Dynamical isobar models and π-N scattering data

W. H. Klink and M. Rogers
Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242
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Isobar amplitudes for pion-nucleon elastic and production reactions are constructed from a dynamical theory. The framework of point form relativistic quantum mechanics is used to derive a relativistic Lippmann-Schwinger equation linking elastic and production channels. The dynamics is contained in a mass operator which is the sum of a free and interacting mass operator. For a separable interacting mass operator, the Lippmann-Schwinger equations can be solved analytically; the resulting matrix equations have determinants whose zeros give the positions of multiple resonances in the same angular momentum channels. Though varying parameters to fit data is highly nonlinear, procedures are developed for fitting elastic and inelastic data. Fits to data are presented for the P_{33} partial wave channel in pion-nucleon scattering.
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I. INTRODUCTION

Over the past 15 years a number of models have been developed to fit pion-nucleon scattering data, including both the elastic and production channels [1–5]. In some of these models, amplitudes are written as Feynman diagrams and procedures developed to enforce inelastic unitarity [1–3]. Other models use an S matrix or modified K matrix approach in which the three particle final states are written as quasi-two-body states [4–5]. Such models are not dynamical in that they do not have a Hamiltonian (or mass operator) to model the resonances. Consequently it is difficult to treat the thresholds for newly opening channels correctly.

Coupled channel models that treat relativity correctly have been developed by Gross and Surya [3] and Fuda [6] for the πN system. In these models resonances such as the Δ resonance are treated as particles. In this paper we develop a coupled channel method for getting $\pi N \to \pi N$ and $\pi N \to \pi\pi N$ amplitudes which is dynamical and in which the direct channel resonances arise as poles in the scattering amplitudes. Since the amplitudes come from a dynamical theory they automatically incorporate inelastic unitarity and correct threshold properties. Relativity is of course crucial, since it is impossible to have genuine production reactions in nonrelativistic quantum mechanics.

The framework for developing a dynamical isobar model is point form relativistic quantum mechanics. There are a number of ways of doing relativistic quantum mechanics for systems with a finite number of degrees of freedom, called by Dirac the instant, front and point forms of relativistic quantum mechanics [7]; each of these forms has various advantages and disadvantages, and all forms are, in any event, more complicated than nonrelativistic quantum mechanics. Aside from being manifestly covariant, a feature of the point form is that it is possible to define so-called velocity states, in which orbital and spin angular momentum can be coupled together exactly as is done nonrelativistically; this property of velocity states, discussed in great detail in the following paper [8], is a key ingredient in developing an isobar model, in which resonances are defined by their masses, widths, and spins.

In point form relativistic quantum mechanics, all interactions are put into the four-momentum operator P^μ, generalizing nonrelativistic quantum mechanics where all interactions are put into the Hamiltonian. Thus in the point form, there are not only interactions in the Hamiltonian $H=P^0$ but also the momentum operator $\vec{P}$. But in contrast to the other forms, all the Lorentz transformations, rotations as well as pure boosts, are kinematic, and hence the unitary operators representing Lorentz transformations are the same as for noninteracting particles.

The problem then is to construct interacting four-momentum operators on the appropriate Hilbert space, which in this paper is taken to be the direct sum of two- and three-particle Hilbert spaces, namely, $\mathcal{H}_{\pi N} \oplus \mathcal{H}_{\pi\pi N}$. In contrast to the previous paper [9], where P^μ was constructed from local currents, here we use the so-called Bakamjian-Thomas construction [10], in which $P^\mu = MV^\mu$, the product of the mass operator and four-velocity operator, satisfying $V^\mu V_\mu = I$, the identity operator. The four-velocity operator is purely kinematic, so all the dynamics resides in the mass operator M.

The mass operator can be written as a sum of free and interacting mass operators $M = M_{\rm fr} + V$, where $M_{\rm fr}$ is given from representations of the Poincaré group, while V contains the dynamics. The time-dependent Schrödinger equation is generalized to $P^\mu \psi_x = i\hbar(\partial\psi_x/\partial x_\mu)$, which becomes a mass operator eigenvalue problem when $P^\mu = MV^\mu$:

$$M\psi = \omega\psi, \quad \psi \in \mathcal{H}_{\pi N} \oplus \mathcal{H}_{\pi\pi N}.$$

Though this equation is relativistic, when velocity state variables are used the structure is very similar to the nonrelativistic case.

In particular, scattering states are associated with the continuous part of the spectrum of M, in which case it is possible to rewrite the above equation as a relativistic Lippmann-Schwinger equation. The derivation of the relativistic Lippmann-Schwinger in point form relativistic quantum mechanics is carried out in Sec. II.

