

Effective theory for neutron-deuteron scattering: Energy dependence

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We report on results of the effective theory method applied to neutron-deuteron scattering. We extend previous results in the $J=3/2$ channel to nonzero energies and find very good agreement with the experiment without any parameter fitting. [S0556-2813(98)50808-4]

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Since the establishment of QCD as the theory of the strong interactions very little progress has been made in understanding nuclear forces from first principles. Many phenomenological models have been developed with great success, but they all suffer from shortcomings, among them ambiguities in using nucleon-nucleon scattering information in the calculation of other processes, difficulty in relating them to the underlying QCD, and especially, a lack of a systematic expansion in a small parameter. The effective field theory approach has the promise of solving these difficulties [1]. The role of the small parameter is played by the ratio of the typical momentum scale Q in the problem of the scale associated with the physics left out of the effective theory. In the case of nucleon interactions up to momenta of the order of 300 MeV one can build an effective theory containing only nucleons and pions (and delta isobars). The scale of the physics left out is $\sim m_\rho$ and the expansion parameter is $\sim Q/m_\rho$. This idea was elaborated in a large number of works in the last few years [2]. Subtle problems regarding the naturalness of the shallow nuclear bound states, renormalization, and power counting in the presence of pion exchange are nowadays a subject of intense discussion [3,4]. However, such problems can be bypassed in those nuclear processes where the typical momentum scale is small compared to the pion mass. In this case one is allowed to use an effective theory without explicit pions, contact forces being all that remain. That is what happens in deuteron physics, since the typical momentum scale in a deuteron is given by the inverse of the 3S_1 scattering length, $1/a_t \ll m_\pi$. This situation arises because the nuclear potential is fine tuned so that there is a bound state close to threshold with energy $\sim 1/(Ma_t^2)$, much smaller than other scales in the problem like $\sim 1/(Mr_{0t}^2) \sim m_\pi^2/M$ (we take the effective range in the 3S_1 channel $r_{0t} \sim 1/m_\pi$ for power counting purposes). Attempts at model-independent approaches in this energy range have a long history [5]. When this approach is applied to nucleon-nucleon scattering up to momenta $\sim 1/a_t$ the effective range expansion is reproduced. The first nontrivial application is thus in the three-nucleon sector. In this Rapid

Communication we report results of this approach in the case of neutron-deuteron scattering in the $J=3/2$ channel below deuteron breakup. We perform an expansion on powers of r_{0t}/a_t and pr_{0t} , where p is the typical momentum of the process, keeping terms up to order $(r_{0t}/a_t)^2, (pr_{0t})^2$ (we take $r_{0t} \sim 1/m_\pi$). Results in extraordinary agreement for the quartet scattering length were previously reported in Ref. [6]. Here we extend this calculation to finite energy.

In the $J=3/2, I=1/2$ channel the spins of all three nucleons are aligned and all two-body s -wave interactions are in the spin triplet, isospin singlet channel. (For this reason we will drop from now on the subscript in a_t and r_{0t} .) The effective Lagrangian restricted to this channel is given by [4]

$$\begin{aligned} \mathcal{L} = & N^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2M} + \dots \right) N + C_0 (N^\dagger \tau_2 \vec{\sigma} \sigma_2 N)^2 \\ & + C_2 [(N^\dagger \tau_2 \vec{\sigma} \sigma_2 \nabla N)(N^\dagger \tau_2 \vec{\sigma} \sigma_2 \nabla N) \\ & - 3(N^\dagger \tau_2 \vec{\sigma} \sigma_2 N)(N^\dagger \tau_2 \vec{\sigma} \sigma_2 \nabla^2 N) + \text{H.c.}] + \dots, \quad (1) \end{aligned}$$

where M is the nucleon mass, C_n are constants related to the two-body force terms containing n derivatives, and the dots stand for higher-order terms including relativistic corrections, higher-derivative terms, three-body forces, etc. The constants C_n are determined by nucleon-nucleon scattering data. It turns out that, using dimensional regularization and minimal subtraction, $C_0 \sim a/M$, $C_2 \sim r_0(r_0 a)/M$, $C_4 \sim r_0(r_0 a)^2/M + \dots$ and so on (ellipses stand for terms suppressed by powers of r_0/a). The leading pieces in each one of these terms form a geometric series that can be conveniently summed to all orders by the introduction of a field of baryon-number two [7]

$$\begin{aligned} \mathcal{L} = & N^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2M} + \dots \right) N + \vec{d}^\dagger \cdot \left(-i\partial_0 - \frac{\vec{\nabla}^2}{4M} + \Delta + \dots \right) \vec{d} \\ & - \frac{g}{2} (\vec{d}^\dagger \cdot N \vec{\sigma} \sigma_2 \tau_2 N + \text{H.c.}) + \dots. \quad (2) \end{aligned}$$

