Semirelativistic resonating group method calculations of pion-pion scattering

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Pion-pion scattering is investigated in the framework of the resonating group method. The wave function of an isolated pion, described as a quark-antiquark system interacting through a potential composed of a central and a spin-spin part, is determined using the spinless Salpeter equation. Given these ingredients the kernel of the integrodifferential equation governing the relative motion of the colliding pions is calculated using relativistic kinematics. The corresponding S- and D-wave pion-pion phase shifts are compared with their Galilean counterparts. [S0556-2813(99)01302-3]

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I. INTRODUCTION

The resonating group method (RGM) is the natural way to investigate the scattering of composite particles. This method has been taken up in multiquark physics, for instance, to describe nucleon-nucleon scattering in terms of the underlying quark dynamics. For this process it is possible to justify the use of Galilean kinematics because the constituent quark mass is generally taken equal to about one third of the nucleon mass. For the other systems composed of light quarks, relativistic kinematics ought to be used. It is thus worth evaluating the effects of relativity in the application of the RGM to multiquark problems. To this purpose pion-pion scattering appears as the most suitable process in that the relativistic effects are expected to be especially large; moreover the pion is the simplest multiquark system presenting a color singlet.

The present work aims at making RGM calculation of pion-pion scattering consistent, to some extent, with the requirements of special relativity by substituting the nonrelativistic kinetic energy operator of the ith particle by its relativistic counterpart. Thus the present calculations rely on the Hamiltonian

$$H=T_1+T_{\bar{1}}+T_2+T_{\bar{2}}+V_{1\bar{1}}+V_{2\bar{2}}+V_{1\bar{2}}+V_{2\bar{1}}+V_{12}+V_{\bar{1}\bar{2}} \tag{1}$$

with

$$T_i=\sqrt{m^2+\vec{p}_i^{\,2}}-m, \quad m=m_u=m_d, \tag{2}$$

and

$$V_{ij}=-\frac{3}{16}\tilde{\lambda}_i\cdot\tilde{\lambda}_j(v_{ij}+w_{ij}\vec{s}_i\cdot\vec{s}_j). \tag{3}$$

The matrices $\tilde{\lambda}_i$ are the SU(3) color generators of the ith particle ($-\tilde{\lambda}_i^*$ for antiparticles). The dependence of V_{ij} upon the interparticle distance r_{ij} is taken of the form

$$v_{ij}=-\frac{A}{r_{ij}}+Br_{ij}-C, \quad w_{ij}=V_g\exp\left(-\frac{r_{ij}^2}{r_0^2}\right). \tag{4}$$

We fully realize that the use of a naive two-body potential in four-quark systems is questionable. Actually our efforts center on the technical aspects inherent in the handling of relativistic kinetic energy operators in the framework of the RGM and we believe that, notwithstanding the use of an oversimplified picture of the interaction between the constituent quarks, our calculations are capable of yielding a reliable comparison between semirelativistic RGM phase shifts and their Galilean counterparts.

The method of calculation, namely, the evaluation of the exchange kernel with regard to the semirelativistic Hamiltonian (1) and the procedure used to extract the corresponding scattering phase shifts, is outlined in Sec. II. Our results are discussed in Sec. III. Concluding remarks are presented in Sec. IV.

II. METHOD OF CALCULATION

The wave function of an isolated S-wave pion is given by

$$\psi(r_{q\bar{q}})=\phi(r_{q\bar{q}})[s^{1/2}(q)s^{1/2}(\bar{q})]^0[\tau^{1/2}(q)\tau^{1/2}(\bar{q})]^1 \tag{5}$$

where $s^{1/2}$ and $\tau^{1/2}$ denote the spin and isospin wave functions of a single quark or antiquark. The square brackets in Eq. (5) and in subsequent expressions stand for angular momentum and isospin coupling. The spatial wave function is approximated by

$$\phi(r_{q\bar{q}})=b^{-3/2}\left(\frac{2}{\pi}\right)^{1/4}\exp\left(-\frac{r_{q\bar{q}}^2}{4b^2}\right), \tag{6}$$

where $b=(m\omega)^{-1/2}$ is the oscillator length parameter. The value of b and the corresponding theoretical pion mass are obtained by minimizing the expectation value of the relevant spinless Salpeter Hamiltonian [1,2]. Thus

$$m_\pi=\min_b\int_0^\infty\phi(r_{q\bar{q}})\left[2\sqrt{m^2+\vec{p}_{q\bar{q}}^{\,2}}+v_{q\bar{q}}-\frac{3}{4}w_{q\bar{q}}\right]\times\phi(r_{q\bar{q}})r_{q\bar{q}}^2dr_{q\bar{q}} \tag{7}$$

which in the nonrelativistic limit reduces to

$$m_\pi^{NR}=2m+\min_b\int_0^\infty \phi(r_{ij})\left[-\frac{1}{m}\left(\frac{d^2}{dr_{q\bar q}^2}+\frac{2}{r_{q\bar q}}\frac{d}{dr_{q\bar q}}\right)+v_{q\bar q}-\frac{3}{4}w_{q\bar q}\right]\phi(r_{q\bar q})r_{q\bar q}^2 dr_{q\bar q}. \quad (8)$$

The one-channel RGM integro-differential equation governing the relative motion of the colliding particles is of the form

$$T_{\vec r}\psi(\vec r)+\int K(\vec r,\vec r\,')\psi(\vec r\,')d^3r'=E\psi(\vec r). \quad (9)$$

The RGM wave function describing the two-pion system reads

$$\psi=\Phi(1\bar 1 2\bar 2)-\Phi(1\bar 2 2\bar 1) \quad (10)$$

with

$$\Phi(1\bar 1 2\bar 2)=\sum_{L'M'I'}\phi_{L'M'}^{I'}(1\bar 1 2\bar 2)\frac{f_{L'M'}^{I'}(r)}{r},\quad L'+I'\quad \text{even} \quad (11)$$

and

$$\Phi_{LM}^I(1\bar 1 2\bar 2)=\mathcal{C}(1\bar 1)\mathcal{C}(2\bar 2)\phi(r_{1\bar 1})\phi(r_{2\bar 2})\times S(1\bar 1 2\bar 2)I(1\bar 1 2\bar 2)Y_{LM}(\hat r), \quad (12)$$

where $\mathcal{C}(1\bar 1)$ and $\mathcal{C}(2\bar 2)$ represent color singlets. The vector $\vec r$ is the relative separation of the pions $(1\bar 1)$ and $(2\bar 2)$

$$\vec r=\frac{1}{2}(\vec r_1+\vec r_{\bar 1}-\vec r_2-\vec r_{\bar 2}),\quad r\equiv|\vec r|,\quad \hat r=\frac{\vec r}{r}. \quad (13)$$

