Relativistic simultaneously coupled multiparticle states

W. H. Klink
Department of Physics and Astronomy and Department of Mathematics, University of Iowa, Iowa City, Iowa 52242
(Received 26 January 1998)

To carry out calculations dealing with relativistic multiparticle systems requires making a choice of variables that describe the system. Simultaneously coupled states, wherein n single-particle states are coupled together simultaneously rather than in a stepwise fashion, are defined and the resulting variables compared with stepwise variables. Generalized Racah coefficients that connect stepwise coupled states with simultaneously coupled states are derived for three-particle systems and used to calculate properties of resonances in isobar models. Invariants of simultaneously coupled states include orbital and spin angular momentum variables. It is shown how these variables can be coupled together in exactly the same way as is done nonrelativistically. [S0556-2813(98)05212-1]

PACS number(s): 24.10.Jv, 11.30.Cp, 24.10.Cn

I. INTRODUCTION

To carry out calculations on multiparticle systems, for example, bound-state wave functions or scattering amplitudes, it is necessary to make a choice of variables for the multiparticle system. Often the procedure for obtaining variables is to couple the constituent variables together one at a time, 1 to 2, (1,2) to 3, and so on, resulting in a stepwise coupled scheme. Such a scheme is of obvious utility for defining a two-body potential between particles 1 and 2, but a two-body potential between particles 2 and 3 requires a different stepwise scheme. Then coefficients are needed that connect different stepwise schemes, and often one chooses variables from one stepwise scheme as a standard and relates quantities such as kernels of operators naturally defined in another stepwise scheme by generalized Racah coefficients.

For nonrelativistic quantum mechanical systems such a procedure is workable and has been used in many calculations of three and more body systems [1]. For relativistic multiparticle systems, however, the procedure is more complicated, especially when the constituent particles have spin. One of the goals of this paper is to define states called velocity states having the property that internal variables such as spin and orbital angular momenta of relativistic multiparticle systems can be coupled together exactly as is done nonrelativistically. That is, just as a nonrelativistic n-particle state $|\mathbf{p}_1\mu_1\cdots\mathbf{p}_n\mu_n\rangle$ ($\mathbf{p}_i$ is the momentum of the ith particle and μ_i its spin projection) can be rewritten as $|\mathbf{p},\mathbf{k}_1\mu_1\cdots\mathbf{k}_n,\mu_n\rangle$ (with $\Sigma\mathbf{k}_i=\mathbf{0}$) and then the spins and orbital angular momenta coupled together, so too can a relativistic n-particle state $|p_1\mu_1\cdots p_n\mu_n\rangle$ (p_i is the four-momentum of the ith particle) be rewritten as a velocity state $|v,\mathbf{k}_1\mu_1\cdots\mathbf{k}_n\mu_n\rangle$ (with $\Sigma\mathbf{k}_i=\mathbf{0}$) and the angular momenta coupled together. $\mathbf{p}$ is the total momentum of the nonrelativistic n-particle system, $\mathbf{p}=\Sigma\mathbf{p}_i$, whereas v is the four velocity of the relativistic n-particle system, $v=1/(m_n)\Sigma p_i$, $m_n^2=(\Sigma p_i)^2$, the square of the mass of the n-particle system. Section II reviews the relativistic kinematics needed to define velocity states.

The velocity states written above are not diagonal in the total angular momentum. It is often desirable to construct states that are diagonal in the total angular momentum as well as the overall four-velocity. One way to construct such states is to stepwise couple the n-particle states. For relativistic systems there is, however, another possibility, namely to couple all of the single-particle states together simultaneously to obtain a state labeled by the overall four momentum, the total angular momentum and remaining variables; these remaining variables are invariant variables of the whole system and are obtained from the velocity states by suitable transformations, as shown in Sec. III.

A second goal of this paper is to construct simultaneously coupled states that can serve as a standard relative to any stepwise coupled states. In Sec. IV the coefficients relating any stepwise coupled state to the standard simultaneously coupled state for three-particle systems are computed. Then the coefficients relating different stepwise states are given via the coefficients relating stepwise to simultaneously coupled states. These coefficients are all products of SU(2) Clebsch-Gordan coefficients and Wigner D functions.

Simultaneously coupled states exist only for groups whose irreducible representations can be written as induced representations [2]. This includes the Poincaré group whose representations describe relativistic systems, as well as the Galilei group describing nonrelativistic systems. It does not include groups such as the three-dimensional rotation group SO(3) or other compact groups. Hence, for these groups there are no simultaneously coupled angular momentum states and Racah coefficients simply connect different stepwise schemes [3].

Reference [4] shows how to construct simultaneously coupled states in which the invariant variables include relativistically invariant spin projections; these states, called simultaneously coupled spin projection states are written $|vj\sigma;I(n)rr_1\cdots r_n\rangle$, where σ is a spin projection along a space-fixed axis while r is an invariant spin projection along a body-fixed axis, with $-j\leqslant\sigma$, $r\leqslant j$. $I(n)$ is a set of functionally independent invariant subenergies needed to specify the entire n-particle system. For the three-particle system discussed in Sec. IV, $I(3)$ consists of three variables that can be chosen as $m_{12}=\sqrt{(p_1+p_2)^2}$, m_{13}, and m_{23}, the invariant masses of the 1-2, 1-3, and 2-3 subsystems, respectively. The total invariant mass $m=\sqrt{(p_1+p_2+p_3)^2}$ can be expressed in terms of m_{12}, m_{13}, and m_{23}. The remaining labels in the simultaneously coupled spin projection state are invariant spin projection variables $r_1\cdots r_n$, eigenvalues of operators such as $P_i\cdot W_j$, where P_i and W_i are the four-momentum and Pauli-Lubanski operators of the ith particle. The r_i range between $-j_i$ and j_i, where j_i is the spin of the ith particle. Reference [4] shows how to construct the generalized Racah coefficients connecting stepwise coupled states to simultaneously coupled spin projection states. If the spin projections are helicities, then this procedure generalizes the two-particle coupling scheme of Jacob and Wick [5].

However, spin projection variables are often not as useful in calculations as are orbital and spin angular momenta. In particular, spin-dependent potentials such as spin orbit forces are not conveniently written in spin projection variables. The main goal of this paper is to construct simultaneously coupled states in which the spin projection variables $r,r_1,r_2\cdots r_n$ are replaced by l, the relative orbital angular momentum of the n-particle system r_l, the projection along a body-fixed axis, and spins (collectively denoted by s). Section III shows how to compute the coefficients connecting a simultaneously coupled spin projection state $|vj\sigma,I(n)rr_1\cdots r_n\rangle$ with $|vj\sigma,I(n)lr_ls\rangle$, a simultaneously coupled orbital angular momentum state. Then in Sec. IV the coefficients connecting a stepwise coupled three-particle state to a simultaneously coupled angular momentum state are computed. As an application the kernel of a two-body operator given in stepwise coupled variables is given in simultaneously coupled variables.

It should be noted that states are labeled by a four-velocity rather than the more usual three-momentum. The choice of four-velocity, three-momentum, or light front momentum $p_\perp=p_1+ip_2$, $p_+=p_0+p_3$ corresponds roughly to the different forms of relativistic dynamics first proposed by Dirac, namely, point, instant, and front-form dynamics [6]. While the background for simultaneously coupled states is a point form of relativistic dynamics, it is possible to carry out all the calculations of this paper equally well using three-momenta or front-form momenta. What is lost, however, is relativistic covariance. In the point form of relativistic dynamics, all Lorentz transformations are kinematic and hence the angular momentum coupling carried out in this paper remains valid even in the presence of interactions.

