{H_1}]&=\left[K_{0_i},-\frac{1}{E_\Lambda-E_\eta}\Lambda H_I\eta\right]\\&=-\frac{1}{E_\Lambda-E_\eta}\Lambda[K_{0_i},H_I]\eta\\&=-\frac{1}{E_\Lambda-E_\eta}\Lambda[H_0,K_{I_i}]\eta\\&=-\frac{1}{E_\Lambda-E_\eta}\Lambda(E_\Lambda-E_\eta)K_{I_i}\eta=-\Lambda K_{I_i}\eta.\end{aligned} \tag{79}$$

Next let us verify that A_{H_2} from Eq. (64) solves Eq. (55):

$$\begin{aligned}[K_{0_i},A_{H_2}]&=-\left[K_{0_i},\frac{1}{E_\Lambda-E_\eta}\Lambda H_IA_{H_1}\eta\right]\\&=-\frac{1}{E_\Lambda-E_\eta}\Lambda[K_{0_i},H_IA_{H_1}]\eta\\&=-\frac{1}{E_\Lambda-E_\eta}\Lambda[H_0,K_{I_i}]A_{H_1}\eta\\&\quad-\frac{1}{E_\Lambda-E_\eta}\Lambda H_I[K_{0_i},A_{H_1}]\eta.\end{aligned} \tag{80}$$

Using Eq. (79) and applying H_0 this can be rewritten as

$$\begin{aligned}[K_{0_i},A_{H_2}]&=-\frac{1}{E_\Lambda-E_\eta}\Lambda K_{I_i}(E_\Lambda-H_0)A_{H_1}\eta\\&\quad+\frac{1}{E_\Lambda-E_\eta}\Lambda H_IK_{I_i}\eta.\end{aligned} \tag{81}$$

Further using Eq. (74) we get

$$\begin{aligned}[K_{0_i},A_{H_2}]&=-\frac{1}{E_\Lambda-E_\eta}\Lambda K_{I_i}[(E_\Lambda-H_0)A_{H_1}-H_I]\eta\\&=-\frac{1}{E_\Lambda-E_\eta}\Lambda K_{I_i}(E_\Lambda A_{H_1}-[H_0,A_{H_1}]\\&\quad-A_{H_1}H_0-H_I)\eta\\&=-\Lambda K_{I_i}A_{H_1}-\frac{1}{E_\Lambda-E_\eta}\Lambda K_{I_i}\\&\quad\times(-[H_0,A_{H_1}]-H_I)\eta\\&=-\Lambda K_{I_i}A_{H_1}.\end{aligned} \tag{82}$$

The last step follows from Eq. (51) and proves $A_{H_2}=A_{K_{i_2}}$.

Due to the structure of the set (54)–(56) it turns out that the proof for A_{H_3} and A_{H_4} should also be treated separately. Since the algebra is rather lengthy and the steps used for the general case A_{H_n}, $n\geqslant 5$ include all the ones for the simpler cases $n=3$ and $n=4$ we leave the verification for those simpler cases to the reader.

We embark now in the proof that $A_{H_{n+1}}$, $n\geqslant 4$ is a solution of Eq. (56) provided every A_{H_ν}, $\nu\leqslant n$ is a solution of the set (54)–(56). From Eq. (65) we get

$$[K_{0_i},A_{H_{n+1}}]=\frac{1}{E_\Lambda-E_\eta}\Lambda\left[K_{0_i},\left(-H_IA_{H_n}+\sum_{\nu=1}^{n-1}A_{H_\nu}H_IA_{H_{n-\nu}}\right)\right]\eta. \tag{83}$$