The total spin and isospin wave functions are given by

$$S(1\bar 1 2\bar 2)=[[s^{1/2}(1)s^{1/2}(\bar 1)]^0[s^{1/2}(2)s^{1/2}(\bar 2)]^0]^0, \quad (14)$$

$$I(1\bar 1 2\bar 2)=[[\tau^{1/2}(1)\tau^{1/2}(\bar 1)]^1[\tau^{1/2}(2)\tau^{1/2}(\bar 2)]^1]^I. \quad (15)$$

The partial wave $f_{LM}^I(r)$ satisfies the equation

$$\langle\phi_{LM}^I(1\bar 1 2\bar 2)|H-\mathcal{E}|\psi\rangle=0, \quad (16)$$

in which the integration is carried out over the color, spin, isospin and spatial variables keeping r constant. Using expression (11) of ψ this equation splits into a direct and an exchange term; thus

$$D_{LM}^I-E_{LM}^I=0, \quad (17)$$

with

$$D_{LM}^I=\langle\phi_{LM}^I(1\bar 1 2\bar 2)|H-\mathcal{E}|\phi_{LM}^I(1\bar 1 2\bar 2)\rangle\frac{f_{LM}^I(r)}{r}, \quad (18)$$

$$E_{LM}^I=\sum_{L'M'I'}\int_0^\infty\langle\phi_{LM}^I(1\bar 1 2\bar 2)|H-\mathcal{E}|\phi_{L'M'}^{I'}(1\bar 2 2\bar 1)\rangle f_{L'M'}^{I'}(r')r'dr'. \quad (19)$$

The vector $\vec r\,'$ is of course the relative separation of the pions $(1\bar 2)$ and $(2\bar 1)$

$$\vec r\,'=\frac{1}{2}(\vec r_1+\vec r_{\bar 2}-\vec r_2-\vec r_{\bar 1}). \quad (20)$$

A. Direct term

Owing to the color dependence of the interparticle interaction, only $V_{1\bar 1}$ and $V_{2\bar 2}$ contribute to the direct matrix element. Consequently, in the center-of-mass frame of the two-pion system, D_{LM}^I describes the relative motion of two free pions.

Using Galilean kinematics and Eq. (8), a straightforward calculation results in

$$D_{LM}^{I,NR}=\frac{1}{r}\left(-\frac{1}{2m}\frac{d^2}{dr^2}+\frac{L(L+1)}{2mr^2}+2m_\pi^{NR}-4m-\mathcal{E}\right)f_{LM}^I(r), \quad (21)$$

which is inconsistent in that the mass appearing in the kinetic energy and the centrifugal terms should be equal to m_π^{NR} instead of $2m$. This defect is especially important for pion-pion scattering as the observed pion mass is equal to 0.138 GeV whereas, in nonrelativistic quark models, $2m$ amounts generally to values as large than 0.6 GeV. It is thus essential to incorporate relativistic effects in the description of pion-pion scattering. Therefore, we shall assume that, in the context of the spinless Salpeter equation, D_{LM}^I is given by

$$D_{LM}^I=[2\sqrt{m_\pi+\vec p^{\,2}}-2m_\pi-E]\frac{f_{LM}^I(r)}{r},$$

$$E=\mathcal{E}+4m-2m_\pi, \quad (22)$$

where E and $\vec p$ are the relative energy and momentum of the colliding pions, respectively. Actually, Eq. (22) relies, in addition to the approximations underlying the spinless Salpeter equation [1,2], on the assumption that the constituent particles of the two-pion system interact pairwise through the same scalar potential than the potential used to describe an isolated pion.

B. Exchange term

The color, spin, and isospin matrix elements in the exchange term are given by

$$\langle\mathcal{C}(1\bar 1)\mathcal{C}(2\bar 2)|\mathcal{C}(1\bar 2)\mathcal{C}(2\bar 1)\rangle=\frac{1}{3}, \quad (23)$$

$$\langle\mathcal{C}(1\bar 1)\mathcal{C}(2\bar 2)|-\frac{3}{16}\tilde\lambda_i\cdot\tilde\lambda_j|\mathcal{C}(1\bar 2)\mathcal{C}(2\bar 1)\rangle=\mathcal{C}_{ij}, \quad (24)$$

with

$$\mathcal{C}_{1\bar{1}}=\mathcal{C}_{2\bar{2}}=\mathcal{C}_{1\bar{2}}=\mathcal{C}_{2\bar{1}}=-\mathcal{C}_{12}=-\mathcal{C}_{\overline{12}}=\frac{1}{3}, \tag{25}$$

$$\langle S(1\bar{1}2\bar{2})|S(1\bar{2}2\bar{1})\rangle=\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{Bmatrix}=\frac{1}{2}, \tag{26}$$

$$\langle S(1\bar{1}2\bar{2})|\vec{s}_i\cdot\vec{s}_j|S(1\bar{2}2\bar{1})\rangle=S_{ij},$$

$$S_{1\bar{1}}=S_{2\bar{2}}=S_{1\bar{2}}=S_{2\bar{1}}=-S_{12}=-S_{\overline{12}}=-\frac{3}{8}, \tag{27}$$

$$\langle I(1\bar{1}2\bar{2})|I'(1\bar{2}2\bar{1})\rangle=9\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & 1\\ 1 & 1 & I\end{Bmatrix}\delta_{II'}. \tag{28}$$