In a similar vein all the simultaneously coupled spin projection states of Ref. [4] and the Racah coefficients connecting them to stepwise coupled states are carried out for arbitrary boosts. Boosts are certain Lorentz transformations, coset representatives of the Lorentz group SO(1,3) with respect to the rotation group SO(3). They correspond to different possibilities of relativistic spin. The most popular spin choices are canonical, helicity, and front-form spin; each corresponds to a different choice of boost or coset representative. Though there are advantages to leaving the boosts arbitrary in the Racah coefficients, because canonical spin is often used in applications, and since velocity states are most naturally defined using canonical spin boosts, only canonical spin will be used in the main body of the paper. By choosing a particular boost, all of the rotations appearing in the Wigner D functions can then be computed explicitly, as shown in Sec. IV. The Appendix shows how to carry out the calculations for arbitrary boosts, by introducing the notion of a generalized Melosh rotation [7].

II. REVIEW OF RELATIVISTIC KINEMATICS

The irreducible representation space of the Poincaré group for single particles of mass m and spin of j is the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^3)\otimes V^j$, where V^j is the usual $(2j+1)$-dimensional space. n-particle spaces $\mathcal{H}_n$ are then appropriately symmetrized or antisymmetrized n-fold tensor products of single-particle spaces. Since $\mathcal{H}_n$ is a representation space of the Poincaré group, the actions of space time translations and Lorentz transformations Λ are well defined.

It turns out to be more convenient to specify the Poincaré group action on states rather than wave functions. For a single-particle state the group action is

$$U_a|pj\sigma\rangle=e^{-ip\cdot a}|pj\sigma\rangle,$$

$$U_\Lambda|pj\sigma\rangle=\sum_{\sigma'=-j}^{+j}|\Lambda p,j\sigma'\rangle D^j_{\sigma'\sigma}(R_w). \tag{1}$$

p is a four-momentum vector satisfying $p\cdot p=m^2$ and $p\cdot a:=p^\mu a_\mu=p^Tga$, with g the Minkowski metric $(1,-1,-1,-1)$. The Lorentz transformation $\Lambda\in SO(1,3)$ sends p to Λp and changes the spin projection component σ. $R_w\in SO(3)$ is a Wigner rotation, which is also sometimes written as $R_w(p,\Lambda)$ to emphasize that the rotation R_w depends on p and Λ. R_w is defined as

$$R_w(p,\Lambda):=B^{-1}(\Lambda v)\Lambda B(v),\quad v:=p/m, \tag{2}$$

where $B(v)$ is a boost, a coset representative of SO(1,3) with respect to SO(3), and thus a Lorentz transformation, which is completely specified by the four-velocity v. It is called a boost because it takes the four-momentum of a particle at rest $p^{\text{rest}}=(m,\mathbf{0})$ to the four-momentum p: $p=B(v)p^{\text{rest}}$. This condition does not uniquely specify $B(v)$ so there are many different choices of boosts possible corresponding to different types of relativistic spin. $D^s_{\sigma'\sigma}(\ )$ is a Wigner D function and j is the spin of the particle.

Though there are many different choices for $B(v)$, canonical spin boosts have several properties that can be used to advantage in defining simultaneously coupled states. For that reason only canonical spin boosts will be used in this paper. There are various equivalent definitions of canonical spin boosts:

$$B(v):=R(\hat{v})\Lambda_z(|\mathbf{v}|)R^{-1}(\hat{v}),\quad B^T(v)=B(v) \tag{3a}$$

$$=\begin{bmatrix} v^0 & \mathbf{v}^T \\ \mathbf{v} & I+\dfrac{\mathbf{v}\otimes\mathbf{v}^T}{1+v^0}\end{bmatrix} \tag{3b}$$

$$=[v,v(1),v(2),v(3)]. \tag{3c}$$

$R(\hat{v})\equiv R(\varphi,\theta,0)$ is the rotation specified by Euler angles φ, θ, the azimuthal and polar angles of the unit velocity vector $\hat{v}$. It is embedded into the Lorentz group as $\left(\begin{smallmatrix}1 & 0\\ 0 & R(\hat{v})\end{smallmatrix}\right)$. $\Lambda_z(|\mathbf{v}|)$ is a pure Lorentz transformation along the z axis with components $\cosh\alpha=v^0$ and $\sinh\alpha=|\mathbf{v}|$ given through the four-velocity $v=(v^0,\mathbf{v})$, $(v^0)^2=1+\mathbf{v}\cdot\mathbf{v}$. Multiplying the matrices in Eq. (3a) together gives a second way of writing out $B(v)$, namely, Eq. (3b).

The most useful form of $B(v)$ for this paper is Eq. (3c), where $B(v)$ is written as 4 four-vectors, of which the first is v. In order to be a Lorentz transformation, $v(i)\cdot v(j)=-\delta_{ij}$ and must also be orthogonal to v, $v\cdot v(i)=0$, $i,j=1,2,3$.

If $e^{\mu}_{\nu}:=\delta^{\mu}_{\nu}$, then $v(i)$ for canonical boosts can be written as

$$v^{\mu}(i)=e^{\mu}_{i}-\frac{e_i\cdot v}{1+v^0}\,(v^{\mu}+e^{\mu}_{0}). \tag{4}$$

Boosts other then canonical spin boosts differ in $v(i)$. The Appendix gives $v(i)$ for helicity and front-form boosts.

The property that makes canonical boosts so useful for simultaneously coupled states is that the Wigner rotation of a rotation is the same rotation:

$$R_w(p,R)=R, \tag{5}$$

a result proved in Ref. [4]. Helicity and front-form boosts also have special properties that make them useful in other contexts.

As shown in Ref. [4] the labels of the state $|pj\sigma\rangle$ are all eigenvalues of operators built out of Lie algebra elements of the Poincaré group. Thus p is the eigenvalue of the free four-momentum operator $P^{\mu}_{\rm fr}$, the exponential of U_a in Eq. (1), while j and σ are eigenvalues of the Pauli-Lubanski operator:

$$\begin{aligned} W_{\mu}&=\frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}P^{\nu}_{\rm fr}J^{\alpha\beta},\\ \tilde{W}_{\mu}&=\frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}V^{\nu}J^{\alpha\beta},\\ &=\frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}V^{\nu}\Sigma^{\alpha\beta}\\ V^{\nu}&:=P^{\nu}_{\rm fr}/M_{\rm fr},\qquad M^2_{\rm fr}=P_{\rm fr}\cdot P_{\rm fr}. \end{aligned} \tag{6}$$

The modified Pauli-Lubanski operator $\tilde{W}^{\mu}$ differs from W^{μ} by having the mass taken out. $J^{\alpha\beta}$ are the infinitesimal Lorentz generators and as seen from Eq. (1) can be separated into orbital and spin parts. Because of $\epsilon_{\mu\nu\alpha\beta}$ in Eq. (6), the orbital parts of $J^{\alpha\beta}$ do not contribute to $\tilde{W}^{\mu}$; only the intrinsic spin components $\Sigma^{\alpha\beta}$ contribute. As shown in Ref. [4] $\tilde{W}\cdot\tilde{W}$ is a Casimir invariant with eigenvalue $j(j+1)$, with j the spin of the particle. σ is the eigenvalue of $v(3)\cdot\tilde{W}$, while $[v(1)\pm iv(2)]\cdot\tilde{W}$ act similar to SU(2) raising and lowering operators:

$$\begin{aligned} \tilde{W}\cdot\tilde{W}|pj\sigma\rangle&=j(j+1)|pj\sigma\rangle,\\ v(3)\cdot\tilde{W}|pj\sigma\rangle&=\sigma|pj\sigma\rangle,\\ [v(1)\pm iv(2)]\cdot\tilde{W}|pj\sigma\rangle&=\sqrt{(j\mp\sigma)(j\pm\sigma+1)}|pj\sigma\pm1\rangle. \end{aligned} \tag{7}$$

When $v=(1,0,0,0)$, $B(v)$ is the identity; in that case the $v(i)$ are the same for all boost choices. However, when a particle at rest is boosted to $p=mv$, the direction of quantization will differ depending on the choice of boost, that is, on $v(i)$. This shows that relativistic spin is most naturally defined in the rest frame of the particle.

