Hadronization with a confining equation of state

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We present a fast hadronization model for the constituent quark plasma (CQP) produced in relativistic heavy-ion collisions at SPS (Super Proton Synchrotron). The model is based on rate equations and on a confining equation of state inspired by the string phenomenology. We display the time evolution of the relevant physical quantities during the hadronization process and the final hadron multiplicities. The results indicate that the hadronization of CQP is fast. [S0556-2813(99)08202-3]

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I. INTRODUCTION

The research of ultrarelativistic heavy-ion collisions is from its beginning motivated by the search for quark matter: a bulk of deconfined color charges. The prime attempts to describe the hadronization assumed that this quark matter consists of noninteracting massless quarks and gluons [1–3]. In place of these early images of free quarks and gluons (plasma)—which was based on thermodynamical studies of the pure non-Abelian gauge theory on the lattice—gradually the picture of a quark matter emerges, containing effective propagators and interaction vertices [4–6]. At the characteristic energy scale of CERN SPS (Super Proton Synchrotron) heavy-ion experiments ($\sqrt{s}/2 \approx 10$ GeV/nucleon) the dressed gluons are heavier than the quarks ($M_g \approx 600 - 800$ MeV, $M_q \approx 150 - 300$ MeV) [6], and both values are bigger than the temperature ($T \approx 160$ MeV). Therefore the number of quarks outweight that of the gluons and can be treated nonrelativistically. On the grounds of these theoretical indications we expect that in the CERN heavy-ion experiments not an ideal quark gluon plasma, but a constituent quark plasma (CQP), which contains antiquarks too, is formed in some intermediate state of the reaction (see the ALCOR (algebraic coalescence rehadronization) model, Refs. [9,10]).

For the description of heavy-ion reaction products at SPS a purely hadronic interpretation of bulk experimental results was also suggested [11]. This alternative interpretation, however, has problems in explaining the process of the creation of new hadrons, especially the short time scale. Namely, the hadronic processes have characteristic times of several tens of fm/c [3,12], while the typical heavy-ion reaction time at SPS energy is about $1 - 2$ fm/c [13]. Processes on the quark level, on the other hand, have equilibration times of the order of $0.1 - 2$ fm/c [1,2,14].

The CQP picture has some advantages in comparison with the quark-gluon plasma scenario too: In the hadronization of a quark-gluon plasma the principles of color confinement and entropy generation can be both satisfied only with the assumption of an extreme growth of the reaction volume [15]. Further, a slow first-order transition through near-equilibrium states would need too long a time (up to 50 fm/c), due to the reheating [16].

In this paper we assume that a massive quark matter, the CQP, is formed in heavy-ion reactions at SPS energy, which is in thermal but not necessarily in chemical equilibrium. The evolution of this CQP is followed through a set of coupled time-dependent differential equations for the temperature and chemical composition. The construction of the equations requires relations between the different equilibrium particle numbers at an actual temperature as input. These input values are determined by the equation of state.

In particular we utilize an equation of state motivated by QCD string phenomenology. Since strings are screened in dense quark matter at high temperature this goes over into an ideal gas equation of state in the infinite temperature limit. At finite temperatures the screening of strings longer than the (average) nearest-neighbor distance results in an interaction energy density proportional to a fractional power of the quark density $n^{2/3}$ [7]. This addition also occurs in the free-energy density defining the equation of state (EOS). Although such an additive modification of the ideal gas EOS is formally analogous to a bag constant, the physical origin and the thermodynamical behavior of the string EOS only loosely resembles the behavior of bag models. In fact, a number of important differences occur. One of them is the fact that below a critical temperature, somewhat below the point where the pressure becomes negative, chemical equilibrium cannot be maintained anymore. Here the quark matter clusterizes into color neutral hadrons. In contrast the bag model EOS can be continued for the quark matter down to zero temperature, having negative pressure (which excludes mechanical equilibrium, but not a collapsing system) but always a chemical equilibrium solution.

Furthermore, it is important to note that the string EOS results confinement already in the infinite volume (thermodynamical) limit. This behavior is beyond that which Elze *et al.* have shown in a pioneering work pointing out that the requirement of global color symmetry leads to a reduction of the effective number of degrees of freedom in finite volumes [8].

We mention also here that in the ALCOR model [9,10], the hadronization problem is treated with an algebraic approximation, in contrast to the method of the present paper, where we follow the complete time evolution of the system. The inclusion of the confining equation of state in the present model is also a very important difference from the ALCOR model. To our best knowledge, this type of phenomenological equation of state with confining character has not been discussed in the literature until now.

The paper is structured as follows: in Sec. II we discuss the equation of state of the mixture of hadrons and interact-

ing quarks. In Sec. III we describe in details the dynamics of the hadronization. In Sec. IV we discuss our numerical results. The conclusion is drawn in Sec. V. In the Appendix we collect the relevant hydrodynamical and thermodynamical and statistical expressions used in the paper.

II. EQUATION OF STATE OF THE MIXTURE OF HADRONS AND INTERACTING QUARKS

We assume that at the beginning of the hadronization the matter consists of massive quarks and antiquarks. In the time evolution of the system quarks and antiquarks form diquarks, antidiquarks, mesons, baryons, and antibaryons. We assume that the mixture of all of these particles are in thermal equilibrium which can be characterized by a temperature. For the representation of the interaction of the colored particles we introduce an extra term into the free energy, which is inspired by the string picture.

We note that there is an important difference to the canonical approach to color confinement transition: in an (ideal) mixture of quarks and hadrons the occupied volume, V, is the same for both components, $V_q = V_h = V$, the pressure contributions p_q and p_h are additive. On the other hand, in the application of the Gibbs criteria of a phase coexistence the volumes V_q and V_h are additive $V_q + V_h = \text{constant}$, and the partial pressures are equal, $p_q = p_h = p$, in phase equilibrium. In our physical picture of hadronization there is no phase coexistence and the Gibbs criteria do not apply. Colored particles and color-neutral clusters, prehadrons are distributed in a common reaction volume and chemical reactions convert eventually the quark matter into a pure hadronic matter.

