

## Finite quark mass effects in the improved ladder Bethe-Salpeter amplitudes

K. Naito,\* K. Yoshida, Y. Nemoto, and M. Oka

*Department of Physics, Tokyo Institute of Technology, Meguro, Tokyo 152-8551, Japan*

M. Takizawa†

*Laboratory of Computer Sciences, Showa College of Pharmaceutical Sciences, Machida, Tokyo 194-8543, Japan*

(Received 6 November 1998)

We study the finite quark mass effects of the low-energy QCD using the improved ladder Schwinger-Dyson and Bethe-Salpeter equations which are derived in the manner consistent with the vector and axial-vector Ward-Takahashi identities. The nonperturbative mass-independent renormalization allows us to calculate the quark condensate for a nonzero quark mass. We explicitly show that the partial conservation of axial-vector current relation holds. The key ingredients are the Cornwall-Jackiw-Tomboulis effective action, the generalized Noether current, and the introduction of the regularization function to the Lagrangian. Reasonable values of the pion mass, the pion decay constant, and the quark condensate are obtained with a rather large  $\Lambda_{\text{QCD}}$ . The pion mass square and the pion decay constant are almost proportional to the current quark mass up to the strange quark mass region. It suggests that the chiral perturbation is applicable up to the strange quark mass region. We study the validity of the approximation often used in solving the Bethe-Salpeter equations too. [S0556-2813(99)03703-6]

PACS number(s): 11.10.St, 12.38.-t, 14.65.Bt

### I. INTRODUCTION

A program to derive the observed properties of hadrons nonperturbatively in QCD has been pursued with great intensity but not accomplished yet. The concept of chiral symmetry and its spontaneous breakdown are among the most important aspects of low-energy hadron physics. The spontaneous breakdown of chiral symmetry is believed to be responsible for a large part of the low-lying hadron masses as well as for the emergence of octet pseudoscalar mesons such as Goldstone bosons. In order to explain the observed hadron spectrum, one also needs small, explicitly chiral-symmetry-breaking terms, namely, flavor-dependent current quark mass terms.

The Cornwall-Jackiw-Tomboulis (CJT) effective action approach [1] for composite operators is widely used to study the dynamical symmetry-breaking phenomena in the quantum field theories. The extremum condition for the effective action with respect to the quark propagator leads to the Schwinger-Dyson (SD) equation for the quark propagator on the nonperturbative vacuum and the second variational derivative of the effective action with respect to the quark propagator leads to the Bethe-Salpeter (BS) equation describing the bound states. The advantage of the present approach is that the derived SD and BS equations are consistent with the symmetry of the effective action evaluated in a certain approximation scheme.

The QCD SD equation for the quark propagator has been studied in the improved ladder approximation (ILA) by Higashijima [2] and Miransky [3]. They took the ladder diagrams of one-gluon exchange between  $q$  and  $\bar{q}$  and assumed that the coupling constant is modified according to standard perturbative corrections. It has been shown that the

asymptotic behavior of the solution is consistent with the leading order renormalization group analysis while the infrared gluon exchange breaks chiral symmetry dynamically. Aoki *et al.* solved the BS equation for the  $J^{PC} = 0^{-+} q\bar{q}$  state and confirmed the existence of the Nambu-Goldstone pion in this approximation [4]. The numerical predictions of the pion decay constant  $f_{\pi}$  and the quark condensate  $\langle \bar{\psi}\psi \rangle$  are rather good. It was also shown that the BS amplitude shows the correct asymptotic behavior as predicted by the operator product expansion (OPE) in QCD [5]. The masses and decay constants for the lowest-lying scalar, vector, and axial-vector mesons have been evaluated by calculating the two-point correlation functions for the composite operators  $\bar{\psi}M\psi$ . The obtained values are in good agreement with the observed ones [6].

So far, the current quark mass term has not been introduced in the studies of the BS amplitudes in the ILA. The purpose of the present paper is to investigate the effects of the finite current quark masses on the BS amplitudes for the  $J^{PC} = 0^{-+}$  states. As shown in [2], the asymptotic behavior of the solution of the SD equation for the quark propagator with the finite current quark mass is rather different from that in the chiral limit. Therefore, it is important to study the effects of the finite current quark masses not only on the SD equations but also on the BS amplitudes.

There have been many studies of the pion BS amplitude using the effective models of QCD and/or the approximation schemes of the QCD [7,8]. The advantages of the ILA model are as follows. (i) The model is given in the Lagrangian form so that one is able to apply the CJT effective action formulation and study symmetry properties of the system. (ii) The asymptotic behavior of the solutions of the SD and BS equations is consistent with the renormalization group analysis of QCD. (iii) It has been shown that the ILA model corresponds to the local potential approximation with the ladder part in

\*Electronic address: kenichi@th.phys.titech.ac.jp

†Electronic address: takizawa@ac.shoyaku.ac.jp

the nonperturbative renormalization group approach [9]. (iv) The angular integration in the SD equation can be performed analytically. On the other hand, the disadvantages of the ILA model are as follows. (i) The axial-vector Ward-Takahashi identity is violated [10]. (ii) The quark may not be confined in the color singlet state.

In the finite quark mass case, it has been known that there is a difficulty in defining the quark condensate in the studies of the QCD SD equation. The extraction of the perturbative quark mass contribution in the UV region is not sufficient to remove the UV divergence in the SD equation since the SD equation includes the nonperturbative contribution even in the UV region. Inspired by the idea of Kusaka *et al.* of the nonperturbative renormalization of the fermion mass term in the mass-independent renormalization scheme [11], we propose a novel way to renormalize the SD equation and define the quark condensate with the finite quark mass.

The paper is organized as follows. In Sec. II we explain the ILA model Lagrangian we have used in the present study. In Sec. III the SD equation is derived from the CJT action and the renormalization of the SD equation is discussed in Sec. IV. In Sec. V our method of solving the BS equation for the pseudoscalar meson is presented. In Sec. VI the formulation for the meson decay constant is given and the low-energy theorem is discussed. Section VII is devoted to the numerical results. Finally, summary and concluding remarks are given in Sec. VIII.

## II. IMPROVED LADDER MODEL OF QCD

We work with the following Lagrangian density of the improved ladder approximation model of QCD proposed by Aoki *et al.* [4,6]:

$$\mathcal{L}[\psi, \bar{\psi}] := \mathcal{L}_{\text{free}}[\psi, \bar{\psi}] + \mathcal{L}_{\text{int}}[\psi, \bar{\psi}], \quad (1)$$

$$\mathcal{L}_{\text{free}}[\psi, \bar{\psi}] := \bar{\psi} f(\partial^2)(i \not{\partial} - m_0) \psi. \quad (2)$$

Here the function  $f(\zeta)$  of  $\zeta = \partial^2$  is introduced to provide a cutoff regularization of the ultraviolet divergences of the quark loops. The reason we introduce the cutoff function at the Lagrangian level is to preserve the consistency between the SD and BS equations. If one uses the regularization that is inconsistent between the SD and BS equations, the low-energy relation based on the chiral symmetry should be vio-

lated by the regularization. The function  $f(\zeta)$  should satisfy  $f(\zeta=0)=1$  and  $f(\zeta) \rightarrow \infty$  for  $\zeta \gg \Lambda_{\text{UV}}^2$ . In this paper, we employ the sharp cutoff function

$$f(\zeta) = 1 + M \theta(\zeta - \Lambda_{\text{UV}}^2), \quad M \rightarrow \infty. \quad (3)$$

We introduce the bare mass of quarks,  $m_0$ , which is evaluated at  $\Lambda_{\text{UV}}$  [4,12]. In general  $m_0$  is a diagonal flavor matrix, i.e.,  $m_0 = \text{diag}(m_u, m_d, m_s)$  for  $N_f=3$ . In this paper we deal only with a flavor-independent mass and therefore the case with  $\text{SU}(3)_F$  symmetry.

The interaction term is given by

$$\begin{aligned} \mathcal{L}_{\text{int}}[\psi, \bar{\psi}](x) := & -\frac{1}{2} \int_{pp'qq'} \mathcal{K}^{mm',nn'}(p,p';q,q') \\ & \times \bar{\psi}_m(p) \psi_{m'}(p') \bar{\psi}_n(q) \psi_{n'}(q') \\ & \times e^{-i(p+p'+q+q')x}, \end{aligned} \quad (4)$$

$$\begin{aligned} \mathcal{K}^{mm',nn'}(p,p';q,q') = & \bar{g}^2 \left( \left( \frac{p_E - q'_E}{2} \right)^2, \left( \frac{q_E - p'_E}{2} \right)^2 \right) \\ & \times i D^{\mu\nu} \left( \frac{p+p'}{2} - \frac{q+q'}{2} \right) \\ & \times (\gamma_\mu T^a)^{mm'} (\gamma_\nu T^a)^{nn'}, \end{aligned} \quad (5)$$

where  $\int_p$  denotes  $\int d^4 p / (2\pi)^4$  and  $p_E$  represents the Euclidean momentum. The Fourier transformations of the fields are defined by  $\bar{\psi}(p) = \int d^4 x e^{ipx} \bar{\psi}(x)$  and  $\psi(p) = \int d^4 x e^{ipx} \psi(x)$ . The indices  $m, n, \dots$  are combined indices  $m := (a, i, f)$ ,  $n := (b, j, g), \dots$  with Dirac indices  $a, b, \dots$ , color indices  $i, j, \dots$ , and flavor indices  $f, g, \dots$ .  $T^a$  denotes the generator of the color  $\text{SU}(N_c)$ . According to Higashijima [2] and Miransky [3], we choose a particular set of the momenta that determines the running coupling constant, i.e.,

$$\bar{g}^2(p_E^2, q_E^2) = \theta(p_E^2 - q_E^2) g^2(p_E^2) + \theta(q_E^2 - p_E^2) g^2(q_E^2). \quad (6)$$