In Sec. III the interacting mass operator V is chosen to have a certain separable form, from which we show how to derive isobar amplitudes. In general, of course, the relativis-

tic Lippmann-Schwinger equation cannot be any more easily solved than the nonrelativistic one. But as in the nonrelativistic cases, it is possible to solve the equation analytically for certain types of potentials, including separable potentials. For such potentials, the solutions reduce to linear matrix equations, which when solved, have denominators that have determinants [called $D(\omega)$, see Eq. (58)] whose zeros determine the positions of the s channel resonances.

Using suitably chosen input functions for the separable potential results in a closed form expression for the elastic and production partial wave amplitudes. These input functions contain certain parameters that can be adjusted to fit data. But the relation between the parameters appearing in the input functions and the data is nonlinear, so it is a nontrivial task to fit the data by varying the input parameters.

In Sec. IV we develop a fitting procedure by starting with parameters that have no inelasticity; in this case it is relatively easy to fit the direct channel resonances [the $\Delta(1232)$ and $\Delta(1600)$ as well as the antiresonance at 1400 MeV for the P_{33} case]. Then by slowly increasing the inelasticity, such that at each step the resonance parameters are refit, it is possible to fit the full inelastic data. In this paper we show how our procedure works for partial cross sections and Argand plots for the P_{33} channel.

II. POINT FORM RELATIVISTIC QUANTUM MECHANICS

In this section we review the elements of point form relativistic quantum mechanics needed to formulate the relativistic Lippmann-Schwinger equation, from which the scattering amplitudes are obtained. Details of point form relativistic quantum mechanics are presented in Refs. [8,9,11].

All forms of relativistic quantum mechanics are related to how interactions are put into the ten generators of the Poincaré group. In the point form the six generators of Lorentz transformations are all kinematic, meaning they do not contain any interacting terms. All of the interactions appear in the four-momentum operator. These operators are the generators of space-time displacements, in analogy to the Hamiltonian which produces time translations in nonrelativistic quantum mechanics. A key difference is that in nonrelativistic quantum mechanics the momentum operator that generates spatial displacements contains no interactions, while in point form relativistic quantum mechanics the generator of spatial displacements must contain interactions in order that the entire theory be manifestly covariant.

The generalization of the time-dependent Schrödinger equation in point form relativistic quantum mechanics is

$$P^\mu \psi_x = i\hbar \frac{\partial}{\partial x_\mu} \psi_x , \tag{1}$$

where P^μ is the four-momentum operator and ψ_x is an element of the Hilbert space. In this paper the Hilbert space will be chosen to be the direct sum of a two-particle Hilbert space with a three-particle Hilbert space, with P^μ containing interactions that mix the two spaces, and hence cause production reactions. More generally a four-momentum operator can be constructed that acts on any number of Hilbert spaces. x is a Minkowski space-time point, $x=(ct,\mathbf{x})$, and ψ_x means the wave function of the system at the space-time point x.

As shown in Ref. [9] P^μ must satisfy the Poincaré relations

$$[P^\mu, P^\nu] = 0, \quad \mu,\nu = 0,1,2,3,$$

$$U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^{-1})^\mu{}_\nu P^\nu, \tag{2}$$

where Λ is an arbitrary Lorentz transformation and U_Λ the unitary operator representing Λ. Since the four operators P^μ commute with each other, a mass operator

$$M := \sqrt{P\cdot P} \tag{3}$$

can be defined; the spectrum of this operator gives the bound states (discrete spectrum) and scattering states (continuous spectrum). In this paper we will deal only with the continuous spectrum and rewrite Eq. (1) in the form of a relativistic Lippmann-Schwinger equation.

To construct an interacting four-momentum operator, we begin by reviewing the Poincaré action for a single particle. For a single particle of mass m and spin j, the Poincaré action on a four-momentum eigenstate is given by

$$P^\mu_{\rm fr} |p,j,\sigma\rangle = p^\mu |p,j,\sigma\rangle,$$

$$U_\Lambda |p,j,\sigma\rangle = \sum_{\sigma'} |\Lambda p,j,\sigma'\rangle D^j_{\sigma'\sigma} R_w(p,\Lambda),$$

where $P^\mu_{\rm fr}$ is the noninteracting (free) four-momentum operator, with eigenvalues satisfying

$$p\cdot p := p^\mu p_\mu = m^2. \tag{4}$$

σ is the spin projection, while R_w is a Wigner rotation, an element of the rotation group depending on p and Λ. $D^j_{\sigma'\sigma}[R_w(p,\Lambda)]$ is an SO(3) Wigner D function (see Refs. [8] and [9]).