Using these results the exchange term can be written

$$E^I_{LM}=\sum_{L'M'}\int_0^\infty\langle H-\mathcal{E}\rangle^I_{LML'M'}f^I_{L'M'}(r')r'dr' \tag{29}$$

with

$$\langle H-\mathcal{E}\rangle^I_{LML'M'}=\frac{3}{2}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & 1\\ 1 & 1 & I\end{Bmatrix}\left\langle T+v-\frac{3}{4}w-\mathcal{E}\right\rangle_{LML'M'} \tag{30}$$

and

$$\langle T\rangle_{LML'M'}=\langle T_1+T_{\bar{1}}+T_2+T_{\bar{2}}\rangle_{LML'M'}, \tag{31}$$

$$\langle v\rangle_{LML'M'}=\langle v_{1\bar{1}}+v_{2\bar{2}}+v_{1\bar{2}}+v_{2\bar{1}}-v_{12}-v_{\overline{12}}\rangle_{LML'M'}, \tag{32}$$

$$\langle w\rangle_{LML'M'}=\langle w_{1\bar{1}}+w_{2\bar{2}}+w_{1\bar{2}}+w_{2\bar{1}}+w_{12}+w_{\overline{12}}\rangle_{LML'M'}. \tag{33}$$

In these equation $\langle\mathcal{O}\rangle_{LML'M'}$, where $\mathcal{O}$ is any spatial operator, denotes the matrix element resulting from the integration of the quantity

$$Y^*_{LM}(\hat{r})\phi(r_{1\bar{1}})\phi(r_{2\bar{2}})\mathcal{O}\phi(r_{1\bar{2}})\phi(r_{2\bar{1}})Y_{L'M'}(\hat{r}') \tag{34}$$

over the spatial coordinates keeping r and r' constant. Note that the $9j$ symbol in Eq. (30) and condition (11) imply I and L' even.

1. Energy and potential matrix elements

The calculation of the potential matrix elements obtained using meson wave functions expanded in arbitrary large harmonic oscillator bases has been outlined in Refs. [3,4]. The $0\hbar\omega$ results relevant to the present work can be summarized as follows:

$$\left\langle v-\frac{3}{4}w-\mathcal{E}\right\rangle_{LML'M'}=\frac{4}{b^3\sqrt{\pi}}\delta_{LL'}\delta_{MM'}(V_L-\mathcal{E}\delta_{L0})\times\exp\left(-\frac{r^2+r'^2}{2b^2}\right) \tag{35}$$

with

$$V_L=\frac{4}{b^3\sqrt{\pi}}\delta_{L0}\int_0^\infty\left\{[v_L(r,\rho)+v_L(r',\rho)]-\frac{3}{4}[w_L(r,\rho)+w_L(r',\rho)]\right\}\rho^2\exp\left(-\frac{\rho^2}{b^2}\right)d\rho-\left[v_L(r,r')+\frac{3}{4}w_L(r,r')\right], \tag{36}$$

where the vector $\vec{\rho}$ is the relative separation of the pairs (12) and $(\overline{12})$

$$\vec{\rho}=\frac{1}{2}(\vec{r}_1+\vec{r}_2-\vec{r}_{\bar{1}}-\vec{r}_{\bar{2}}). \tag{37}$$

The functions of r,r' and ρ in Eq. (36) are obtained by expanding the functions v_{ij} and w_{ij} in multipoles; for instance,

$$v_L(r,\rho)=\int_{-1}^1\left(-\frac{A}{|\vec{r}-\vec{\rho}|}+B|\vec{r}-\vec{\rho}|-C\right)P_L(\mu)d\mu, \tag{38}$$

$$w_L(r,\rho)=V_g\int_{-1}^1\exp\left(-\frac{|\vec{r}-\vec{\rho}|^2}{r_0^2}\right)P_L(\mu)d\mu. \tag{39}$$

These integrals, in which μ represents the scalar product $\hat{\rho}\cdot\hat{r}$, can be carried out analytically.

2. Kinetic energy matrix element

We have calculated the kinetic matrix elements using a set of dimensionless variables defined by

$$\vec{x}_i=\frac{\vec{r}_i}{b},\quad \vec{x}=\frac{\vec{r}}{b},\quad \vec{y}=\frac{\vec{r}'}{b},\quad \vec{z}=\frac{\vec{\rho}}{b}. \tag{40}$$

The intrinsic coordinates $\vec{x},\vec{y},\vec{z}$ and the center-of-mass coordinate defined by

$$\vec{x}_{\text{c.m.}}=\frac{1}{2}(\vec{x}_1+\vec{x}_{\bar{1}}+\vec{x}_2+\vec{x}_{\bar{2}}) \tag{41}$$

are connected to the particle coordinates by an orthogonal transformation.

In terms of the dimensionless coordinates (40) the quantity (34) relative to the kinetic energy is given, up to a normalization factor, by

$$Y^*_{LM}(\hat{x})\exp\left(-\frac{y^2+z^2}{2}\right)(T_1+T_{\bar{1}}+T_2+T_{\bar{2}})$$
$$\times\exp\left(-\frac{x^2+z^2}{2}\right)Y_{L'M'}(\hat{y}) \tag{42}$$

with

$$T_i=m\sqrt{1-(mb)^{-2}\vec{\nabla}^2_{\vec{x}_i}}-m. \tag{43}$$

As these operators do not act, in expression (42), upon functions of $\vec{x}_{\text{c.m.}}$, the Laplacian operators $\vec{\nabla}^2_{\vec{x}_i}$ reduce to

$$\vec{\nabla}^2_{\vec{x}_1}=\frac{1}{2}[mb^2T^{NR}+\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{y}}+\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{z}}+\vec{\nabla}_{\vec{y}}\cdot\vec{\nabla}_{\vec{z}}], \tag{44}$$

$$\vec{\nabla}^2_{\vec{x}_2}=\frac{1}{2}[mb^2T^{NR}-\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{y}}-\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{z}}+\vec{\nabla}_{\vec{y}}\cdot\vec{\nabla}_{\vec{z}}], \tag{45}$$

$$\vec{\nabla}^2_{\vec{x}_3}=\frac{1}{2}[mb^2T^{NR}+\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{y}}-\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{z}}-\vec{\nabla}_{\vec{y}}\cdot\vec{\nabla}_{\vec{z}}], \tag{46}$$

$$\vec{\nabla}^2_{\vec{x}_4}=\frac{1}{2}[mb^2T^{NR}-\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{y}}+\vec{\nabla}_{\vec{x}}\cdot\vec{\nabla}_{\vec{z}}-\vec{\nabla}_{\vec{y}}\cdot\vec{\nabla}_{\vec{z}}], \tag{47}$$

where

$$T^{NR}=\frac{1}{2mb^2}(\vec{\nabla}^2_{\vec{x}}+\vec{\nabla}^2_{\vec{y}}+\vec{\nabla}^2_{\vec{z}}) \tag{48}$$

is the intrinsic kinetic energy operator of the two-pion system in the nonrelativistic limit.