This way of introducing the running coupling constant is very natural from the nonperturbative renormalization group approach with a local potential approximation [9]. It is often called the Higashijima-Miransky approximation. The infrared cutoff  $t_{\text{IF}}$  is introduced in the running coupling constant as

$$g^2(p_E^2) := \begin{cases} \frac{1}{\beta_0} \frac{1}{1+t} & \text{for } t_{\text{IF}} \leq t, \\ \frac{1}{2\beta_0} \frac{1}{(1+t_{\text{IF}})^2} \left[ 3t_{\text{IF}} - t_0 + 2 - \frac{(t-t_0)^2}{t_{\text{IF}} - t_0} \right] & \text{for } t_0 \leq t \leq t_{\text{IF}}, \\ \frac{1}{2\beta_0} \frac{3t_{\text{IF}} - t_0 + 2}{(1+t_{\text{IF}})^2} & \text{for } t \leq t_0, \end{cases} \quad (7)$$

$$t := \ln \frac{p_E^2}{\Lambda_{\text{QCD}}^2} - 1, \quad (8)$$

$$\beta_0 := \frac{1}{(4\pi)^2} \frac{11N_C - 2N_f}{3}. \quad (9)$$

Above  $t_{\text{IF}}$ ,  $g^2(p_E^2)$  develops according to the one-loop result of the QCD renormalization group equation and, below  $t_0$ ,  $g^2(p_E^2)$  is kept constant. These two regions are connected by the quadratic polynomial so that  $g^2(p_E^2)$  becomes a smooth function. Here  $N_C$  is the number of colors and  $N_f$  is the number of active flavors. We use  $N_C = N_f = 3$  in our numerical studies. The gluon propagator is given in the Landau gauge

$$iD^{\mu\nu}(k) = \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \frac{-1}{k^2}. \quad (10)$$

### III. SD EQUATION

In order to derive the Schwinger-Dyson equation, we use the formalism of the Cornwall-Jackiw-Tomboulis effective action [1] which is given by

$$\Gamma[S_F] := i\text{TrLn}[S_F] - i\text{Tr}[S_0^{-1}S_F] + \Gamma_{\text{loop}}[S_F]. \quad (11)$$

The last term of Eq. (11) is the residual term. Multiplying by a factor  $i$ ,  $i\Gamma_{\text{loop}}[S_F]$  is given by the sum of all Feynman amplitudes of two-loop or higher-loop two-particle irreducible vacuum diagrams in which every bare quark propagator

$$S_0(x, y) = \int_q e^{-iq(x-y)} \frac{1}{f(-q^2)} \frac{i}{\not{q} - m_0} \quad (12)$$

is replaced by a full one

$$S_F(x, y) = \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle. \quad (13)$$

The SD equation is the stability condition of the CJT action

$$\frac{\delta\Gamma[S_F]}{\delta S_{Fmn}(x, y)} = 0. \quad (14)$$

Throughout this paper, we employ the lowest-order (lowest-loop) expansion of the  $\Gamma_{\text{loop}}[S_F]$  as

$$\begin{aligned} \Gamma_{\text{loop}}[S_F] = & -\frac{1}{2} \int d^4x \mathcal{K}^{m_1 m_2, n_1 n_2} (i\partial_{x_1}, i\partial_{x_2}; i\partial_{y_1}, i\partial_{y_2}) \\ & \times [S_{Fm_2 m_1}(x_2, x_1) S_{Fn_2 n_1}(y_2, y_1) \\ & - S_{Fm_2 n_1}(x_2, y_1) S_{Fn_2 m_1}(y_2, x_1)] |_* , \end{aligned} \quad (15)$$

where the asterisk means taking  $x_1, x_2, y_1, y_2 \rightarrow x$  after all the derivatives are operated. This leads to the ILA model, where the SD equation is given in momentum space by

$$\begin{aligned} iS_F^{-1}(q) - iS_0^{-1}(q) + C_F \int_p \bar{g}^2(q_E^2, p_E^2) \\ \times iD^{\mu\nu}(p-q) \gamma_\mu S_F(p) \gamma_\nu = 0, \end{aligned} \quad (16)$$

with

$$C_F = \frac{\text{tr}[T^a T^a]}{N_C} = \frac{N_C^2 - 1}{2N_C}. \quad (17)$$

Now we introduce the regularized propagator as

$$S_0^R(q) := f(-q^2) S_0(q) = \frac{i}{\not{q} - m_0}, \quad (18)$$

$$S_F^R(q) := f(-q^2) S_F(q). \quad (19)$$

Then the SD equation (16) becomes

$$\begin{aligned} iS_F^{R-1}(q) - iS_0^{R-1}(q) + \frac{C_F}{f(-q^2)} \int_p \frac{1}{f(-p^2)} \bar{g}^2(q_E^2, p_E^2) \\ \times iD^{\mu\nu}(p-q) \gamma_\mu S_F^R(p) \gamma_\nu = 0, \end{aligned} \quad (20)$$

in which one finds that the integral is cut off at  $p_E^2 = -p^2 = \Lambda_{\text{UV}}^2$  due to the function  $f(-p^2)$ . Substituting the general form of the SD solution

$$S_F^R(q) = \frac{i}{A(q^2)\not{q} - B(q^2)}, \quad (21)$$

we obtain a set of integral equations

$$A(q^2) = 1 + \frac{iC_F}{q^2 f(-q^2)} \int_p \frac{\bar{g}^2(q_E^2, p_E^2)}{f(-p^2)} \frac{3(p^2 + q^2)(pq) - 4(pq)^2 - 2p^2 q^2}{(q-p)^4} \frac{A(p^2)}{p^2 A^2(p^2) - B^2(p^2)}, \quad (22)$$

$$B(q^2) = m_0 + \frac{iC_F}{f(-q^2)} \int_p \frac{\bar{g}^2(q_E^2, p_E^2)}{f(-p^2)} \frac{1}{(p-q)^2} \frac{-3B(p^2)}{p^2 A^2(p^2) - B^2(p^2)}. \quad (23)$$

After a Wick rotation, we obtain

$$A(-q_E^2) \equiv 1 \quad (24)$$

from Eq. (22). This is another advantage of the Higashijima-Miransky approximation, where the running coupling constant is defined so as to make the wave function renormalization  $Z_2$  unity [4,10]. Then we find an integral equation for  $B(-q_E^2)$  as

$$B(-q_E^2) = m_0 + \frac{3C_F}{16\pi^2} \int_0^{\Lambda_{UV}^2} dp_E^2 g^2(q_E^2, p_E^2) \times \frac{p_E^2}{\max\{q_E^2, p_E^2\}} \frac{B(-p_E^2)}{p_E^2 + B^2(-p_E^2)}. \quad (25)$$

#### IV. RENORMALIZATION OF THE QUARK MASS

In this section we discuss the renormalization of the quark mass. The operator product expansion analysis shows that in the asymptotic region the QCD quark mass function  $B(-q_E)$  for three quark flavors behaves as follows [13]:

$$B(-q_E^2) = m_R(\mu^2) \left[ \frac{g^2(q_E^2)}{g^2(\mu^2)} \right]^{4/9} - \xi_R(\mu^2) \frac{g^2(q_E^2)}{3q_E^2} \left[ \frac{g^2(q_E^2)}{g^2(\mu^2)} \right]^{-4/9}, \quad (26)$$

where  $m_R(\mu^2)$  is the current quark mass renormalized at  $\mu^2$  and  $\xi_R(\mu^2) := \langle \bar{\psi}\psi \rangle_R$  is the quark condensate renormalized at  $\mu^2$ . The improved ladder model of QCD is the model which is constructed so as to reproduce the QCD asymptotic behavior. Therefore we introduce the renormalization condition of the quark mass so that the solution of the SD equation can be interpreted as the QCD quark mass function in the asymptotic region.

The quark mass function calculated in the effective model of QCD can be expressed in a similar fashion as Eq. (26):

$$B(-q_E^2) = m_R(\mu^2) F(q_E^2, \mu^2) - \xi_R(\mu^2) G(q_E^2, \mu^2). \quad (27)$$

Then we introduce the renormalization condition

$$F(\mu^2, \mu^2) = 1, \quad (28)$$

which is equivalent to

$$\left. \frac{\partial B(-\mu^2)}{\partial m_R(\mu^2)} \right|_{m_R(\mu^2)=0} = 1. \quad (29)$$

This mass-independent renormalization condition for the SD equation was first proposed by Kusaka *et al.* [11]. The mass renormalization constant  $Z_m = Z_m(\Lambda_{UV}^2, \mu^2)$  is introduced by

$$m_0(\Lambda_{UV}^2) = Z_m^{-1} m_R(\mu^2). \quad (30)$$

It should be noted here that the renormalization constant is independent of mass in this renormalization scheme. It will

be explicitly shown later in this section. By substituting  $m_0$  by Eq. (30), the SD equation (25) becomes

$$B(-q_E^2) = Z_m^{-1} m_R(\mu^2) + \int_0^{\Lambda_{UV}^2} dp_E^2 K(q_E^2, p_E^2) \frac{B(-p_E^2)}{p_E^2 + B^2(-p_E^2)}, \quad (31)$$

with

$$K(q_E^2, p_E^2) := \frac{3C_F}{16\pi^2} g^2(q_E^2, p_E^2) \frac{p_E^2}{\max\{q_E^2, p_E^2\}}. \quad (32)$$

By differentiating this equation with respect to  $m_R(\mu^2)$  and taking  $m_R(\mu^2) = 0$ , one obtains

$$\begin{aligned} & \left. \frac{\partial B(-q_E^2)}{\partial m_R(\mu^2)} \right|_{m_R(\mu^2)=0} \\ &= Z_m^{-1} + \int_0^{\Lambda_{UV}^2} dp_E^2 K(q_E^2, p_E^2) \\ & \quad \times \frac{p_E^2 - B_0^2(-p_E^2)}{[p_E^2 + B_0^2(-p_E^2)]^2} \left. \frac{\partial B(-p_E^2)}{\partial m_R(\mu^2)} \right|_{m_R(\mu^2)=0}. \end{aligned} \quad (33)$$

The renormalization condition (29) leads to

$$\begin{aligned} Z_m^{-1} &= 1 - \int_0^{\Lambda_{UV}^2} dp_E^2 K(\mu^2, p_E^2) \\ & \quad \times \frac{p_E^2 - B_0^2(-p_E^2)}{[p_E^2 + B_0^2(-p_E^2)]^2} \left. \frac{\partial B(-p_E^2)}{\partial m_R(\mu^2)} \right|_{m_R(\mu^2)=0}. \end{aligned} \quad (34)$$

Here  $B_0(-q_E^2)$  is the solution of the SD equation in the chiral limit, namely,

$$\begin{aligned} B_0(-q_E^2) &= \frac{3C_F}{16\pi^2} \int_0^{\Lambda_{UV}^2} dp_E^2 g^2(q_E^2, p_E^2) \\ & \quad \times \frac{p_E^2}{\max\{q_E^2, p_E^2\}} \frac{B_0(-p_E^2)}{p_E^2 + B_0^2(-p_E^2)}. \end{aligned} \quad (35)$$

Now the combination of Eqs. (33) and (34) yields an integral equation for  $\partial B(-q_E^2)/\partial m_R(\mu^2)|_{m_R(\mu^2)=0}$ . Equation (34) explicitly shows that the mass renormalization constant  $Z_m$  does not depend on the quark mass.

