Variational Tamm-Dancoff treatment of quantum chromodynamics. II. A semianalytic treatment of the hadrons in the valence quark approximation

J. R. Spence and J. P. Vary
Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011
(Received 6 October 1998)

Approximate two-fermion interactions and Bethe-Salpeter equations for hadron spectra are obtained by making a Fock space expansion of the quantum chromodynamics (QCD) mass eigenstate. Following a partial wave decomposition, we perform an average over quartic gluon vertices and a root mean square (rms) average over cubic vertices. The expansions in terms of quark and gluon configurations are truncated. Equations for the gluon eigenvalues and eigenvectors are derived and approximate solutions are obtained analytically. Using values for the QCD coupling constant and the quark rest masses obtained from a relativistic two- and three-constituent quark model as starting values, the resulting algebraic eigenvalue equations for the mesons and the light baryons are then solved using a multiscale expansion in harmonic oscillator eigenfunctions. Through an approximate but analytic treatment we demonstrate that, at large distances our formalism gives rise to an effective two fermion potential exhibiting linear confinement. That is, we do not introduce any phenomenological confinement interaction. With minor adjustments of the quark rest masses, our results compare favorably with the results of the phenomenological two- and three-constituent quark models and with experiment. [S0556-2813(99)00703-7]

PACS number(s): 12.39.Ki, 12.38.Lg

I. INTRODUCTION

There is considerable interest in obtaining an accurate description of the mass spectra and amplitudes of elementary particles derived from QCD. In a previous work we adopted an approach to the heavy mesons which begins with a Tamm-Dancoff treatment [1] of the quantized gluonic degrees of freedom, evaluates an effective potential for the quarks moving in the gluon fields, and solves for the quark motion via relativistic wave equations. In the present work, we treat both the mesons and the light baryons within a considerably simpler method which nevertheless retains the dynamical features of our previous work.

We extend previously developed methods of solution [2–4] of the Bethe-Salpeter [5,6] integral equation to encompass the approximate treatment of quantum field theories by the Tamm-Dancoff approach [1]. We adopt an approximate QCD Hamiltonian [7], introduce a variational Tamm-Dancoff treatment, and obtain results with quark masses and strong-coupling constant input from a constituent quark model. We then adjust the constituent quark masses in a fit to selected experimental data.

It is especially significant that we do not add a phenomenological confinement interaction to the QCD Hamiltonian. Our primary results are simply that the resulting hadron mass eigenvalues closely resemble the experimental hadron spectrum and that the derived interaction of the quarks exhibits confinement behavior. Our approximations, methods of solution and results for mass eigenstates of the mesons and light baryons form the focus of the present effort. The heavy baryons, those containing one or more quarks other than *u* or *d* quarks, will be treated in a later work.

Our general procedure involves making a Fock space expansion of the QCD mass eigenstate followed by partial wave decompositions of the Fock space components. Making a variational ansatz for the gluon amplitudes casts the problem of finding the mass eigenstates into the form of a multiconfigurational Hartree-Fock [10,11] problem. The resulting coupled algebraic equations are then solved in a Born-Oppenheimer approximation scheme.

The distinctions between the present work and Ref. [4] are very significant at the level of implementation of our approach. Here, in order to make our implementation more facile and easier to understand, we have developed and used a more extensive set of analytical and semianalytical tools

A variational treatment has been invoked to derive the mean-field equations, which we then solve via a Born-Oppenheimer type of approximation method. We use the constituent quark masses and phenomenological coupling constant to set the scale for the nonlinear gluon equation which we then solve to obtain the effective interaction of the quarks. This effective interaction exhibits a confining behavior. We analyze the gluon equation to exhibit how the nonlinear character arising from the non-Abelian nature of the problem provides this confining behavior.

Discussions of our methods separate naturally into two parts. First, we introduce an approximate decomposition of our treatment of QCD into two- and three-particle wave equations. This step includes such topics as renormalization and management of divergences on a computer. Second, we solve these relativistic few-body wave equations [2–4,9,10] using established methods in a sequence of coordinate and momentum space calculations.

We organize our presentation along the following lines. In Sec. II we present our ansatz for the truncated Tamm-Dancoff spaces. Then in Sec. III we introduce and discuss our multistep variational approach. Section IV includes details on approximations we introduce to achieve a calculable framework along with our gauge treatment and the successive steps to obtain solutions to the resulting equations.

In Sec. V we derive an approximate analytic solution of the cubic gluon wave equation and exhibit how confinement arises in a natural way. We then show in Sec. VI how we obtain an effective quark-quark interaction. Section VII presents the methods we use to solve the resulting quark wave equations which follow the methods of Refs. [8,9]. Finally, in Sec. VIII, we present our results and conclusions.

II. THE MODEL SPACE ANSATZ

We select the center of momentum (CM) frame and introduce our ansatz for the basis states in the form

$$|\Phi;m\rangle \approx \sum_{\nu=0}^{N} z_{m,\nu}|g^{(\nu)};m\rangle,|\Psi_{q(\nu)};m\rangle, \tag{1}$$

where the kets $|\Psi_{q(\nu)};m\rangle$ correspond to either normalized $q\bar{q}$ or normalized qqq states depending on the system under investigation. The kets $|g^{(\nu)};m\rangle$ are normalized ν gluon states. The label m represents a complete set of commuting observables for the eigenstate. The upper limit N is the maximum number of gluons included in any Fock space term and the $z_{m,\nu}$'s are complex expansion coefficients.

Note that although we are including states with zero gluons all of our states contain either a physical $q\bar{q}$ pair or qqq triplet. Our point of view is that, with the exception of glueballs, the presence of one or more such states is a defining feature of a physical hadron. When invoking the mean-field approximation, we assume the quark motion is governed by a potential generated by the gluons and vice versa. In addition, all states in Eq. (1) are taken to be at rest in the same frame. That is, in our ansatz we do not allow for the relative motion of the quark center of mass relative to the hadron center of mass.