It is possible to afford a meaning to the unusual operator (43) through the Fourier transform (FT) of the function on which is acts; for instance,

$$\sqrt{1-(mb)^{-2}\vec{\nabla}^2_{\vec{\xi}}}f(\vec{\xi})$$
$$=\frac{1}{(2\pi)^{3/2}}\int e^{i\vec{k}\cdot\vec{\xi}}\sqrt{1+(mb)^{-2}k^2}\,\text{FT}[f(\vec{\xi})]d^3k \tag{49}$$

with

$$\text{FT}[f(\vec{\xi})]=\frac{1}{(2\pi)^{3/2}}\int e^{-i\vec{k}\cdot\vec{\xi}'}f(\vec{\xi}')d^3\xi'. \tag{50}$$

In this respect the harmonic oscillator bases are quite convenient as the normalized eigenstates $|nlm\rangle_{\vec{\xi}}$ of the three-dimensional harmonic oscillator satisfy the equation

$$\text{FT}[|nlm\rangle_{\vec{\xi}}]=(-i)^{2n+l}|nlm\rangle_{\vec{k}} \tag{51}$$

which for $n=l=m=0$ reduces to the well known result

$$\text{FT}\left[\exp\left(-\frac{\xi^2}{2}\right)\right]=\exp\left(-\frac{k^2}{2}\right). \tag{52}$$

Using Eqs. (44)–(47), (50), and (52) as well as the Hermitian property of the kinetic energy operator, the matrix element $\langle T_1\rangle_{LML'M'}$ is equal to the integral over $\hat{x},\hat{y},\vec{z},\vec{k},\vec{k}',\vec{k}''$, and $\vec{k}'''$ of the quantity

$$\frac{m}{64\pi^9}Y^*_{LM}(\hat{x})Y_{L'M'}(\hat{y})\{\sqrt{1+(2mb)^{-2}K_1^2}-1\}$$
$$\times\exp\Big[i(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{y}+(\vec{k}''-\vec{k}''')\cdot\vec{z})$$
$$-\frac{1}{2}(k^2+k'^2+k''^2+k'''^2)\Big] \tag{53}$$

with

$$\vec{K}_1=\vec{k}+\vec{k}'+\vec{k}''. \tag{54}$$

The matrix elements relative to the operator $T_{\bar{1}},T_2$, and $T_{\bar{2}}$ are obtained in the same way with

$$\vec{K}_{\bar{1}}=\vec{k}-\vec{k}'-\vec{k}'', \tag{55}$$

$$\vec{K}_2=\vec{k}+\vec{k}'-\vec{k}'', \tag{56}$$

$$\vec{K}_{\bar{2}}=\vec{k}-\vec{k}'+\vec{k}''. \tag{57}$$

The integration of expression (53) over $\hat{x},\hat{y}$, and $\vec{z}$ is readily carried out and yields the result

$$\langle T_1\rangle_{LML'M'}$$
$$=\frac{2m}{\pi^4}i^{L-L'}\int d^3k\int d^3k' I_1(\vec{k},\vec{k}')j_L(kx)j_{L'}(k'y)$$
$$\times\exp\left(-\frac{k^2+k'^2}{2}\right)Y_{LM}(\hat{k})Y_{L'M'}(\hat{k}') \tag{58}$$

with

$$I_1(\vec{k},\vec{k}')=\int d^3k''\{\sqrt{1+(2mb)^{-2}K_1^2}-1\}\exp(-k''^2). \tag{59}$$

This integral can be recast in the form

$$I_1(\vec{k},\vec{k}')=\exp(-\kappa^2)\int_0^\infty dK_1\{\sqrt{1+(2mb)^{-2}K_1^2}-1\}K_1^2$$
$$\times\exp(-K_1^2)\int d\hat{K}_1\exp(2\vec{\kappa}\cdot\vec{K}_1) \tag{60}$$

in which

$$\vec{\kappa}=\vec{k}+\vec{k}',\quad \kappa=\sqrt{k^2+k'^2+2kk'\zeta},\quad \zeta=\hat{k}\cdot\hat{k}'. \tag{61}$$

The integral over $\hat{K}_1$ yields

$$\int d\hat{K}_1\exp(2\vec{\kappa}\cdot\vec{K}_1)=\frac{2\pi}{\kappa K_1}\sinh(2\kappa K_1). \tag{62}$$

Finally the integration over $\hat{k}$ and $\hat{k}'$ in Eq. (58) can be carried out through the multipole expansion of $I_1(\vec{k},\vec{k}')$ with regard to ζ

$$I_1(\vec{k},\vec{k}) = 4\pi^2 \sum_\lambda I_1^\lambda(k,k') \sum_\mu Y_{\lambda\mu}(\hat{k}) Y^*_{\lambda\mu}(\hat{k}') \tag{63}$$

with

$$I_1^\lambda(k,k') = \frac{1}{2\pi} \int_{-1}^{+1} I_1(\vec{k},\vec{k}') P_\lambda(\zeta) d\zeta. \tag{64}$$