Using Eq. (73) this can be rewritten as

$$[K_{0_i},A_{H_{n+1}}]=\frac{1}{E_\Lambda-E_\eta}\Lambda\Bigg(-[H_0,K_{I_i}]A_{H_n}-H_I[K_{0_i},A_{H_n}]+\sum_{\nu=1}^{n-1}[K_{0_i},A_{H_\nu}]H_IA_{H_{n-\nu}}+\sum_{\nu=1}^{n-1}A_{H_\nu}[H_0,K_{I_i}]A_{H_{n-\nu}}+\sum_{\nu=1}^{n-1}A_{H_\nu}H_I[K_{0_i},A_{H_{n-\nu}}]\Bigg)\eta. \quad (84)$$

The first term in Eq. (84) is changed as

$$-\frac{1}{E_\Lambda-E_\eta}\Lambda[H_0,K_{I_i}]A_{H_n}\eta=-\Lambda K_{I_i}A_{H_n}\eta-\frac{1}{E_\Lambda-E_\eta}\Lambda K_{I_i}(E_\eta-H_0)A_{H_n}\eta. \quad (85)$$

Inserting this into Eq. (84) and using the assumption that A_{H_ν}, $\nu\leqslant n$ solves the set (54)–(56) we get

$$\begin{aligned}[K_{0_i},A_{H_{n+1}}]=&-\Lambda K_{I_i}A_{H_n}+\frac{1}{E_\Lambda-E_\eta}\Lambda\Bigg(-K_{I_i}(E_\eta-H_0)A_{H_n}+\Bigg[H_I\Lambda' K_{I_i}A_{H_{n-1}}-H_I\sum_{\nu=1}^{n-2}A_{H_\nu}K_{I_i}A_{H_{n-1-\nu}}\Bigg]\\&+\Bigg[-K_{I_i}\eta' H_IA_{H_{n-1}}-K_{I_i}A_{H_1}H_IA_{H_{n-2}}+\sum_{\nu=3}^{n-1}\Bigg(-K_{I_i}A_{H_{\nu-1}}+\sum_{\nu'=1}^{\nu-2}A_{H_{\nu'}}K_{I_i}A_{H_{\nu-1-\nu'}}\Bigg)H_IA_{H_{n-\nu}}\Bigg]\\&+\sum_{\nu=1}^{n-1}A_{H_\nu}[H_0,K_{I_i}]A_{H_{n-\nu}}+\Bigg[\sum_{\nu=1}^{n-3}A_{H_\nu}H_I\Bigg(-K_{I_i}A_{H_{n-\nu-1}}+\sum_{\nu'=1}^{n-\nu-2}A_{H_{\nu'}}K_{I_i}A_{H_{n-\nu-1-\nu'}}\Bigg)-A_{H_{n-2}}H_IK_{I_i}A_{H_1}\\&-A_{H_{n-1}}H_IK_{I_i}\Bigg]\Bigg)\eta+\frac{1}{E_\Lambda-E_\eta}\Lambda(K_{I_i}\Lambda' H_IA_{H_{n-1}}-K_{I_i}\Lambda' H_IA_{H_{n-1}})\eta. \quad (86)\end{aligned}$$

The square brackets are inserted to group the terms together resulting from the commutators with K_{0i}. Moreover we added a zero at the end and used the identity

$$A_{H_n}H_I\Lambda K_{I_i}=A_{H_n}H_IK_{I_i} \quad (87)$$

which is valid because A_H has an η projector on the right. Equation (86) can be simplified by means of the two identities

$$\begin{aligned}-K_{I_i}\Lambda H_I+H_I\Lambda K_{I_i}-K_{I_i}\eta H_I&=-K_{I_i}H_I+H_I\Lambda K_{I_i}\\&=-H_I\eta K_{I_i} \quad (88)\end{aligned}$$

and

$$\frac{1}{E_\Lambda-E_\eta}\Lambda\Bigg(K_{I_i}(H_0-E_\eta)A_{H_n}+K_{I_i}\Lambda' H_IA_{H_{n-1}}-K_{I_i}\sum_{\nu=3}^{n-1}A_{H_{\nu-1}}H_IA_{H_{n-\nu}}-K_{I_i}A_{H_1}H_IA_{H_{n-2}}\Bigg)\eta=0. \quad (89)$$