In the spirit of the mean-field approximation we further restrict consideration to the case in which only one quark state (either $q\bar{q}$ or qqq) is present and assume that the kets $|\Psi_{q(\nu)};m\rangle$ for fixed m, may be taken to be the same for all gluon ν kets $|g^{(\nu)};m\rangle$ with this general quark ket to be denoted $|\Psi_q;m\rangle$. These approximations constitute what we shall term the ''valence quark approximation.'' The assumed form of the hadron mass eigenstate may now be written $|G;m\rangle|\Psi_q;m\rangle$ where

$$|G;m\rangle = \sum_{\nu=0}^{N} z_{m,\nu}|g^{(\nu)};m\rangle. \tag{2}$$

III. THE VARIATIONAL METHODS

In general terms, the QCD Hamiltonian may be written

$$H_{\mathrm{QCD}} = T_Q + H_G + H_{\mathrm{INT}}, \tag{3}$$

where T_Q denotes the kinetic energy for the quarks, H_G is the Hamiltonian for the gluons, and H_{INT} is for the interaction of quarks with gluons. Using the assumed decomposition of $|\Phi;m\rangle$ into the products of an expansion coefficient, a quark ket and a gluon ket, we now cast the problem as three coupled variational calculations in order to compute the approximate eigenvalues and eigenvectors of H_{QCD}.

In Ref. [4] we found that the total energy of the gluons becomes very large and negative as the number of gluons is allowed to increase and hence, to treat the gluon sector, we now solve for the average energy per constituent [11] (counting the quark ket as one constituent). We then search for the stationary points of $\langle\Phi;m|H_{\mathrm{QCD}}|\Phi;m\rangle$ subject to the constraint

$$\delta\langle\Phi;m|N_+|\Phi;m\rangle = 0 \tag{4}$$

with $N_+ = N+1$, and N denoting the gluon number operator. When implementing this constraint, we hold fixed the state vector of the quark sector from the first variation. We define the square root of N_+ to be given by

$$\langle g^{(\nu')};m|N_+{}^{1/2}|g^{(\nu)};m\rangle = \delta_{\nu,\nu'}\sqrt{\nu+1} \tag{5}$$

and we define

$$|\overline{\Phi};m\rangle = N_+{}^{1/2}|\Phi;m\rangle, \tag{6}$$

$$N_+{}^{-1/2}H_{\mathrm{QCD}}N_+{}^{-1/2} = \overline{H}_{\mathrm{QCD}}. \tag{7}$$

The new variational condition becomes

$$\delta\langle\overline{\Phi};m|\overline{H}_{QCD}|\overline{\Phi};m\rangle = 0. \tag{8}$$

with the condition

$$\delta\langle\overline{\Phi};m|\overline{\Phi};m\rangle = 0. \tag{9}$$

Clearly, this is equivalent to the variational of H_{QCD} divided by the expectation value of N_+. For simplicity of notation we shall now suppress the bar on all the quantities defined above. Further, for these calculations, the number of quarks plus antiquarks will be held fixed at 2 or 3 depending on the system. Only the number of gluons is allowed to vary in the Fock space states.

We make a further simplifying approximation [4] consisting of assuming that each $|g^{(\nu)};m\rangle$ is composed of ν identical single gluon states, and these single gluon states are the same for all gluon kets. Note that the m dependence is retained.

IV. THE MEAN-FIELD EQUATIONS

First variation: expansion coefficients

The expansion coefficients $z^*_{m,\nu}$ are varied while the $z_{m,\nu}$'s and both quark and gluon bras and kets are treated as fixed. This yields the equation

$$E_m z_{m,\mu} = \sum_{\nu=0}^{N} \langle\Psi_q;m|\langle g^{(\mu)};m|H_{\mathrm{QCD}}|g^{(\nu)};m\rangle|\Psi_q;m\rangle z_{m,\nu}. \tag{10}$$

This is a straightforward Ritz variational calculation, treating the expansion coefficients as parameters while keeping other quantities fixed, and yields the usual finite matrix diagonalization problem. Upon convergence, one obtains the expansion coefficients of the total state vector as governed by the coupled mean fields of both the quarks and gluons.

Second variation: quarks

Next, we continue using the Ritz variational principle and obtain two coupled equations which are to describe, in the center of momentum (CM) frame, the quark state vector $|\Psi_q;m\rangle$ and, gluon state vector $|g^{(\nu)};m\rangle$, interacting by the term $H_{\rm INT}$. In other words, to determine the quark kets and their mean-field energies we invoke the variational condition

$$\delta\langle\Psi_q;m|\hat{H}_{\rm QCD}|\Psi_q;m\rangle=0, \tag{11}$$

$$\delta\langle\Psi_q;m|\Psi_q;m\rangle=0, \tag{12}$$

where $\hat{H}_{\rm QCD}$ signifies an expectation value with respect to fixed gluon components and fixed expansion coefficients. In the later stages of the calculations, the total state vectors $|\Phi;m\rangle$ are orthonormalized. At the present stage we obtain an equation of the form

$$E_m|\Psi_q;m\rangle=T_Q|\Psi_q;m\rangle+\sum_{\mu,\nu} z_{m,\mu}z^*{}_{m,\nu}\langle g^{(\mu)};m|T_G+H_{G3}+H_{G4}+H_{\rm INT}|g^{(\nu)};m\rangle|\Psi_q;m\rangle, \tag{13}$$

which symbolizes the quarks moving in the mean-field of the gluons with a mean-field energy E_m, interaction Hamiltonian $H_{\rm INT}$, gluon free particle energy T_G, and quark free particle energy T_Q. In addition, H_{G3} and H_{G4} represent contributions arising from the three and four gluon vertices.

Third variation: gluons

We take as our ansatz for the gluon kets the color singlet forms [12–14]

$$|g^{(\nu)};m\rangle=N_{(m,\nu)}\mathrm{Tr}(\hat{\eta}^\dagger{}_{(m,\nu)})^\nu|\varnothing\rangle. \tag{14}$$

The gluon creation operators, $\hat{\eta}^\dagger{}_{(m,\nu)}$ are regarded as three by three matrices in the color indices and the traces of the products are taken to construct color singlet states [9,10].

Here $N_{(m,\nu)}$ is a normalization factor, $|\varnothing\rangle$ is the gluon vacuum, and each $\eta^\dagger$ denotes a gluon creation operator given in terms of the gluon field creation operator $\hat{a}^\dagger_{(a,b)\vec{q},\lambda}$ by

$$\hat{\eta}^\dagger_{(m,\nu)}=\sum_{\lambda=\pm1}\sum_{a,b=1}^{3}\int\frac{d^3q}{\sqrt{2q}}\phi(\vec{q})_{(a,b)m,\nu,\lambda}\hat{a}^\dagger_{(a,b)\vec{q},\lambda}. \tag{15}$$

In solving the variational equations by iteration, we develop a generalized polynomial series for $\phi(\vec{q})$. The color indices are specified by a and b.

In the usual mean-field approximation, we retain only those terms in the summation over ν where a single gluon emerges.