During this hadronization process the reaction volume expands and cools. Due to the change of the multiparticle composition this expansion is not adiabatic: some heat can be produced (or consumed) by quarkochemical processes. The expansion law of an ideal mixture follows from Eqs. (A5) and (A20) in the Appendix:

$$\sum_i m_i \dot{N}_i + \frac{3}{2} T \sum_i \dot{N}_i + \frac{3}{2} \dot{T} \sum_i N_i + T \frac{\dot{V}}{V} \sum_i N_i = 0. \quad (1)$$

Due to the foregoing hadronization the number of particles decreases,

$$\sum_i \dot{N}_i < 0, \quad (2)$$

therefore this process reheats the system. Cooling effects are due to the expansion ($\dot{V}/V = \partial_\mu u^\mu > 0$) and rest mass creation ($\Sigma_i m_i \dot{N}_i > 0$). The latter is possible with satisfying Eq. (2) only if hadron masses are larger than the sum of their constituents.

At the present level, however, the physical process of color confinement by the hadronization is not yet taken into account. In particular, due to the dilution during the expansion it happens that not all quarks or diquarks find a partner to hadronize. In order to avoid this effect we supplement the model by the following confinement principle: all particles carrying color charge (quarks, diquarks, antiquarks, and antidiquarks) will be penalized by a free-energy contribution stemming from strings. The number of strings is proportional to a weighted sum of the number of color charges,

$$Q = \sum_i q_i N_i . \quad (3)$$

Here it is $q_i = 0$ for hadrons, $q_i = 1/2$ for quarks and antiquarks, and $q_i = 3/4$ for diquarks and antidiquarks. The higher effective charge of diquarks reflects a possibly higher number of in-medium partners, to which a string is stretched.

The average length L of a string depends on the density of colored objects, as $L = n_c^{-1/3}$, where

$$n_c = \sum_{i \in c} N_i / V, \quad (4)$$

where the summation $i \in c$ excludes color neutral particles (hadrons). So the free energy of the ideal quark matter–hadron matter mixture, $F = F_{\rm id} + \Delta F$, is supplemented by the following contribution of strings:

$$\Delta F = \sigma_s n_c^{-1/3} Q, \quad (5)$$

with the effective string tension $\sigma_s \approx 1.0$ GeV/fm. We note here that if all colored particles carried the same color charge, the effective string tension would be equal for all, and the interaction free energy would become

$$\Delta F = \sigma_{s,\rm eff} n_c^{2/3} V. \quad (6)$$

A mathematically more rigorous derivation of such an additional free-energy density in the nearest-neighbor approximation to strongly interacting systems applied to the string potential can be found in Ref. [7].

In order to achieve baryon production and diquark elimination properly we shall, however, need to use the more complicated ansatz (5).

This additional free energy comprises the nonideality of the equation of state we use. Since this addition is proportional to the volume V and the rest depends on densities only, it satisfies thermodynamical consistency requirements [17] due to its construction.

While there is no new contribution to the entropy,

$$S = S_{\rm id}, \quad (7)$$

the pressure, the energy, and the chemical potentials of colored ($q_i \neq 0$) particles receive important modifications:

$$p = p_{\rm id} - \frac{1}{3} \sigma_s n_c^{-1/3} \frac{Q}{V},$$

$$E = E_{\rm id} + \sigma_s n_c^{-1/3} Q,$$

$$\mu_i = \mu_{i,\rm id} + \sigma_s n_c^{-1/3} \left(q_i - \frac{1}{3} \overline{q} \right), \quad (8)$$

with $\overline{q} = Q/(V n_c)$. Hadronic chemical potentials have no modifications at all.

This nonideal completion of the equation of state influences both the expansion and cooling and the changes of

particle composition. Since the entropy is not changed by the introduction of the interaction term [see Eq. (7)] therefore,

$$T\,dS = T\,dS_{\rm id}. \tag{9}$$

From this equation, using Eq. (A7) we obtain

$$dE + p\,dV - \sum_i \mu_i\,dN_i = (dE + p\,dV)_{\rm id} - \sum_i \mu_{i,{\rm id}} dN_i. \tag{10}$$

Applying Eq. (A5), the nonideal cooling law becomes

$$(dE + p\,dV)_{\rm id} + \sum_i (\mu_i - \mu_{i,{\rm id}})dN_i = 0. \tag{11}$$

Accordingly Eq. (1) is supplemented by a new generic term due to the nonideal equation of state

$$\frac{\dot T}{T} = -\frac{2}{3}\frac{\dot V}{V} - \frac{\sum_i \dot N_i}{\sum_i N_i} - \frac{2}{3}\frac{\sum_i (m_i/T)\dot N_i}{\sum_i N_i} - \frac{2}{3}\frac{\sum_i (\mu_i/T - \mu_{i,{\rm id}}/T)\dot N_i}{\sum_i N_i}. \tag{12}$$

The additional term has the form

$$\sum_i \frac{\mu_i - \mu_{i,{\rm id}}}{T}\dot N_i = \frac{\sigma_s n_c^{-1/3}}{T}\sum_{i\in c}\left(q_i - \frac{1}{3}\overline{q}\right)\dot N_i. \tag{13}$$

This term is negative if color charges are eliminated from the mixture. Therefore, color confinement, causing an extra suppression of equilibrium numbers of quarks and alike particles, reheats the expanding fireball as well as the ''normal'' chemistry of the ideal quark-hadron mixture. The more one suppresses color charges with respect to an ideal mixture the more reheating occurs during hadronization. This, of course, works against the hadronization process. The only physical effect besides a fast expansion—which has, however, kinematical limits stemming from scaling relativistic expansion—that can cool the mixture sufficiently is rest-mass production. The color charge eliminating hadronization, therefore, must be accompanied by the production of heavy hadron resonances. (Rest-mass production due to quark pair or gluon creation would not reduce color.) Since in our equation of state we have an explicit interaction energy between the quarks, our effective quark masses should be less than that given in Ref. [6]. We shall use the following values: $m_u = [m_{u0}^2 + m_{\rm th}^2]^{1/2}, m_d = [m_{d0}^2 + m_{\rm th}^2]^{1/2}, m_s = [m_{s0}^2 + m_{\rm th}^2]^{1/2}$, with thermal mass $m_{\rm th} = 0.15$ GeV and $m_{u0} = m_{d0} \approx 0, m_{s0} = 0.15$ GeV. The clusters have a mass according to the average mass extra to the summed valence quark masses of the two lowest-lying hadron multiplets: the pseudoscalar and vector meson nonets and the baryon octet and decouplet, respectively.

