Pion charge form factor in point form relativistic dynamics

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(Received 26 June 1998)

Point form relativistic quantum mechanics, a constituent quark model with an oscillator wave function, and a point form impulse approximation are used to calculate the charge form factor of the pion. The resulting form factor fits available data, and in the ultrarelativistic (infinite binding) limit drops off as $1/Q^2$ for high Q^2. [S0556-2813(98)00812-7]

PACS number(s): 13.40.Gp, 12.39.Ki, 14.40.Aq

Measurements conducted at TJNAF[1] (E-93-021) have renewed interest in the study of the pion charge form factor. Point form relativistic quantum mechanics provides a fully covariant and nonperturbative method of calculating such form factors at all values of Q^2. This article demonstrates how a simple point form model suffices to describe currently available data and makes predictions for high-Q^2 behavior.

Perturbative QCD calculations predict that $Q^2F_\pi(Q^2)$ should become constant at high Q^2 [1,2]. However, the range at which this result is valid has been disputed [3]. A variety of refinements have been put forward to account for the low- and intermediate-Q^2 data, often by linking a ''soft'' nonperturbative bound-state wave function to ''hard'' perturbative high-Q^2 calculations. Examples of these may be found in Refs. [4–9].

Relativistic quantum mechanics, being nonperturbative, has had more success in the low to intermediate range. Calculations of F_π using the front and the instant forms of dynamics have recently been published [10–12]. Point form relativistic quantum mechanics differs from the other two forms in that interactions are put into the four-momentum operators, while all Lorentz transformations remain kinematic. Of the various forms of relativistic dynamics, therefore, only the point form is manifestly covariant. Furthermore, spin and orbital angular momenta may be coupled in the same way as is done in nonrelativistic quantum mechanics.

Using a simple interacting four-momentum operator $\hat{P}^\mu$ and an electromagnetic current operator consistently related to $\hat{P}^\mu$ [see Eq. (8)], we will show that a pion charge form factor built out of quark-antiquark constituents gives a good description of low-, intermediate-, and high-Q^2 properties.

In order to have the correct relativistic properties, the four-momentum operator must satisfy

$$[\hat{P}^\mu,\hat{P}^\nu]=0,$$

$$U_\Lambda \hat{P}^\mu U_\Lambda^{-1}=\Lambda_\nu^{-1\ \ \mu}\hat{P}^\nu, \tag{1}$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ on the model Hilbert space.

[1]Formerly CEBAF.

For a single particle of mass m and spin j the transformation properties of the state $|pj\sigma\rangle$, where p is the four momentum (satisfying $p\cdot p=E^2-\mathbf{p}^2=m^2$) and σ is the spin projection, are

$$\hat{P}^\mu_{\rm free}|pj\sigma\rangle=p^\mu|pj\sigma\rangle,$$

$$U_\Lambda|pj\sigma\rangle=\sum_{\sigma'}|\Lambda pj\sigma'\rangle D^j_{\sigma'\sigma}[R_W(p,\Lambda)], \tag{2}$$

with $R_W(p,\Lambda)$ a Wigner rotation given by $R_W(p,\Lambda)=B^{-1}(\Lambda p)\Lambda B(p)$, and with $B(p)$ a canonical spin boost. If $\Lambda_z(\alpha)$ is a boost along the z-axis, with tanh $\alpha=p_z/E$, then the Wigner D functions take the form

$$D_{1/2\ 1/2}\{R_W[p,\Lambda_z(\alpha)]\}=N_D\left[(E+m)\cosh\frac{\alpha}{2}+p_z\sinh\frac{\alpha}{2}\right],$$

$$D_{1/2-1/2}\{R_W[p,\Lambda_z(\alpha)]\}=N_D\left[p_\perp e^{-i\phi}\ \sinh\frac{\alpha}{2}\right],$$

$$D_{-1/2\ 1/2}=-D^*_{1/2-1/2},\quad D_{-1/2-1/2}=D_{1/2\ 1/2},$$

$$N_D=\frac{1}{\sqrt{(E+m)(E'+m)}}. \tag{3}$$

(See Ref. [13] for details and discussion.)