In the usual mean-field approximation, we assume that the single-gluon state vector is the same for all ν-gluon configurations. The resulting equation is

$$\omega_{m,\nu}|\phi^{(\nu)};m\rangle=T_G|\phi^{(\nu)};m\rangle+\langle g^{(\nu-1)};m|\langle\Psi_q;m|H_{\rm INT}|\Psi_q;m\rangle+H_{G3}|g^{(\nu)};m\rangle. \tag{16}$$

At this stage of the development, we consider the case $\nu=2$ with a goal of developing a gluon state which typifies the state involved in an interaction between a pair of fermions. Ultimately we will insert this typical gluon state into a ladder series to develop the full dynamics of gluon-mediated interaction.

Dropping labels where they are no longer needed, we arrive at

$$\omega_m|\phi;m\rangle=T_G|\phi;m\rangle+\langle g^{(1)};m|\langle\Psi_q;m|H_{\rm INT}|\Psi_q;m\rangle+H_{G3}|g^{(2)};m\rangle \tag{17}$$

where, in keeping with our notation

$$|\phi;m\rangle=\langle g^{(1)};m|g^{(2)};m\rangle. \tag{18}$$

Finally, with our approximations, we also now identify

$$|\phi;m\rangle=|g^{(1)};m\rangle. \tag{19}$$

Thus, we have arrived at a nonlinear gluon wave equation whose solution we seek in order to provide an effective gluon exchange interaction. Note that this equation could be obtained from the equations in Ref. [4] by omitting the quartic interaction term.

We are seeking the soft gluon, long-range limit where we may put

$$\overline{\omega}_m\approx\epsilon_0=\langle\Psi_q;m|T_Q|\Psi_q;m\rangle\approx\langle\Psi_q;m|\gamma_0 m_i|\Psi_q;m\rangle. \tag{20}$$

It should be noted that this is our critical scale-setting step within our approach. Our quark mass is a phenomenological parameter expected to be in the range of the usual constituent quark mass values.

The pair of coupled mean-field equations for quarks and gluons has been derived here by minimizing the expectation value of the total QCD Hamiltonian subject to certain normalization conditions. The pair also has a form reminiscent of a pair of equations obtained through a reduction of the Bethe-Salpeter equation, known as Salpeter's equation, which ladders the kernels $H_{G3}+H_{G4}$ and $H_{\rm INT}$ in terms of the quark and gluon momenta

$$T_Q=\sum_{i=1}^{n}(m_i^2+p_i^2)^{1/2}, \tag{21}$$

$$T_G=\sum_{i=1}^{\nu}q_i. \tag{22}$$

V. APPROXIMATE ANALYTIC SOLUTION OF THE GLUON WAVE EQUATION

Our procedure for computing the gluon wave function and fermion-fermion potential is based upon employing a series to sum ladders or portions of ladders with one gluon exchange (OGE) rungs and a sequence to sum ladders or segments thereof having cubic rungs. Note that the treatment of the OGE ladders by themselves is a largely solved matter [12–14], and our evaluation of their contributions used standard approximation methods of atomic physics [15–18], evaluating expansions of the OGE interactions in the coordinate representation. We also included the effects of the running of the coupling constant at lowest order using a leading log approximation.

The nonlinearities due to the cubic vertex H_{G3}, dominate in the confinement region. We approximately solve the cubic gluon wave equation in its nonlinear regime and then use the resulting solutions as our ansatz for iterating Eq. (17) until convergence is obtained.

The cubic gluon wave equation: first iteration

Previously, in Ref. [4], we argued that confinement could be understood by considering an aspect of the non-Abelian character involving the cubic gluon self-couplings. In the present work we establish an improved and more detailed picture of how these processes lead to a confining potential. We begin by assuming a static and classical gluon field for each of the three gluons at the vertex. Then, we consider our gluon wave equation for this process in the mean-field treatment, writing the total gluon wave function as the product of the three individual gluon wave functions, and including the H_{INT} term. Next we use conservation of three momentum to write the third gluon field as a convolution of the first and second gluon field accompanied by factors associated with the Feynman diagram for a cubic vertex.

With the understanding (justified below) of what results from averaging over Lorentz indicies in the mean-field treatment, we take these gluon fields to be single-component objects rather than four vectors. Next we place the gluons on shell and impose conservation of three momentum and energy on the gluon momenta yielding the equations $\vec{p}_1=\vec{p}_2+\vec{p}_3$ and $p_1=p_2+p_3$ which imply $p_2p_3=\vec{p}_2\cdot\vec{p}_3$ and hence that the momentum of the incoming (outgoing) particle p_1 is collinear with the momenta of the remaining particles $\vec{p}_2$ and $\vec{p}_3$. This reduces the integral to one dimension and leaves us with the following form of the colored wave equation:

$$\begin{aligned}\epsilon A_A(p)=pA_A(p)+\frac{4\pi G}{(2\pi)^3}\int_0^\infty dq q^2 A_B(p-q)\\ \times A_C(q)U^{ABC}(p,q,r)\\ +\frac{4\pi}{(2\pi)^3}\int_0^\infty dq q^2\langle\Psi_q m|H_{\mathrm{INT}}|\Psi_q;m\rangle A_A(q)\end{aligned}\tag{23}$$

with

$$\begin{aligned}U^{ABC}(p,q,r)=-igf^{ABC}[(r-q)_\mu\delta_{\nu\rho}+(q-p)_\rho\delta_{\mu\nu}\\ +(p-r)_\nu\delta_{\rho\mu}];\quad p+q+r=0,\end{aligned}\tag{24}$$

where A,B,C represent color indices (1-8), ρ,μ,ν are the Lorentz indices and f^{ABC} represents the structure constants of $\mathrm{SU}(3)_C$.