III. DYNAMICS OF HADRONIZATION

A. Initial state

The initial energy density — distributed along the beam direction between $-\tau_0 \sinh\eta_0$ and $\tau_0 \sinh\eta_0$ — can be related to the center-of-mass bombarding energy $\sqrt{s}$ in the experiment

$$\varepsilon_0 = \frac{\sqrt{s}}{\pi R_0^2 \tau_0 2\,{\rm sh}\,\eta_0}. \tag{14}$$

On the other hand, the initial invariant volume dual to $d\tau$ at constant $\tau = \tau_0$ is given by

$$V_0 = \pi R_0^2 \tau_0 2\,\eta_0. \tag{15}$$

The initial internal energy (i.e., the energy without the collective flow of a fluid cell) at $\tau = \tau_0$ is, therefore, less than $\sqrt{s}$ for finite η_0:

$$E_0 = \varepsilon_0 V_0 = \frac{\eta_0}{{\rm sh}\,\eta_0}\sqrt{s}. \tag{16}$$

At the CERN SPS experiment $\eta_0 \approx 1.75$ (due to some stopping), $R_0 \approx 7$ fm, $\tau_0 \approx 0.8$ fm and we obtain $V_0 \approx 431$ fm^3 and $E_0 \approx 2.13$ TeV. Compared to the total energy of about $\sqrt{s} = 3.4$ TeV (carried by about 390 participating nucleons in a central Pb-Pb collision) approximately two-thirds of the energy is invested into the rest mass of newly produced particles and thermal motion and one-third into the flow.

Comparing this with an alternative expression for the thermal energy of an ideal massive quark matter

$$E_0 = \sum_i N_i(0)\left(m_i + \frac{3}{2}T\right), \tag{17}$$

one can estimate the initial temperature at the beginning of hadronization. Using our standard values for the incoming quark numbers [18] $N_u(0) = 544, N_d(0) = 626$, further, assuming that 400 $u\overline{u}$, 400 $d\overline{d}$ and, with $f_s = 0.21$, 168 $s\overline{s}$ quark antiquark pairs are created in one central collision, we get from the above equation $T_0 = 0.18$ GeV. We use these numbers for the newly produced quark pairs in order to arrive the experimentally measured hadron and strange particle numbers.

B. Hadronization processes

Chemical equilibrium is not supposed initially, rather a definite oversaturation of quarks in the reaction volume. The initially missing color-neutral hadron states—mesons and baryons—are formed due to quark fusion processes in a nonrelativistic Coulomb potential. The rates for different flavor compositions differ mainly due to the different reduced masses of quark antiquark or quark diquark pairs. First of all this influences the Bohr radius in the Coulomb potential [9]. Of course, the presence of a medium—which establishes the necessary momentum balance after the fusion—also influences the hadronization rate. The cross section for such a $2 \to 1$ process in a medium is

$$\sigma=\left(\frac{\rho}{a}\right)^3\frac{16M^2\sqrt{\pi}\alpha^2}{(\vec{p}^2+1/a^2)^2} \tag{18}$$

with $a=1/(\alpha m)$ Bohr radius of the $1s$ state in the Coulomb potential and ρ is the Debye screening length [9]. Here $\vec{p}$ is the relative momentum of the hadronizing precursors, m is their reduced mass, and M is the total mass.

The coupling constant we are using is a function of the relative momentum according to the formula

$$\frac{1}{\alpha(p)}=\frac{1}{\alpha_0}+\beta_{\rm QCD}\ln\frac{p^2}{\Lambda^2_{\rm QCD}} \tag{19}$$

as long as $\alpha(p)<\alpha_0$. At smaller momenta it levels off at α_0. Here

$$\beta_{\rm QCD}=\frac{1}{4\pi}\left(\frac{11}{3}-2N_F\right) \tag{20}$$

is the β parameter of the one-loop β function occurring in the charge renormalization process in perturbative QCD. The function (19) is a good phenomenological approximation and has the correct infrared and ultraviolet asymptotics. The value $\alpha_0\approx 1.4$ is taken in order to fit the gluon condensate strength in vacuum.

The medium effect is parametrized in all cross sections by $\rho\approx 0.2-0.3$ fm in a flavor-independent way. If the hadronization process is fast enough, the prehadron numbers (i.e., hadron numbers before resonance decay) become proportional to the two-body reaction rates. Since quark fusion does not change the number of valence quarks during the hadronization the fast hadronization limit leads to the ALCOR model [9].

The relative momenta are taken from a random Gaussian distribution,

$$d\mathcal{P}(\vec{p})\propto e^{-p^2/2mT}d^3p, \tag{21}$$

at temperature T and reduced mass m. This method allows us both to simulate thermally averaged hadronization rates,

$$R=\left\langle \sigma\frac{|\vec{p}|}{m}\right\rangle, \tag{22}$$

or to follow the event by event variation of the quark-hadron composition. For a fast estimate of thermally averaged rates, about $3-5$ points in the relative phase space suffice for each reaction at each time instant.

Although the above corrections to the equation of state help to reduce the equilibrium ratio of free quarks to those confined in hadrons, total color confinement would only occur after a long time when the mixture is cool and dilute. This is obviously not the case in relativistic heavy-ion collisions. Therefore, we also apply a *dynamic confinement* mechanism in our model: the medium screening length ρ occurring in the hadronization cross section will be related to strings pulled by color charges trying to leave the reaction zone. This way the screening length ρ is increased as the color density decreases: we keep, however, the product $\rho^3 N_c/V$ constant

$$\rho(t)=\rho(0)*[N_c(0)/N_c(t)]^{1/3}. \tag{23}$$

Here N_c stands for the number of colored objects (each in triplet or antitriplet representation),

$$N_c=N_Q+N_{\bar{Q}}+N_D+N_{\bar{D}}. \tag{24}$$

As we shall discuss in the next chapter both color confinement mechanisms are necessary in order to achieve a pure hadronic composition in a short time by transchemical processes, while satisfying the requirements of the entropy growth and energy conservation.

C. Reaction network

What remains to specify the model is the system of rate equations describing the transformation of quark matter into hadronic matter. We consider $N_F=3$ light quark flavors u, d, and s.