The quark-antiquark Hilbert space on which the pion bound state is constructed is the tensor product of two single particle states with spins $j_1=j_2=\frac{1}{2}$. Since under Lorentz transformations the Wigner rotations for the two states are in general different, the spins of the two-particle tensor product state cannot be coupled together. But velocity states, defined by

$$v^\mu:=\frac{(p_1^\mu+p_2^\mu)}{\sqrt{(p_1+p_2)^2}},$$

$$|v\mathbf{k}\mu_1\mu_2\rangle:=\sum_{\sigma_1\sigma_2}|p_1\sigma_1\rangle|p_2\sigma_2\rangle D^{1/2}_{\sigma_1\mu_1}\{R_W[k_1,B(v)]\}$$

$$D^{1/2}_{\sigma_2\mu_2}\{R_W[k_2,B(v)]\}, \tag{4}$$

where $p_i = B(v)k_i$, $k_1 = (\sqrt{m_1^2+\mathbf{k}^2},\mathbf{k})$, $k_2 = (\sqrt{m_2^2+\mathbf{k}^2}, -\mathbf{k})$, have the property that under Lorentz transformations,

$$U_\Lambda |v\mathbf{k}\mu_1\mu_2\rangle = \sum_{\mu_1'\mu_2'} |\Lambda v R_W \mathbf{k}\mu_1'\mu_2'\rangle D^{1/2}_{\mu_1'\mu_1}(R_W) D^{1/2}_{\mu_2'\mu_2}(R_W). \tag{5}$$

What is significant is that now $R_W = R_W(v,\Lambda)$ is the same for both D functions, as well as for the rotation on the internal momentum $\mathbf{k}$. This means that the spin and orbital angular momentum states can be coupled exactly as in nonrelativistic quantum mechanics [13].

The transformation from individual two-particle variables to velocity state variables results in a new measure:

$$\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} \to \int \mathcal{M}^3 \frac{d^3v}{2v_0}\frac{d^3k}{2\omega_1\omega_2}, \tag{6}$$

where $\mathcal{M} = \omega_1 + \omega_2$ is the two-particle mass. In the equal-mass case that we will examine here, the measure reduces to $\int (d^3v/2v_0) 4\omega d^3k$.

In the point form, interacting four-momentum operators can be constructed from an interacting mass operator [13,14] by writing

$$\hat{P}^\mu = \hat{M}\hat{V}^\mu,$$

$$\hat{M}\phi_\pi(v\mathbf{k}\mu_1\mu_2) = m_\pi \phi_\pi(v\mathbf{k}\mu_1\mu_2), \tag{7}$$

where the four velocity operator $\hat{V}^\mu$ acting on a velocity state, Eq. (4), gives v^μ, the four velocity of the state. If $\hat{M}$ commutes with $\hat{V}^\mu$ and is a Lorentz scalar, then the point form equations, Eqs. (1), are automatically satisfied.

The electromagnetic current operator must transform in a way consistent with the interacting representation of the Poincaré group, namely,

$$[\hat{P}^\mu, \hat{J}^\nu(x)] = i\frac{\partial \hat{J}^\nu}{\partial x^\mu} \quad \text{(translational covariance)}, \tag{8}$$

$$U_\Lambda \hat{J}^\mu(x) U_\Lambda^{-1} = \Lambda_\nu^{-1\ \mu} \hat{J}^\nu(\Lambda x) \quad \text{(Lorentz covariance)}, \tag{9}$$

$$\frac{\partial \hat{J}^\mu}{\partial x^\mu} = 0 \quad \text{(current conservation)}. \tag{10}$$