The second one is just a consequence of Eq. (53). Using Eqs. (88) and (89) we find

$$\begin{aligned}[K_{0_i},A_{H_{n+1}}]=&-\Lambda K_{I_i}A_{H_n}+\frac{1}{E_\Lambda-E_\eta}\Lambda\\&\times\Bigg[-H_I\eta' K_{I_i}A_{H_{n-1}}-H_I\sum_{\nu=1}^{n-2}A_{H_\nu}K_{I_i}A_{H_{n-1-\nu}}\\&+\sum_{\nu=3}^{n-1}\sum_{\nu'=1}^{\nu-2}A_{H_{\nu'}}K_{I_i}A_{H_{\nu-1-\nu'}}H_IA_{H_{n-\nu}}\\&+\sum_{\nu=1}^{n-1}A_{H_\nu}[H_0,K_{I_i}]A_{H_{n-\nu}}+\sum_{\nu=1}^{n-3}A_{H_\nu}H_I\\&\times\Bigg(-K_{I_i}A_{H_{n-\nu-1}}+\sum_{\nu'=1}^{n-\nu-2}A_{H_{\nu'}}K_{I_i}A_{H_{n-\nu-1-\nu'}}\Bigg)\\&-A_{H_{n-2}}H_IK_{I_i}A_{H_1}-A_{H_{n-1}}H_IK_{I_i}\Bigg]\eta. \quad (90)\end{aligned}$$

Next we exchange the orders of summation

$$\begin{aligned}&\sum_{\nu=3}^{n-1}\sum_{\nu'=1}^{\nu-2}A_{H_{\nu'}}K_{I_i}A_{H_{\nu-1-\nu'}}H_IA_{H_{n-\nu}}\\&=\sum_{\nu'=1}^{n-3}\sum_{\nu=1}^{n-\nu'-2}A_{H_{\nu'}}K_{I_i}A_{H_\nu}H_IA_{H_{n-\nu'-1-\nu}} \quad (91)\end{aligned}$$

and

$$\sum_{\nu=1}^{n-3} A_{H_\nu} H_I \sum_{\nu'=1}^{n-\nu-2} A_{H_{\nu'}} K_{I_i} A_{H_{n\text{-}\nu\text{-}1\text{-}\nu'}}$$
$$=\sum_{\nu=1}^{n-3} A_{H_\nu} H_I \sum_{\nu'=1+\nu}^{n-2} A_{H_{\nu'\text{-}\nu}} K_{I_i} A_{H_{n\text{-}1\text{-}\nu'}}$$
$$=\sum_{\nu'=2}^{n-2} \sum_{\nu=1}^{\nu'-1} A_{H_\nu} H_I A_{H_{\nu'\text{-}\nu}} K_{I_i} A_{H_{n\text{-}1\text{-}\nu'}}. \tag{92}$$

Using Eq. (91) we can group together some terms from Eq. (90) taking Eq. (74) and the set (51)–(53) into account:

$$\sum_{\nu=3}^{n-1} \sum_{\nu'=1}^{\nu-2} A_{H_{\nu'}} K_{Ii} A_{H_{\nu\text{-}1\text{-}\nu'}} H_I A_{H_{n\text{-}\nu}}$$
$$-\sum_{\nu'=1}^{n-3} A_{H_{\nu'}} K_{I_i} H_I A_{H_{n\text{-}\nu'\text{-}1}} - A_{H_{n\text{-}2}} K_{I_i} H_I A_{H_1}$$
$$-A_{H_{n\text{-}1}} K_{I_i} H_I$$
$$=\sum_{\nu'=1}^{n-3} \sum_{\nu=1}^{n-\nu'-2} A_{H_{\nu'}} K_{I_i} A_{H_\nu} H_I A_{H_{n\text{-}\nu'\text{-}1\text{-}\nu}}$$
$$-\sum_{\nu'=1}^{n-3} A_{H_{\nu'}} K_{I_i} H_I A_{H_{n\text{-}\nu'\text{-}1}} - A_{H_{n\text{-}2}} K_{I_i} H_I A_{H_1}$$
$$-A_{H_{n\text{-}1}} K_{I_i} H_I$$
$$=+\sum_{\nu'=1}^{n-3} A_{H_{\nu'}} K_{I_i} [H_0, A_{H_{n\text{-}\nu'}}] + A_{H_{n\text{-}2}} K_{I_i} [H_0, A_{H_2}]$$
$$+A_{H_{n\text{-}1}} K_{I_i} [H_0, A_{H_1}]$$
$$=\sum_{\nu'=1}^{n-1} A_{H_{\nu'}} K_{I_i} [H_0, A_{H_{n\text{-}\nu'}}]. \tag{93}$$