Next we absorb a factor of ϵ into each A, perform a color average and replace $f^{ABC}A_B(p-q)A_C(q)$ by

$$1/8[f^{ABC}f^{ABC}A(p-q)A(q)]=1/8[24A(p-q)A(q)].\tag{25}$$

We drop the color indices A, B, and C, and for the zeroeth order drop both the free particle energy term $pA_A(p)$ and the quark-gluon interaction term

$$\langle\Psi_q;m|H_{\mathrm{INT}}|\Psi_q;m\rangle A_C(q).\tag{26}$$

Finally we replace the terms in the factor U by their rms average over the quantities $(p\text{-}q),(q\text{-}r)$, and $(r\text{-}p)$, using

$$[(p-q)^2+(q-r)^2+(r-p)^2]^{1/2}=(p^2+q^2+r^2)^{1/2}/2\tag{27}$$

$$\approx(q^2 3/2)^{1/2}=q\sqrt{3/2}.\tag{28}$$

This replaces the four vectors by three vectors as in making an instantaneous approximation

$$(3q^2/2)^{1/2}=\sqrt{3/2}|\mathbf{q}|.\tag{29}$$

When we employ the pole dominance argument below and invoke the symmetry of the boson wave functions we find that, in the center of momentum, the averaged value of U becomes

$$\beta q=3\sqrt{3/2}q.\tag{30}$$

This results in a first-order rms averaged mean-field single gluon wave equation

$$A(p)=\frac{12\pi\beta}{\epsilon^2(2\pi)^3}\int_0^\infty dq q^3 A(p-q)A(q).\tag{31}$$

We now note that by performing a similar rms color average over each color factor f^{ABC} and then by averaging over spin and Lorentz indicies on the quartic vertex

$$\begin{aligned}&ig^2[f^{NAC}f^{NBD}(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\gamma\beta})\\ &\quad+f^{NAD}f^{NBC}(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta})+f^{NAB}f^{NCD}\\ &\quad\times(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma})]\\ &\quad\to ig^2 f^{NAC}f^{NBD}f^{NAC}f^{NBD}[(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\gamma\beta})\\ &\quad+(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta})+(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma})].\end{aligned}$$

Then, by averaging over $\alpha,\beta,\gamma,$ and δ

$$\to\frac{1}{256}\sum_{\alpha,\beta,\gamma,\delta}ig^2 f^{NAC}f^{NBD}f^{NAC}f^{NBD}(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\gamma\beta})=0.$$

With this understanding we discard the quartic gluon terms in the gluon mean-field wave equation.

We take as an ansatz for the gluon fields $A(p)=Cp^\alpha$, with C and α constants, and we substitute this form into the above equation. With the simplifications from above the mean-field equation becomes

$$A(p)=\frac{12\pi G\beta}{\epsilon^2(2\pi)^3}\int_0^\infty dq q^3 A(p-q)A(q),\tag{32}$$

and with our ansatz

$$p^\alpha\approx 12\pi\overline{C}/(2\pi)^3\epsilon^2\int_0^\infty dq q^3(p-q)^\alpha q^\alpha,\tag{33}$$

where $\overline{C}=G\times\beta\times C$.

Our purpose is to solve this approximate mean-field equation for α as $p\rightarrow 0$. The solution can be deduced by dimensional analysis where, in powers of momentum k, we observe

$$k^{\alpha}\approx k^{2\alpha+4} \tag{34}$$

leading to $\alpha=-4$. However, further attention to the approximate evaluation of the integral is desirable to obtain information about $\overline{C}$. A more rigorous treatment is given in Refs. [2] and [3]. Continuing to concentrate on the limit $p\rightarrow 0$ for the first iteration, we will assume for simplicity, the integral is dominated by the pole at $p=q$, so that

$$p^{\alpha}=\frac{-12\pi\overline{C}(p-0)^{\alpha+1}p^{\alpha+3}}{(2\pi)^3\epsilon^2(\alpha+1)}. \tag{35}$$

This yields the result as $p\rightarrow 0$

$$p^{\alpha}=\frac{-12\pi\overline{C}}{(2\pi)^3(\alpha+1)\epsilon^2}p^{2\alpha+4}. \tag{36}$$

Thus, we have seen through a simplified analysis that each gluon in the process follows a dependence on its momentum to the $\alpha=-4$ power as its momentum goes to zero. The constant of proportionality being

$$\overline{C}=-\frac{(\alpha+1)\epsilon^2(2\pi)^3}{12\pi}. \tag{37}$$

The cubic gluon wave equation: second iteration

Having solved the rms averaged gluon wave equation in first approximation and obtained asymptotic large r solutions of the form $A^{(1)}(p)=\overline{C}/p^4$ we now iterate with the rms averaged gluon equation

$$\epsilon A(p)=pA(p)+\frac{12\pi\beta}{\epsilon(2\pi)^3}\int_0^{\infty}dqq^3A(p-q)A(q). \tag{38}$$

Because we are interested in those soft, low-energy gluons with $\epsilon+\epsilon_0\approx 0$ the gluon wave equation may be approximated

$$-\epsilon_0A(p)=pA(p)+\frac{-12\pi\beta}{\epsilon_0(2\pi)^3}\int_0^{\infty}dqq^3A(p-q)A(q). \tag{39}$$

Making the pole dominance approximation as before and using the expression for the first iteration gluon wave function we find for the second iteration single gluon wave function

$$A^{(2)}(p)=\frac{\overline{C}}{(1+p/\epsilon_0)p^4}. \tag{40}$$

The normalization ensures that the expectation value of the total $A(p)$ is (trivially)

$$\frac{\langle\Phi;m|A(p)|\Phi;m\rangle}{\langle\Phi;m|\Phi;m\rangle}=A(p) \tag{41}$$

because this $A(p)$ is the same for each Fock space component.

The cubic gluon wave equation: third iteration

We have subtracted off the perturbative OGE contribution to the gluon wave function as our first approximation and obtained first and second iterates of the nonperturbative gluon wave equations. That is, we have solved approximately for $A^{(1)}(p)$ and $A^{(2)}(p)$, from

$$A^{(1)}(p)=\frac{12\pi G\beta}{\epsilon^2(2\pi)^3}\int_0^{\infty}dqq^3A^{(1)}(p-q)A^{(1)}(q) \tag{42}$$

and

$$-\epsilon_0A^{(2)}(p)=pA^{(2)}(p)-\frac{12\pi\beta}{\epsilon_0(2\pi)^3}\times\int_0^{\infty}dqq^3A^{(1)}(p-q)A^{(1)}(q). \tag{43}$$

We initiated our iterative sequence of approximations for the nonperturbative portion of the gluon wave function with an ansatz, $A^{(1)}(p)$, which satisfies the gluon wave equation in the strong-coupling (nonlinear) domain. With such a starting point we may expect that a standard iteration scheme as

$$-\epsilon_0A^{(n+1)}(p)=pA^{(n)}(p)-\frac{12\pi\beta}{\epsilon_0(2\pi)^3}\times\int_0^{\infty}dqq^3A^{(n)}(p-q)A^{(n)}(q) \tag{44a}$$

or

$$(p+\epsilon_0)A^{(n+1)}(p)=\frac{12\pi\beta}{\epsilon_0(2\pi)^3}\int_0^{\infty}dqq^3A^{(n)}(p-q)A^{(n)}(q);\quad n=2,3,4,\ldots \tag{44b}$$

will be at worst semiconvergent.