In this way we obtain in terms of r and r'

$$\langle T_1 \rangle_{LML'M'} = \frac{8m}{\pi^2 b^3} \delta_{LL'} \delta_{MM'} T_L \tag{65}$$

with

$$T_L = \int_0^\infty dk \int_0^\infty dk' (kk')^2 I_1^L(k,k') j_L\left(\frac{kr}{b}\right) j_L\left(\frac{k'r'}{b}\right) \times \exp\left(-\frac{k^2+k'^2}{2}\right) \tag{66}$$

in which, according to Eqs. (60), (62) and (64),

$$I_1^L(k,k') = \int_{-1}^{+1} d\zeta P_L(\zeta) \frac{\exp(-\kappa^2)}{\kappa} \times \int_0^\infty dK_1 \{\sqrt{1+(2mb)^{-2}K_1^2} - 1\} K_1 \times \exp(-K_1^2) \sinh(2\kappa K_1). \tag{67}$$

Note that Eqs. (54)–(57), (60), and (64) imply

$$I_1^L(k,k') = I_2^L(k,k') = (-1)^L I_{\bar{1}}^L(k,k') = (-1)^L I_{\bar{2}}^L(k,k') \tag{68}$$

and, since L', and therefore L, are restricted to even values

$$\langle T \rangle_{LML'M'} = 4 \langle T_1 \rangle_{LML'M'} = \frac{32m}{\pi^2 b^3} \delta_{LL'} \delta_{MM'} T_L . \tag{69}$$

In the nonrelativistic limit, that is to say, when

$$\sqrt{1+(2mb)^{-2}K_1^2} - 1 \simeq \frac{K_1^2}{8m^2 b^2}, \tag{70}$$

the integrals (66) and (67) can be carried out analytically so that the quantity (66) amounts in this case to

$$T_L = \frac{\pi\sqrt{\pi}}{16m^2 b^2} \left[\frac{15}{2} - \frac{1}{b^2}(r^2+r'^2)\right] \exp\left(-\frac{r^2+r'^2}{2b^2}\right) \delta_{L,0} \tag{71}$$

in agreement with the results reported in Ref. [4].

3. Scattering phase shifts

Using the above results Eq. (9) reads

$$[2\sqrt{m_\pi^2 + \vec{p}^2} - 2m_\pi - E] \frac{f_L^I(r)}{r} + \frac{1}{r} \int_0^\infty K_L^I(r,r') f_L^I(r') dr' = 0 \tag{72}$$

with

$$K_L^I(r,r') = -\frac{48mrr'}{\pi^2 b^3} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & I \end{Bmatrix} \times \left[T_L + \frac{\pi\sqrt{\pi}}{8m}(V_L - \mathcal{E}\delta_{L0}) \exp\left(-\frac{r^2+r'^2}{2b^2}\right)\right], \tag{73}$$

which implies

$$K_L^2(r,r') = -2K_L^0(r,r'). \tag{74}$$

According to Eq. (73) it follows also that for non zero values of L the exchange kernels do not depend upon $\mathcal{E}$ and thus, upon the dipion mass defined by

$$m_{\pi\pi} = 2m_\pi + E = \mathcal{E} + 4m. \tag{75}$$

The application of the operator $\sqrt{m_\pi^2 + \vec{p}^2}$ upon the terms of Eq. (72) leads to the Schrödinger-like equation

$$\left[-\frac{1}{m_\pi}\frac{d^2}{dr^2} + \frac{L(L+1)}{m_\pi r^2} - E\left(1+\frac{E}{4m_\pi}\right)\right] f_L^I(r) + \left(1+\frac{E}{4m_\pi}\right) \int_0^\infty K_L^I(r,r') f_L^I(r') dr' + \int_0^\infty \tilde{K}_L^I(r,r') f_L^I(r') dr' = 0 \tag{76}$$

in which the additional kernel $\tilde{K}_L^I(r,r')$ is given by

$$\tilde{K}_L^I(r,r') = \frac{r}{2}[\sqrt{1-(m_\pi b)^{-2}\vec{\nabla}_r^2} - 1] \frac{K_L^I(r,r')}{r}. \tag{77}$$

Using Eqs. (49) and (50), expression (77) yields, after integration over the angular variables

$$\tilde{K}_L^I(r,r') = \frac{1}{\pi b} \int_0^\infty dk \left[\sqrt{1+\frac{k^2}{m_\pi^2 b^2}} - 1\right] \times \sin\left(\frac{kr}{b}\right) \mathrm{FT}[K_L^I(r,r')] \tag{78}$$

with

$$\mathrm{FT}[K(r,r')] = \int_0^\infty dr'' \sin\left(\frac{kr''}{b}\right) K_L^I(r'',r'). \tag{79}$$

The scattering phase shifts δ_L^I as functions of the relative energy of the colliding pions can be extracted by the usual method which consists in solving Eq. (76) numerically up to

TABLE I. Parameters of the interparticle potential determined using relativistic (*R*) and Galilean (*G*) kinematics.

	A	B (GeV^2)	C (GeV)	V_g (GeV)	r_0 (GeV^{-1})	m (GeV)
R	0.752	0.184	0.455	1.12	3.07	0.171
G	0.583	0.169	0.827	2.82	2.11	0.324

$r=R$, with R larger than the range of the exchange kernels, and in fitting the solution obtained in this way to its asymptotic form

$$N_L \sin\left(ar - \frac{L\pi}{2} + \delta_L^I \right) \tag{80}$$

with

$$a = \sqrt{m_\pi E\left(1 + \frac{E}{4m_\pi}\right)}. \tag{81}$$

III. RESULTS

The parameters of the interquark potential used in our calculations were determined so that the variational bounds of the corresponding spinless Salpeter Hamiltonian obtained in extended harmonic oscillator bases compare satisfactorily with the observed masses of a great variety of mesons. As shown in Table I, the potential obtained in this way differs significantly from its nonrelativistic counterpart. In particular, the conversion from Galilean to relativistic kinematics reduces considerably the constituent quark mass.

The variational bounds for the pion mass obtained in various bases, using both relativistic and Galilean kinematics, are presented in Table II together with the corresponding values of the oscillator length parameter and the amplitude of the $0\hbar\omega$ component. From these figures it appears that the extension of the harmonic oscillator base, though essential to improve the theoretical pion mass, has little effect upon the general trend of its wave function and, consequently, upon the RGM exchange kernel associated to pion-pion scattering, as it has been verified explicitly in our nonrelativistic calculations [4]. Therefore, a one-Gaussian description of the pion wave function is justified and, accordingly, only the $N=0$ values displayed in Table II will be used in our numerical calculations.