There are $N_F(N_F+1)/2=6$ possible diquark flavors and the same number of antidiquark flavors. The number of quark antiquark flavor combinations is $N_F^2=9$, while that of quark or antiquark triplet combinations is $N_F(N_F+1)(N_F+2)/6=10$. In the hadronizing quark matter we deal with altogether $2*3+2*6+9+2*10=47$ sorts of particles.

Let us generally denote quarks by Q, diquarks by D, mesons by M and baryons by B. The possible fusion reactions are

$$\begin{aligned} Q+Q&\rightarrow D,\\ \bar{Q}+\bar{Q}&\rightarrow \bar{D},\\ Q+\bar{Q}&\rightarrow M,\\ Q+D&\rightarrow B,\\ \bar{Q}+\bar{D}&\rightarrow \bar{B}. \end{aligned} \tag{25}$$

Accordingly the number of reaction channels, $2*10+9+2*6=41$ is $2N_F=6$ less than the number of particle sorts. Therefore, in chemical equilibrium, when the above reactions are balanced by the respective decays, the number of all particles are underdetermined.

This is fortunately not a problem, because the hadronization reactions under consideration do not create or annihilate elementary quark flavors, they just produce new combinations. There are, therefore, exactly $2N_F=6$ quark and antiquark flavor numbers, which are conserved by these reactions. They are determined by the initial state.

Our model is completed by the system of rate equations. Considering a general reaction of type

$$i+j\rightarrow k,$$

we account for the changes

$$dN_i=dN_j=-A\,dt,\quad dN_k=+A\,dt, \tag{26}$$

cumulatively in each reaction. Here

$$A=R_{ij\rightarrow k}N_iN_j(1-e^{\mu_k/T-\mu_j/T-\mu_i/T}) \tag{27}$$

with a thermally averaged rate $R_{ij\to k}$ [cf. Eq. (22)]. The changes stemming from different reactions accumulate to a total change of each particle sort in a time-step dt.

Inspecting the expression (27) it is transparent that an extra increase of a chemical potential suppresses the production or enhances the decay of the corresponding particle. This behavior ensures that

$$\dot{S}=-\sum_i \frac{\mu_i}{T}\dot{N}_i>0, \tag{28}$$

for any complicated network of reactions. The approach to chemical equilibrium always produces entropy.

D. Chemical equilibrium

Chemical equilibrium is defined by the requirement that all chemical rates vanish [cf. Eq. (27)]. It leads to relations like

$$\mu_i^{\rm eq}+\mu_j^{\rm eq}=\mu_k^{\rm eq} \tag{29}$$

for each reaction channel. The correspondence between equilibrium chemical potentials and equilibrium number densities is, however, in the general case not as simple as for a mixture of ideal gases. From Eq. (8) we obtain an implicit equation for the equilibrium densities of colored particles

$$n_i^{\rm eq}=n_i^{\rm th}\exp\left[\frac{\mu_i^{\rm eq}-b_i(n_c^{\rm eq})-m_i}{T}\right], \tag{30}$$

where $n_c^{\rm eq}$ is the color charge density in equilibrium, [see Eq. (4)], $n_i^{\rm th}=N_i^{\rm th}/V$ from Eq. (A17) and

$$b_i(n_c)=\sigma_s n_c^{-1/3}\left(q_i-\frac{1}{3}\overline{q}\right). \tag{31}$$

We call the attention to the fact that no solution of Eq. (30) exists below a critical temperature.

Since the number of reactions is less than the number of particle sorts, the equilibrium state is not fully determined by these conditions alone. The missing information is contained in the value of conserved numbers.

Applying Eq. (8), the nonequilibrium chemical potentials, and hence the essential factors, $e^{-\mu_i/T}$, in the chemical rates [Eq. (27)] can be expressed as

$$e^{-\mu_i/T}=e^{-\mu_i^{\rm eq}/T}\cdot e^{[b_i(N^{\rm eq})-b_i(N)]/T}\cdot N_i^{\rm eq}/N_i. \tag{32}$$

In the combinations appearing in the detailed balance factor of the rate equations using Eq. (29) we obtain

$$1-e^{(\mu_k-\mu_i-\mu_j)/T}=1-\frac{N_i^{\rm eq}}{N_i}\frac{N_j^{\rm eq}}{N_j}\frac{N_k}{N_k^{\rm eq}}e^{(\Delta\mu_k-\Delta\mu_i-\Delta\mu_j)/T} \tag{33}$$

with

$$\Delta\mu_i=b_i(N)-b_i(N^{\rm eq}). \tag{34}$$

The corrections $b_i(N)$ in the nonequilibrium chemical potentials may in general depend on the number densities of several other components on the mixture.

At this point we note that the extra $e^{-\Delta\mu_i/T}$ factors occur for nonideal equations of state where the correction to the free energy density is a nonlinear function of the number densities.

E. Detailed rate equations

With the purpose of better understanding in mind we enumerate certain types of the rate equations given in the above compressed form in Eqs. (26),(27). Let $\{u,d,s,\dots\}$ be an ordered set of flavor indices. We denote quarks by flavor indices i, j, or k, the antiquarks by their overlined versions, diquarks consisting of a flavor pair satisfying $i\leqslant j$ by $[ij]$, a meson made from a flavor i quark and a flavor $\overline{j}$ antiquark by $[i\overline{j}]$, and finally a baryon made from quark flavors i, j, and k satisfying $i\leqslant j\leqslant k$ by $[ijk]$. In fusion reactions we create primarily mesons, (anti)diquarks and (anti)baryons.