Computation of approximations to the third iterate, $A^{(3)}(p)$, indicates, for the first time, logarithmic divergences and the consequent need for renormalization. We write the integral appearing on the right-hand side of Eqs. (44a),(44b) as

$$\begin{aligned}I^{(3)}&=\frac{12\pi\beta}{\epsilon_0(2\pi)^3}\int_0^{\infty}dqq^3A^{(2)}(p-q)A^{(2)}(q)\\&=\frac{12\pi\beta}{\epsilon_0(2\pi)^3}\frac{\overline{C}^2}{\epsilon_0^4}\int_0^{\infty}dqq^3\\&\quad\times\frac{1}{(1+(p-q)/\epsilon_0)(p-q)^4(1+q/\epsilon_0)q^4}\\&=\frac{3\overline{C}}{\epsilon_0^3}[I_1^{(3)}+I_2^{(3)}+I_3^{(3)}].\end{aligned} \tag{45}$$

We now invoke pole dominance.

(1) For the pole at $p=q$:

$$I_1^{(3)} \approx \int_0^\infty dq \frac{1}{(p-q)^4(1+p/\epsilon_0)p}. \tag{46a}$$

(2) For the pole at $q=0$:

$$I_2^{(3)} \approx \int_0^\infty dq \frac{1}{(1+p/\epsilon_0)p^4(1+q/\epsilon_0)q}. \tag{46b}$$

(3) For the remaining poles at $q=\epsilon_0+p$ and $q=-\epsilon_0$:

$$I_3^{(3)} \approx \int_0^\infty dq \frac{1}{(1+(p-q)/\epsilon_0)\epsilon_0^4(1+q/\epsilon_0)(\epsilon_0+p)}. \tag{46c}$$

Treating the integral $I_1^{(3)}$ as if it were a nonsingular integral as with the first iteration calculation again gives

$$I_1^{(3)} = \frac{1}{3(1+p/\epsilon_0)p^4} \tag{47}$$

so that $3\overline{C}/\epsilon_0$ times this term is equal to $A^{(2)}(p)$. Term $I_2^{(3)}$ is a logarithmically divergent constant times $I_1^{(3)}$ and will be shown to be the second term in a subseries obtained by iterating a portion of the recursion relation for the $A^{(n)}$. This subseries of divergent terms will then be seen to sum to zero.

Writing

$$I^{(3)} = -6I_1^{(1)} + 9I_1^{(1)} + 3I_1^{(2)} + 3I_3^{(3)},$$

$$I^{(3)} = -6I_1^{(1)} + 3I_1^{(1)}[1+\ln(0)] + 3I_3^{(3)},$$

we recognize

$$3I_1^{(1)}[1+\ln(0)] + 3I_3^{(3)}$$

as the first iterate of an equation which will occur in the continued iteration of our gluon wave equation

$$I = 3I_1 + \ln(0)I$$

which sums to

$$I = \frac{3I_1^{(1)}}{1-\ln(0)}.$$

Having thus eliminated the log divergence $3I_1^{(1)}[1+\ln(0)]$ we obtain $I^{(3)} = -6I_1^{(1)} + 3I_3^{(3)}$.

We can further simplify by using Eq. (43b) and making the approximation $\epsilon_0 = p$ in order to obtain

$$A^{(3)}(p) = \frac{6\pi\beta}{\epsilon_0^2(2\pi)^3}\int_0^\infty dq q^3 A^{(2)}(p-q)A^{(2)}(q)$$

$$= A^{(2)}(p) + \frac{3\overline{C}}{\epsilon_0^2} I_3^{(3)}.$$

Using partial fractions and treating the integrals as nonsingular casts $I_3^{(3)}$ into the convergent form

$$I_3^{(3)} \approx \int_0^\infty dq \left[\frac{1}{(1+(p-q)/\epsilon_0)} + \frac{1}{1+(q/\epsilon_0)} \frac{1}{\epsilon_0^4(\epsilon_0+p)(2\epsilon_0+p)} \right]$$

$$= \ln(1+p/\epsilon_0)\frac{1}{\epsilon_0^2(\epsilon_0+p)(2\epsilon_0+p)}.$$

Linearization of the gluon wave equation

So far we have considered only a spherically symmetric, and colorless solution of the cubic gluon equation. We now indicate a construction for a basis set of approximate eigenstates of the linearized cubic gluon wave equation. This basis contains a version of the confinement state discussed earlier but the remaining states in this basis are L_2 normalizable. The ability of nonlinear wave equations to support more solutions than corresponding linear equations is well known from mean-field calculations in atomic and nuclear physics.

We replace $A(p)$ by $A(p)_{n,l,m} + A(p)$ in the second-order gluon wave equation. We take $A(p)_{n,l,m} = R(p)_{n,l}Y_{l,m}(\theta,\phi)$ to be an ordinary momentum space wave function with variables separated in spherical coordinates. Treating $A_{n,l,m}$ as small we expand about $A(p)$ and linearize obtaining

$$\epsilon A_{n,l,m}(p) = pA_{n,l,m}(p) + \frac{24\pi\beta}{\epsilon(2\pi)^3} \times \int_0^\infty dq q^3 A(p-q)A_{n,l,m}(q). \tag{48}$$

We will now linearize this equation and convert to a coordinate representation through the adoption of an approximate form for $\widetilde{A}(r)$ provided below. This results in our linearized equation

$$(\epsilon-\epsilon_0)\widetilde{A}(\vec{r})_{n,l,m} = \widetilde{\epsilon}\widetilde{A}(\vec{r})_{n,l,m} = -\nabla^2\widetilde{A}(\vec{r})_{n,l,m} + \frac{kr^2}{2[1+A_0 m_1 m_2 r^2]^{1/2}}\widetilde{A}(\vec{r})_{n,l,m}. \tag{49}$$

The quantities m_i refer to the constituent quark masses.

The crucial points to be noted here are that the quantities k and A_0 will be determined *ab initio* and that for small r the linearized cubic gluon wave equation is a harmonic oscillator. Hence, we note that approximate solutions for the $\widetilde{A}(\vec{r})_{n,l,m}'s$ might be obtained by the methods employed with the quark wave equations below.