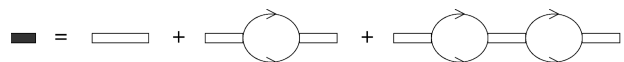


FIG. 1. Dressed dibaryon propagator.

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More generally, if the dibaryon field \vec{d} is integrated out, the Lagrangian (2) is recovered as long as Δ and g are appropriate functions of C_0 and C_2 . This resummation is by no means necessary, since for momenta of the order $p \sim 1/a$ the resummed terms are subleading, but it is a convenient way of computing higher-order corrections.

The numerical values of g and Δ can be determined if we consider the dressed dibaryon propagator (Fig. 1). The linearly divergent loop integral is set to zero in dimensional regularization and the result is

$$iS(p) = \frac{1}{p^0 - \vec{p}^2/4M - \Delta + (Mg^2/2\pi)\sqrt{-Mp^0 + \vec{p}^2/4 - i\epsilon} + i\epsilon}. \quad (3)$$

This propagator is, up to a constant, the scattering matrix of two nucleons in the 3S_1 channel

$$T(k) = \frac{4\pi}{M} \frac{1}{-(2\pi\Delta/Mg^2) + (2\pi/M^2g^2)k^2 - ik}, \quad (4)$$

where k^2/M is the energy in the center-of-mass frame. This result is just the familiar effective range expansion, from what we can infer about the proper values for the constants g and Δ . Using $a = 5.42$ fm and $r_0 = 1.75$ fm [8], we find

$$g^2 = \frac{4\pi}{M^2 r_0} = 1.6 \times 10^{-3} \text{ MeV}^{-1}, \quad (5)$$

$$\Delta = \frac{2}{Mar_0} = 8.7 \text{ MeV}. \quad (6)$$

From Eqs. (3), (5), and (6) we see why it is necessary to resum the bubble graphs in Fig. 1 to all orders for $p \sim 1/a$: the term in the square root coming from the unitarity cut is of the same order as Δ . On the other hand, as mentioned before, the kinetic term of the dibaryon is smaller than the other terms in Eq. (3) and is resummed for convenience only. Notice that the propagator (3) has two poles, one at $p^0 = \vec{p}^2/4M - B$ (the deuteron pole), another at $p^0 = \vec{p}^2/4M - B_{\text{deep}}$ (unphysical deep pole), and a cut along the positive real axis starting at $p^0 = \vec{p}^2/4M$.

Let us now turn to neutron-deuteron scattering. The simplest diagram contributing to this process is the first diagram in Fig. 2. For momenta of the order of $p \sim 1/a$ it contributes $\sim Mg^2/p^2 \sim a^2/Mr_0$. The one-loop graph mixes different orders of the expansion, since it involves the dibaryon propagator $g^2/(\Delta + p^2/M) \sim (a/M)[1 + \mathcal{O}(r_0/a) + \dots]$; it gives a contribution $\sim g^4 M^2/p\Delta \sim (a^2/Mr_0)[1 + \mathcal{O}(r_0/a) + \dots]$. It is easy to see that the remaining graphs in Fig. 2 give con-

tributions of the same order, which means that an infinite number of diagrams contribute to the leading orders.