Since parity is conserved in the strong interactions, it is necessary to know what the parity operation, $\pi:=\mathrm{diag}(1,-1,-1,-1)$ does to velocity states. For single-particle states

$$\begin{aligned} U_{\mathcal{P}}|pj\sigma\rangle&=U_{\mathcal{P}}u_{B(v)}|p^{\rm rest}j\sigma\rangle=U_{B(\mathcal{P}v)}U_{R_{\mathcal{P}}}U_{\mathcal{P}}|p^{\rm rest}j\sigma\rangle\\ &=\eta|\mathcal{P}p,j\sigma\rangle, \end{aligned} \tag{8}$$

where $\mathcal{P}B(v)=\mathcal{P}B(v)\mathcal{P}\mathcal{P}=B(\mathcal{P}v)R_{\mathcal{P}}\mathcal{P}$; for canonical spin boosts, $R_{\mathcal{P}}$ is the identity rotation. η is the intrinsic parity of the particle.

n-particle states are defined as n-fold tensor products of single-particle states:

$$|p_1j_1\sigma_1\cdots p_nj_n\sigma_n\rangle:=|p_1j_1\sigma_1\rangle\cdots|p_nj_n\sigma_n\rangle \tag{9}$$

and their transformation properties are inherited from the single-particle transformation properties. As a first step to constructing simultaneously coupled states, we define velocity states as

$$\begin{aligned} |v\,\mathbf{k}_i\mu_i\rangle&:=U_{B(v)}|k_1j_1\mu_1\cdots k_nj_n\mu_n\rangle\\ &=\sum_{\sigma_i}|p_1j_1\sigma_1\cdots p_nj_n\sigma_n\rangle\prod_{i=1}^{n}D^{j_i}_{\sigma_i\mu_i}\{R_w[k_i,B(v)]\} \end{aligned} \tag{10}$$

with $p_i=B(v)k_i$, $\Sigma_{i=1}^{n}\mathbf{k}_i=\mathbf{0}$, $k_i=(\omega_i,\mathbf{k}_i)$, $\omega_i=\sqrt{m_i^2+\mathbf{k}_i\cdot\mathbf{k}_i}$. Under Lorentz transformations velocity states transform as

$$\begin{aligned} U_{\Lambda}|v\,\mathbf{k}_i\mu_i\rangle&=U_{\Lambda}U_{B(v)}|k_1j_1\mu_1\cdots k_nj_n\mu_n\rangle\\ &=U_{B(\Lambda v)}U_{R_w}|k_1j_1\mu_1\cdots k_nj_n\mu_n\rangle\\ &=\sum_{\mu_i'}U_{B(\Lambda v)}|R_wk_1,j_1\mu_1'\cdots R_wk_nj_n\mu_n'\rangle\\ &\quad\times\prod D^{j_i}_{\mu_i'\mu_i}[R_w(k_i,R_w)]\\ &=\sum_{\mu_i'}|\Lambda v,R_w\mathbf{k}_i,\mu_i'\rangle\prod D^{j_i}_{\mu_i'\mu_i}(R_w). \end{aligned} \tag{11}$$

Here use has been made of the rotation property of canonical boosts, namely, $R_w(k_i,R_w)=R_w$. As seen in the Appendix the definition of velocity states must be modified if boosts other than canonical boosts are used. Equation (11) states that under a Lorentz transformation v goes to Λv as expected, while the internal momenta $\mathbf{k}_i$ all undergo the same (Wigner) rotation. Moreover, the spin components also undergo the same (Wigner) rotation, which means that orbital and spin angular momentum can be coupled together exactly as is done nonrelativistically to obtain the total angular momentum j of the n-particle system. How this is done for simultaneously coupled orbital angular momentum states is the subject of the next section.

Under parity, a velocity state transforms as

$$U_{\mathcal{P}}|v\,\mathbf{k}_i\mu_i\rangle = U_{\mathcal{P}}U_{B(v)}|k_1 j_1\mu_1,\ldots,k_n j_n\mu_n\rangle$$
$$= U_{B(\mathcal{P}v)}U_{\mathcal{P}}|k_1 j_1\mu_1,\ldots,k_n j_n\mu_n\rangle$$
$$= \prod_{i=1}^{\eta} \eta_i|\mathcal{P}v,\mathcal{P}\mathbf{k}_i,\mu_i\rangle. \tag{12}$$

III. SIMULTANEOUSLY COUPLED ORBITAL ANGULAR MOMENTUM STATES

The simultaneously coupled orbital angular momentum states to be constructed in this section have labels $v j \sigma$, the four-velocity, total angular momentum *j*, and component σ that describe the external features of the system and internal labels *l*, the relative orbital angular momentum r_l, the component of *l* along a body-fixed axis, as well as $I(n)$, the functionally independent set of subenergies and spin labels $\boldsymbol{s}$. All of the internal variables are relativistic invariants, meaning that under a Lorentz transformation they remain unchanged. The mass of the *n*-particle system is actually an ''external'' invariant, but it is more convenient to treat it as a function of variables in $I(n)$. The mathematical machinery needed to decompose an *n*-fold tensor product of single-particle states is discussed in Ref. [4], and references therein. But the simultaneously coupled spin states constructed in Ref. [4] have internal variables that include the invariant spin projections. In this section the goal is to construct simultaneously coupled orbital states $|v j \sigma, I(n) l r_l s\boldsymbol{s}\rangle$.

To that end, define

$$|v\, l m_l r_l I(n)\mu_i\rangle := \int_{\mathrm{SO}(3)} dR D^{l*}_{m_l r_l}(R)|v, R\mathbf{k}_i(\mathrm{st})\mu_i\rangle, \tag{13}$$

where the velocity state $|v, R\mathbf{k}_i(\mathrm{st})\mu_i\rangle$ has as variables $\mathbf{k}_i = R\mathbf{k}_i(\mathrm{st})$, with $\mathbf{k}_i(\mathrm{st})$ a set of standard (body-fixed) vectors satisfying $\Sigma \mathbf{k}_i(\mathrm{st}) = \mathbf{0}$. In the next section which deals with three-particle systems, the standard vectors will be chosen so that $\hat{k}_1(\mathrm{st}) = \hat{z}$ defines the body-fixed *z* axis, while $\hat{k}_2(\mathrm{st}) \neq \hat{z}$ fixes the *x*-*z* plane. All invariant spin projections such as r_l are specified relative to such a body-fixed axis. Under a Lorentz transformation the orbital angular momentum state Eq. (13) transforms as

$$U_\Lambda|v\, l m_l r_l I(n)\mu_i\rangle$$
$$= \sum_{\mu_i'} \int_{\mathrm{SO}(3)} dR D^{l*}_{m_l r_l}(R)|\Lambda v, R_w R\mathbf{k}_i(\mathrm{st})\mu_i'\rangle$$
$$\times \prod D^{j_i}_{\mu_i'\mu_i}(R_w)$$
$$= \sum_{m_l'\mu_i'} |\Lambda v, l m_l' r_l I(n)\mu_i'\rangle D^{l}_{m_l' m_l}(R_w) \prod D^{j_i}_{\mu_i'\mu_i}(R_w). \tag{14}$$

If all the intrinsic spins j_i were zero, the state (13) would transform as a particle of spin *l*, as seen by comparing Eq. (14) with Eq. (1). Thus *l* is identified as the relativistic orbital angular momentum of the *n*-particle system and, as with r_l, the projection of *l* along a body-fixed axis, is a relativistic invariant.