The rate equations for these reactions are given as follows:

$$\begin{aligned}
V\frac{dN_i}{dt}=&-\sum_{j\geqslant i}R_{i+j\to[ij]}\left(N_iN_j-\frac{N_i^{\rm eq}N_j^{\rm eq}}{N_{[ij]}^{\rm eq}}N_{[ij]}e^{(\Delta\mu_{[ij]}-\Delta\mu_i-\Delta\mu_j)/T}\right)\\
&-R_{i+i\to[ii]}\left(N_iN_i-\frac{N_i^{\rm eq}N_i^{\rm eq}}{N_{[ii]}^{\rm eq}}N_{[ii]}e^{(\Delta\mu_{[ii]}-\Delta\mu_i-\Delta\mu_i)/T}\right)\\
&-\sum_j R_{i+\overline{j}\to[i\overline{j}]}\left(N_iN_{\overline{j}}-\frac{N_i^{\rm eq}N_{\overline{j}}^{\rm eq}}{N_{[i\overline{j}]}^{\rm eq}}N_{[i\overline{j}]}e^{(\Delta\mu_{[i\overline{j}]}-\Delta\mu_i-\Delta\mu_{\overline{j}})/T}\right)\\
&-\sum_{[jk]}R_{i+[jk]\to[ijk]}\left(N_iN_{[jk]}-\frac{N_i^{\rm eq}N_{[jk]}^{\rm eq}}{N_{[ijk]}^{\rm eq}}N_{[ijk]}e^{(\Delta\mu_{[ijk]}-\Delta\mu_i-\Delta\mu_{[jk]})/T}\right)
\end{aligned} \tag{35}$$

for quarks

$$V\frac{dN_{[ij]}}{dt}=R_{i+j\to[ij]}\left(N_iN_j-\frac{N_i^{\rm eq}N_j^{\rm eq}}{N_{[ij]}^{\rm eq}}N_{[ij]}e^{(\Delta\mu_{[ij]}-\Delta\mu_i-\Delta\mu_j)/T}\right)_{i\leqslant j}$$
$$-\sum_k R_{[ij]+k\to[ijk]}\left(N_{[ij]}N_k-\frac{N_{[ij]}^{\rm eq}N_k^{\rm eq}}{N_{[ijk]}^{\rm eq}}N_{[ijk]}e^{(\Delta\mu_{[ijk]}-\Delta\mu_{[ij]}-\Delta\mu_k)/T}\right) \tag{36}$$

for diquarks,

$$V\frac{dN_{[i\bar{j}]}}{dt}=R_{i+\bar{j}\to[i\bar{j}]}\left(N_iN_{\bar{j}}-\frac{N_i^{\rm eq}N_{\bar{j}}^{\rm eq}}{N_{[i\bar{j}]}^{\rm eq}}N_{[i\bar{j}]}e^{(\Delta\mu_{[i\bar{j}]}-\Delta\mu_i-\Delta\mu_{\bar{j}})/T}\right) \tag{37}$$

for mesons and finally

$$V\frac{dN_{[ijk]}}{dt}=R_{i+[jk]\to[ijk]}\left(N_iN_{[jk]}-\frac{N_i^{\rm eq}N_{[jk]}^{\rm eq}}{N_{[ijk]}^{\rm eq}}N_{[ijk]}e^{(\Delta\mu_{[ijk]}-\Delta\mu_i-\Delta\mu_{[jk]})/T}\right)_{i\leqslant j\leqslant k}$$
$$+R_{j+[ik]\to[ijk]}\left(N_jN_{[ik]}-\frac{N_j^{\rm eq}N_{[ik]}^{\rm eq}}{N_{[ijk]}^{\rm eq}}N_{[ijk]}e^{(\Delta\mu_{[ijk]}-\Delta\mu_j-\Delta\mu_{[ik]})/T}\right)_{i<j\leqslant k}$$
$$+R_{k+[ij]\to[ijk]}\left(N_kN_{[ij]}-\frac{N_k^{\rm eq}N_{[ij]}^{\rm eq}}{N_{[ijk]}^{\rm eq}}N_{[ijk]}e^{(\Delta\mu_{[ijk]}-\Delta\mu_k-\Delta\mu_{[ij]})/T}\right)_{i\leqslant j<k} \tag{38}$$

for baryons. We obtain similar equations for antiquarks, antidiquarks, and antibaryons.

We would like to point out that the novel appearance of the above rate equations is a consequence of the equation of state we are using. This form goes beyond the usual expression multilinear in $N_i/N_i^{\rm eq}$, and reflects the fact that the hadronization reactions are strongly influenced by the presence of colored particles not taking part in the particular two-body process directly.

Especially the rate equation for baryon production is complicated. The conditions indicated as lower indices at the bracket expressions or in the summation signs rule out multiple counting of baryons which contain equal flavor quarks. For example, the reaction $u+[uu]\to[uuu]$ makes only one $[uuu]$ baryon (pre-Δ^{++}) and not three. On the other hand, $[uds]$ can be constructed from three different quark-diquark fusion processes:

$$u+[ds]\to[uds],$$
$$d+[us]\to[uds],$$
$$s+[ud]\to[uds]. \tag{39}$$

The only easy and transparent notation we found for these reactions is the compact Eq. (26). Since we calculate the evolution of the number of a given type of quark group (e.g., uud), at the final time we distribute this number between the corresponding multiplets — the lowest lying pseudoscalar and vector nonets for the mesons and lowest lying octet and decouplet for baryons (anti-baryons) — according to the spin degeneracy. (E.g., the number of the uud quark group is distributed between the p^+ and Δ^+ in the ratio two to four, while uuu populates only the Δ^{++} resonance.)

F. Hadronic decays

The set of rate equations describes the time evolution of the number of all involved particles. In order to get the final hadron numbers we integrate these equations until the number of colored particles becomes negligible. This way we obtain a number of hadronic resonances (in the present version the vector meson nonet and baryon decouplet). Finally hadronic decays are taken into account with the dominant branching ratios obtained from the Particle Data Table [19]. We assume that secondary hadron-hadron interactions have a negligible effect on the finally observed hadronic composition.

The time evolution of the entropy and temperature is obtained by simultaneous integration of Eqs. (A19) and (1). The structure of the rate equations containing both creation and decay terms ensures that the entropy never decreases during the transition [26]. On the other hand, the energy conservation is established by the effective cooling law Eq. (1). The numerical study of this complex system of equations is presented in the next section.