In order to obtain the quark mass function with the renormalized current quark mass, one first calculates  $B_0(-q_E^2)$  by solving the SD equation in the chiral limit, Eq. (35). Next  $Z_m$  and  $\partial B(-q_E^2)/\partial m_R(\mu^2)|_{m_R(\mu^2)=0}$  are obtained by solving Eqs. (33) and Eq. (34). Finally Eq. (31) is solved to find  $B(-q_E^2)$ .

Let us now propose the following definition of the quark condensate:

$$\xi_R(\mu^2) := Z_m^{-1} \xi_0(\Lambda_{UV}^2), \quad (36)$$

$$\xi_0(\Lambda_{UV}^2) := - \left( \int^{\Lambda_{UV}} \frac{d^4 q}{(2\pi)^4} \text{tr}[S_F^R(q)] - \int^{\Lambda_{UV}} \frac{d^4 q}{(2\pi)^4} \text{tr}[S_F^R(q)_{\text{pert}}] \right), \quad (37)$$

$$S_F^R(q) := \frac{i}{\not{q} - B(q^2)}, \quad (38)$$

$$S_F^R(q)_{\text{pert}} := \left. \frac{\partial S_F^R(q)}{\partial m_R(\mu^2)} \right|_{m_R(\mu^2)=0} m_R(\mu^2). \quad (39)$$

In order to avoid the divergence originated by the perturbative quark mass contribution to the quark condensate in the UV region, the perturbative quark mass contribution should be subtracted. The key point of our definition of the quark condensate is Eq. (39); namely, the perturbative quark mass contribution is defined using the fully calculated quark mass function  $B(q^2)$ . The subtraction of the perturbative quark mass contribution obtained by the operator product expansion approach in the UV region is not sufficient. Our definition of the quark condensate has the desirable property:

$$m_0(\Lambda_{UV}^2) \xi_0(\Lambda_{UV}^2) = m_R(\mu^2) \xi_R(\mu^2). \quad (40)$$

## V. BS EQUATION FOR PSEUDOSCALAR MESONS

The homogeneous BS equation is given by

$$\frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(x, y) \delta S_{F'n'm'}(y', x')} \chi_{n'm'}(y', x'; P_B) = 0. \quad (41)$$

Here the BS amplitude is defined by

$$\chi_{nm}(y, x; P_B) := \langle 0 | T \psi_n(y) \bar{\psi}_m(x) | P_B \rangle \quad (42)$$

for a  $q\bar{q}$  state  $|P_B\rangle$ . The normalization condition is  $\langle P_B | P'_B \rangle = (2\pi)^3 2P_{B0} \delta^3(P_B - P'_B)$  and  $P_B := (\sqrt{M_B^2 + \mathbf{P}_B^2}, \mathbf{P}_B)$  is the on-shell momentum. Equation (41) is expressed in momentum space as

$$\begin{aligned} & S_F^{-1}(q_+) \chi(q; P_B) S_F^{-1}(q_-) \\ &= -iC_F \int_k \bar{g}^2(q_E^2, k_E^2) iD^{\mu\nu}(q-k) \gamma_\mu \chi(k; P_B) \gamma_\nu, \end{aligned} \quad (43)$$

with

$$q_+ = q + \frac{P_B}{2}, \quad q_- = q - \frac{P_B}{2}, \quad (44)$$

where the Fourier transformation of the BS amplitude is defined by

$$\chi_{nm}(y, x; P_B) = e^{-iP_B X} \int_q e^{-iq(y-x)} \chi_{nm}(q; P_B), \quad X = \frac{y+x}{2}. \quad (45)$$

We introduce the regularized BS amplitude by

$$\chi_{nm}^R(q; P) := f(-q^2) \chi_{nm}(q; P) f(-q^2), \quad (46)$$

then Eq. (43) is rewritten as

$$\begin{aligned} & S_F^{R-1}(q_+) \chi^R(q; P_B) S_F^{R-1}(q_-) \\ &= -iC_F \int_k \frac{1}{f(-k_+^2) f(-k_-^2)} \\ &\quad \times \bar{g}^2(q_E^2, k_E^2) iD^{\mu\nu}(q-k) \gamma_\mu \chi^R(k; P_B) \gamma_\nu. \end{aligned} \quad (47)$$

We see again that the integral equation is regularized correctly.

The BS amplitude for the pseudoscalar meson can be written in terms of four scalar amplitudes,

$$\begin{aligned} \chi_{nm}^R(k; P) = \delta_{ji} \frac{(\lambda^\alpha)_{gf}}{2} & \left[ \left( \phi_S(k; P) + \phi_P(k; P) \not{k} + \phi_Q(k; P) \not{P} \right. \right. \\ & \left. \left. + \frac{1}{2} \phi_T(k; P) (\not{P} \not{k} - \not{k} \not{P}) \right) \gamma_5 \right]_{ba}, \end{aligned} \quad (48)$$

where  $\lambda^\alpha$  denotes the flavor matrix. Substituting this into Eq. (47), we obtain coupled integral equations for four scalar amplitudes. The explicit form is rather complicated and given in the Appendix. The integral equations can be written down formally as

$$\phi_{\mathcal{A}}(q; P_B) = \int_k \mathcal{M}_{\mathcal{A}\mathcal{B}}(q, k; P_B) \phi_{\mathcal{B}}(k; P_B), \quad (49)$$

where  $\mathcal{A}, \mathcal{B}$  denotes  $S, P, Q, T$ . Among the four dimensional integrations  $d^4 k$ , two of the integrations can be performed analytically after the Wick rotation and we set the total momentum  $P_{BE} = (M_E, 0, 0, 0)$ . Then we obtain

$$\begin{aligned} \phi_{\mathcal{A}}(q_R, q_\theta; M_E) = \int_{(k_R, k_\theta) \in I} & dk_R dk_\theta \mathcal{M}_{\mathcal{A}\mathcal{B}}(q_R, q_\theta; k_R, k_\theta; M_E) \\ & \times \phi_{\mathcal{B}}(k_R, k_\theta; M_E), \end{aligned} \quad (50)$$

where

$$k_E^2 = k_R^2, \quad k_E P_E = k_R M_E \sin k_\theta, \quad (51)$$

and the integral region is given by

$$I := \left\{ (k_R, k_\theta) \left| k_R^2 \pm k_R M_E \sin k_\theta + \frac{M_E^2}{4} \leq \Lambda_{UV}^2 \right. \right\}. \quad (52)$$

This integral region is determined uniquely by the cutoff function in Eq. (47) and is consistent with the SD equation (25).

There is no solution of Eq. (50) for a real  $M_E$  because  $M_E^2$  is a Euclidean meson mass whose square is negative  $M_E^2 = -M_B^2$ . Since the SD equation can be solved only for the spacelike region  $q_E^2 \geq 0$ , the region  $M_E^2 < 0$  is not accessible. Instead of solving Eq. (50), we convert it into an eigenvalue equation for a fixed  $M_E^2 \geq 0$ , given by

$$\lambda \phi_A(q_R, q_\theta; M_E) = \int dk_R dk_\theta \mathcal{M}_{AB}(q_R, q_\theta; k_R, k_\theta; M_E) \times \phi_B(k_R, k_\theta; M_E), \quad (53)$$

where  $\lambda$  is an eigenvalue that is equal to unity for solutions of Eq. (50). We solve Eq. (53) numerically using the iteration procedure. When we iterate Eq. (53), the eigenfunction associated with the maximum absolute eigenvalue dominates. Then we obtain the maximum eigenvalue and its eigenfunction.

In the numerical calculations, we use the discretization of the continuous variable  $(q_R, q_\theta)$  and come across a problem that the kernel  $K_{AB}$  diverges at the point  $(q_R, q_\theta) = (k_R, k_\theta)$ . This divergence, which originates from the gluon propagator, does not cause a real divergence. We may remove this divergence by carefully choosing the discretization points in the iteration procedure.

Once we obtain (the largest absolute)  $\lambda$  as a function of  $M_E^2 \geq 0$ , then we extrapolate  $\lambda(M_E^2)$  to the timelike region  $M_E^2 < 0$  and look for the on-shell point where  $\lambda(-M_B^2) = 1$ .

Since this extrapolation is the most ambiguous procedure in our calculation, we will later consider another function which is similarly extrapolated to the on-shell value and compare the predicted values  $M_B^2$  obtained in the two independent extrapolations.