Though *l*, r_l, and $I(n)$ are relativistic invariants, m_l and μ_i are not. But since the arguments of the Wigner *D* functions that push around m_l and μ_i are all the same, all the spins $j_1 \cdots j_n$ can be coupled together to give the total spin *s*, and then *s* coupled to *l* to give the total angular momentum *j*. A simultaneously coupled orbital angular momentum state is defined as

$$|v j \sigma, I(n) l r_l s\boldsymbol{s}\rangle := \sum \langle l m_l s m_s | j \sigma(l_s)\rangle$$
$$\times \langle j_1\mu_1 \ldots j_n\mu_n | s m_s(\boldsymbol{s})\rangle |v\, l m_l r_l I(n)\mu_i\rangle. \tag{15}$$

$\langle|\rangle$ are SU(2) Clebsch-Gordan coefficients and $\boldsymbol{s}$ is a degeneracy parameter specifying how the *n* spins $j_1 \cdots j_n$ are coupled to form a total spin *s*.

It is now straightforward to show that

$$U_\Lambda|v j \sigma, I(n) l r_l s\boldsymbol{s}\rangle = \sum |\Lambda v, j\sigma', I(n) l r_l s\boldsymbol{s}\rangle D^{j}_{\sigma'\sigma}(R_w)$$
$$U_a|v j \sigma, I(n) l r_l s\boldsymbol{s}\rangle = e^{-i m_n v\cdot a}|v j \sigma, I(n) l r_l s\boldsymbol{s}\rangle \tag{16}$$

indicating that the simultaneously coupled orbital state, Eq. (15), transforms as a particle of spin *j* and mass m_n, with $m_n = \sqrt{(\Sigma k_i)^2}$ a function of the invariants $I(n)$ only. The states (15) are called simultaneously coupled orbital angular momentum states because there are no subsystem orbital or total angular momentum variables.

Under parity the simultaneously coupled orbital angular momentum states transform as

$$U_{\mathcal{P}}|v j \sigma, I(n) l r_l s\boldsymbol{s}\rangle$$
$$= \prod_{i=1}^{n} \eta_i|\mathcal{P}v, j\sigma, I^{\mathcal{P}}(n) l r_l' s\boldsymbol{s}\rangle D^{l}_{r_l r_l'}[R_y(\mathcal{P})]$$
$$= \prod_{i=1}^{n} \eta_i(-1)^{l+r_l}|\mathcal{P}v, j\sigma, I^{\mathcal{P}}(n) l, -r_l s\boldsymbol{s}\rangle, \tag{17}$$

where $I^{\pi}(n)$ is the parity transformed set of invariants.

To conclude this section, we compute the Racah coefficients that connect simultaneously coupled orbital angular momentum states to simultaneously coupled spin projection states; these coefficients are needed in Sec. IV to compute the Racah connecting stepwise coupled states to simultaneously coupled orbital angular momentum states.

Now the simultaneously coupled orbital angular momentum states, Eq. (15), can be written as

$$|vj\sigma,I(n)lr_l s\boldsymbol{s}\rangle=\sum \langle lm_l sm_s|j\sigma(ls)\rangle\langle j_1\mu_1\cdots j_n\mu_n|sm_s(\boldsymbol{s})\rangle\int_{\mathrm{SO}(3)} dR D^{l*}_{m_l r_l}(R)U_{B(v)}|Rk_i(\mathrm{st})j_i\mu_i\rangle=\sum \langle lm_l sm_s|j\sigma(ls)\rangle$$
$$\times\langle j_1\mu_1\cdots j_n\mu_n|sm_s(\boldsymbol{s})\rangle U_{B(v)}\int_{\mathrm{SO}(3)} dR D^{l*}_{m_l r_l}(R)U_R|k_i(\mathrm{st})j_i r_i\rangle\prod D^{j_i}_{r_i\mu_i}(R^{-1})=\sum \langle lr_l sr_s|jr(ls)\rangle$$
$$\times\langle j_1 r_1\cdots j_n r_n|sr_s(\boldsymbol{s})\rangle U_{B(v)}\int_{\mathrm{SO}(3)} dR D^{j*}_{\sigma r}(R)U_R|k_i(\mathrm{st})j_i r_i\rangle. \tag{18}$$

In the last line of Eq. (18) intertwining properties of Clebsch-Gordan coefficients have been used to rewrite the Wigner D functions.

The n-particle state of standard four-vectors $k_i(\mathrm{st})$ can now be coupled to give a simultaneously coupled spin projection state [see Ref. [4], Eq. (3.5)]. The result is

$$|k_i(\mathrm{st})j_i r_i\rangle=\sum_{j'r'}|\mathbf{0}j'r'I(n)r_1\cdots r_n\rangle$$

$$U_R|k_i(\mathrm{st})j_i r_i\rangle=\sum_{j'r'\sigma'}|\mathbf{0}j'\sigma'r'I(n)r_1\cdots r_n\rangle D^{j'}_{\sigma'r'}(R). \tag{19}$$

Using the orthogonality properties of D functions integrated over $R\in \mathrm{SO}(3)$ then gives

$$|vj\sigma,I(n)lr_l s\boldsymbol{s}\rangle=\sum_{rr_ir_s}\langle lr_l sr_s|jr(ls)\rangle \times\langle j_1 r_1\cdots j_n r_n|sr_s(\boldsymbol{s})\rangle|vj\sigma,I(n)rr_i\rangle \tag{20}$$

with inverse

$$|vj\sigma,I(n)rr_i\rangle=\sum_{lr_l s s r_s}\langle jr(ls)|lr_l sr_s\rangle \times\langle sr_s(\boldsymbol{s})|j_1 r_1\ldots j_n r_n\rangle|vj\sigma,I(n)lr_l s\boldsymbol{s}\rangle. \tag{21}$$

For two-particle states this becomes

$$|vj\sigma,m_{12}ls\rangle =\sum_{r_1r_2r_s}\langle l0sr_s|jr_s(ls)\rangle|j_1r_1j_2r_2|sr_s\rangle|vj\sigma,m_{12}r_1r_2\rangle,$$

$$|vj\sigma,m_{12}r_1r_2\rangle =\sum_{lsr_s}\langle jr_s(ls)|l0sr_s\rangle\langle sr_s|j_1r_1j_2r_2\rangle|vj\sigma,m_{12}ls\rangle. \tag{22}$$

For example, a four-particle spin state with invariant labels $r,r_1r_2r_3r_4$ is connected to a simultaneously coupled orbital state with invariant labels l, r_l, and s along with stepwise coupled spin labels that could be chosen, for example, to be s_{12} and s_{34}, the spin of the 12 and 34 particles, respectively. Equations (20) and (21) show that the Clebsch-Gordan coefficients linking invariant spin labels to invariant orbital labels themselves depend on only invariant labels.