VI. THE EFFECTIVE QUARK-QUARK INTERACTION

To make effective use of an existing technology for approximating analytic solutions of the quark wave equation while retaining the essential physics, we seek to cast the second iteration single gluon wave function of Eq. (40) into the form

$$\tilde{A}(r)=\frac{kr^2}{2[1+A_0m_1m_2r^2]^{1/2}} \tag{50}$$

with k being a spring constant, k and A_0 to be determined. The key physics is retained since p^{-4} goes into $r/8\pi$ when transformed to coordinate space, and p^{-5} into $r^2/24(4\pi)$. This means that correspondence at both small and large r of the second iteration gluon wave function and its approximating coordinating function can be insured by choosing

$$k=m_{12}\omega_q^2=\tilde{\omega}_{q\bar{q}}^2=\frac{m_{12}\bar{C}}{24(4\pi)(2\pi)^3};\quad A_0=1/6^2 \tag{51}$$

with $m_{12}=m_1+m_2$.

We obtain an expression for the $q\bar{q}$ potential by inserting a factor of g_s for each vertex as in [3] yielding the nonperturbative potential

$$V_{\rm NP}(r)=(4\pi)^2\alpha_s^2\frac{\omega_{q\bar{q}}^2r^2}{2[1+A_0m_1m_2r^2]^{1/2}}. \tag{52}$$

We define a ''cubic ladders approximation'' consisting of ignoring all additional diagrams (other than OGE ladders). The potential for such cubic ladders is in the standard manner $V_{\rm NP}(r)$. With such a definition the problem of computing the Fock space expansion coefficients for the gluon wave functions is now circumvented. We note also that the r used here denotes the distance from the cubic gluon vertex to one of the fermion-gluon vertices. If r is to denote the fermion-fermion distance r must be replaced by $r/2$ on the right-hand side of Eq. (52).

A small point remains in the qqq case in that two of the quarks are interacting while the third remains a spectator. However, this spectator plays a dynamical role in that its color must be different than either of the active participants. Following the standard rule of averaging over initial states and summing over final, $V_{\rm NP}(r)$ must be reduced by a factor of 1/3 for the qqq case.

VII. THE QUARK WAVE EQUATIONS

At this stage we can sumarize our fermion-fermion interaction kernel as

$$V(r)=V_{\rm OGE}(r)+V_{\rm NP}(r), \tag{53}$$

where $V_{\rm NP}(r)$ is given by our variational treatment of the gluon dynamics (triple gluon coupling) in Eq. (52), and $V_{\rm OGE}(r)$ represents a semirelativistic (Foldy-Wouthysen) treatment of the one gluon exchange. Now, for the actual computation of quark eigenenergies and eigenfunctions we adopt a particular (Foldy-Wouthysen) treatment of the one-gluon exchange. In particular our $V_{\rm OGE}(r)$ is the same as the one-photon exchange from the literature [15–18] with the appropriate color weighting factors [7] and the strong-coupling constant α_s.

We choose a simple and convenient form of the scale dependence of the strong coupling $\alpha_s(Q^2)$ from previous investigations [8]. In particular,

$$Q^2=(m_1+m_2)^2.$$

Inspired by the approach of Mittal and Mitra [8], we can reduce the two-fermion problem as a state dependent, but conveniently soluble Bethe-Salpeter equation in an instantaneous approximation. We map $V_{\rm NP}(r)$ to a state-dependent harmonic-oscillator potential $V_{\rm HO}$. The rules for the mapping will be given below.

Now, for the actual computation of quark eigenenergies and eigenfunctions we adopt a particular form of the instantaneous approximation to the Bethe-Salpeter equation:

$$1/2M(4m_q^2-M^2-4\nabla^2{}_r)\tilde{\psi}=V_{\rm OGE}\tilde{\psi}+V_{\rm HO}\tilde{\psi}. \tag{54}$$

Note there are no additional phenomenological terms and that the interactions are both governed by the strong-coupling constant $\alpha_s(Q^2)$. In the momentum representation the harmonic-oscillator portion of the equation (i.e., leaving out $V_{\rm OGE}$), may be written in the reduced form:

$$\mathbf{q}^2\gamma_M^2\phi-\frac{1}{2}M\omega_{q\bar{q}}^2\nabla_q^2\phi-1/4F_{\rm HO}(M)\Omega_M\phi=0. \tag{55}$$

Defining

$$F_{\rm HO}(M)=(M^2-4m_q^2)\Omega_M^{-1}-1/4\Omega_MM^{-2}\gamma_M^{-2}\times(2\mathbf{J}\cdot\mathbf{S}-3-Q_N), \tag{56}$$

$$\Omega_M=4(2M)^{1/2}\omega_{q\bar{q}}\gamma_M;\quad \gamma_M^2=1+8M^{-1}m_q^{-2}\omega_{q\bar{q}}^2, \tag{57}$$

$$8\beta_M^2\gamma_M^2\equiv\Omega_M;\quad \omega_{q\bar{q}}^2=m_{12}\tilde{\omega}^2.$$

For equal mass m_q quarks, under harmonic confinement alone, the meson mass eigenvalues are determined by the algebraic equation

$$F_{\rm HO}=N+\frac{3}{2};\quad N=2n+l;\quad l=0,1,2\ \ldots;$$

$$n=0,1,2,\ \ldots. \tag{58}$$

Here N is the principal quantum number, l is the orbital angular momentum quantum number, and n is the radial quantum number.

In the case of unequal quark masses the following changes are made:

$$\bar{\tau}=\tau/\sqrt{2-\tau},\quad \tau=4m_1m_2/(m_1+m_2)^2,$$

$$\omega_{q\bar{q}}^2=m_{12}\bar{\tau}\tilde{\omega}^2,$$

$$\Omega_M=4(M\tau)^{1/2}\omega_{q\bar{q}}\gamma_M,$$

$$\gamma_M^2=1+8\tilde{\omega}^2M^{-1}m_{12}^{-2}\bar{\tau}.$$

The mass eigenvalues for a baryon composed of three equal mass quarks again under harmonic confinement alone, satisfy a similar set of algebraic relations.

TABLE I. Light and $s\bar{s}$ quarkonium masses in MeV with kernel as given in the text. The quark masses used are as explained in the text.