Let $\hat{J}^\mu(x) = e^{i\hat{P}\cdot x}\hat{J}^\mu(0)e^{-i\hat{P}\cdot x}$; then if $\hat{J}^\mu(0)$ transforms as a Lorentz four vector, $\hat{J}^\mu(x)$ will satisfy Eqs. (8) and (9). Current conservation now becomes

$$[\hat{P}_\mu, \hat{J}^\mu(0)] = 0. \tag{11}$$

For eigenstates of $\hat{P}^\mu$ (such as the pion bound state) current conservation can be written as

$$\langle p'|[\hat{P}_\mu, \hat{J}^\mu(0)]|p\rangle = (p'_\mu - p_\mu)\langle p'|\hat{J}^\mu(0)|p\rangle, \tag{12}$$

where $p' = m_\pi v_f$ and $p = m_\pi v_{in}$. If a standard (Breit) frame is chosen, with

$$p'(st) = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}, \quad p(st) = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}, \quad E = \sqrt{m_\pi^2 + p^2}, \tag{13}$$

and if we set

$$p = \frac{m_\pi}{2m_q}\frac{Q}{2}, \tag{14}$$

(where the factor $m_\pi/2m_q$ assures the correct nonrelativistic limit), then the pion charge form factor

$$F_\pi(Q^2) = \langle p'(st)|\hat{J}^{\mu=0}(0)|p(st)\rangle \tag{15}$$

and current conservation, Eq. (12), implies that $\langle p'(st)|\hat{J}^{\mu=3}(0)|p(st)\rangle = 0$.

In the point form impulse approximation [13], $\hat{J}^\mu(0)$ is chosen to be a one-body quark (or antiquark) electromagnetic current operator:

$$\langle p'(st)|\hat{J}^\mu(0)|p(st)\rangle = \langle p'(st)|\hat{J}^\mu_{\text{free}}(0)|p(st)\rangle, \quad \text{for } \mu = 0,1,2. \tag{16}$$

Specifically, the charge matrix element when particle one is struck becomes

$$\langle p_1'\sigma_1'|\hat{J}_1^0(0)|p_1\sigma_1\rangle = e_1 \delta_{\sigma_1'\sigma_1}. \tag{17}$$

Furthermore, we assume structureless quarks bound by an interacting mass operator such that the wave function takes the form

$$\psi_\pi(v\mathbf{k}\mu_1\mu_2) = 2v_0\delta^3(v - v_{in})\frac{1}{\sqrt{4\omega}}\left(\frac{1}{b^2\pi}\right)^{3/4} e^{-\mathbf{k}^2/2b^2}. \tag{18}$$

With these choices, we can now write the pion charge form factor explicitly:

$$\begin{aligned} F_\pi(Q^2) = \sum_{\mu_1\mu_2\mu_1'\mu_2'} \int\int & 4\omega' d^3k' 4\omega d^3k \psi_\pi^*(\mathbf{k}'\mu_1'\mu_2') \\ & \times \langle v'(st)\mathbf{k}'\mu_1'\mu_2'|\hat{J}^{\mu=0}_{\text{fr}}(0) \\ & \times |v(st)\mathbf{k}\mu_1\mu_2\rangle \psi_\pi(\mathbf{k}\mu_1\mu_2), \end{aligned} \tag{19}$$

where the matrix element is

$$\begin{aligned} & \langle v'(st)\mathbf{k}'\mu_1'\mu_2'|\hat{J}^{\mu=0}_{\text{fr}}(0)|v(st)\mathbf{k}\mu_1\mu_2\rangle \\ & = e_1 \frac{1}{\sqrt{4\omega 4\omega'}} \delta^3[k_2' - B^{-1}(v_f)B(v_{in})k_2] \\ & \quad \times D^{1/2}_{\mu_1'\mu_1}\{R_W[k_1, B^{-1}(v_f)B(v_{in})]\} \\ & \quad \times D^{1/2}_{\mu_2'\mu_2}\{R_W[k_2, B^{-1}(v_f)B(v_{in})]\} + \{1 \leftrightarrow 2\}, \end{aligned} \tag{20}$$

TABLE I. Five pairs of parameters that fit the low-Q^2 data.

	i	ii	iii	iv	v
m_q (GeV)	0.29	0.24	0.23	0.22	0.22
m_q^2/b^2	1.0	0.25	0.10	0.05	0.0

where the factor preceding the δ function is that appropriate to the new measure [Eq. (6)], and the Wigner D functions come from Eq. (4).