Similarly the expression (92) can be grouped together with two more terms from Eq. (90)

$$\sum_{\nu=1}^{n-3} A_{H_\nu} H_I \sum_{\nu'=1}^{n-\nu-2} A_{H_{\nu'}} K_{I_i} A_{H_{n\text{-}\nu\text{-}1\text{-}\nu'}} - \Lambda H_I \eta' K_{I_i} A_{H_{n\text{-}1}} - \Lambda H_I \sum_{\nu'=1}^{n-2} A_{H_{\nu'}} K_{I_i} A_{H_{n\text{-}1\text{-}\nu'}}$$
$$=\sum_{\nu'=2}^{n-2} \left(-\Lambda H_I A_{H_{\nu'}} + \sum_{\nu=1}^{\nu'-1} A_{H_\nu} H_I A_{H_{\nu'\text{-}\nu}} \right) K_{I_i} A_{H_{n\text{-}1\text{-}\nu'}} - \Lambda H_I \eta' K_{I_i} \Lambda A_{H_{n\text{-}1}} - \Lambda H_I A_{H_1} K_{I_i} A_{H_{n\text{-}2}}$$
$$=\sum_{\nu'=2}^{n-2} [H_0, A_{H_{\nu'+1}}] K_{I_i} A_{H_{n\text{-}1\text{-}\nu'}} + [H_0, A_{H_1}] K_{I_i} A_{H_{n\text{-}1}} + [H_0, A_{H_2}] K_{I_i} A_{H_{n\text{-}2}}$$
$$=\sum_{\nu'=3}^{n-1} [H_0, A_{H_{\nu'}}] K_{I_i} A_{H_{n\text{-}\nu'}} + [H_0, A_{H_1}] K_{I_i} A_{H_{n\text{-}1}} + [H_0, A_{H_2}] K_{I_i} A_{H_{n\text{-}2}}$$
$$=\sum_{\nu'=1}^{n-1} [H_0, A_{H_{\nu'}}] K_{I_i} A_{H_{n\text{-}\nu'}}. \tag{94}$$

Again we used the set (51)–(53) several times. Inserting all that into Eq. (90) that expression simplifies greatly and leads to the desired result

$$[K_{0_i}, A_{H_{n+1}}] = -\Lambda K_{I_i} A_{H_n}$$
$$+\frac{1}{E_\Lambda - E_\eta} \Lambda \sum_{\nu=1}^{n-1} [A_{H_\nu} K_{I_i} [H_0, A_{H_{n\text{-}\nu}}]$$
$$+A_{H_\nu} [H_0, K_{I_i}] A_{H_{n\text{-}\nu}}$$
$$+[H_0, A_{H_\nu}] K_{I_i} A_{H_{n\text{-}\nu}}] \eta$$
$$=-\Lambda K_{Ii} A_{H_n}$$
$$+\frac{1}{E_\Lambda - E_\eta} \Lambda \sum_{\nu=1}^{n-1} A_{H_\nu} \eta' K_{I_i} \Lambda' A_{H_{n\text{-}\nu}}$$
$$\times (E_{\Lambda'} - E_\eta + E_{\eta'} - E_{\Lambda'} + E_\Lambda - E_{\eta'}) \eta$$
$$=-\Lambda K_{I_i} A_{H_n} + \sum_{\nu=1}^{n-1} A_{H_\nu} K_{I_i} A_{H_{n\text{-}\nu}}. \tag{95}$$

This is our final result stating that $A_{H_{n+1}}$ solves Eq. (56) for $n \geqslant 4$ provided that A_{H_ν}, $\nu = 1,2,3,4$ is a solution of Eqs. (54)–(56). This, however, is the induction assumption and has been shown before.