TABLE II. Optimal values of the theoretical pion mass obtained in various $N\hbar\omega$ harmonic oscillator bases together with the corresponding values of the oscillator length parameter and the amplitude A_0 of the $N=0$ component of the pion wave function using relativistic (*R*) and Galilean (*G*) kinematics.

	m_π (GeV)		b (GeV^{-1})		A_0	
N	R	G	R	G	R	G
0	0.217	0.195	1.23	1.60	1.000	1.000
4	0.163	0.146	1.12	1.51	0.982	0.987
8	0.147	0.140	1.04	1.46	0.967	0.980
18	0.138	0.138	1.04	1.46	0.965	0.980

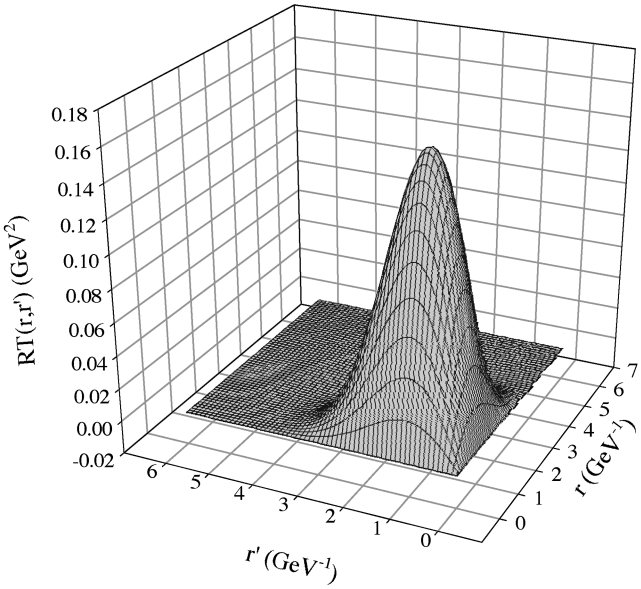

FIG. 1. Contribution of the kinetic energy to the semirelativistic exchange kernel $K_0^0(r,r')$.

We have calculated the contribution of the kinetic energy to the exchange kernel in the $L=I=0$ channel, namely, the quantity obtained by turning of V_0 and $\mathcal{E}$ in $K_0^0(r,r')$ and using expression (66) of T_0 as well as the R values of m and b displayed in Tables I and II. This contribution, termed $RT(r,r')$, and its Galilean counterpart $GT(r,r')$ calculated using expression (71) of T_0, and the G values of m and b are presented in Figs. 1 and 2. It is seen that these kinetic exchange kernels are quite similar: They have practically the same shape and reach their largest value for r located between 1.0 and 1.6 GeV^{-1}. As shown in Figs. 3 and 4, the same holds true with respect of the potential exchange kernels $RV(r,r')$ and $GV(r,r')$ obtained by turning of T_0 and $\mathcal{E}$ in $K_0^0(r,r')$.

We wish to stress that this striking similarity of the R and

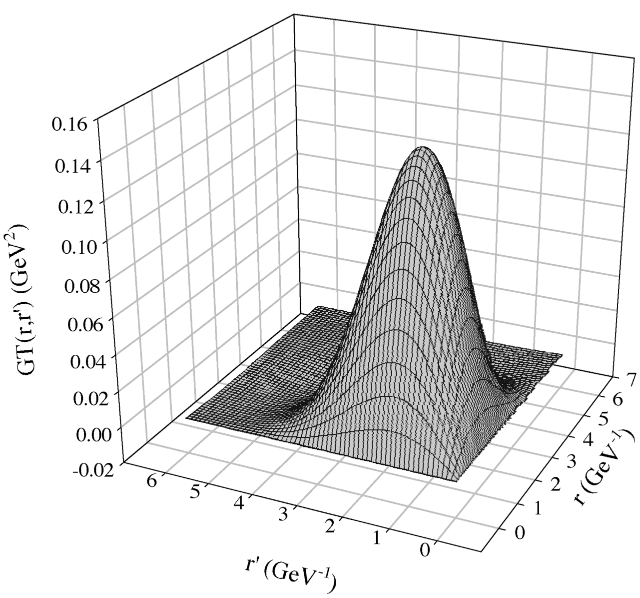

FIG. 2. Contribution of the kinetic energy to the Galilean exchange kernel $K_0^0(r,r')$.

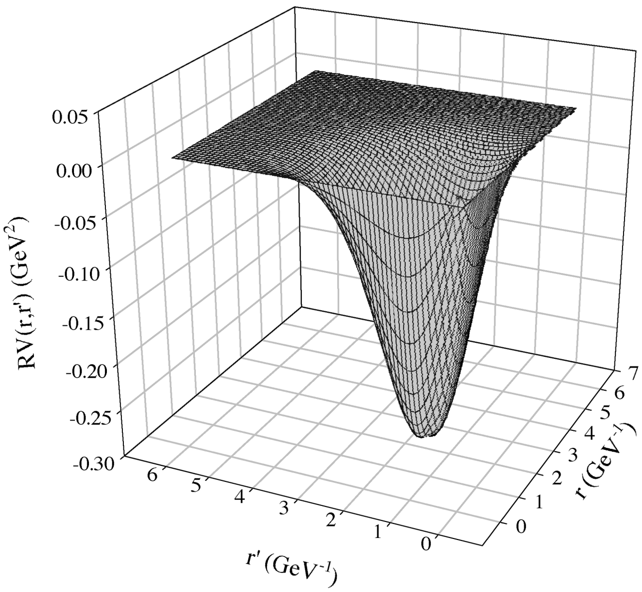

FIG. 3. Contribution of the potential energy to the semirelativistic exchange kernel $K_0^0(r,r')$.