## VI. DECAY CONSTANT AND LOW-ENERGY RELATION

To obtain the decay constant, we need the normalization of the BS amplitude. The normalization condition of the BS amplitude is derived from the inhomogeneous BS equation

$$\frac{1}{i} \frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(x, y) \delta S_{Fn'm'}(y', x')} G_{C;n'm'm''n''}^{(2)}(y' x'; x'' y'') = \delta_{m''m} \delta_{n''n} \delta(x'' - x) \delta(y - y''), \quad (54)$$

where

$$G_{C;nmm'n'}^{(2)}(yx; x'y') := \langle 0 | T \psi_n(y) \bar{\psi}_m(x) \psi_{m'}(x') \bar{\psi}_{n'}(y') | 0 \rangle - \langle 0 | T \psi_n(y) \bar{\psi}_m(x) | 0 \rangle \langle 0 | T \psi_{m'}(x') \bar{\psi}_{n'}(y') | 0 \rangle. \quad (55)$$

In momentum space, Eq. (54) gives

$$i \int_q \frac{1}{f(-q_+^2) f(-q_-^2)} \chi_{n_1 m_1}^R(q; P_B) \bar{\chi}_{m_2 n_2}^R(q; P_B) \frac{\partial}{\partial P^\mu} [S_{F n_2 n_1}^{R-1}(q_+) S_{F m_1 m_2}^{R-1}(q_-)] + i \int_q \frac{-(q_+)_\mu f'(-q_+^2) f(-q_-^2) - (q_-)_\mu f'(-q_-^2) f(-q_+^2)}{f^2(-q_+^2) f^2(-q_-^2)} \times \chi_{n_1 m_1}^R(q; P_B) S_{F m_1 m_2}^{R-1}(q_-) \bar{\chi}_{m_2 n_2}^R(q; P_B) S_{F n_2 n_1}^{R-1}(q_+) = -2P_\mu, \quad P \rightarrow P_B. \quad (56)$$

In the case of the sharp cutoff function (3), the second term on the left-hand side (LHS) of Eq. (56) does not contribute and the integral region in the first term is determined uniquely.

Let us now turn to a discussion of the axial-vector Ward-Takahashi (WT) identity. It has been found that the axial-vector WT identity is violated in the Higashijima-Miransky (HM) approximation [14]. Of course the Goldstone theorem holds in this case because the HM approximation respects global chiral symmetry. The chiral WT identity in the ladder approximation has been carefully studied in [10]. The reason for the violation of the axial-vector WT identity is that the HM approximation breaks local chiral symmetry. As shown in [10], the improved ladder approximation of the SD and BS equations preserves the WT identity for the axial-vector vertex if and only if one uses the gluon momentum square as the argument of the running coupling constant. However, in this

case the renormalization factor  $Z_2$  of the quark wave function deviates from unity in the Landau gauge. In order to avoid such problems, the authors of [10] have introduced a nonlocal gauge so that the gauge parameter in the gluon propagator becomes a momentum-dependent function. On the other hand, we have proposed another way to recover the axial-vector WT identity [15]. In this approach the axial-vector current is modified so as to become the correct Noether current of the effective model of QCD. The advantage of this approach is that it is applicable to all the effective models of QCD which respect global chiral symmetry. According to [15], the meson decay constant can be expressed as follows:<sup>1</sup>

<sup>1</sup> $f_\pi$  in this paper corresponds to  $\tilde{f}_\pi$  in [15].

$$f_B = \lim_{P \rightarrow P_B} \frac{1}{P^2} \int_q \text{tr} \left[ \bar{\chi}^R(q; P_B) \left\{ i \gamma_5 \frac{\lambda^\alpha}{2} \left( \frac{f(-q_-^2) + f(-q_+^2)}{2} \right) \not{P} + [f(-q_+^2) - f(-q_-^2)] \not{q} + E^\alpha(q; P) \right\} \right], \quad (57)$$

$$E_{mn}^\alpha(q; P) := \int_k \left[ \left\{ \mathcal{K}^{n'n, mm'} \left( -k, q - \frac{P}{2}; -q - \frac{P}{2}, k + P \right) - \mathcal{K}^{n'n, mm'} \left( -k, q - \frac{P}{2}; -q + \frac{P}{2}, k \right) \right\} \times \left( i \gamma_5 \frac{\lambda^\alpha}{2} S_F(k) \right)_{m'n'} + \left\{ \mathcal{K}^{n'n, mm'} \left( -k + P, q - \frac{P}{2}; -q - \frac{P}{2}, k \right) - \mathcal{K}^{n'n, mm'} \left( -k, q + \frac{P}{2}; -q - \frac{P}{2}, k \right) \right\} \times \left( S_F(k) i \gamma_5 \frac{\lambda^\alpha}{2} \right)_{m'n'} \right]. \quad (58)$$

The on-shell value  $f_B(M_E^2 = -M_B^2)$  is obtained again by extrapolation from the spacelike region  $M_E^2 > 0$  to the on-shell point  $M_E^2 = -M_B^2$ . For the neutral pion ( $B = \pi^0$ ),  $\alpha = 3$  and so on in Eq. (57).

As shown in Ref. [15], the WT identity for the axial-vector vertex leads to the following relation in the improved ladder approximation model of QCD:

$$M_B^2 f_B = -2m_R(\mu^2) \mathcal{E}_B(\mu^2), \quad (59)$$

with

$$\mathcal{E}_B(\mu^2) := Z_m^{-1}(\mu^2) i \int_q \frac{f(-q_-^2) + f(-q_+^2)}{2} \times \text{tr} \left[ \bar{\chi}^R(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2} \right]. \quad (60)$$

This relation is satisfied for a finite quark mass. In the case of the chiral limit, we obtain

$$f_B \mathcal{E}_B(\mu^2) = \xi_R(\mu^2) \quad \text{where} \quad \xi_R := \langle \bar{\psi} \psi \rangle_R. \quad (61)$$

One can treat Eq. (61) as an approximated relation of the leading term of the expansion of  $m_R$  for a finite quark mass. Substituting Eq. (61) into Eq. (59), one obtains the Gell-Mann–Oakes–Renner (GMOR) mass formula

$$M_B^2 f_B^2 \approx -2m_R \langle \bar{\psi} \psi \rangle_R |_{\text{chiral limit}}. \quad (62)$$

For finite  $m_R > 0$ , Eq. (59) is an exact relation, while the violation of the GMOR formula is incurred by the violation of Eq. (61).

We define  $\mathcal{R}$  by

$$\mathcal{R}(M_E^2) := \frac{-M_E^2 f_B(M_E^2)}{-2m_R \mathcal{E}_B}. \quad (63)$$

The relation  $\mathcal{R}(-M_B^2) = 1$  must be satisfied due to Eq. (59). We use this condition to make the extrapolation more reliable.

## VII. NUMERICAL RESULTS

### A. Parameters of the model

The parameters of the improved ladder model of QCD are the current quark mass  $m_R$  for up and down quarks (the isospin symmetry is assumed throughout this paper); the scale parameter of QCD,  $\Lambda_{\text{QCD}}$ ; the infrared cutoff  $t_{\text{IF}}$  for the running coupling constant; the smoothness parameter  $t_0$ ; and the ultraviolet cutoff  $\Lambda_{\text{UV}}$ . We take  $t_0 = -3$  throughout this paper, which is the same value used in Ref. [4]. In Ref. [4] they have shown that the numerical results are quite insensitive to the choice of smoothness parameter in their studies of the BS amplitudes in the chiral limit. As for the ultraviolet cutoff  $\Lambda_{\text{UV}}$ , we shall show that the physical observables depend on it rather weakly after our renormalization procedure described in Sec. IV, if we use reasonably large values of  $\Lambda_{\text{UV}}$ . Of course, as we are treating not the full QCD but its approximation, we expect that weak dependences remain in our numerical studies. Thus we only have three physically relevant parameters, namely, the current quark mass,  $\Lambda_{\text{QCD}}$ , and the infrared cutoff.

We choose  $\Lambda_{\text{QCD}}$  at about 0.6 (GeV). It is rather large compared with the value obtained from analyses in deep inelastic scattering. In the framework of this model, however, one must employ a large value of  $\Lambda_{\text{QCD}}$  in order to obtain sufficiently strong dynamical chiral symmetry breaking. It may be an indication of the limitation of the improved ladder approach. Other nonperturbative interactions between quarks may solve this discrepancy. One candidate for such nonperturbative interactions is the instanton-induced interaction proposed by 't Hooft [16]. There have been many studies of the roles of the instanton in low-energy QCD such as the instanton liquid model [17], the generalized Nambu–Jona-Lasinio (NJL) model [18] with 't Hooft instanton-induced

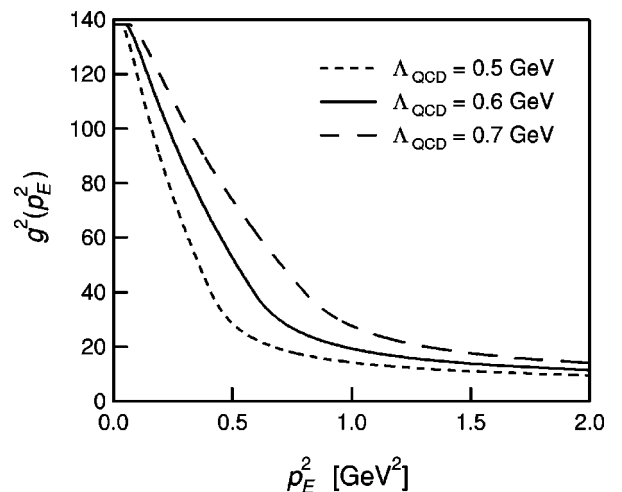


FIG. 1.  $q_E^2$  dependence of  $g^2(q_E^2)$  for various  $\Lambda_{\text{QCD}}$  with  $t_{\text{IF}} = -0.5$ .