IV. NUMERICAL RESULTS AND DISCUSSION

In this section we present the results of the numerical solution of the set of the $41+2$ coupled differential equations. In the figures the calculated time evolution of the temperature, entropy, internal energy, and the different particle and antiparticle numbers are presented for a 158 GeV/nucleon Pb+Pb central collision. For the numerical solution the initial condition and the values of the parameters has to

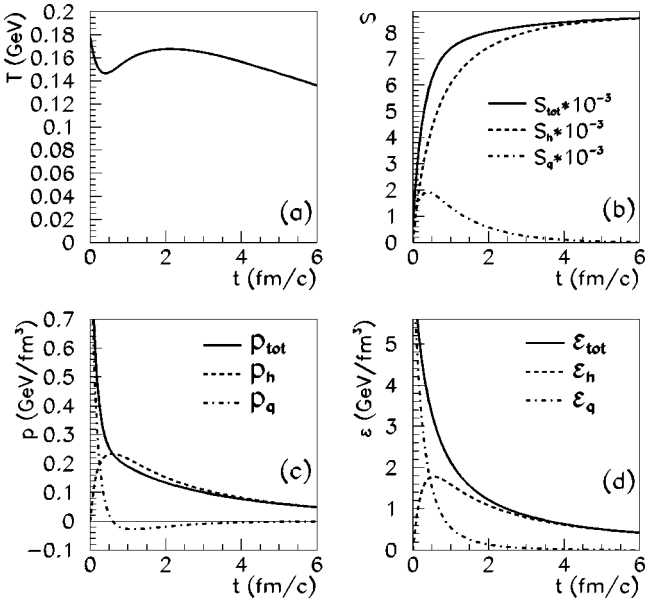

FIG. 1. The time evolution of the temperature T (a), the entropy S (b), the pressure p (c), and the energy density ε (d) of the system together with the partial contributions of quarks and hadrons.

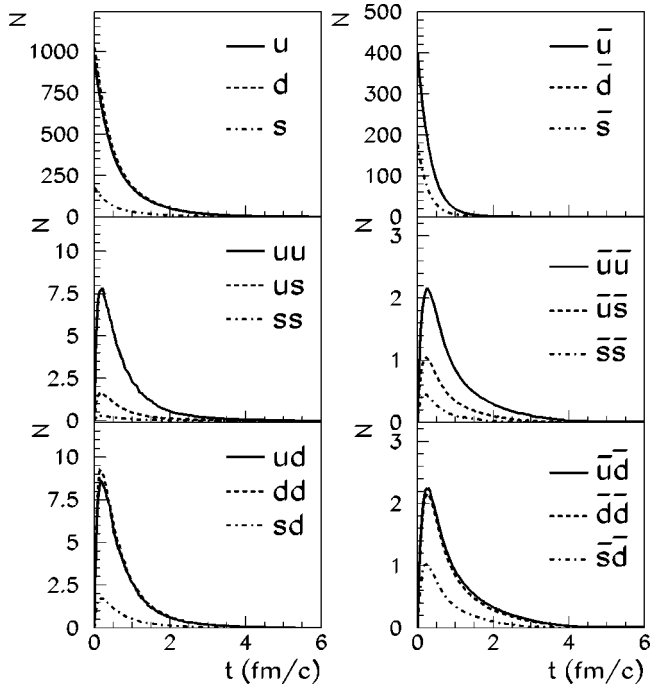

FIG. 2. The time evolution of the number of colored particles. The line styles of different flavor compositions are indicated in the respective figures.

be specified. For the parameters describing the initial state we used those given in Sec. III A, while for the parameters determining the dynamics of the hadronization we used the following values: $\rho=0.2$ fm, $\alpha=1.4$. Both the quarks and diquarks vanished only in the case, if we used the value $q_d=1.5$ for the effective diquark color string tension. The time evolution of different quantities calculated with this initial condition and parameters are shown in the figures.

From Fig. 1(a) we can see that at the beginning of the hadronization there is a rapid decrease in the temperature due to the rest mass formation of the hadrons. Shortly after that, the reheating starts as an effect of color confinement [see Eqs. (28) and (32)]. Equation (28) shows that by removing two colored objects to produce a colorless hadron, the associated string energy is also removed, and it has to appear in the thermal energy. Since this energy is inversely proportional to the one-third root of the color density, this effect is stronger at smaller color density. Finally, as the hadronization is completed, the expansion leads to the cooling of the system. Figure 1(b) shows, that the total entropy is monotonically increasing during the hadronization, while the contribution of colored particles (CQP) is gradually eliminated. In Fig. 1(c) one can observe an interesting pattern in the time evolution of the pressure. The partial pressure of the interacting CQP rapidly decreases as the number of quarks decrease. As the color density drops, this pressure becomes even negative. The increasing hadron partial pressure, however, overcompensates this negative value. At later times with the expansion of the ideal hadron gas the total pressure decreases. Due to this interplay, the total pressure falls steeply in the early period, it stays for a short while almost constant in the reheating period, and finally drops slowly at the late expansion stage. Assuming an initial condition leading to slower hadronization, this balance is not so effective and in the time evolution of pressure a local minimum appears. The partial and total internal energy evolution, displayed in Fig. 1(d), shows that the hadronization is completed at 2 fm/c after the beginning of the process. The decrease of the internal energy is compensated by the work of pressure while making the flow [cf. Eq. (A5)], the total energy density drops due to the volume expansion like $\epsilon \propto 1/\tau$ at very late times only, when $p=0$.

Figure 2 shows the time evolution of different colored particles. The quark and antiquarks numbers are monotonically decreasing functions of time. The diquarks are produced from the quarks, and then they contribute rapidly to the formation of baryons. That keeps their number always on a low level. In Fig. 3 the evolution of hadron numbers is shown. Here we did not take into account the decay of the resonances. Actually these time evolutions are the evolution of the constituent quark clusters, i.e., prime hadronic resonances, from which the experimentally observed particles emerge. It is clear from these figures that the mesons are formed faster than the baryons. This is understandable, if we consider that the mesons are produced in a one-step process, while the baryons are formed in a two-step process: first the formation of diquarks and in a diquark quark reaction the formation of the baryon. In the ALCOR model the ratio of hadronic species are determined by the ratio of steepnesses of these curves. Since these curves do not cross each other, one can understand why the algebraic ALCOR approach to the solution of rate equations is a good approximation. The difference that at the very beginning the meson curves increase linearly with the time and the baryon curves start with a quadratic form, was taken into account in the ALCOR model by one single common factor, the so-called baryon suppression factor.