IV. RACAH COEFFICIENTS FOR THREE-PARTICLE STATES

In this section we show how to connect stepwise coupled three-particle states to simultaneously coupled orbital states. The generalization from three-particles to n-particles is straightforward but tedious; moreover for more than three-particles there are many different possible stepwise schemes. The strategy for computing Racah coefficients for orbital states is as follows. In the previous section the Racah coefficients connecting simultaneously coupled spin projection states with simultaneously coupled orbital states were shown to be products of Clebsch-Gordan coefficients. In Ref. [4] the Racah coefficients connecting simultaneously coupled spin projection states with stepwise coupled spin projection states were shown to be products of Wigner D functions. Combining these two results shows that the Racah coefficients connecting simultaneously coupled orbital states with stepwise coupled orbital states are products of Clebsch-Gordan coefficients and Wigner D functions.

Using Eq. (20) and Eq. (3.1) of Ref. [4] gives

$$|vj\sigma,I(3)l'r'_l s'\boldsymbol{s}\rangle =\sum \langle l'r'_l s'r'_s|jr(l'j')\rangle\langle j_1r'_1j_2r'_2j_3r'_3|s'r'_s(\boldsymbol{s})\rangle U_{B(v)} \times\int dR D^{j*}_{\sigma r}(R)U_R|k_1(\mathrm{st})r'_1,k_2(\mathrm{st})r'_2,k_3(\mathrm{st})r'_3\rangle. \tag{23}$$

If now particle 1 is coupled to particle 2 and then 1-2 coupled to particle 3 and the orthogonality properties of the D function used, Eq. (23) becomes

$$|vj\sigma,I(3)l'r'_l s's'_{12}\rangle =\sum \langle l'r'_l s'r'_s|jr(l's')\rangle\langle j_1r'_1j_2r'_2j_3r'_3|s'r'_s(s'_{12})\rangle \times D^{j_{12}}_{r_{12}r_1+r_2}[R^{-1}(\hat{k}^{12}_3)R(\hat{k}^{12}_1)]D^{j_1^*}_{r'_1r_1}[R(\hat{k}^1_2)] \times D^{j_2^*}_{r'_2r_2}[R(\hat{k}^2_1)]D^{j_3^*}_{r'_3r_3}[R(\hat{k}^3_{12})]D^{j}_{rr_{12}+r_3}\{R[\hat{k}_3(\mathrm{st})]\} \times|vj\sigma,mm_{12}j_{12}r_{12}r_1r_2r_3\rangle. \tag{24}$$

The arguments of the various D functions are always specified by unit vectors obtained by deboosting the previously chosen standard vectors to specific reference frames. The notation $k_i^{(j)}$ means the four vector $k_i(\mathrm{st})$ deboosted to the frame where $\mathbf{k}_j=\mathbf{0}$:

$$k_i^{(j)}:=B^{-1}(v_j)k_i(\mathrm{st}),$$

$$k_i^{(i)}:=B^{-1}(v_i)k_i(\mathrm{st})=\begin{pmatrix} m_i \\ \mathbf{0}\end{pmatrix}. \tag{25}$$

Thus $k_2^{(1)}=B^{-1}(v_1)k_2(\mathrm{st})$ is the deboosted four momentum of particle 2 in the rest frame of particle 1. k_3^{12} means the four momentum of particle 3 in the $1+2$ rest frame; that is, where

$$k_1(\mathrm{st})+k_2(\mathrm{st})=B(v_{12})\begin{pmatrix} m_{12} \\ \mathbf{0}\end{pmatrix} \tag{26}$$

with m_{12} the mass of the two-particle system, $m_{12}^2=[k_1(\mathrm{st})+k_2(\mathrm{st})]^2$. With these definitions the Euler angles of the various D functions in Eq. (24) can always be written as functions of the invariants $I(3)=\{m_{12},m_{13},m_{23}\}$.

Since one of the applications of simultaneously coupled states is to the $N\pi\pi$ system in which the nucleon spin is $\frac{1}{2}$ and the pions are spinless, we choose the nucleon as particle 1 and compute $D^{j_1}[R(\hat{k}_2^1)]$, where $\hat{k}_2^1$ has azimuthal and polar angles φ_2^1 and θ_2^1, respectively. Now $k_2^{(1)}=B^{-1}[k_1(\mathrm{st})/m_1]k_2(\mathrm{st})$. But

$$\cos\,\theta_1^2=-\frac{k_2^1\cdot e_3}{|\mathbf{k}_2^1|}=-\frac{1}{|\mathbf{k}_2^1|}\,B^{-1}\left(\frac{k_1(\mathrm{st})}{m_1}\right)k_2(\mathrm{st})\cdot e_3=-\frac{1}{|\mathbf{k}_2^1|}\,k_2(\mathrm{st})\cdot B\left(\frac{k_1(\mathrm{st})}{m_1}\right)e_3=-\frac{1}{|\mathbf{k}_2^1|}\,k_2(\mathrm{st})\cdot v_1(3)$$

$$=-\frac{1}{|\mathbf{k}_2^1|}\,k_2(\mathrm{st})\cdot\left[e_3-\frac{e_3\cdot v_1}{1+v_1^0}\,(v_1+e_0)\right]=\frac{1}{|\mathbf{k}_2^1|}\left[k_2(\mathrm{st})_z-\frac{k_1(\mathrm{st})_z}{m_1+E_1(\mathrm{st})}\left(E_2(\mathrm{st})+\frac{k_1(\mathrm{st})\cdot k_2(\mathrm{st})}{m_1}\right)\right]. \tag{27}$$

All of these momenta can be written as invariants. Thus

$$m_{12}^2=(k_1^{(1)}+k_2^{(1)})^2=m_1^2+m_2^2+2m_1E_2^1=m_1^2+m_2^2+2m_1\sqrt{m_2^2+(\mathbf{k}_2^1)^2},$$

$$\sqrt{m_2^2+(\mathbf{k}_2^1)^2}=\frac{1}{2m_1}\,[m_{12}^2-m_1^2-m_2^2],$$

$$m_{23}^2=[k_1(\mathrm{st})+k_2(\mathrm{st})+k_3(\mathrm{st})-k_1(\mathrm{st})]^2=m_{123}^2+m_1^2-2m_{123}E_1(\mathrm{st}),$$

$$E_1(\mathrm{st})=\sqrt{m_1^2+[\mathbf{k}_1(\mathrm{st})]^2}=\frac{1}{2m_{123}}\,(m_{123}^2+m_1^2-m_{23}^2) \tag{28}$$

and the like. The azimuthal angle

$$\cos\,\varphi_2^1\sin\,\theta_2^1=-\frac{1}{|\mathbf{k}_2^1}\,k_2^1\cdot e_1,$$

$$\sin\,\varphi_2^1\sin\,\theta_2^1=-\frac{1}{|\mathbf{k}_2^1|}\,k_2^1\cdot e_2 \tag{29}$$

can be computed in a similar manner.

The rotation $R^{-1}(\hat{k}_3^{12})R(\hat{k}_1^{12})$ appearing as the argument in the $D^{j_{12}}(\ )$ function differs from the other rotations in that—as will be shown—it is independent of the boost that is chosen. We will show this for the middle Euler angle as it is the only one relevant for Racah coefficients. The azimuthal Euler angles are all canceled off by the azimuthal angles appearing in the D^{j_1}, D^{j_2}, and D^{j} functions.