Other contributions are suppressed by at least three powers of r_0/a or pr_0 [6]. For instance, the effect of the subleading (not resummed) piece of C_4 is to generate the shape parameter ($\sim k^4$) in the effective range expansion of the nucleon-nucleon interaction. Its typical size is $\sim k^4 r_0^3$ compared to the leading piece $\sim 1/a$ and is thus also suppressed by $(r_0/a)^3$. Likewise, p -wave interactions, unaffected by the existence of a shallow s -wave bound state, arise from a term in the Lagrangian with two derivatives and a coefficient of the order $\sim 1/Mm_\pi^3$. We conclude then that a diagram made out of the substitution of one of the dibaryon propagators in a diagram in Fig. 2 by a p -wave interaction vertex would be suppressed by $(r_0/a)^3$ in comparison to the leading order. Three-body force terms have to contain at least two derivatives since in the $J=3/2$ channel all the spins are up and Fermi statistics forbids the placement of all three nucleons in an s -wave. The natural size of the coefficient of the six nucleon, two derivative term that produces such a three-body force is $\sim 1/Mm_\pi^6$. This term is generated, upon integration of the dibaryon field, by a term containing two dibaryon fields, two nucleon fields, and two derivatives with a coefficient of the order of $\sim r_0^5/Ma^4$. Thus contributions coming from the three-body force are suppressed in relation to the leading-order graphs by $(r_0/a)^6$.

A calculation accurate up to corrections of order $(r_0/a)^3$ is possible by summing the diagrams of Fig. 2. Fortunately, the interaction mediated by the s -channel dibaryon generates a very simple, local and separable potential between nucleons. It is well known that the three-body problem with separable two-body interactions reduces to an equivalent two-body problem. In our case the equation to be solved can be read off Fig. 2, and an integration over the energy inside the loop gives [6]

$$\begin{aligned} & \left[-\frac{3(\vec{p}^2 - \vec{k}^2)}{8M^2g^2} + \frac{1}{4\pi} \left(\sqrt{\frac{3}{4}(\vec{p}^2 - \vec{k}^2) + MB} - \sqrt{MB} \right) \right] \frac{t(\vec{p}, \vec{k})}{\vec{p}^2 - \vec{k}^2 - i\epsilon} \\ & = \frac{-1}{(\vec{p} - \vec{k}/2)^2 + MB} - \int \frac{d^3l}{(2\pi)^3} \frac{1}{\vec{l}^2 - \vec{l} \cdot \vec{p} + \vec{p}^2 - \frac{3}{4}\vec{k}^2 + MB} \frac{t(\vec{l}, \vec{k})}{\vec{l}^2 - \vec{p}^2 - i\epsilon}, \end{aligned} \quad (7)$$

where B is the deuteron binding energy. Since we are interested only in s -wave scattering, we should project this equation into its $L=0$ component. The result is

$$\begin{aligned} & \frac{3}{2} \left[-\eta + \frac{1}{\sqrt{\frac{3}{4}(x^2-y^2)+1+1}} \right] a(x,y) \\ &= -\frac{1}{xy} \ln \left[\frac{(x+y/2)^2+1}{(x-y/2)^2+1} \right] - \frac{2}{\pi x} \int_0^\infty dz z \ln \left(\frac{z^2+x^2+1-\frac{3}{4}y^2+xz}{z^2+x^2+1-\frac{3}{4}y^2-xz} \right) \frac{a(z,y)}{z^2-y^2-i\epsilon}, \end{aligned} \quad (8)$$

where we use the dimensionless quantities $x=p/\sqrt{MB}$, $y=k/\sqrt{MB}$, $z=l/\sqrt{MB}$, and $a(x,y)=(\sqrt{MB}/4\pi)t_{L=0}(p,k)$, and $\eta=\sqrt{MB}r_0/2$. For finite values of k this equation is complex even below threshold ($3k^2/4=B$) because of the $i\epsilon$ prescription. It is convenient for the numerical treatment to use the real K matrix defined by

$$K(x,y) = \frac{a(x,y)}{1+iy a(y,y)}, \quad (9)$$

which satisfies the equation

$$K(x,y) = -h(x,y,y) - \frac{2}{\pi} \int_0^\infty dz z^2 h(x,y,z) \frac{\mathcal{P}}{z^2-y^2} K(z,y), \quad (10)$$

with

$$h(x,y,z) = \frac{1}{xz \tilde{f}(x,y)} \ln \left(\frac{z^2+x^2+1-\frac{3}{4}y^2+xz}{z^2+x^2+1-\frac{3}{4}y^2-xz} \right),$$

$$\tilde{f}(x,y) = \frac{3}{2} \left[-\eta + \frac{1}{\sqrt{\frac{3}{4}(x^2-y^2)+1+1}} \right]. \quad (11)$$

The phase shifts can be obtained directly from the on-shell K matrix:

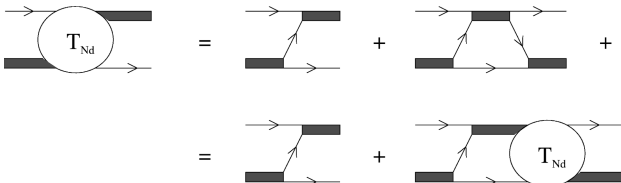


FIG. 2. Graphs contributing up to order $(r_0/a)^2$.

$$k \cot \delta = \frac{\sqrt{MB}}{K(k/\sqrt{MB}, k/\sqrt{MB})}. \quad (12)$$

Defining $f(x,y)$ by the equation

$$f(x,y) = \frac{h(x,y,y)}{h(y,y,y)} - \frac{2}{\pi} \int_0^\infty dz z^2 \left[h(x,y,z) - \frac{h(x,y,y)}{h(y,y,y)} h(y,y,z) \right] \frac{f(z,y)}{z^2-y^2}, \quad (13)$$

the on-shell K matrix can be obtained by

$$K(y,y) = -h(y,y,y) \left[1 + \frac{2}{\pi} \int_0^\infty dz (z^2 h(y,y,z) f(z,y) - y^2 h(y,y,y) f(y,y)) \frac{1}{z^2-y^2} \right]^{-1}. \quad (14)$$

Rewriting Eq. (8) this way greatly simplifies its numerical solution, for now the integrand is regular and the principal value can be dropped from Eqs. (13) and (14).

We have solved Eqs. (13) and (14) numerically and the result for the phase shifts for energies up to the breakup point is shown in Fig. 3. The data points at finite energy were taken from the phase shift analysis in [9] and the much more precise (nearly) zero-energy point from [10]. Also plotted is

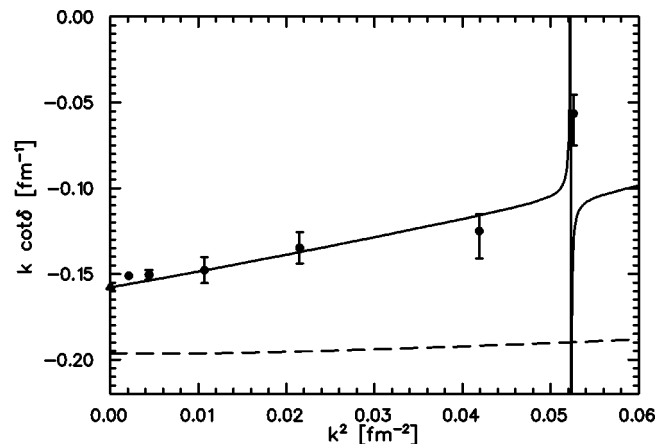


FIG. 3. $k \cot \delta$ in the $J=3/2$ channel to order $(r_0/a)^0$ (dashed line) and $(r_0/a)^2$ (solid line). Circles are from the phase shift analysis in Ref. [9] and the triangle is from Ref. [10].

the result of the leading-order calculation obtained by setting $\eta=0$, in which case our equations reduce to the case studied in Ref. [11].

We expect errors in our calculation to be of the order $(r_0/a)^3, (kr_0)^3$ compared to the leading order. These errors are smaller than the experimental errors in the finite energy case and of the same order as the experimental uncertainty in the case of the more precise measurement near $k=0$, where we find $^4a_{th}=6.33\pm 0.10$ fm [6] compared to $^4a=6.35\pm 0.02$ fm [10].

Our results seem to deviate from a simple effective range type expansion only around the pole at ~ 0.05 fm⁻². (A pole in $k \cot \delta$ corresponds to a zero in the scattering matrix, which does not carry any special meaning.) This pole does not appear in potential model calculations (e.g., [12]), and presumably will be smoothed out by higher-order terms that we have not yet included. It is interesting that the only “experimental” point in this region seems to indicate some structure there, but more experimental input would be necessary to confirm the behavior we predict.

The calculation of higher-order corrections involves the knowledge of further counterterms like the ones giving rise to p -wave interactions, etc. These parameters can be determined either by fitting other experimental data or by matching with another effective theory, involving explicit pions, valid up to higher energies. If more precise experimental data, particularly at zero-energy, appear, we would be facing a unique situation where precision calculations in strong-interaction physics can be carried out [13] and tested.

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