We conclude that

$$A_{K_{in}} = A_{H_n}, \quad n \geqslant 1 \tag{96}$$

and hence

$$A_{K_i} = A_H. \tag{97}$$

Together with Eqs. (69) and (70) we arrive at the important result that A_H solves the set of all Eqs. (34)–(37) and all indices on the A's can be omitted. In practice one will use the recursion relations (63)–(65) for calculating A since they are easier to solve than the ones resulting from Eqs. (54)–(56) and further the conditions (57) and (58) are not specific enough.

IV. APPLICATION

We now want to outline how the results of Sec. III can be applied in the case of nucleon-nucleon scattering. We take η to be the projector on the subspace of two nucleons. For a simple field theoretical example we refer to Ref. [10]. There two real scalar fields Φ and Φ' are introduced,

$$\Phi(x)=\frac{1}{\sqrt{2\pi}^3}\int d^3q\frac{1}{\sqrt{2E_{\boldsymbol{q}}}}(a_{\boldsymbol{q}}e^{-iqx}+a_{\boldsymbol{q}}^{\dagger}e^{iqx}),$$

$$\Phi'(x)=\frac{1}{\sqrt{2\pi}^3}\int d^3q\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}(b_{\boldsymbol{q}}e^{-iqx}+b_{\boldsymbol{q}}^{\dagger}e^{iqx}), \quad (98)$$

where

$$E_{\boldsymbol{q}}\equiv\sqrt{\boldsymbol{q}^2+m_0^2},$$

$$\omega_{\boldsymbol{q}}\equiv\sqrt{\boldsymbol{q}^2+\mu_0^2}. \quad (99)$$

These two fields are interacting via

$$\mathcal{L}_I(x)=g\Phi^2(x)\Phi'(x), \quad (100)$$

g being the coupling constant. Φ and Φ' refer to scalar ‘‘nucleons’’ and mesons and m_0 and μ_0 are the bare nucleon and meson masses respectively. In Ref. [10] the effective generators of this model are calculated up to second order using Eqs. (38) and (50). The result is

$$\eta\tilde{P}_m\eta=\eta\left[\int d^3q\, a_{\boldsymbol{q}}^{\dagger}q_m a_{\boldsymbol{q}}\right]\eta, \quad (101)$$

$$\eta\tilde{J}_m\eta=\eta\left[i\int d^3q\, a_{\boldsymbol{q}}^{\dagger}\left(\frac{\partial}{\partial q_k}q_l-\frac{\partial}{\partial q_l}q_k\right)a_{\boldsymbol{q}}\right]\eta, \quad (102)$$

$$\eta\tilde{H}\eta=\eta\Bigg[\int d^3q a_{\boldsymbol{q}}^{\dagger}E_{\boldsymbol{q}}a_{\boldsymbol{q}}-\frac{g^2}{4(2\pi)^3}\int d^3q_1d^3q_2d^3q_1'd^3q_2'a_{\boldsymbol{q}_1'}^{\dagger}a_{\boldsymbol{q}_2'}^{\dagger}a_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}\frac{\delta^3(\boldsymbol{q}_1+\boldsymbol{q}_2-\boldsymbol{q}_1'-\boldsymbol{q}_2')}{\sqrt{E_{\boldsymbol{q}_1'}E_{\boldsymbol{q}_2'}E_{\boldsymbol{q}_1}E_{\boldsymbol{q}_2}}}$$
$$\times\left(\frac{1}{\omega^2_{\boldsymbol{q}_2-\boldsymbol{q}_2'}-(E_{\boldsymbol{q}_2}-E_{\boldsymbol{q}_2'})^2}+\frac{1}{\omega^2_{\boldsymbol{q}_1-\boldsymbol{q}_1'}-(E_{\boldsymbol{q}_1}-E_{\boldsymbol{q}_1'})^2}\right)\Bigg]\eta+\mathcal{O}(g^4), \quad (103)$$