In our model we calculate the number of all produced hadrons. In Table I the hadron numbers obtained with the Transchemistry model (in the first row for one-dimensional, and in the second for three-dimensional flow results) and

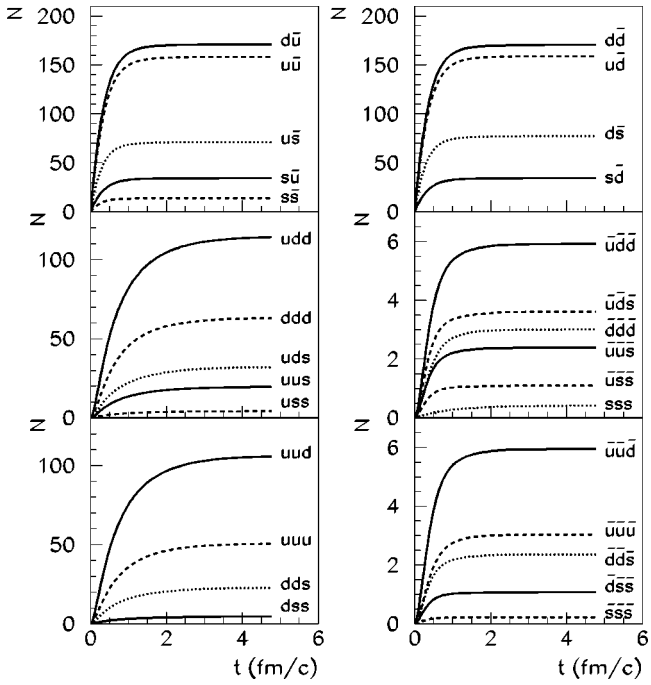

FIG. 3. The time evolution of the number of color neutral clusters consisting of quarks and/or antiquarks indicated on the figures.

those obtained with the ALCOR [18] and the RQMD [20] models are shown together with the few published experimental data. The present results of the ALCOR model are slightly different from the previously published ones [18], because here we considered the feeding of Λ^0 (and $\overline{\Lambda^0}$) particle multiplicities from $\Sigma^0,\Xi^-,\Xi^0,\Omega^-$ (and $\overline{\Sigma^0}$, $\overline{\Xi}^+$, $\overline{\Xi}^0$, $\overline{\Omega}^+$, respectively), as it was obtained in the experiments. The ALCOR fit parameters to this new case are: $N_{q\bar{q}}=391$, $N_{s\bar{s}}=172$, and $\alpha_{\rm eff}=0.97$ comparing to the earlier $N_{q\bar{q}}=398$, $N_{s\bar{s}}=175$, and $\alpha_{\rm eff}=1.03$.

Table II shows a comparison for the multistrange baryon ratios. While in many cases there are intriguing agreements, in some other cases there are some discrepancies. The reason for this may originate from two sources: (i) the experimental data referred to here are the production ratios in the overlap window of the detector acceptances. Thus, if the momentum distribution of the two particle sorts is not the same, then these ratios are not equal to the total number ratios. (ii) These calculated values are more sensitive to the simplifying assumption that the hadronization happens into the lowest-lying baryon octet and decouplet and the two lowest-lying meson nonet. We have committed this simplification, because the inclusion of other hadron multiplets would have multiplied the workload in the computer simulation. In our model we do not have such parameters with which the different particle ratios could be manipulated independently. It is also not possible to distinguish between the rapidity distributions of baryons and antibaryons in the present version of the transchemistry model. Thus due to simplifications which can possibly be overridden in a future extension of this work. Nonetheless the degree of the overall harmony between our results and the presently known experimental data seems to be promising and thus it could initiate the search for further signatures of the CQP state.

V. CONCLUSION

In this paper we presented a new model for the hadronization of a constituent quark plasma (CQP) based on rate equations in a quark matter–hadron matter mixture. The color confinement was taken into account by using consistently a plausible equation of state motivated by the string model. The role of the different physical processes entering into the hadronization was discussed. Our results presented in the figures clearly show a very fast hadronization. This is mainly a consequence of the large hadronization rates, $R_{ij\to k}$ and of the fact that in the initial state the system is very far from equilibrium. Observing the shape of the time evolution of different hadron multiplicities, it became understandable, why the simpler algebraic approximation, applied in the ALCOR model, works so well. The comparison with the existing experimental data indicate that it is possible, that in the Pb Pb collision at SPS a piece of matter is formed, inside which the massive quarks and antiquarks interact with a stringlike mean field.

Finally we emphasize that these types of phenomenological investigations are necessary, as long as the hadronization of the quark matter as a nonequilibrium, nonstatic, nonperturbative process, cannot be described with the methods of the elementary field theory.

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APPENDIX: THE RELEVANT HYDRODYNAMICAL AND THERMODYNAMICAL EXPRESSIONS

The familiar energy-momentum conservation

$$\partial_\nu T^{\mu\nu}=0 \tag{A1}$$

with

$$T^{\mu\nu}=(\varepsilon+p)u^\mu u^\nu - p g^{\mu\nu}, \tag{A2}$$

being the energy-momentum tensor of a perfect fluid with flow four-velocity u_μ, local energy density ε, and pressure p, has two interesting projections: one is parallel to the four velocity,

$$u_\mu \partial_\nu T^{\mu\nu}=\partial_\nu(\varepsilon u^\nu)+p(\partial_\nu u^\nu)=0, \tag{A3}$$

and the other is orthogonal to this

$$[u^\mu(u^\nu\partial_\nu)-\partial^\mu]p+(\varepsilon+p)(u^\nu\partial_\nu)u^\mu=0. \tag{A4}$$

The former—expressing local energy conservation—can be cast into the suggestive form

$$dE+p\,dV=0 \tag{A5}$$

TABLE I. Total hadron multiplicities for Pb+Pb collision at 158 GeV/nucleon bombarding energy. The displayed experimental results are from the NA49 Collaboration. Theoretical results are from the Transchemistry, ALCOR, and RQMD [20] (''ropes+no rescattering'' version) model. Here Λ^0 like denotes the sum $\Lambda^0+\Sigma^0+\Xi^-+\Xi^0+\Omega^-$.