The middle Euler angle is

$$\cos\,\beta_{3,1}=\hat{z}\cdot R^{-1}(\hat{k}_3^{12})R(\hat{k}_1^{12})\hat{z}=\hat{k}_3^{12}\cdot\hat{k}_1^{12}. \tag{30}$$

But

$$m_{13}^2=(k_3^{12}+k_1^{12})^2=m_3^2+m_1^2+2k_3^{12}\cdot k_1^{12}=m_1^2+m_3^2$$
$$+2(E_3^{12}E_1^{12}-|\mathbf{k}_3^{12}||\mathbf{k}_1^{12}|\cos\,\beta_{3,1}),$$

$$m_{123}^2=(k_1^{12}+k_2^{12}+k_3^{12})^2=m_{12}^2+m_3^2+2m_{12}E_3^{12},$$

$$m_2^2=(k_1^{12}+k_2^{12}-k_1^{12})^2=m_{12}^2+m_1^2-2m_{12}E_1^{12}, \tag{31}$$

so

$$\cos\,\beta_{3,1}=\frac{1}{2|\mathbf{k}_3^{12}||\mathbf{k}_1^{12}|}\,[2E_3^{12}E_1^{12}+m_1^2+m_3^2-m_{13}^2],$$

$$E_3^{12}=\frac{1}{2m_{12}}\,[m_{123}^2-m_{12}^2-m_3^2],$$

$$E_1^{12}=\frac{1}{2m_{12}}\,[m_{12}^2+m_1^2-m_2^2]; \tag{32}$$

thus all rotations appearing in the D functions of Eq. (24) can be written as functions of $I(3)$. Moreover, rotations of the form $R^{-1}(\ )R(\ )$ are independent of the boost, whereas rotations of the form $R(\ )$ depend on the boost that is chosen (in this case canonical spin boosts).

Equation (24) gives the Racah coefficients connecting $|vj\sigma,I(3)l'r_l's's_{12}'\rangle$ with the stepwise coupled spin projection state $|vj\sigma,mm_{12}j_{12}r_{12}r_1r_2r_3\rangle$. Using the results of Eq. (22), with the variables r_1 and r_2 replaced by l_{12} and s_{12}, the orbital and spin angular momentum of the 1-2 system, and the variables r_{12} and r_3 replaced by l and s, the orbital and spin angular momentum of the three-particle system, gives the desired Racah coefficients

$$|vj\sigma,I(3)l'r_l's's_{12}'\rangle=\sum\ \langle l'r_l's'r_s'|jr(l's')\rangle\langle j_1r_1'j_2r_2'j_3r_3'|s'r_s'(s_{12}')\rangle d^{j_{12}}_{r_{12}r_1+r_2}(\cos\beta_{3,1})$$
$$\times\ D^{j_1^*}_{r_1'r_1}[R(\hat{k}_2^1)]D^{j_2^*}_{r_2'r_2}[R(\hat{k}_1^2)]D^{j_3^*}_{r_3'r_3}[R(\hat{k}_{12}^3)]D^{j}_{rr_{12}+r_3}\{R[\hat{k}_3(\mathrm{st})]\}\langle j_{12}\overline{r}(l_{12}s_{12})|l_{12}0s_{12}\overline{r}\rangle$$
$$\times\langle s_{12}\overline{r}(j_1j_2)|j_1r_1j_2r_2\rangle\langle j\tilde{r}(l_s)|l0s\overline{r}\rangle\langle s\overline{r}|j_{12}r_{12}j_3r_3\rangle|vj\sigma,mm_{12}lsj_{12}l_{12}s_{12}\rangle. \tag{33}$$

The Racah coefficient is considerably simplified if only particle 1 (the ''nucleon'') has spin while particles 2 and 3 are spinless. Then the degeneracy labels s and s_{12} disappear and

$$|vj\sigma,I(3)l'r_l'\rangle=\sum\ \langle l'r_l'j_1r_1'|jr(l's_1)\rangle d^{j_{12}}_{r_{12}r_1}(\cos\beta_{3,1})D^{j_1^*}_{r_1'r_1}[R(\hat{k}_2^1)]D^{j}_{rr_{12}}\{R[\hat{k}_3(\mathrm{st})]\}\langle j_{12}r_1(l_{12}s_1)|l_{12}0s_1r_1\rangle$$
$$\times\langle jr_{12}(lj_{12})|l0j_{12}r_{12}\rangle|vj\sigma,mm_{12}lj_{12}l_{12}\rangle. \tag{34}$$

Equation (34) is already suitable for describing resonances in the final state for the $N\pi$ system. In the reaction $N\pi\to N\pi\pi$, if the $N\pi$ system results from the decay of a Δ resonance, then $j_{12}=\frac{3}{2}$ while m_{12} is the ''mass'' of the Δ resonance. The threshold factor for Δ decay is given by $|\mathbf{k}_1^{12}|^{l_{12}}$. In the direct channel the $N\pi\to N\pi\pi$ reaction is dominated by angular momentum $j=\frac{3}{2}$, with m the ''mass'' of the Δ resonance and the threshold factor $|\mathbf{k}_3(\mathrm{st})|^l$. Thus the variables appearing in the stepwise coupled state of Eq. (34) are the natural variables for resonances in the 1-2 system.

However, if there are also resonances in the π-π or 2-3 system, the variables appearing in the stepwise scheme, Eq. (34), are of little use. The relevant variables can be obtained by interchanging variables pertaining to particles 1 and 3 in Eq. (33) to give a 2-3 coupling scheme, and then setting the spins of particles 2 and 3 equal to 0. The result is

$$|vj\sigma,I(3)l'r_l'\rangle=\sum\ \langle l'r_l'j_1r_1'|jr(l'j_1)\rangle d^{j_{23}}_{r_{23}0}(\cos\beta_{1,3})$$
$$\times D^{j_1^*}_{r_1'r_1}[R(\hat{k}_{23}^1)]D^{j}_{rr_1+r_{23}}[\hat{k}_1(\mathrm{st})]$$
$$\times\langle j\tilde{r}(ls)|l0s\tilde{r}\rangle$$
$$\times\langle j\tilde{r}|j_{23}r_{23}j_1r_1\rangle|vj\sigma,mlsm_{23}j_{23}\rangle, \tag{35}$$

which is now suitable for describing a π-π resonance, say the ρ with $j_{23}=1$. The relevant threshold factor is $|\mathbf{k}_2^{23}|^{j_{23}}$ for the decay, while the direct channel threshold factor is $|\mathbf{k}_1(\mathrm{st})|^l$.

But experimentally final-state resonances are seen in Dalitz plots, where the relevant variables are m_{12}, m_{23}, and m_{13}, that is, the set $I(3)$. If the invariant mass m is held fixed, cross sections in m_{12} and m_{23} show bands giving the positions and widths of resonances. Then the natural variables to use are lr_l and $I(3)$.

In the model for πN production given in Ref. [8], the final-state resonances are described by a reduced three-body function $u_{3j}[lr_lI(3)]$. But for πN resonances the natural variables to use are $u_{3j}(lmm_{12}j_{12}l_{12})$ while for $\pi\pi$ resonances $u_{3j}(mlsm_{23}j_{23})$. Simple choices for these resonance functions then result in $u_{3j}[lr_lI(3)]$, in which all the spin information about the resonances is given through the Racah coefficients. For example, for the ρ resonance, $j_{23}=1$, $j_1=\frac{1}{2}$, and $j=\frac{3}{2}$ for the D functions in Eq. (35).