$$\eta\tilde{K}_m\eta=\eta\Bigg[\frac{i}{2}\int d^3q\, a_{\boldsymbol{q}}^{\dagger}\left(E_{\boldsymbol{q}}\frac{\partial}{\partial q_m}+\frac{\partial}{\partial q_m}E_{\boldsymbol{q}}\right)a_{\boldsymbol{q}}-\frac{g^2}{4(2\pi)^3}\int d^3q_1d^3q_2d^3q_1'd^3q_2'a_{\boldsymbol{q}_1'}^{\dagger}a_{\boldsymbol{q}_2'}^{\dagger}a_{\boldsymbol{q}_1}a_{\boldsymbol{q}_2}\frac{1}{\sqrt{E_{\boldsymbol{q}_1'}E_{\boldsymbol{q}_2'}E_{\boldsymbol{q}_1}E_{\boldsymbol{q}_2}}}$$
$$\times\int d^3k\Bigg(\delta^3(\boldsymbol{q}_2-\boldsymbol{q}_2'-\boldsymbol{k})\frac{i}{\omega_{\boldsymbol{k}}^2-(E_{\boldsymbol{q}_2}-E_{\boldsymbol{q}_2'})^2}\frac{\partial}{\partial k_m}\delta^3(\boldsymbol{q}_1-\boldsymbol{q}_1'+\boldsymbol{k})$$
$$+\delta^3(\boldsymbol{q}_1-\boldsymbol{q}_1'-\boldsymbol{k})\frac{i}{\omega_{\boldsymbol{k}}^2-(E_{\boldsymbol{q}_1}-E_{\boldsymbol{q}_1'})^2}\frac{\partial}{\partial k_m}\delta^3(\boldsymbol{q}_2-\boldsymbol{q}_2'+\boldsymbol{k})\Bigg)\Bigg]\eta+\mathcal{O}(g^4). \quad (104)$$

In Eqs. (103) and (104) we neglected vacuum and self energy terms which should be treated by suitable renormalization procedures and also nucleon antinucleon annihilation contributions are not included. In Ref. [10] it is shown by explicit verification that the effective generators given above fulfil the Lie algebra up to defects of order g^4. This has to be the case, since the unitary transformation conserves the commutation relations, and because of the nonlinearity present in Eqs. (13) and (14) the error has to be of order $\mathcal{O}(g^4)$, if the effective generators are determined up to order $\mathcal{O}(g^2)$ only. Also the operator products (matrix multiplications) up to that order are well defined. It will be interesting to pursue the expansion of the effective generators to order $\mathcal{O}(g^4)$ to see the emergence of vertex corrections (form factors) and to see which role they play for the fulfilment of the algebra. Also renormalization will be an interesting issue. This will be left to a forthcoming article. Renormalization for the η space corresponding to one nucleon has already been carried through in lowest order $[\mathcal{O}(g^2)]$ and leads nicely to the expected mass renormalization [17]. We find due to self-energies expressions for $\tilde{H}$ and $\tilde{K}_i$ where bare masses are replaced by the physical masses in second order.

It is important to note that the operators (101)–(104) are well defined Hilbert space operators. This is obvious because of the Yukawa-type forces in $\tilde{H}$ and $\tilde{K}_i$. Note that this is not the case for the original generators defined through matrix elements in the full Fock space.