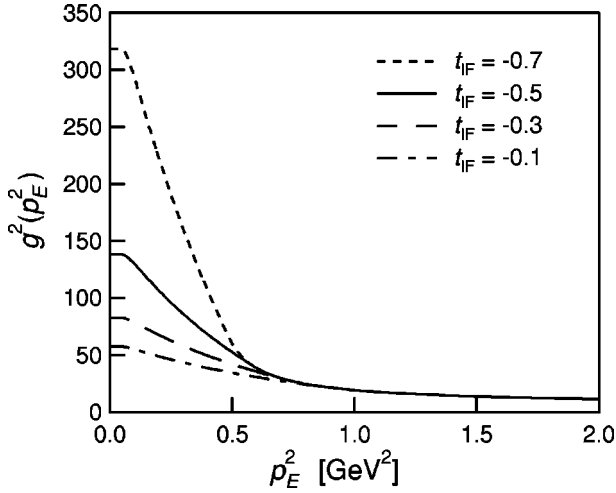


FIG. 2.  $q_E^2$  dependence of  $g^2(q_E^2)$  for various  $t_{IF}$  with  $\Lambda_{QCD} = 0.6$  GeV.

interaction [19], and the effects of the instanton in baryon sector [20]. Recent studies of the  $\eta$ -meson properties in the generalized NJL model with the 't Hooft instanton-induced interaction have shown [21] that the contribution from the 't Hooft instanton-induced interaction to the dynamical mass of the up and down quark mass is 44% of that from the usual  $U_L(3) \times U_R(3)$  invariant four-quark interaction. The introduction of the 't Hooft instanton-induced interaction to the ILA model seems to be interesting and such an attempt is now in progress [22].

We employ  $t_{IF}$  at about  $-0.5$  due to Ref. [4]. The running coupling constant for various  $\Lambda_{QCD}$  and  $t_{IF}$  is shown in Figs. 1 and 2.

**B. SD equation**

We discuss solutions of the SD equation in this subsection.  $B(-q_E^2)$  as solutions of Eq. (25) for various values of  $m_R$  are shown in Fig. 3. In the chiral limit, i.e.  $m_R=0$ , we find a nontrivial solution  $B(-q_E^2)$  which is nonzero for  $q_E \ll 1$  GeV. Note that Eq. (25) has also a trivial solution

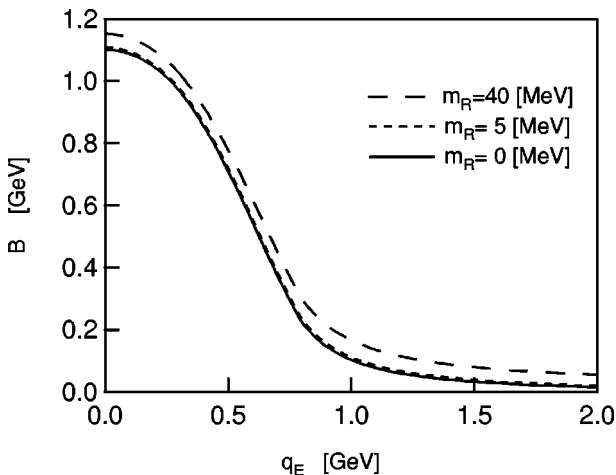


FIG. 3. Quark mass function  $B(-q_E^2)$  as function of  $q_E$ . Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

TABLE I.  $t_{IF}$  dependence of quark condensate in the chiral limit. Other model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV, and  $\mu^2=4$  GeV<sup>2</sup>.

$t_{IF}$	-0.5	0.0	0.5	1.0	1.5	2.0
$-\langle\bar{\psi}\psi\rangle_R^{1/3}$ [MeV]	259	240	187	116	43	2

$B(-q_E^2)=0$ , if  $m_R=0$ . The existence of the nontrivial solution indicates that chiral symmetry is broken dynamically.  $B(-q_E^2)$  decreases quickly to zero for  $q_E \gg 1$  GeV. The asymptotic behavior is consistent with the OPE result as shown in Ref. [2]. The range of nonzero values of  $B(-q_E^2)$  is determined by the  $q_E^2$  dependence of the coupling constant,  $g(q_E^2)$  defined in Eq. (7). The results shown in Fig. 3 correspond to the choice  $\Lambda_{QCD}=0.6$  GeV. When  $\Lambda_{QCD}$  decreases, the range of  $B(-q_E^2)$  decreases accordingly and therefore the chiral symmetry breaking is weakened. But the  $\Lambda_{QCD}$  determines the scale of the system. The order parameter  $\langle\bar{\psi}\psi\rangle$  is almost proportional to the  $\Lambda_{QCD}^3$  and chiral symmetry breaking always occurs for smaller  $\Lambda_{QCD}$ . The parameter  $t_{IF}$  determines the strength of the coupling constant. Table I shows the  $t_{IF}$  dependence of the condensate in the chiral limit. As can be seen from Table I, chiral symmetry breaking does not occur for larger  $t_{IF}$ . The asymptotic behavior of  $B(-q_E^2)$  with finite current quark mass is rather different from that in the chiral limit. It can be seen clearly from the log-log plot of the quark mass function  $B(-q_E^2)$  given in Fig. 4.

We next discuss the quark condensate. The quark condensates  $-\langle\bar{\psi}\psi\rangle_R^{1/3}$  calculated for the various quark masses are shown in Fig. 5. Since in our definition of the quark condensate given in Eqs. (36)–(39) the perturbative quark mass contribution is subtracted,  $-\langle\bar{\psi}\psi\rangle_R^{1/3}$  decreases as  $m_R$  increases. Similar behavior is observed in the QCD sum rule approach.  $\langle\bar{\psi}\psi\rangle_R$  for  $m_R=120$  MeV is about 78% of  $\langle\bar{\psi}\psi\rangle_R$  for  $m_R=5$  MeV. This is in reasonable agreement with the QCD sum rule result [23]:  $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle=0.8 \pm 0.1$ .

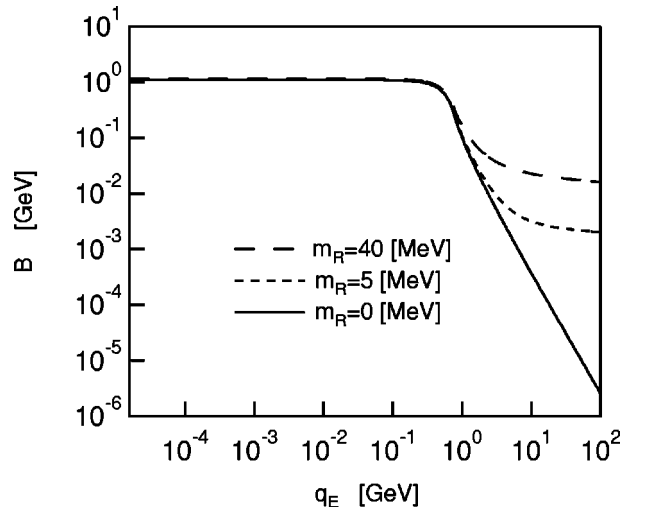


FIG. 4. Log-log plot of quark mass function  $B(-q_E^2)$ . Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.



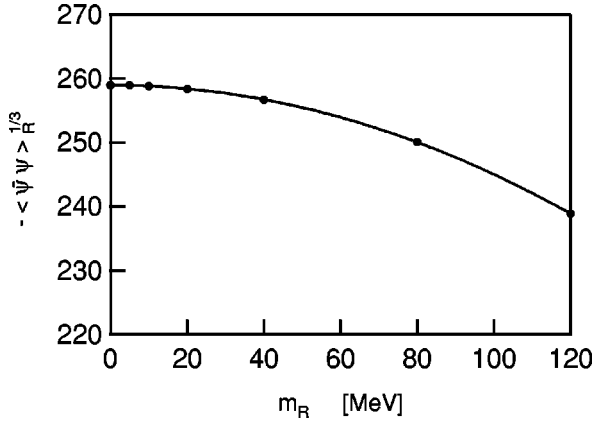


FIG. 5.  $m_R$  dependence of  $-\langle\bar{\psi}\psi\rangle_R^{1/3}$ . Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

### C. BS equation

Let us now turn to the discussion of the solutions of the BS equation. The eigenvalues  $\lambda(M_E^2)$  of the BS equation are shown in Fig. 6. One sees that the massless solution  $\lambda(M_E^2=0)=1$  appears in the chiral limit. This is a result of the Nambu-Goldstone (NG) theorem. For a nonzero quark mass, we need to extrapolate  $\lambda(M_E^2)$  to the timelike  $M_E^2<0$ . We have fitted  $\lambda(M_E^2)$  by a quadratic function using the method of least squares and extrapolated  $\lambda(M_E^2)$  to the timelike region to find the point at which  $\lambda(M_E^2)$  becomes unity. Figure 6 implies that the extrapolation length is longer for a larger quark mass. In order to reduce the ambiguity in the extrapolation procedure, we also evaluate the ratio  $\mathcal{R}$  defined by Eq. (63) as a function of  $M_E^2$ . Because of the exact relation (59),  $\mathcal{R}$  must hit  $\mathcal{R}(-M_\pi^2)=1$  at the on mass shell of the pion. The value  $\mathcal{R}$  is shown in Fig. 7. We fit the graph with the linear function using the methods of least squares. We show our calculated results of the pion mass  $M_\pi$  determined by the above-mentioned two conditions, i.e.,  $\lambda=1$  and  $\mathcal{R}=1$ , for various values of the quark mass  $m_R$  in Table II. The ambiguity by the extrapolation procedure is reasonably small up to the strange quark mass region. We plot our results of  $M_\pi^2$