Meson	NJLS	$M_{\rm expt}$	$M_{\rm calc}$	$M_{\rm calc} - M_{\rm expt}$
π	0000	140	140*	+0
ρ	0101	770	546	−224
b_1	1110	1232	1182	−50
a_2	1211	1320	1283	−37
π'	2000	1300	1366	+66
ρ'	2101	1450	1535	+85
π_2	2220	1670	1780	+110
ρ_3	2321	1690	1802	+112
f_4	3431	2030	2009	−21
ϕ	0101	1020	1021	+1
f_1	1111	1420	1380	−40
f_2'	1211	1525	1572	+47
ϕ	2101	1680	1772	+92

$$F_{\rm HO}(M) = (M^2 - m_0^2)\Omega_B{}^{-1} - 1/4\Omega_B M^{-2}\gamma_B^{-2} \times\left(\frac{2}{3}\mathbf{J}\cdot\mathbf{S} - \frac{3}{2} - \bar{Q}_N - Q_N'\right), \quad (59)$$

$$\Omega_B = 9(2M)^{1/2}\omega_{qq}\gamma_B; \quad \gamma_B^2 = 1 + \frac{81}{8}M^{-1}m_0^{-2}\omega_{qq}^2, \quad (60)$$

$$m_0 = 3m_q; \quad \omega_{qq}^2 = \frac{1}{3}\omega_q^2; \quad F_{\rm HO}(M) = N+3. \quad (61)$$

Here $Q_N, \bar{Q}_N, Q_N'$ are given by

$$4Q_N = 2/3(N^2+3N-3) - (N+3)^2; \quad 6\bar{Q}_N = 8Q_N, \quad (62)$$

$$6Q_N' \approx -4/3(1+m_1m_2/M^2)[7/Q_N - 9/2(N+3/2)^2 + 9/2], \quad (63)$$

$$27\beta_B^2\gamma_B^2 \equiv \Omega_B.$$

The case of three unequal mass particles is nontrivial.

Solution of the $q\bar{q}$ equations

In order to obtain accurate analytic approximate solutions to the quark wave equations with our approximate potential the following scheme is employed [8]:

(1) The quantity

$$\omega_q^2 r^2/(1+m_1m_2A_0r^2)^{1/2} \quad (64)$$

was replaced by

$$r^2\omega_q^2\langle[1/(1+m_1m_2A_0r^2)^{1/2}]\rangle, \quad (65)$$

which was further approximated by

TABLE II. Heavy quarkonium masses in MeV with kernel as given in the text. The quark masses used were fitted to the state indicated by an asterisk.

Meson	NJLS	$M_{\rm expt}$	$M_{\rm calc}$	$M_{\rm calc} - M_{\rm expt}$
η_c	0000	2980	3048	+68
J/ψ	0101	3097*	3096	−1
χ_0	1011	3415	3461	+46
χ_1	1111	3511	3556	+45
χ_2	1211	3526	3700	+174
η_c	2000	3590	3542	−52
ψ	2101	3686	3569	−117
ψ	2121	3770	3779	+79
ψ	4101	4040	4022	−18
ψ	4121	4159	4174	+15
ψ	6101	4415	4445	+30
Υ	0101	9460*	9460	+0
^3P	1011	9860	9719	−141
^3P	1111	9892	9744	−148
^3P	1211	9913	9781	−132
Υ	2101	10023	9951	−72
$2\,^3P$	3011	10232	9960	−272
$2\,^3P$	3111	10255	10046	−209
$2\,^3P$	3211	10268	10072	−196
Υ	4101	10355	10268	−87
Υ	6101	10573	10598	+25

$$r^2\omega_q^2[1/(1+m_1m_2A_0\langle r^2\rangle)^{1/2}]. \quad (66)$$

(2) Using the properties of the relativistic harmonic oscillator defined above, obtained the following coupled equation which were solved iteratively:

$$(1+A_0m_1m_2r^2)^{1/2} \rightarrow [1+A_0m_1m_2(N+3/2)/\beta_N^2]^{1/2} = \sigma_n, \quad (67)$$

$$\gamma\beta_N^2(2\sigma_n)^{1/2} = \omega_0(2mM\alpha_s^2)^{1/2}, \quad (68)$$

$$F_{\rm HO} + 4m\omega_0^2\alpha_s^2 Q_N' = 2\beta_N^2\gamma^2(N+3/2). \quad (69)$$

This defined a running spring constant, differing for the different eigenstates, and computed self-consistently so that (to within the obvious approximations), the expectation value of the potential in an approximate eigenstate is the same as in the corresponding exact state. The Coulomb portion of the remaining OGE potentials was treated by direct diagonalization in a ten-dimensional subspace of the resulting basis. Following the diagonalization the remaining portions of the OGE potential were incorporated by first-order perturbation theory.

Solution of the qqq equations

Our procedure for the three-quark system is similar to our procedure for the $q\bar{q}$ system. We divide the effective two-body potential which we have obtained by our analytic procedures into a one gluon exchange component and the ana-

TABLE III. Light baryons in the nucleon channel: masses in MeV with kernel as given in the text. Quark mass used was fitted to the state indicated by an asterisk.

Baryon	$T(J^P)$	M_{expt}	M_{calc}	$M_{\mathrm{calc}} - M_{\mathrm{ext}}$
P_{11}	1/2(1/2+)	938*	943	+5
P_{11}	1/2(1/2+)	1440	1390	−50
D_{13}	1/2(3/2−)	1520	1535	+15
S_{11}	1/2(1/2−)	1535	1532	−3
S_{11}	1/2(1/2−)	1650	1777	−127
D_{15}	1/2(5/2−)	1675	1653	−22
F_{15}	1/2(5/2+)	1680	1679	−1
D_{13}	1/2(3/2−)	1700	1790	−90
P_{11}	1/2(1/2+)	1710	1674	−36
P_{13}	1/2(3/2+)	1720	1675	−45
G_{17}	1/2(7/2−)	2190	2201	+11
H_{19}	1/2(9/2+)	2220	2107	−113
G_{19}	1/2(9/2−)	2250	2123	−127
$I_{1,11}$	1/2(11/2−)	2600	2320	−280

TABLE IV. Light baryons in the delta channel: masses in MeV with kernel as given in the text.

Baryon	$T(J^P)$	M_{expt}	M_{calc}	$M_{\mathrm{calc}} - M_{\mathrm{expt}}$
P_{33}	3/2(3/2+)	1232	1198	−34
P_{33}	3/2(3/2+)	1600	1647	+47
S_{31}	3/2(1/2−)	1620	1629	+9
D_{33}	3/2(3/2−)	1700	1627	−73
S_{31}	3/2(1/2−)	1900	1930	+30
F_{35}	3/2(5/2+)	1905	1836	−69
P_{31}	3/2(1/2+)	1910	1870	−40
P_{33}	3/2(3/2+)	1920	1912	−8
D_{35}	3/2(5/2−)	1930	1916	−14
F_{37}	3/2(7/2+)	1950	1916	−34
$H_{3,11}$	3/2(11/2+)	2420	2349	+71

lytic confining component. We then take this confining component and map it onto a state-dependent but equivalent oscillator potential. We do this mapping by computing the expectation value of the confining two-body potential in a given state of two-body motion and then compute the same expectation value of the oscillator in that oscillator state. This provides us with a more convenient form of our confining potential for further calculations of the three-quark spectroscopy with the more complete Hamiltonian.