Pb+Pb	NA49	TrCHEM.(1)	TrCHEM.(3)	ALCOR	RQMD
h^-	680[a]	677.0	669.1	679.8	
π^+		581.7	571.4	590.6	692.9
π^0		617.0	609.2	605.9	724.9
π^-		613.1	602.8	622.0	728.8
K^+	76[d]	79.58	78.29	78.06	79.0
K^0		79.58	78.29	78.06	79.0
$\overline{K^0}$		39.47	39.79	34.66	50.4
K^-	{32}[b]	39.47	39.79	34.66	50.4
p^+		158.7	162.2	153.2	199.7
n^0		175.8	179.5	170.5	217.6
Σ^+		8.38	8.32	9.16	12.9
Σ^0		9.79	9.97	9.76	13.1
Σ^-		9.79	9.45	10.39	13.3
Λ^0		46.79	47.06	48.85	35.3
Ξ^0		4.40	4.21	4.89	4.2
Ξ^-		4.43	4.25	4.93	4.2
Ω^-		0.42	0.36	0.62	
$\overline{p}^-$		8.98	11.25	6.24	27.9
$\overline{n}^0$		8.93	11.21	6.24	27.9
$\overline{\Sigma}^-$		1.00	1.21	0.91	4.6
$\overline{\Sigma}^0$		1.09	1.37	0.91	4.6
$\overline{\Sigma}^+$		0.99	1.19	0.91	4.6
$\overline{\Lambda}^0$		5.24	6.41	4.59	10.7
$\overline{\Xi}^0$		1.08	1.25	1.12	2.0
$\overline{\Xi}^+$		1.08	1.25	1.12	2.0
$\overline{\Omega}^+$		0.22	0.23	0.35	
K_S^0	{54}[b,c]	59.66	59.04	56.36	63.5
$p^+-\overline{p}^-$	{145}[a]	149.7	151.0	147.0	171.8
Λ^0 like	{50±10}[b]	56.58	57.03	69.07	56.8
$\overline{\Lambda}^0$ like	{8±1.5}[b]	6.34	7.77	8.12	19.3

[a]Estimated result (Ref. [21]).
[b]Reference [22].
[c]Reference [23].
[d]Estimated from $\{K^-\}$ and $\{K_S^0\}$.

using an infinitesimal volume element, $dV=(\partial_\mu u^\mu)V\,d\tau$, the total internal energy, $E=\varepsilon V$, and the parametric derivation with respect to the time variable in the comoving fluid cell,

$$\frac{d}{d\tau}=u^\nu\partial_\nu. \tag{A6}$$

Starting from the first-principals theorem of thermodynamics

$$dE=T\,dS-p\,dV+\sum_i \mu_i\,dN_i, \tag{A7}$$

TABLE II. Strange baryon and antibaryon ratios measured by WA97 Collaboration [24,25] and obtained from Transchemistry and the ALCOR model for Pb+Pb collision at 158 GeV/nucleon bombarding energy. The experimental data are the production ratios in midrapidity at $p_T>0$ GeV.

Pb+Pb	WA97	TrCHEM.(1)	TrCHEM.(3)	ALCOR
$\overline{\Lambda}^0/\Lambda^0$	0.128±0.012	0.112	0.136	0.117
$\overline{\Xi}^+/\Xi^-$	0.266±0.028	0.243	0.293	0.227
$\overline{\Omega}^+/\Omega^-$	0.46±0.15	0.529	0.625	0.564
Ξ^-/Λ^0	0.093±0.007	0.078	0.078	0.071
$\overline{\Xi}^+/\overline{\Lambda}^0$	0.195±0.023	0.170	0.160	0.138
Ω^-/Ξ^-	0.195±0.028	0.095	0.085	0.125

and using Eq. (A5) we arrive at the following rate of change of the total entropy:

$$\dot{S} = -\sum_i \frac{\mu_i}{T}\dot{N}_i, \tag{A8}$$

where the ''dot'' stands for the time derivative ($\dot{S} = dS/dt$). The chemical potentials μ_i can generally be derived from the equation of state.

We consider a flow pattern given by the Bjorken flow

$$u_\mu = \left(\frac{t}{\tau}, \frac{z}{\tau}, 0, 0\right). \tag{A9}$$

Here

$$\tau^2 = t^2 - z^2, \tag{A10}$$

and the four-velocity is normalized to $u_\mu u^\mu = 1$ and its four divergence is

$$\partial_\mu u^\mu = \frac{1}{\tau}. \tag{A11}$$

The energy density ϵ and the pressure p depend only on τ and the flow is stationary. Corrections stemming from a mild radial flow also can be worked out, but we do not use these corrections here. The effective dimensionality of the flow is 1 in this case.

Since we deal with a system of rate equations for particle species numbers N_i occupying a common reaction volume V at temperature T, the generating thermodynamical functional is the free energy, $F(V,T,N_i)$. The chemical potentials are given by

$$\mu_i = \frac{\partial F}{\partial N_i}, \tag{A12}$$

the pressure by

$$p = -\frac{\partial F}{\partial V}, \tag{A13}$$

the entropy by

$$S = -\frac{\partial F}{\partial T}, \tag{A14}$$

and finally the internal energy is

$$E = F + TS. \tag{A15}$$

As a starting point we consider a mixture of ideal gases of massive quarks, diquarks, mesons, and baryons, and their respective antiparticles. The corresponding free energy is

$$F_{\rm id} = \sum_i TN_i\left(\ln\frac{N_i}{N_{i,\rm th}} - 1\right) + \sum_i m_i N_i, \tag{A16}$$

with Maxwell-Boltzmann statistics for the nonrelativistic massive matter:

$$N_i^{\rm th} = Vd_i \int \frac{d^3p}{(2\pi)^3} e^{-p^2/2m_iT}. \tag{A17}$$

Here $d_i = (2s_i+1)c_i$ are spin and color degeneracy factors. The chemical potentials in an ideal-gas mixture are

$$\mu_{i,\rm id} = T\ln\frac{N_i}{N_i^{\rm th}} + m_i. \tag{A18}$$

The total entropy is given by

$$S_{\rm id} = \sum_i N_i\left(\frac{5}{2} - \ln\frac{N_i}{N_i^{\rm th}}\right). \tag{A19}$$

The energy and pressure of such an ideal, nonrelativistic mixture is given by

$$E_{\rm id} = \sum_i \left(m_i + \frac{3}{2}T\right)N_i,$$

$$p_{\rm id} = \sum_i N_i\frac{T}{V}. \tag{A20}$$

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