If the three-body potential arises from two-body interactions, the Racah coefficients derived in Eq. (33) can be used to write the kernel in simultaneously coupled variables. Let

$$\langle v_{12}'j_{12}'\sigma_{12}'m_{12}'l_{12}'s_{12}'|V_{12}|v_{12}j_{12}\sigma_{12}m_{12}l_{12}s_{12}\rangle$$
$$=\delta^3(v_{12}'-v_{12})\,\delta_{j_{12}'j_{12}}\delta_{\sigma_{12}'\sigma_{12}}\langle m_{12}'l_{12}'s_{12}'|V_{12}^{j_{12}}|m_{12}l_{12}s_{12}\rangle \tag{36}$$

be the two-body kernel for the 1-2 system on the two-particle Hilbert space. If on the three-particle space

$$\langle v'j'\sigma',m'm_{12}'l's'j_{12}'l_{12}'s_{12}'|V_{12}|vj\sigma,mm_{12}lsj_{12}l_{12}s_{12}\rangle$$
$$=\delta^3(v'-v)\,\delta_{j'j}\delta_{\sigma'\sigma}\langle m_{12}'l_{12}'s_{12}'|V_{12}^{j_{12}}|m_{12}l_{12}s_{12}\rangle, \tag{37}$$

then

$$\langle\overline{v}'\overline{j}'\overline{\sigma}',\overline{I}'(3)\overline{l}'\overline{r}_l'\overline{s}'\overline{s}_{12}'|V_{12}|\overline{v}\overline{j}\overline{\sigma},\overline{I}(3)\overline{l}\overline{r}_l\overline{s}\overline{s}_{12}\rangle$$

$$=\sum\!\!\!\!\!\!\int R^{\underline{j'}m'}_{\overline{I'(3)l'},\overline{r'_l}\overline{s'}\overline{s'_{12}},m'_{12}l's'j'_{12}l'_{12}s'_{12}}$$
$$\times\langle v'j'\sigma',m'm'_{12}l's'j'_{12}l'_{12}s'_{12}|$$
$$\times V_{12}|vj\sigma,mm_{12}lsj_{12}l_{12}s_{12}\rangle R^{jm}_{m_{12}lsj_{12}l_{12}s_{12},\overline{I}(3)\overline{l}\overline{r_l ss_{12}}}, \tag{38}$$

where the Racah coefficients $R^{jm}_{-\,-}$ are

$$R^{jm}_{m_{12}lsj_{12}l_{12}s_{12},\overline{I}(3)\overline{l}\overline{r_l ss_{12}}} := \langle vj\sigma,mm_{12}lsj_{12}l_{12}s_{12}|vj\sigma,\overline{I}(3)\overline{l}\overline{r_l ss_{12}}\rangle, \tag{39}$$

as given in Eq. (35).

V. CONCLUSION

Though multiparticle states diagonal in the four momentum and spin projection variables of all the particles are natural to use for scattering amplitudes and cross sections, such variables do not exhibit the spin properties of a system such as the total angular momentum or the orbital angular momentum of some subsystem. Spin variables are important for expressing conservation laws such as angular momentum conservation, or expressing properties of subsystems, such as the spin and mass of resonances, or the threshold behavior of a resonance.

Even when the overall four-momentum and angular momentum variables are diagonal there are many possible choices for the remaining invariant (under Lorentz transformations) variables. The goal of this paper has been to compute the (Racah) coefficients that connect these different possibilities. Unlike the Racah coefficients for angular momentum, wherein the coefficients connect different stepwise couplings, for the Poincaré group there is also the possibility of simultaneous coupling, where all the particle variables are coupled at once to produce an overall four-momentum, and angular momentum state [see Eq. (15)].

Simultaneously coupled states provide a natural standard with which to connect to any stepwise coupled state. Thus, the strategy in this paper has been to define Racah coefficients as the coefficients connecting any stepwise coupled state to the simultaneously coupled state. Then the more usual Racah coefficients connecting one stepwise coupled state to any other is the product of two such Racah coefficients. As seen in Sec. IV these Poincaré Racah coefficients are products of SU(2) Clebsch-Gordan coefficients and Wigner D functions; they are computed explicitly for three-particle states in Eq. (33).

There are actually two different types of simultaneously coupled states. One called simultaneously coupled spin projection states, has invariant labels that include the spin projections of all the constituent particles. Racah coefficients for such variables are treated in detail in Ref. [4]. The other possibility, called simultaneously coupled orbital states, has invariant labels that include the total orbital angular momentum, as well as the total and subsystem spins [see Eq. (15)]. Such variables are important in resonance analysis and isobar models and are the variables of main interest in this paper. Racah coefficients connecting these two types of simultaneously coupled states are given in Eqs. (20), (21), and (22).

The main goal of this paper has been to compute Racah coefficients connecting stepwise orbital states to simultaneously coupled orbital states. As shown in Sec. IV such coefficients are of obvious utility in isobar models, where a multiparticle reaction proceeds through a series of resonances. Each resonance has its associated mass, spin, and orbital labels. But the variables describing resonances in different channels are not compatible and (Racah) coefficients are needed to connect them. Examples are given in Sec. IV for a three-particle system with variables for a ''Δ'' resonance as against a ''ρ'' resonance.

The simultaneous coupled states analyzed in this paper will be of most use in the point form of relativistic dynamics [6]. In particular specifying the dynamics in terms of a mass operator means that interaction parts of the mass operator can be chosen as kernels in simultaneously or stepwise coupled variables. An explicit example is given in Ref. [8] where a separable potential couples a two-body to a three-body Hilbert space to model π-N production data.

The Racah coefficients computed in Sec. IV involve Wigner D functions, in which the Euler angles are functions of relativistic invariants. By making a definite choice of boost, namely, a canonical boost [see Eq. (3)], it is possible to compute the angle dependence explicitly, as shown in examples such as Eqs. (27)–(33).

Racah coefficients connect different choices of invariant variables. It is also important to have the Clebsch-Gordan coefficients connecting n-fold single-particle states to angular momentum states. Since simultaneously coupled orbital states have been chosen as a standard in this paper, we collect the various formulas given throughout the paper to link states $|p_1\sigma_1\cdots p_n\sigma_n\rangle$ [Eq. (9)] to $|vj\sigma,I(n)lr_l s\boldsymbol{s}\rangle$ [Eq. (15)]. The result is

$$|vj\sigma,I(n)lr_l s\boldsymbol{s}\rangle = \sum_{\substack{m_l m_s\\ \mu_1\cdots\mu_n}} \langle lm_l sm_s|j\sigma(ls)\rangle$$
$$\times\langle j_1\mu_1\cdots j_n\mu_n|sm_s(\boldsymbol{s})\rangle$$
$$\times\int_{\mathrm{SO}(3)} dR D^{l*}_{m_l r_l}(R)|p_1\sigma_1\cdots p_n\sigma_n\rangle$$
$$\times\prod_{i=1}^{n} D^{j_i}_{\sigma_i\mu_i}\{R_w[Rk_i(\mathrm{st}),B(v)]\},$$

where

$$p_i = B(v)Rk_i(\mathrm{st}), \quad \Sigma\mathbf{k}_i(\mathrm{st}) = \mathbf{0},$$

$$p = \sum_i p_i = B(v)m_n,$$

$$m_n^2 = \left(\sum_i p_i\right)^2 = \left(\sum_i k_i(\mathrm{st})\right)^2.$$

The Clebsch-Gordan coefficients converting n single-particle states to a simultaneously coupled orbital state are then

$$\langle p_1\sigma_1\cdots p_n\sigma_n|vj\sigma,I(n)lr_l\mathbf{s}\rangle$$

$$=\sum_{\substack{m_l m_s\\ \mu_1\cdots\mu_n}}\langle lm_l sm_s|j\sigma(ls)\rangle\langle j_1\mu_1\cdots j_n\mu_n|sm_s(\mathbf{s})\rangle D^{l*}_{m_l r_l}(R)\times\prod_{i=1}^{n}D^{j_i}_{\sigma_i\mu_i}\{R_w[Rk_i(\mathrm{st}),B(v)]\};$$

$R_w[Rk_i(\mathrm{st}),B(v)]$ is a Wigner rotation whose arguments can be computed along the lines given in Sec. IV.