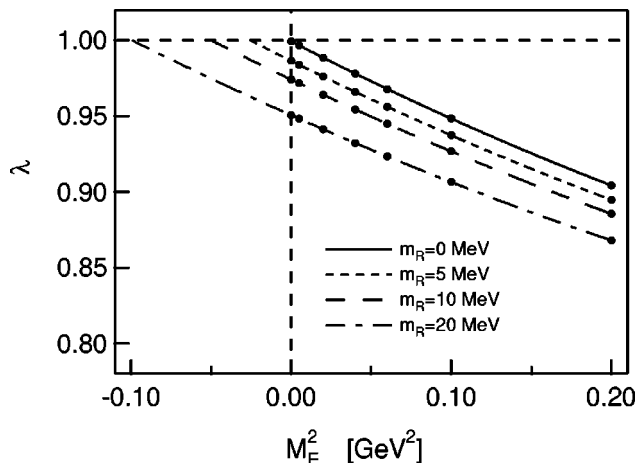


FIG. 6. Eigenvalues of the BS equation  $\lambda(M_E^2)$ . Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

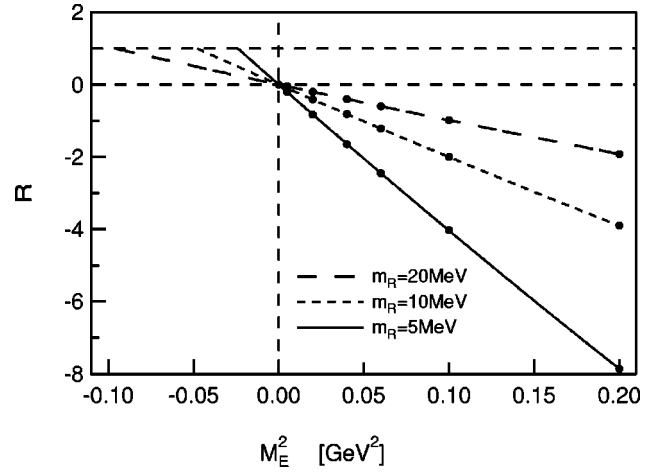


FIG. 7. The value  $\mathcal{R}(M_E^2)$ . Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

obtained by the condition  $\lambda=1$  as a function of  $m_R$  in Fig. 8. The  $M_\pi^2$  seems to be almost a linear function of  $m_R$  up to  $m_R\sim 40$  MeV. This is suggested by the GMOR formula

$$M_\pi^2 \simeq \left( \frac{-2\langle\bar{\psi}\psi\rangle_R}{f_\pi^2} \right)_{\text{chiral limit}} m_R. \quad (64)$$

The deviation from linear dependence at  $m_R=120$  MeV is about 9%.

We next discuss the pion decay constant. As mentioned in Sec. VI, we can calculate  $f_\pi(M_E^2)$  only for the timelike  $M_E^2$  and therefore the on-shell value of the decay constant  $f_\pi(M_E^2=-M_\pi^2)$  can be obtained again by the extrapolation. The  $M_E^2$  dependences of the pion decay constants in the chiral limit and in the case of  $m_R=5$  MeV are shown in Fig. 9.

To estimate the effect of  $E^\alpha(q;P)$  in Eq. (57), we plot the naive value  $f_\pi^N$  which is defined by neglecting  $E^\alpha(q;P)$  term from Eq. (57). It seems to be a good approximation that  $f_\pi(M_E^2)$  is a linear function of  $M_E^2$ . Therefore we fit the curve by the linear function using the method of least squares and make an extrapolation to the timelike  $M_E^2$  for finite  $m^2$ . On the other hand, we fit  $f_\pi^N$  by the quadratic function using the least-squares method to extrapolate to the on-shell point  $M_E^2=-M_\pi^2$ . Our results for  $m_R=5$  MeV are  $f_\pi$

TABLE II. Pion masses determined by the two conditions  $\lambda=1$  and  $\mathcal{R}=1$  for various values of  $m_R$ . All the entries are in units of MeV. Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

$m_R$	0	5	10	20	40	80	120
$M_\pi$ ( $\lambda=1$ )	0.0	159.1	222.0	312.9	444.4	639.8	800.9
$M_\pi$ ( $\mathcal{R}=1$ )	0.0	154.5	218.5	309.0	436.9	616.1	749.7

<sup>2</sup>It should be noted that the decay constant at small (positive)  $M_E^2$  suffers from numerical uncertainty and thus it deviates from the straight line. We do not use these points in our extrapolation procedure.

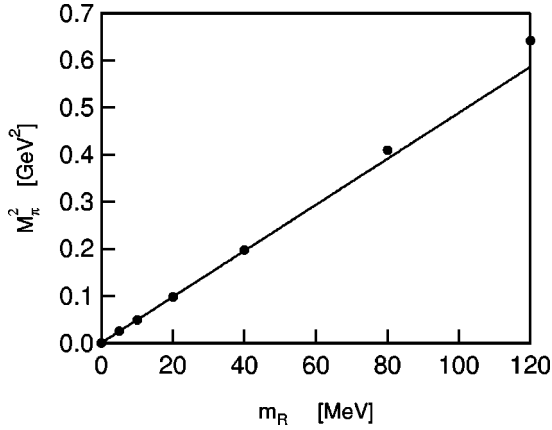


FIG. 8.  $M_\pi^2$  as a function of  $m_R$ . The dots represent our calculated results and the line represents a linear fit of the first four points. Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

=88 MeV and  $f_\pi^N=155$  MeV; so the contribution of the  $E^\alpha(q;P)$  term is remarkable. A similar result has been found in the ILA model which respects the axial-vector WT identity by using the gluon momentum square as the argument of the running coupling constant and the nonlocal gauge [10]. The condition  $\mathcal{R}(-M_\pi^2)=1$  is the direct consequence of the axial-vector WT identity and therefore it has been proved numerically that our definition of the decay constant is consistent with the axial-vector WT identity from the fact that the pion mass determined by the condition  $\lambda=1$  is almost the same as that determined by the condition  $\mathcal{R}=1$ .

We plot the quark mass dependence of the decay constant in Fig. 10.  $f_\pi$  almost linearly depends on  $m_R$  in our case.  $f_\pi$  at  $m_R=40$  MeV is about 15% bigger than  $f_\pi$  at  $m_R=5$  MeV and  $f_\pi$  at  $m_R=80$  MeV is about 32% bigger than  $f_\pi$  at  $m_R=5$  MeV. Since the observed  $f_K/f_\pi=1.23$ , the  $m_R$  dependence of the decay constant seems to be reasonable though we have not solved the kaon BS equation. This  $m_R$  dependence of  $f_\pi$  is similar to that obtained in chiral perturbation theory (ChPT) [24] though the chiral logarithmic term has not been seen in our numerical result. It is understand-

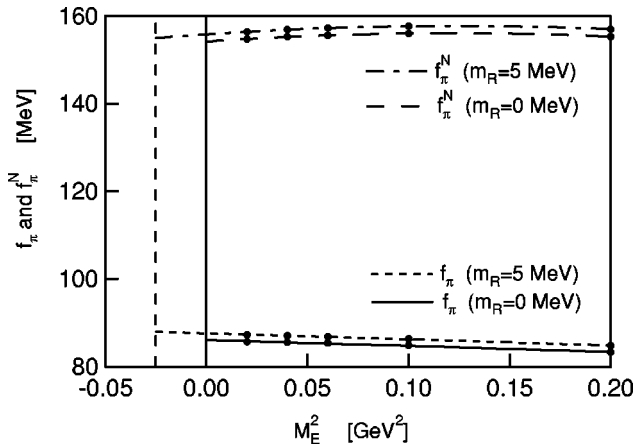


FIG. 9.  $M_E^2$  dependence of  $f_\pi$  and  $f_\pi^N$ . The vertical dashed line represents the point  $M_E^2=-M_\pi^2$  for  $m_R=5$  MeV. Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

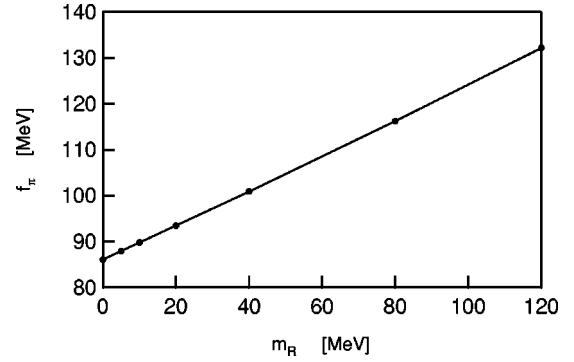


FIG. 10.  $m_R$  dependence of  $f_\pi$ . Model parameters are  $\Lambda_{UV}=100$  GeV,  $\Lambda_{QCD}=0.6$  GeV,  $t_{IF}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

able because the Goldstone boson loop contribution is not taken into account explicitly in our approach. On the other hand, the quark-antiquark structure is included explicitly and finite quark mass effects are fully taken into account without performing the perturbative expansion with respect to the quark mass.

Let us now discuss the  $\Lambda_{QCD}$  and the  $t_{IF}$  dependences. Table III shows the  $\Lambda_{QCD}$  dependences of the pion mass, the pion decay constant, and the quark condensate. As shown in Table III, all quantities with mass dimension 1 are roughly proportional to  $\Lambda_{QCD}$ . It is understandable since the only scale of the theory is  $\Lambda_{QCD}$  if one can neglect the current quark mass. Table IV shows the  $t_{IF}$  dependence. For  $t_{IF}$  below  $-0.7$  the coupling constant becomes very steep and our numerical procedure is not sufficiently accurate. Although we have not performed a fine-tuning of the model parameters, it is clear from Tables I, III, and IV that one can fit the model parameters so as to reproduce the observed values of  $M_\pi$  and  $f_\pi$  and the empirically determined value of the quark condensate.

We study the  $\Lambda_{UV}$  dependence by changing the value of  $\Lambda_{UV}$  from 10 GeV to 1000 GeV. It causes less than 1% changes of the  $M_\pi$ ,  $f_\pi$ , and  $\langle\bar{\psi}\psi\rangle_R$ . This stability indicates that our nonperturbative renormalization procedure works well.