Let us now draw our attention to the formulation of scattering theory. We assume a more realistic field theoretical model that guarantees baryon conservation. For this purpose we define $|\Psi_E\rangle^{(+)}$ to be a scattering state developing from two initially free nucleons described by $|\Psi_E\rangle^0$

$$|\Psi_E\rangle^{(+)}=\lim_{\epsilon\to 0} i\epsilon \frac{1}{E-H+i\epsilon}|\Psi_E\rangle^0. \tag{105}$$

We perform an Okubo transformation

$$|\tilde{\Psi}_E\rangle \equiv U|\Psi_E\rangle^{(+)}=\lim_{\epsilon\to 0} i\epsilon U\frac{1}{E-H+i\epsilon}U^\dagger U|\Psi_E\rangle^0$$
$$=\lim_{\epsilon\to 0} i\epsilon \frac{1}{E-\tilde{H}+i\epsilon}U|\Psi_E\rangle^0. \tag{106}$$

From Eq. (31) we have

$$|\tilde{\Psi}_E\rangle=\lim_{\epsilon\to 0} i\epsilon \frac{1}{E-\tilde{H}+i\epsilon}[1+F(A)]|\Psi_E\rangle^0 \tag{107}$$

with

$$F(A)=-1+(1+A^\dagger A)^{-1/2}-(1+AA^\dagger)^{-1/2}A. \tag{108}$$

We expand $F(A)$ and get

$$F(A)=-A-\frac{1}{2}A^\dagger A+\frac{1}{2}AA^\dagger A+\frac{3}{4}A^\dagger AA^\dagger A$$
$$-\frac{3}{4}AA^\dagger AA^\dagger A+\cdots. \tag{109}$$

As long as $F(A)\,\eta|\Psi_E\rangle^0$ is no eigenstate of the full effective Hamiltonian or of the free Hamiltonian and has a finite norm, we can conclude

$$|\tilde{\Psi}_E\rangle=\lim_{\epsilon\to 0} i\epsilon \frac{1}{E-\tilde{H}+i\epsilon}|\Psi_E\rangle^0 \tag{110}$$

and hence

$$|\tilde{\Psi}_E\rangle \equiv \eta|\tilde{\Psi}_E\rangle \tag{111}$$

since $\tilde{H}$ is block diagonal. According to Eq. (110) we can calculate $|\tilde{\Psi}_E\rangle$ in this case from an ''effective Lippmann Schwinger equation'' and therefore write

$$|\tilde{\Psi}_E\rangle^{(+)}\equiv|\tilde{\Psi}_E\rangle, \tag{112}$$

$$V_{\rm eff}\equiv \tilde{H}-H_0, \tag{113}$$

$$|\tilde{\Psi}_E\rangle^{(+)}=|\Psi_E\rangle^0+\frac{1}{E-H_0+i\epsilon}V_{\rm eff}|\tilde{\Psi}_E\rangle^{(+)}. \tag{114}$$

Note that H_0 is not transformed. What are the conditions allowing for our assumption (110)? We concentrate on the first term of Eq. (109)

$$F(A)\cong -A. \tag{115}$$

From the discussion in Sec. III we know that

$$A=\sum_{\nu=1}^{\infty} A_\nu g^\nu \tag{116}$$

and from Eqs. (63)–(65) we see that overall energy denominators are a common feature of A_ν:

$$A_\nu=\Lambda a_\nu \eta \frac{1}{E_\Lambda - E_\eta}. \tag{117}$$

We again used our notation (62) and a_ν is a string of operators. According to Eqs. (107), (115), and (117) we have the following contributions to the second part in Eq. (107):

$$\lim_{\epsilon\to 0} i\epsilon \frac{1}{E-\tilde{H}+i\epsilon}\int_\Lambda d\Lambda|\Lambda\rangle\langle\Lambda|\frac{a_\nu}{E_\Lambda-E}|\Psi_E\rangle^0$$
$$=\lim_{\epsilon\to 0} i\epsilon(G_0+G_0V_{\rm eff}G_0+\cdots)$$
$$\times\int_\Lambda d\Lambda|\Lambda\rangle\langle\Lambda|\frac{a_\nu}{E_\Lambda-E}|\Psi_E\rangle^0. \tag{118}$$

Since we assumed an interaction conserving the baryon number the Λ states in Eq. (118) contain two nucleons and an arbitrary number of mesons and $N\bar{N}$ pairs. Then we distinguish two cases.