To conclude it should be noted that no normalization conventions for the various states have been given. All of the various coupled states are built out of products of single-particle states and hence inherit the normalizations chosen for single-particle states. These normalizations are worked out in detail for covariantly normalized single-particle states in the Appendix of Ref. [8]. But since the Racah coefficients connecting different states, all diagonal in v,j,σ, are products of SU(2) Clebsch-Gordan coefficients and Wigner D functions, the normalization of one such state relative to another is fixed by the normalization of the Clebsch-Gordan coefficients and D functions.

ACKNOWLEDGMENT

W.H.K. was supported in part by the U.S. DOE.

APPENDIX: VELOCITY STATES FOR ARBITRARY BOOSTS; HELICITY AND FRONT FORM VELOCITY STATES

Velocity states, states in which all the internal variables transform under the same (Wigner) rotation were defined for canonical spin in Eq. (10). This definition must be modified for boosts other than canonical boosts, and in particular for helicity and front form boosts. In place of Eq. (10), write

$$|v\mathbf{k}_i\mu_i\rangle:=\sum_{\mu_i'}U_{B(v)}|k_i\mu_i'\rangle\prod_{i=1}^{n}D^{j_i}_{\mu_i'\mu_i}[B^{-1}(k_i)B_c(k_i)],\tag{A1}$$

where $B_c(k_i)$ is a canonical boost defined in Eq. (3), $B(k_i)$ is any other choice of boost, and the combination $B^{-1}(k_i)B_c(k_i)$ is a rotation, a generalized Melosh rotation [7]. Under a Lorentz transformation, the velocity state transforms as

$$\begin{aligned}U_\Lambda|v\mathbf{k}_i\mu_i\rangle&=U_\Lambda U_{B(v)}\sum_{\mu_i'}|k_i\mu_i'\rangle\prod_{i=1}^{n}D^{j_i}_{\mu_i'\mu_i}[B^{-1}(k_i)B_c(k_i)]\\&=U_{B(\Lambda v)}\sum_{\mu_i'\mu_i''}|R_wk_i,\mu_i''\rangle\prod_{i=1}^{n}D^{j_i}_{\mu_i''\mu_i'}[R_w(k_i,R_w)]D^{j_i}_{\mu_i'\mu_i}[B^{-1}(k_i)B_c(k_i)]\\&=U_{B(\Lambda v)}\sum_{\mu_i''}|R_wk_i,\mu_i''\rangle\prod_{i=1}^{n}D^{j_i}_{\mu_i''\mu_i}[B^{-1}(R_wk_i)B_c(R_wk_i)R_w]=\sum_{\mu_i'}|\Lambda v,R_w\mathbf{k}_i\mu_i'\rangle\prod_{i=1}^{n}D^{j_i}_{\mu_i'\mu_i}(R_w),\end{aligned}\tag{A2}$$

which is the same as Eq. (11). Thus, with the aid of the generalized Melosh rotation, it is possible to define velocity states for any type of boost.

Besides canonical spin the boosts most often used are helicity and front form boosts. They are distinguished by the choice of $v^\mu(i)$ given in Eq. (3c), three spacelike unit vectors making up the last three columns of the boost matrix $B(v)$. For canonical spin the $v^\mu(i)$ are given in Eq. (4).

Helicity boosts are defined by

$$B_H(v):=R(\hat{v})\Lambda_z(|\mathbf{v}|)=\begin{bmatrix}v^0&0&0&|\mathbf{v}|\\\mathbf{v}&\hat{v}_1&\hat{v}_2&v_0\hat{v}\end{bmatrix},$$

$$v_H^\mu(1)=\begin{pmatrix}0\\\hat{v}_1\end{pmatrix},\quad\hat{v}_1=\begin{pmatrix}\cos\theta\cos\varphi\\\cos\theta\sin\varphi\\-\sin\theta\end{pmatrix},$$

$$v_H^\mu(2)=\begin{pmatrix}0\\\hat{v}_2\end{pmatrix},\quad\hat{v}_2=\begin{pmatrix}-\sin\varphi\\\cos\varphi\\0\end{pmatrix},$$

$$v_H^\mu(3)=\begin{pmatrix}|\mathbf{v}|\\v_0\hat{v}\end{pmatrix},\quad\hat{v}=\begin{pmatrix}\sin\theta\cos\varphi\\\sin\theta\sin\varphi\\\cos\theta\end{pmatrix},$$

$$v_H(i)\cdot v_H(j)=-\delta_{ij}.\tag{A3}$$

The Melosh rotation is given by

$$\begin{aligned}R_M(v)&=B_H^{-1}(v)B_c(v)\\&=[R(\hat{v})\Lambda_z(|\mathbf{v}|)]^{-1}R(\hat{v})\Lambda_z(|\mathbf{v}|)R^{-1}(\hat{v})=R^{-1}(\hat{v}).\end{aligned}\tag{A4}$$

Front form boosts are usually defined using SL(2,C) rather than SO(1,3). But it is possible to define $v_F^\mu(i)$ in terms of a null vector $n^\mu=\binom{1}{\hat{n}}$. Then

$$v_F^\mu(3) := \frac{n^\mu}{n\cdot v} - v^\mu,$$

$$v_F^\mu(i) := n_i^\mu - \frac{n_i\cdot v}{n\cdot v}\, n^\mu, \quad i=1,2, \tag{A5}$$

where $n_i^\mu = \binom{0}{\hat{n}_i}$, such that $n_i\cdot n_j = -\delta_{ij}$, $n\cdot n_i = 0$. It is readily checked that $v_F(i)\cdot v_F(j) = -\delta_{ij}$, and $v\cdot v_F(i) = 0$ as required.

The corresponding Melosh rotation is

$$R_M(v) = B_F^{-1}(v) B_c(v) = g \begin{pmatrix} v^T \\ v_F^T(1) \\ v_F^T(2) \\ v_F^T(3) \end{pmatrix} g[v\, v_c(1) v_c(2) v_c(3)]$$

$$= g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & v_F(i)\cdot v_c(j) & & \end{pmatrix},$$

$$R_M(v)_{ij} = -v_F(i)\cdot v_c(j), \tag{A6}$$

which is straightforward but tedious to work out using Eqs. (4) and (A5).

[1] See, for example, W. Glöckle, *The Quantum Mechanical Three Body Problem* (Springer Verlag, Berlin, 1983), especially Chap. 3, Sec. 6.5.

[2] For a definition of induced representation, see for example, R. E. Warren and W. H. Klink, J. Math. Phys. **11**, 1155 (1970); G. Mackey, *The Theory of Induced Representations* (University of Chicago Press, Chicago, 1955).

[3] See, for example, L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics* (Addison-Wesley, Reading, MA, 1981); also W. H. Klink and T. Ton-That, J. Math. Phys. **37**, 6468 (1996).

[4] W. H. Klink, Ann. Phys. (N.Y.) **213**, 31 (1992).

[5] M. Jacob and A. Wick, Ann. Phys. (N.Y.) **7**, 401 (1959).

[6] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949); B. D. Keister and W. N. Polyzou, in *Advances in Nuclear Physics*, edited by J. W. Negele and E. W. Vogt (Plenum, New York, 1991), Vol. 20, p. 225.

[7] H. D. Melosh, Phys. Rev. D **9**, 1095 (1974); see also F. Coester, *Progress in Particle Nuclear Physics*, edited by A. Faessler (Pergamon, New York, 1992), Vol. 29.

[8] W. H. Klink and M. Rogers, Phys. Rev. C **58**, 3605 (1998), preceding paper.