#### D. Approximation

Finally we discuss the approximation often used in solving the BS equation. The approximation in which one neglects the  $\phi_P, \phi_Q, \phi_T$  terms on the RHS in the BS equation (49) is often used in the literature [25]. The resulting eigenvalues are shown in Fig. 11. While this approximation gives the massless NG boson in the chiral limit, it underestimates the pion mass for finite quark mass. The decay constants

TABLE III.  $\Lambda_{QCD}$  dependences of  $M_\pi$ ,  $f_\pi$ , and  $-\langle\bar{\psi}\psi\rangle_R^{1/3}$ . Other model parameters are  $\Lambda_{UV}=100$  GeV,  $t_{IF}=-0.5$ ,  $\mu^2=4$  GeV<sup>2</sup>, and  $m_R=5$  MeV.

$\Lambda_{QCD}$ [GeV]	$M_\pi$ [MeV]	$f_\pi$ [MeV]	$-\langle\bar{\psi}\psi\rangle_R^{1/3}$ [MeV]
0.5	152	74	223
0.6	159	88	259
0.7	166	102	293

TABLE IV.  $t_{\text{IF}}$  dependences of  $M_\pi$ ,  $f_\pi$ , and  $-\langle\bar{\psi}\psi\rangle_R^{1/3}$ . Other model parameters are  $\Lambda_{\text{UV}}=100$  GeV,  $\Lambda_{\text{QCD}}=0.6$  GeV,  $\mu^2=4$  GeV<sup>2</sup>, and  $m_R=5$  MeV.

$t_{\text{IF}}$	$M_\pi$ [MeV]	$f_\pi$ [MeV]	$-\langle\bar{\psi}\psi\rangle_R^{1/3}$ [MeV]
-0.3	149	91	256
-0.5	159	88	259
-0.7	(181)	(74)	(253)

obtained from the approximated BS amplitude are shown in Table V. The approximation overestimates the pion decay constant by about 30% while the axial-vector WT identity  $(M_\pi^{\text{approx}})^2 f_\pi^{\text{approx}} = -2m_R \mathcal{E}_\pi^{\text{approx}}$  is preserved. It is seen that the effect of  $E^\alpha(q;P)$  is very small in this case, and  $f_\pi^{\text{approx}} \simeq f_\pi^{N\text{approx}}$ . Therefore, violation of the axial-vector WT identity or that of the exact relation for the partial conservation of axial-vector current (PCAC) incurred by neglecting the  $E^\alpha(q;P)$  effect is very small.

The above  $f_\pi^{\text{approx}}$  is calculated in a similar way as in the approximation discussed in Refs. [15,16]. The following is shown in theorem 2 of [26]. If the interaction is locally chiral invariant, the approximation of taking the wave function renormalization of the quark propagator to be 1 and at the same time neglecting the  $\phi_P$ ,  $\phi_Q$ , and  $\phi_T$  terms on the RHS of the BS equation preserves the low-energy relations. Theorem 2 cannot be applied to the present case since the interaction term of the ILA model breaks the local chiral symmetry. However, one can prove that the low-energy relation holds if one neglects the  $\phi_P$  and  $\phi_Q$  terms in  $E^\alpha(q;P)$  as well as the  $\phi_P$ ,  $\phi_Q$ , and  $\phi_T$  terms on the RHS of the BS equation by following the same argument of the proof of theorem 2 in [26].

### VIII. CONCLUSION

We have solved the Schwinger-Dyson equation for the quark propagator and the Bethe-Salpeter equation for the pion in the improved ladder approximation of QCD. We have carefully treated the consistency of the equations in order to preserve the low-energy relations associated with chiral symmetry by using the Cornwall-Jackiw-Tomboulis

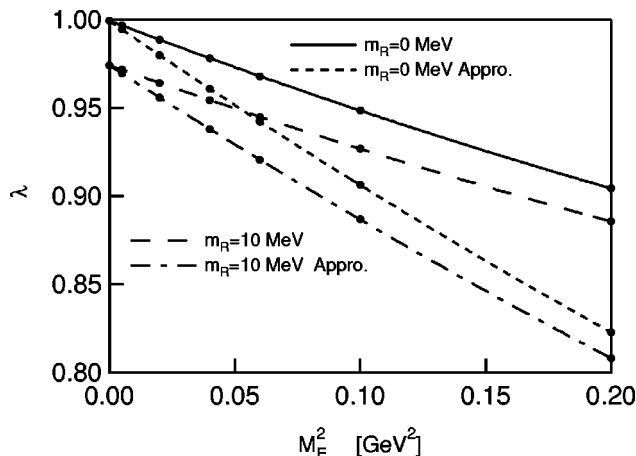


FIG. 11. Eigenvalues of the BS equations with and without the approximation described in the text. Model parameters are  $\Lambda_{\text{UV}}=100$  GeV,  $\Lambda_{\text{QCD}}=0.6$  GeV,  $t_{\text{IF}}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

TABLE V. Result with and without the approximation. All entries are in units of MeV. Model parameters are  $\Lambda_{\text{UV}}=100$  GeV,  $\Lambda_{\text{QCD}}=0.6$  GeV,  $t_{\text{IF}}=-0.5$ , and  $\mu^2=4$  GeV<sup>2</sup>.

$m_R$	$M_\pi$	$M_\pi^{\text{approx}}$	$f_\pi$	$f_\pi^{\text{approx}}$	$f_\pi^N$	$f_\pi^{N\text{approx}}$
0	0	0	86	115	154	116
5	159	119	88	117	156	117
10	222	166	90	118	158	119

effective action approach. We have introduced the finite quark mass term in order to study effects of explicit chiral symmetry breaking on the low-energy relations. Because of the difference in the asymptotic behavior of the quark mass function for a finite quark mass, the nonperturbative mass-independent renormalization has been introduced and the quark condensate for finite quark mass is calculated. In solving the SD and BS equations, we have not taken any further approximation such as an expansion of BS amplitudes in the Gegenbauer polynomials.

We have obtained reasonable values of  $M_\pi$ ,  $f_\pi$ , and  $\langle\bar{\psi}\psi\rangle_R$  with a rather large value of  $\Lambda_{\text{QCD}}$ . It may indicate the limitation of the improved ladder approach. The pion mass  $M_\pi$  grows as the quark mass  $m_R$  increases. Up to the strange quark mass region  $M_\pi^2$  seems to be proportional to quark mass  $m_R$  almost as predicted by the GMOR relation

$$M_\pi^2 = \left( \frac{-2\langle\bar{\psi}\psi\rangle_R}{f_\pi^2} \right)_{\text{chiral limit}} m_R. \quad (65)$$

We have found that the  $f_\pi$  also grows as  $m_R$  increases almost linearly. The  $m_R$  dependences of  $M_\pi^2$  and  $f_\pi$  are similar to those obtained in chiral perturbation theory. It suggests that chiral perturbation is applicable up to the strange quark mass region.

We have studied the effect of the  $E^\alpha(q;P)$  term in the true decay constant. We have found that it is significantly large for various input parameters. Therefore, in the framework of the improved ladder approximation,  $E^\alpha(q;P)$  plays an essential role to keep the chiral property.

We have further shown the result of the approximation neglecting  $\phi_P(q;P)$ ,  $\phi_Q(q;P)$ , and  $\phi_T(q;P)$  terms on the RHS of the BS equations. This approximation is very useful and makes the calculation easy greatly. But the result gives a smaller pion mass. This suggests that the simple picture of the  $\phi_S(q;P)$  dominance in the BS equation is not so good, at least in the present model.

So far, we have studied the symmetric  $q\bar{q}$  systems,  $u\bar{u}$ ,  $d\bar{d}$ , etc. It is interesting to extend the present formulation to asymmetric systems like the kaon. It is also interesting to introduce the  $U_A(1)$  breaking interaction to this framework and to study the  $\eta\text{-}\eta'$  systems. Such attempts are in progress.

### ACKNOWLEDGMENTS

The authors would like to thank Kensuke Kusaka for useful discussions. This work was supported in part by Grants-in-Aid for Scientific Research Nos. (A)1)08304024 and

(C)(2)08640356 of the Ministry of Education, Science, Sports and Culture of Japan.

**APPENDIX**

Here we write down the BS equation explicitly. In this section the total momentum is denoted by  $P$  instead by  $P_B$  for simplicity. First we define the regularized amputated BS amplitude  $\hat{\chi}^R(q;P)$  by

$$\hat{\chi}^R(q;P) := S_F^{R-1} \left( q + \frac{P}{2} \right) \chi^R(q;P) S_F^{R-1} \left( q - \frac{P}{2} \right), \quad (\text{A1})$$

which can be expressed in terms of

$$\hat{\chi}_{nm}^R(q;P) = \delta_{ji} \frac{(\lambda^a)_{gf}}{2} \left[ \left( \hat{\phi}_S(q;P) + \hat{\phi}_P(q;P) \not{q} + \hat{\phi}_Q(q;P) \not{P} + \frac{1}{2} \hat{\phi}_T(q;P) (\not{P} \not{q} - \not{q} \not{P}) \right) \gamma_5 \right]_{ba}. \quad (\text{A2})$$