We make one further approximation for the solution of the three-quark system in Jacobi coordinates. In order to simplify further for these initial applications, we angle average the confining part of the interaction over the angle between the two Jacobi coordinates.

VIII. RESULTS AND CONCLUSIONS

We present in Table I the experimental and theoretical meson masses for the light mesons and the $s-\overline{s}$ mesons along with their quantum numbers. We have adjusted the light quark mass and the strange quark mass to fit the mass of the π and the ϕ, respectively. The resulting values are $m_u = m_d = 0.330$ GeV and $m_s = 0.516$ GeV, respectively. For the strong-coupling constant we have taken the conventional leading log form:

$$\alpha_s(Q^2) = \frac{12\pi}{(33-2N_f)\ln(Q^2/\lambda^2)}, \tag{70}$$

where N_f equals the number of flavors up to and including the mass of the highest quark mass involved in that state, and $\lambda = 0.25$ GeV. We have taken the same λ used in Ref. [8] and we have fixed Q^2 to the constituent quark masses via $Q^2 = (m_1 + m_2)^2$.

As may be expected, we obtain a π-ρ splitting which is too small compared to experiment, a general characteristic of constituent quark models. On the other hand, the remaining masses are in reasonable accord with the data and the overall description is competitive with results from more phenomenological approaches.

Table II presents our results for the heavy quarkonium states where the constituent quark masses have been adjusted to the J/ψ and the Υ mesons. The resulting quark masses are $m_c = 1.538$ GeV and $m_b = 4.830$ GeV.

Overall the agreement between theory and experiment for the heavy quarkonium states is good considering the only adjustable parameters are the heavy quark masses. The usual freedom attached to the coupling parameters is absent in this calculation as they have been fixed by the theory as described above. Clearly, one can the improve the fit significantly in Table II by a minimization of an overall chisquare or rms deviation with respect to the constituent quark masses. We have not attempted such fits at the present time but will present results of various fitting strategies along with certain improvements to our approach in a future effort.

In Table III, we present the results for the light baryons in

TABLE V. Flavored quarkonium masses in MeV with kernel as given in the text.

Meson	NJLS	M_{expt}	M_{calc}	$M_{\mathrm{calc}} - M_{\mathrm{expt}}$
$K(498)$	0000	98	616	+118
$K^*(892)$	0101	892	881	−11
$K_1(1270)$	1011	1270	1239	−31
$K_1(1400)$	1111	1400	1369	−31
$K^*(1410)$	1211	1410	1562	+152
$K_2(1770)$	2220	1770	1986	+216
$K_3^*(1780)$	2321	1780	2006	+226
$K_4^*(2045)$	2431	2045	2432	+387
$D(1869)$	0000	1869	1717	−152
$D^*(2010)$	0101	2010	1730	−280
$F(1968)$	0000	1968	1946	−22
$F^*(2112)$	0101	2112	1975	−137
$B(5279)$	0000	5279	5194	−85

the nucleon channel. Here, we have again adjusted the light quark mass to fit the mass of the lightest baryon—the proton with the result $m_u = m_d = 0.380$ GeV. The typical deviation between the theoretical and experimental masses is of the order of 50 MeV until the higher spin states are reached above 2 GeV. We note that the theoretical masses tend to be systematically lower than the experimental values. This implies that adjusting the quark mass to obtain a fit to the full spectrum will improve the overall results considerably.

Nevertheless, even at this stage, the overall results for the light baryons in the nucleon channel are very encouraging and are also competitive with results from more phenomenological approaches.

In Table IV we present our results for the light baryons in the 3-3 resonance or delta channel. No additional parameter adjustments have been made. The comparison between theory and experiment is very good, even better than the results for the nucleon channel (Table III).

Finally, in Table V, we present results for the mixed flavor meson states. Again, there are no additional parameters adjusted to produce the results of this table. Comparing the results of Table I and Table V for the meson states depending on the strange quark mass, one estimates that lowering the strange quark mass will provide an improved overall description of these meson states.

Overall, the results produced with our approach appear to give a good representation of the experimental data. Several improvements are envisioned and will be reported in a future work. Applications to additional experimental observables are enabled and will be addressed as well.

ACKNOWLEDGMENTS

This work was performed with partial support by the U.S. Department of Energy under Grant No. DE-FG02-87ER-40371, Division of High Energy and Nuclear Physics. Some of the calculations were performed with a grant of supercomputer time at the National Energy Research Supercomputer Center.

[1] I. Tamm, J. Phys. (USSR) **9**, 449 (1945); S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[2] J.R. Spence and J.P. Vary, Phys. Rev. D **35**, 2191 (1987).
[3] J.R. Spence and J.P. Vary, Phys. Rev. C **47**, 1282 (1993).
[4] J.R. Spence and J.P. Vary, Phys. Rev. C **52**, 1668 (1995).
[5] H.A. Bethe and E.E. Salpeter, Phys. Rev. **82**, 309 (1951); **84**, 1232 (1951).
[6] E.E. Salpeter, Phys. Rev. **87**, 328 (1952).
[7] T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, Chur, Switzerland, 1981).
[8] Anuradha Mittal and A.N. Mitra, Phys. Rev. Lett. **57**, 290 (1986).
[9] A.N. Mitra and I. Santhanam, Z. Phys. C **8**, 33 (1981).
[10] D.R. Hartree, Proc. Cambridge Philos. Soc. **24**, 111 (1928).
[11] A.L. Fetter and J.D Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971).
[12] Charles B. Thorn, Phys. Rev. D **20**, 1435 (1979).
[13] Masa-aki Sato, Phys. Rev. D **29**, 2952 (1984).
[14] J. Finger, D. Horn, and J.E. Mandula, Phys. Rev. D **20**, 3253 (1979).
[15] Joseph J. Kubis, Phys. Rev. D **6**, 547 (1972).
[16] K. Erkelenz, Phys. Rep., Phys. Lett. **13C**, 191 (1974).
[17] R. Machleidt, K. Holinde, and Ch. Elster, Phys. Rep. **179**, 1 (1987).
[18] H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957).