(1) $E<2m_0+\mu_0$. In this case E_Λ can never be equal to E and therefore the denominator cannot vanish. We end up with integrals being finite after renormalization. Consequently the multiplication with $i\epsilon$ gives zero.

(2) $E\geqslant 2m_0+\mu_0$. In this case the energy denominator can vanish and the formal expression in Eq. (117) will have two contributions. One is a principal value integral in Eq. (118) which leads to a finite result and therefore does not contribute. The second part is proportional to $\delta(E_\Lambda-E)$ giving a discrete contribution at $E_\Lambda=E$. The corresponding eigenstate to H_0 leads to an $1/i\epsilon$ coming from G_0 in Eq. (118) and therefore gives a nonvanishing contribution.

The additional terms in Eq. (109) do not have the structure (117) and hence do not contribute. We conclude that our assumption (110) is only valid at energies below the meson production threshold. The result is that effective scattering states in the η space are equivalent to those in the full space and can be obtained as solutions of an effective Lippmann Schwinger equation for scattering at energies below the threshold of particle production. If we go to higher energies the solution of this Eq. (110) is no longer equivalent to the original scattering states since Eq. (110) neglects nonvanishing contributions coming from Eq. (108). In this case the η space has to be augmented by including an additional meson. This is also left to a future investigation.

V. SUMMARY

The matrices of the ten generators of the Poincaré group (2)–(5) with respect to Fock space states are nondiagonal and connect states for nucleons, antinucleons, and mesons. The generators can be constructed according to often used Lagrange densities. The matrices can be blockdiagonalized at the same time by a single unitary transformation. Thereby the blocks are defined by two projection operators which span the Fock space, one refering to a fixed number of nucleons and the other to the rest of the space. As a consequence the resulting unitarily transformed generators act in these two spaces separately and specifically we gained effective generators in the space of N nucleons which are representations of the Poincaré algebra. This result has been proven using a (formal) power series expansion in the coupling constant. We see the importance of that result in the existence proof. Clearly in practice this series has to be truncated as is usually done in evaluating NN forces in low orders of meson exchanges. We would also like to note that our proof in Sec. III relies only on the forms (2)–(5) for the ten Hermitian generators together with interactions (6) being linear in a coupling constant and the additional requirements $\eta H_I \eta = \eta K_{Ii} \eta = 0$.

Results of numerical studies in Refs. [12–14] using the effective generators Eqs. (101)–(104) are promising. They show for instance that the relativistic energy-momentum relation of a two body state is rather well fulfilled if one solves the Schrödinger equation using the effective Hamiltonian in frames where the total momentum of the two-body system is different from zero. They also show that contributions to the relativistic Hamiltonian which remain undetermined in the scheme of a $1/c^2$ expansion of the Poincaré generators [5] can now be determined for a given field theory and are different from zero. In fact in the numerical examples studied [12] they are as important as those enforced by the Poincaré algebra. Thus the scheme discussed in this article provides interesting structural insight.

In addition one can pose now various questions. One of the effective generators, the Hamiltonian in the space of N nucleons, will contain NN but also many body forces. Will they fulfil the cluster separability? Since the effective generators are constructed in a power series expansion in the coupling constant one encounters in all orders g^n with $n \geqslant 4$ meson exchange diagrams together with vertex corrections for instance. The question then arises whether the Poincaré algebra for the ten effective generators, which is fulfilled in each order in g, requires all the terms of a certain order in g or whether subgroups fulfil the algebra separately. In the first case the Poincaré algebra would impose conditions on the acceptable vertex corrections which would play the role of strong form factors. Further investigations of that type are planned including renormalization.

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