The BS equation (49) reads

$$\hat{\phi}_A(q;P) = \int_k K_{AB}(q,k;P) \phi_B(k;P). \quad (\text{A3})$$

The components of the kernel are given explicitly by

$$K_{SS}(q,k;P) = i C_F \bar{g}^2(q,k) \frac{-3}{(q-k)^2}, \quad (\text{A4})$$

$$K_{PP}(q,k;P) = \frac{i C_F \bar{g}^2(q,k)}{P^2 q^2 - (Pq)^2} \left\{ \frac{P^2(qk) - (Pq)(Pk)}{(q-k)^2} + \frac{2(qk-k^2)[P^2 q^2 - (Pq)^2 + (Pq)(Pk) - P^2(qk)]}{(q-k)^4} \right\}, \quad (\text{A5})$$

$$K_{PQ}(q,k;P) = \frac{i C_F \bar{g}^2(q,k)}{P^2 q^2 - (Pq)^2} \frac{2(Pq-Pk)[P^2 q^2 - (Pq)^2 + (Pq)(Pk) - P^2(qk)]}{(q-k)^4}, \quad (\text{A6})$$

$$K_{QP}(q,k;P) = \frac{i C_F \bar{g}^2(q,k)}{P^2 q^2 - (Pq)^2} \left\{ \frac{(Pk)q^2 - (Pq)(qk)}{(q-k)^2} + \frac{2(qk-k^2)((Pq)(qk) - (Pk)q^2)}{(q-k)^4} \right\}, \quad (\text{A7})$$

$$K_{QQ}(q,k;P) = \frac{i C_F \bar{g}^2(q,k)}{P^2 q^2 - (Pq)^2} \left\{ \frac{P^2 q^2 - (Pq)^2}{(q-k)^2} + \frac{2(Pq-Pk)[(Pq)(Pk) - (qk)q^2]}{(q-k)^4} \right\}, \quad (\text{A8})$$

$$K_{TT}(q,k;P) = \frac{i C_F \bar{g}^2(q,k)}{P^2 q^2 - (Pq)^2} \frac{1}{(q-k)^4} \{ (k^2 - q^2)[(Pq)(Pk) - P^2(qk)] + 2(Pq-Pk)[(Pk)q^2 - (Pq)(qk)] - 2(qk-k^2)[P^2 q^2 - (Pq)^2] \}, \quad (\text{A9})$$

and other components are zero. The relations between  $\phi_A(q;P)$  and  $\hat{\phi}_A(q;P)$  are given by

$$\begin{aligned} \phi_S(q;P) = & \frac{1}{\Delta} \left[ \left\{ q^2 - \frac{P^2}{4} - B(q_+^2)B(q_-^2) \right\} \hat{\phi}_S(q;P) + \left\{ q^2 [B(q_+^2) - B(q_-^2)] - \frac{Pq}{2} [B(q_-^2) + B(q_+^2)] \right\} \hat{\phi}_P(q;P) \right. \\ & \left. + \left\{ (Pq)[B(q_+^2) - B(q_-^2)] - \frac{P^2}{2} [B(q_-^2) + B(q_+^2)] \right\} \hat{\phi}_Q(q;P) + [P^2 q^2 - (Pq)^2] \hat{\phi}_T(q;P) \right], \quad (\text{A10}) \end{aligned}$$

$$\begin{aligned} \phi_P(q;P) = & \frac{1}{\Delta} \left[ [B(q_+^2) - B(q_-^2)] \hat{\phi}_S(q;P) + \left\{ q^2 + \frac{P^2}{4} - B(q_+^2)B(q_-^2) \right\} \hat{\phi}_P(q;P) + 2(Pq) \hat{\phi}_Q(q;P) \right. \\ & \left. - \left\{ (Pq)[B(q_-^2) + B(q_+^2)] + \frac{P^2}{2} [B(q_-^2) - B(q_+^2)] \right\} \hat{\phi}_T(q;P) \right], \quad (\text{A11}) \end{aligned}$$

$$\begin{aligned} \phi_Q(q;P) = & \frac{1}{\Delta} \left[ -\frac{1}{2} [B(q_+^2) + B(q_-^2)] \hat{\phi}_S(q;P) - \frac{Pq}{2} \hat{\phi}_P(q;P) - \left\{ q^2 + \frac{P^2}{4} + B(q_+^2)B(q_-^2) \right\} \hat{\phi}_Q(q;P) \right. \\ & \left. + \left\{ q^2 [B(q_-^2) + B(q_+^2)] + \frac{Pq}{2} [B(q_-^2) - B(q_+^2)] \right\} \hat{\phi}_T(q;P) \right], \quad (\text{A12}) \end{aligned}$$

$$\begin{aligned} \phi_T(q;P) = & \frac{1}{\Delta} \left[ \hat{\phi}_S(q;P) - \frac{1}{2} [B(q_-^2) - B(q_+^2)] \hat{\phi}_P(q;P) + [B(q_+^2) + B(q_-^2)] \hat{\phi}_Q(q;P) \right. \\ & \left. + \left\{ -q^2 + \frac{P^2}{4} - B(q_-^2)B(q_+^2) \right\} \hat{\phi}_T(q;P) \right], \end{aligned} \quad (\text{A13})$$

where

$$\Delta := [q_+^2 - B^2(q_+^2)][q_-^2 - B^2(q_-^2)]. \quad (\text{A14})$$

- 
- [1] J. M. Cornwall, R. Jackiw, and E. Tomboulis, *Phys. Rev. D* **10**, 2428 (1974).
- [2] K. Higashijima, *Phys. Rev. D* **29**, 1228 (1984).
- [3] V. A. Miransky, *Sov. J. Nucl. Phys.* **38**, 280 (1984).
- [4] K. Aoki, M. Bando, T. Kugo, M. G. Mitchard, and H. Nakatani, *Prog. Theor. Phys.* **84**, 683 (1990).
- [5] K.-I. Aoki, M. Bando, T. Kugo, and M. Mitchard, *Prog. Theor. Phys.* **85**, 355 (1991).
- [6] K.-I. Aoki, T. Kugo, and M. Mitchard, *Phys. Lett. B* **266**, 467 (1991).
- [7] For a review, C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
- [8] For a recent study, P. Maris and C. D. Roberts, *Phys. Rev. C* **56**, 3369 (1997), and references therein.
- [9] K.-I. Aoki, in *Proceedings of the International Workshop on Perspectives of Strong Coupling Gauge Theories*, edited by J. Nishimura and K. Yamawaki (World Scientific, Singapore, 1997), p. 171.
- [10] T. Kugo and M. Mitchard, *Phys. Lett. B* **282**, 162 (1992).
- [11] K. Kusaka, H. Toki, and S. Umisedo (private communication).
- [12] C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
- [13] H. D. Politzer, *Nucl. Phys.* **B117**, 397 (1976).
- [14] P. Jain and H. J. Munczek, *Phys. Rev. D* **44**, 1873 (1991).
- [15] K. Naito, K. Yoshida, Y. Nemoto, M. Oka, and M. Takizawa, “Dynamical Chiral Symmetry Breaking in Effective Models of QCD in the Bethe-Salpeter Approach,” *hep-ph/9805243*, *Phys. Rev. C*. (to be published).
- [16] G. 't Hooft, *Phys. Rev. D* **14**, 3432 (1976).
- [17] E. V. Shuryak, *Nucl. Phys.* **B203**, 93 (1982); **B203**, 116 (1982); D. I. Diakonov and V. Yu. Petrov, *ibid.* **B272**, 457 (1986); M. A. Nowak, J. J. M. Verbaarschot, and I. Zahed, *ibid.* **B324**, 1 (1989); T. Schäfer and E. V. Shuryak, *Phys. Rev. D* **53**, 6522 (1996); **54**, 1099 (1996); *Rev. Mod. Phys.* **70**, 323 (1998).
- [18] Y. Nambu and G. Jona-Lasinio, *Phys. Rev.* **122**, 345 (1961); **124**, 246 (1961).
- [19] T. Kunihiro and T. Hatsuda, *Phys. Lett. B* **206**, 385 (1988); T. Hatsuda and T. Kunihiro, *Z. Phys. C* **51**, 49 (1991); *Phys. Rep.* **247**, 221 (1994); V. Bernard, R. L. Jaffe, and U.-G. Meissner, *Nucl. Phys.* **B308**, 753 (1988); Y. Kohyama, K. Kubodera, and M. Takizawa, *Phys. Lett. B* **208**, 165 (1988); M. Takizawa, K. Tsushima, Y. Kohyama, and K. Kubodera, *Prog. Theor. Phys.* **82**, 481 (1989); *Nucl. Phys.* **A507**, 611 (1990); H. Reinhardt and R. Alkofer, *Phys. Lett. B* **207**, 482 (1988); R. Alkofer and H. Reinhardt, *Z. Phys. C* **45**, 275 (1989); S. Klimt, M. Lutz, U. Vogl, and W. Weise, *Nucl. Phys.* **A516**, 429 (1990); U. Vogl, M. Lutz, S. Klimt, and W. Weise, *Prog. Part. Nucl. Phys.* **27**, 195 (1991).
- [20] E. V. Shuryak and J. L. Rosner, *Phys. Lett. B* **218**, 72 (1989); M. Oka and S. Takeuchi, *Phys. Rev. Lett.* **63**, 1780 (1989); *Nucl. Phys.* **A524**, 649 (1991); S. Takeuchi and M. Oka, *Phys. Rev. Lett.* **66**, 1271 (1991); O. Morimatsu and M. Takizawa, *Nucl. Phys.* **A554**, 635 (1993); S. Takeuchi, *Phys. Rev. Lett.* **73**, 2173 (1994); *Phys. Rev. D* **53**, 6619 (1996).
- [21] M. Takizawa and M. Oka, *Phys. Lett. B* **359**, 210 (1995); **364**, 249(E) (1995); Y. Nemoto, M. Oka, and M. Takizawa, *Phys. Rev. D* **54**, 6777 (1996); M. Takizawa, Y. Nemoto, and M. Oka, *ibid.* **55**, 4083 (1997).
- [22] K. Naito, K. Yoshida, Y. Nemoto, M. Oka, and M. Takizawa (unpublished).
- [23] L. J. Reinders, H. Rubinstein, and S. Yazaki, *Phys. Rep.* **127**, 1 (1985).
- [24] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158**, 142 (1984); *Nucl. Phys.* **B250**, 465 (1985).
- [25] M. R. Frank and C. D. Roberts, *Phys. Rev. C* **53**, 390 (1996); C. Savkli and F. Tabakin, *Nucl. Phys.* **A628**, 645 (1998).
- [26] K. Naito and M. Oka, “Approximation of the Schwinger-Dyson and the Bethe-Salpeter Equations and Chiral Symmetry of QCD,” *hep-ph/9805258*, *Phys. Rev. C*. (to be published).