G kernels does not justify the use of Galilean kinematics. Indeed, the relation (70) breaks down in the integral (67), so that expression (66) of T_L and its nonrelativistic limit (71) yield quite different results: Taking $m=0.171$ GeV,$b=1.23$ GeV^{-1}, and $r=1.1$ GeV^{-1} the values of T_0 calculated using either expression are equal to 0.166 and 0.627, respectively. Actually, the great difference between expressions (66) and (71) of T_L is, surprisingly enough, largely compensated in $RT(r,r')$ and $GT(r,r')$ by the difference between the R and G values of m and b. A similar compensation reduces the effect of the difference between the R and G interparticle interactions in $RV(r,r')$ and $GV(r,r')$.

The numerical calculation of the integrals (66) and (67) is a computer time consuming task: For given values of m and b, the calculation of T_L as a function of r and r' demands

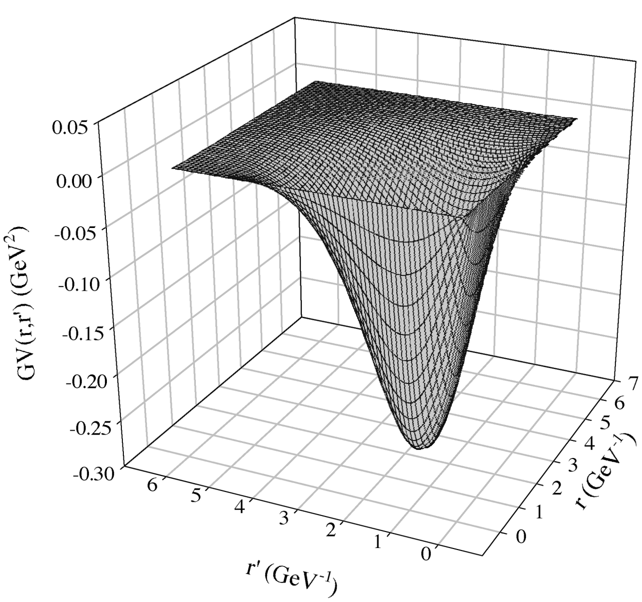

FIG. 4. Contribution of the potential energy to the Galilean exchange kernel $K_0^0(r,r')$.

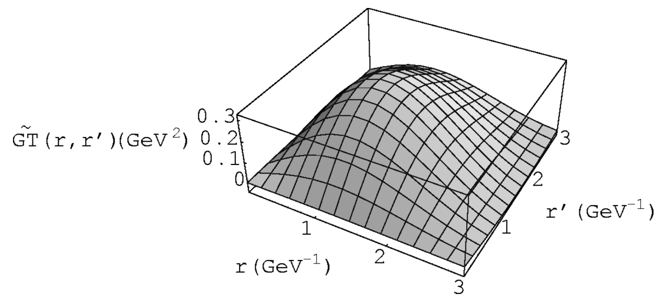

FIG. 5. Additional exchange kernel $\widetilde{GT}(r,r')$ (see text).

about three hours on a Pentium 120 MHz. The extraction of the scattering phase shifts from Eq. (76) requires in addition the calculation of the kernel $\tilde{K}_L^I(r,r')$. We have avoided this last step using a Fourier grid Hamiltonian method to solve Eq. (72) directly [5,6]. It is nevertheless worth estimating the relative importance of the kernels $K_L^I(r,r')$ and $\tilde{K}_L^I(r,r')$ and, especially, to compare their kinetic energy content. As $RT(r,r')$ is similar to $GT(r,r')$ it is enough to calculate the integral (78) using $GT(r,r')$ as input. In this case the integral over r'' can be carried out analytically, yielding

$$\widetilde{GT}(r,r')=\frac{r'\exp(-r'^2/2b^2)}{6\pi\sqrt{2}mb^4}\int_0^\infty\left[\sqrt{1+\frac{k^2}{m_\pi^2b^2}}-1\right]\times\left(\frac{9}{2}-\frac{r'^2}{b^2}+k^2\right)\sin\left(\frac{kr}{b}\right)\exp\left(-\frac{k^2}{2}\right)kdk. \tag{82}$$

This kernel is displayed in Fig. 5. Note its slight asymmetry: $G\tilde{T}(r,r')$ reaches its maximum value for $r=1.16$ GeV^{-1} and $r'=1.41$ GeV^{-1}. This value amounts to 0.319 GeV^2, that is to say about twice the maximum value of $GT(r,r')$, which reflects the inadequacy of Galilean kinematics irrespective of the value of the relative energy of the colliding pions. The magnitude of the additional kernel $\tilde{K}_L^I(r,r')$ in Eq. (76) is thus closely connected to the relativistic nature of the isolated pions. In this respect it is worth noting that a similar RGM calculation of deuteron-deuteron scattering leads to a ratio of max$\{\widetilde{GT}(r,r')\}$ to max$\{GT(r,r')\}$ equal roughly to 10^{-2}.

We present in Fig. 6 the S-wave phase shifts extracted from Eqs. (72) and (73). These results are obtained using the $N=0$ theoretical pion mass of 0.217 GeV. The substitution of this mass by the observed value of 0.138 GeV has very little effect upon these phase shifts, which is another argument justifying the $N=0$ approximation of the pion wave functions. When the calculations are carried out using Galilean kinematics as well for the relative motion of the pions than for the exchange kernels as in our earlier calculations [3,4], the trend of these phase shifts as functions of the dipion mass is completely modified. In particular, the isoscalar relativistic shifts increase smoothly with increasing $m_{\pi\pi}$ whereas their Galilean counterparts present an undesirable bump around 0.5 GeV. It is worth noting that the $I=2$ phase shifts obtained using relativistic kinematics are in agreement with the available experimental data [7] contrary to the calculated $I=0$ shifts which are much too small. This