The four-velocity operator is defined to be

$$V^\mu := M_{\rm fr}^{-1} P^\mu_{\rm fr} \tag{5}$$

and satisfies $V\cdot V = I$, the identity operator. In what is called the Bakamjian-Thomas construction [10] interactions are introduced by perturbing the free mass operator $M_{\rm fr} := \sqrt{P_{\rm fr}\cdot P_{\rm fr}}$ so that the four-momentum operator becomes

$$P^\mu = M V^\mu, \tag{6}$$

where M is the sum of a free and interacting mass operator

$$M = M_{\rm fr} + V, \tag{7}$$

which must satisfy

$$[V^\mu, M] = 0,$$

$$U_\Lambda M U_\Lambda^{-1} = M, \tag{8}$$

in order that

$$[P^\mu, P^\nu] = [MV^\mu, MV^\nu] = 0, \tag{9}$$

$$U_\Lambda P^\mu U_\Lambda^{-1} = U_\Lambda M V^\mu U_\Lambda^{-1} = M(\Lambda^{-1})^\mu{}_\nu V^\nu = (\Lambda^{-1})^\mu{}_\nu P^\nu, \tag{10}$$

as required in Eq. (2).

An n-particle state is the product of n one-particle states. From this it follows that

$$P^\mu_{\rm fr}|p_1,j_1,\sigma_1,\ldots,p_n,j_n,\sigma_n\rangle = \sum_{i=1}^n p_i^\mu |p_1,j_1,\sigma_1,\ldots,p_n,j_n,\sigma_n\rangle,$$

$$U_\Lambda|p_1,j_1,\sigma_1,\ldots,p_n,j_n,\sigma_n\rangle = \sum_{i=1}^n |\Lambda p_1,j_1,\sigma_1,\ldots,\Lambda p_n,j_n,\sigma_n\rangle \prod_i D^j_{\sigma_i'\sigma_i} R_w(p_i,\Lambda). \tag{11}$$

Each of the Wigner D functions in Eq. (11) have different arguments so they cannot be coupled together.

As shown in Ref. [8] it is possible to define velocity states that have the property that the spins can be coupled together, as is done nonrelativistically. Set

$$|v,k_i,\mu_i\rangle = |p_1,j_1,\sigma_1,\ldots,p_n,j_n,\sigma_n\rangle \prod_{i=1}^n D^j_{\sigma_i\mu_i}\{R_w[k_i,B(v)]\}, \tag{12}$$

where $B(v)$ is a boost, a special Lorentz transformation carrying p_i to $k_i = B^{-1}(v)p_i$, with $\Sigma \mathbf{k}_i = 0$. Then as shown in Ref. [8],

$$P^\mu_{\rm fr}|v,\mathbf{k}_i,\mu_i\rangle = m_n v^\mu |v,\mathbf{k}_i\mu_i\rangle,$$

$$U_\Lambda|v,\mathbf{k}_i,\mu_i\rangle = |\Lambda,R_w\mathbf{k}_i,\mu_i'\rangle \prod_{i=1}^n D^j_{\mu_i',\mu_i}(R_w). \tag{13}$$

That is, under a Lorentz transformation Λ, the overall four velocity v (which satisfies $v\cdot v = 1$) goes to Λv, while the internal momenta $\mathbf{k}_i$ (satisfying $\Sigma_i \mathbf{k}_i = 0$) are all rotated by the same (Wigner) rotation, which is also the same rotation appearing in the D functions $D^j_{\mu_i',\mu_i}(R_w)$. Thus for velocity states, all the spins j_i can be coupled together to form a state with overall spin s, as is done nonrelativistically. m_n is the mass of the noninteracting n-particle system, given by

$$m_n = \sum_{i=1}^n \sqrt{m_i^2 + \mathbf{k}_i^2}. \tag{14}$$

The action of the various operators on velocity states is

$$M_{\rm fr}|v,\mathbf{k}_i,\mu_i\rangle = m_n|v,\mathbf{k}_i\mu_i\rangle,$$

$$V^\mu|v,\mathbf{k}_i,\mu_i\rangle = v^\mu|v,\mathbf{k}_i\mu_i\rangle$$

$$P^\mu_{\rm fr} = M_{\rm fr}V^\mu. \tag{15}$$

To construct an interacting mass operator that satisfies Eq. (8), it is easiest to convert the velocity states $|v,\mathbf{k}_i,\mu_i\rangle$ to velocity states $|v,j,\sigma;\xi\rangle$ of definite angular momentum j,σ and degeneracy parameters ξ. If the kernel of the mass operator satisfies