This integral [Eq. (19)] can be computed numerically once the values of the two parameters (the oscillator strength b and the "free quark" mass m_q) are supplied. Once one parameter is fixed (for example, the mass) then the low-Q^2 (charge radius) data restricts the other parameter to a small range of values. However, slight variations in either parameter can have significant effects on the shape of the form factor in the intermediate-Q^2 region.

For example, Table I displays different constituent quark masses and oscillator strengths that can be fit to the charge radius data. The form factors these produce are nearly indistinguishable below 0.15 GeV^2.

In the intermediate region displayed in Fig. 1, however, these same parameters give rise to distinctly different values of the form factor. The larger ratios of mass to binding energy ($m_q^2/b^2 = 1.0$ and 0.25) give curves that fall short of most of the intermediate-range data. Even choosing one particular mass for the constituent quarks (such as 0.22 GeV) allows considerable variation, the form factors rising as m_q/b decreases to the ultrarelativistic limit ($m_q/b = 0$; i.e., infinite binding strength). When data are published from the TJNAF probe of the $0-3$ GeV^2 region, the results should constrain the parameters much further.

Figure 2 extends the calculations into the high-Q^2 region, far beyond present measurements. However, QCD calculations based on pion decay predict a form factor $F_\pi(Q^2) \propto 1/Q^2$. In the ultrarelativistic case ($m_q = 0.22$ GeV, $m_q/b = 0$) our model shows a $1/Q^2$ dependence, albeit at a higher value than QCD methods generally give; in the other cases, $Q^2 F_\pi(Q^2)$ drops at various rates toward zero.

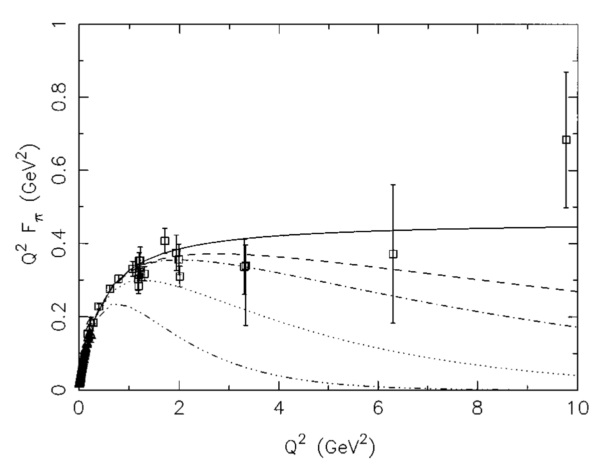

FIG. 1. The pion form factor in the intermediate region, computed using the parameters in Table I. The legend is i = dash-dots; ii = dots; iii = dash-dot; iv = dashes; v = solid line. Data are from Refs. [15,16].

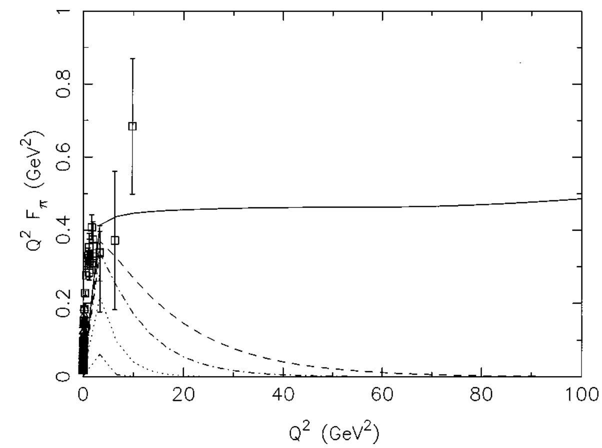

FIG. 2. The pion form factor in the high-Q^2 region, computed using the same parameters and legend as Fig. 1.