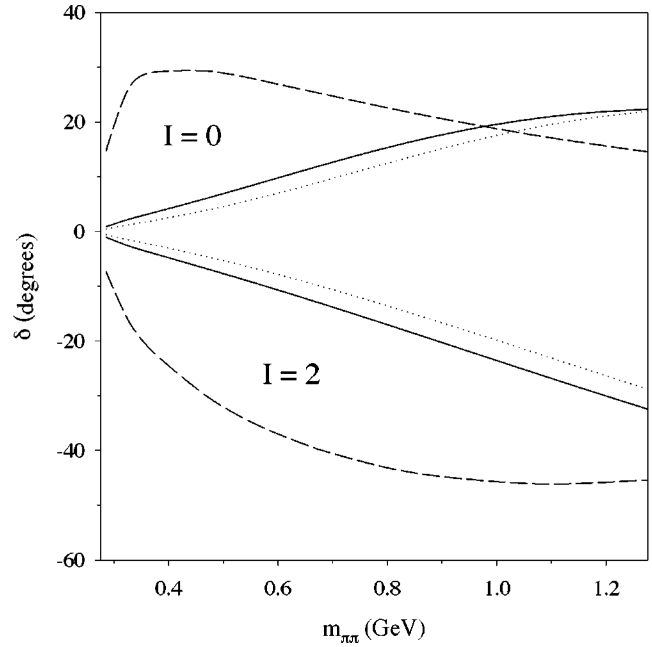

FIG. 6. $I=0$ and 2 S-wave pion-pion phase shifts as functions of $m_{\pi\pi}$ calculated using (a) relativistic kinematics for both the exchange kernel and the π-π relative motion with $m_\pi=0.217$ GeV (solid lines) and $m_\pi=0.138$ GeV (dotted lines). (b) Galilean kinematics for both the exchange kernel and the π-π relative motion with $m_\pi=0.195$ GeV (dashed lines).

feature might originate from the annihilation processes which are not incorporated in the present model. Indeed, the description of such processes through isospin-dependent interquark potentials derived from instanton effects [8] suggests that such processes are effective mainly in the $I=0$ channel.

As illustrated in Figs. 7 and 8 both semirelativistic and Galilean exchange kernels are considerably smaller for $L=2$ than for $L=0$. Accordingly, the corresponding phase shifts are extremely small: Up to $E=1$ GeV they do not

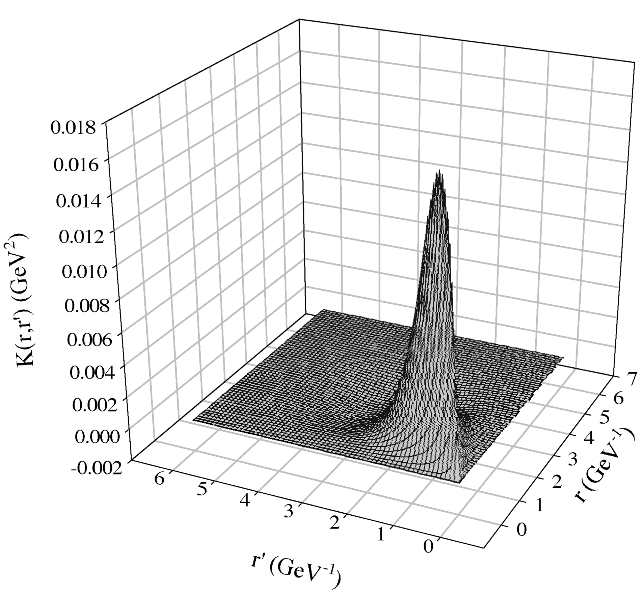

FIG. 7. Exchange kernel in $L=2$, $I=0$ channel obtained using relativistic kinematics.

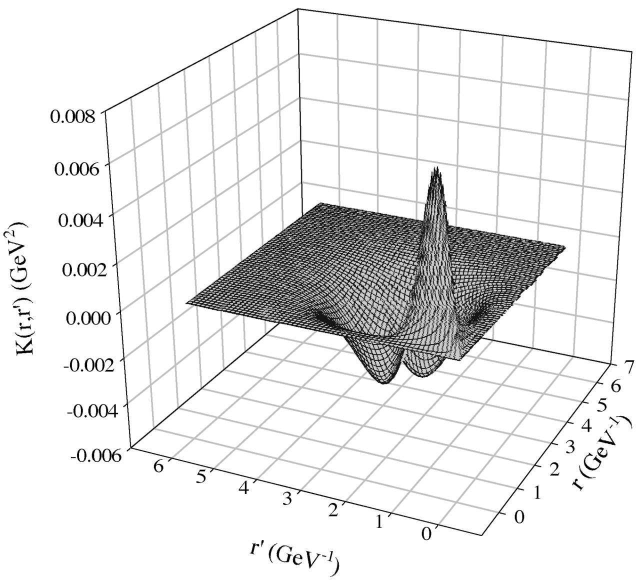

FIG. 8. Same as Fig. 7 but using Galilean kinematics.

exceed a few degrees, which compares satisfactorily with experiment.

IV. CONCLUDING REMARKS

We have presented a method to incorporate relativistic kinematics in the RGM treatment of pion-pion scattering. Both the description of the individual pions and their scattering rely on the assumptions underlying the spinless Salpeter equation. We found that the relativistic and Galilean exchange kernels of the RGM integrodifferential equation are quite similar, which indicates that the relativistic effects are to a large extent taken into account, in the Galilean kernel, through the constituent quark mass and the parameters defining the interquark potential. Furthermore, our calculations show that these kernels bear closely upon the relativistic nature of the pion, as revealed by the magnitude of the additional kernel $\tilde{K}_L^I(r,r')$ appearing in the Schrödinger-like equation (76) deduced from the semirelativistic equation (72). Consequently, the large relativistic effects exhibited by the corresponding phase shifts originate not only from the relative motion of the colliding pions but also from the relativistic dynamics governing their internal structure.

A real understanding of pion-pion scattering requires obviously elaborate interparticle interactions. For instance, lattice calculations indicate that in multiquark systems the interactions cannot be reduced to pairwise potentials. In this context, it is worth noting that, though it is hard to bring multiquark potentials into play in RGM calculations, this effort does not concern the kinetic energy part of the total Hamiltonian and, accordingly, the procedure proposed to make such calculations consistent with the requirements of the special relativity remains applicable. On the other hand, the description of the annihilation processes and the coupling of the pion-pion system to other sectors require the extension of this procedure to coupled channel calculations and the handling of a semirelativistic four-body Hamiltonian for quarks of unequal masses. The semirelativistic calculations outlined in the present work might be a useful step towards the achievement of these improvements which are essential to make RGM calculations of pion-pion scattering reliable.

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