It may be argued that a harmonic oscillator wave function is an unrealistic ansatz; frequently a power law falloff is preferred [17,18]. Note, however, that we may apply a unitary transformation affecting only the magnitude of the relative momentum k of the wave function without altering the spectroscopic or the covariant properties of the model. If we choose

$$U_{\mathrm{QCD}}\Psi(k) = \sqrt{\rho}\,\Psi[f(k)], \quad f(k) = \left[n \ln\left(\frac{k^2+a^2}{a^2}\right)\right]^{1/2},$$

$$\rho = \frac{f'(k)\mu[f(k)]}{\mu(k)} \tag{21}$$

(where μ is the measure) then the harmonic oscillator wave functions are unitarily transformed:

$$U_{\mathrm{QCD}} e^{-k^2} = \sqrt{\frac{m^2 + n \ln[(k^2+a^2)/a^2]}{m^2+k^2}} \times \sqrt{\frac{n}{k(k^2+a^2)\sqrt{n \ln\left(\frac{k^2+a^2}{a^2}\right)}}}\left(\frac{a^2}{k^2+a^2}\right)^n, \tag{22}$$

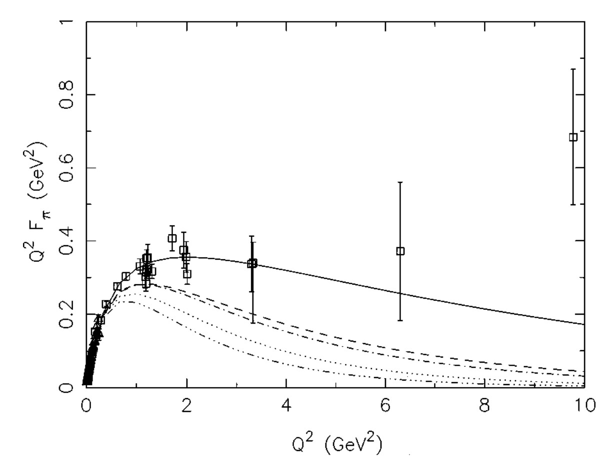

FIG. 3. The pion form factor in the intermediate region, using unitarily transformed wave functions, with $m_q^2/a^2b^2 = 0.1$. Untransformed harmonic oscillator ($m_q = 0.23$ GeV): solid line; $n = 0.5$ ($m_q = 0.22$ GeV): dashed; $n = 1$ ($m_q = 0.23$ GeV): dot-dash; $n = 2$ ($m_q = 0.24$ GeV): dotted; $n = 3$ ($m_q = 0.25$ GeV): dash-dots.

and we may substitute these new wave functions into Eq. (19).

Figure 3 shows the intermediate-Q^2 form factors (using a typical ratio of $m_q^2/a^2b^2 = 0.1$) for the untransformed oscillator as well as for the unitarily transformed functions for $n = 0.5,1,2,3$. Note that as n increases so does m_q, and the form factor dies off more quickly.

We conclude that point form relativistic quantum mechanics provides a useful framework for calculating the pion form factor from a constituent quark model; it is both conceptually and computationally straightforward. We find that a naive oscillator model combined with the point form impulse approximation [Eq. (16)] suffice to explain currently available data. The formalism can easily handle other wave functions, such as the QCD-inspired wave functions [Eq. (22)]. However, beyond noting that in the ultrarelativistic limit the point form model also predicts a $1/Q^2$ falloff, we do not attempt here to connect point form relativistic quantum mechanics and QCD.

We would like to thank F. Coester and W. N. Polyzou for many helpful discussions and suggestions. This work was supported by the Department of Energy, Nuclear Physics Division, under Contract No. DE-FG02-86ER40286.

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