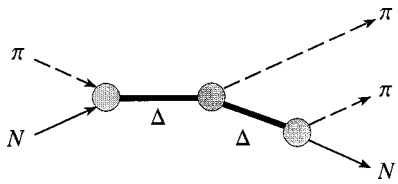

FIG. 1. An example of the isobar mechanism.

$$\langle v',j',\sigma',\xi'|M|v,j,\sigma,\xi\rangle = (1+|v|^2)^{1/2}\delta^3(v-v')\,\delta_{j,j'}\delta_{\sigma,\sigma'}\langle\xi'|M|\xi\rangle, \tag{16}$$

then M will automatically satisfy Eq. (8).

There are a number of different coupling schemes for combining n-particles with spins $j_1,\sigma_1,\ldots,j_n,\sigma_n$ into overall j and σ states. A coupling scheme is chosen to provide a set of kinematic variables naturally suited to describing the phenomena under consideration. Since the isobar model assumes an intermediate quasi-two-body state, it is natural to use a stepwise coupling which combines the spins of the two particles in the final-state resonance. If there are final-state resonances among different combinations of the final-state particles, things are more complicated since the scattering amplitudes must be expressed in a common set of variables so that any interaction written in terms of a two-body cluster, other than the one corresponding to the chosen set of kinematic variables in which the partial-wave amplitudes are parametrized, has to be transformed to those variables via Racah coefficients. How this is done is described in Ref. [8]. In this paper we will consider only resonances between two specific final-state particles, specifically $\pi N \to \pi\Delta, \Delta \to \pi N$, and higher mass resonances in the same (Δ) channel. (See Fig. 1.)

Since isobar analysis is generally done in terms of partial waves, the natural variables should include total spin j, orbital angular momentum l, combined spin of the (i,j) cluster of final-state particles $j_{i,j}$, and their relative orbital angular momentum $l_{i,j}$. With this choice of variables the states for the two-body Hilbert space are (neglecting isospin)

$$|v,j,\sigma;\omega,l\rangle,$$

where v, j, σ, and l are as defined above and ω is the eigenvalue of the free mass operator. For the three-body Hilbert space, the variables are

$$|v,j,\sigma;\omega,l,\omega_{12},j_{12},l_{12}\rangle,$$

where all other variables are as defined above and ω_{12} is the invariant mass of the 1-2 cluster. The details of the construction and normalization of these states are left to the Appendix. Parametrizing the kernel of the mass operator in terms of these variables naturally generates a generalized partial-wave expansion as the scattering amplitudes in these variables are just the partial-wave amplitudes. This also is shown in the Appendix.

The coupled-channel Hilbert space for single pion production is

$$\mathcal{H} := (\mathcal{H}_N \otimes \mathcal{H}_\pi) \oplus (\mathcal{H}_N \otimes \mathcal{H}_\pi \otimes \mathcal{H}_\pi). \tag{17}$$

The first part $(\mathcal{H}_N \otimes \mathcal{H}_\pi)$ is just the space of two-body states (now including isospin)

$$|v, I, I_z, j, \sigma; \omega, l\rangle$$

with wave functions

$$\psi_2(v, I, I_z, j, \sigma; \omega, l) = \langle v, I, I_z, \sigma; \omega, l | \psi_2 \rangle. \tag{18}$$

The second part $(\mathcal{H}_N \otimes \mathcal{H}_\pi \otimes \mathcal{H}_\pi)$ is the space of the three-body states

$$|v, I, I_z, j, \sigma; \omega, l, I_{12}, \omega_{12}, j_{12}, l_{12}\rangle$$

with wave functions

$$\psi_3(v, I, I_z, j, \sigma; \omega, l, \omega_{12}, I_{12}, j_{12}, l_{12}) = \langle v, I, I_z, j, \sigma; \omega, l, \omega_{12}, I_{12}, j_{12}, l_{12} | \psi_3 \rangle. \tag{19}$$

A state in the full model space is denoted by

$$| \; \rangle = \begin{pmatrix} |v, I, I_z, j, \sigma; \omega, l\rangle \\ |v, I, I_z, j, \sigma; \omega, l, \omega_{12}, I_{12}, j_{12}, l_{12}\rangle \end{pmatrix}, \tag{20}$$

and the wave functions have the form

$$\Psi = \begin{pmatrix} \psi_2(v, I, I_z, j, \sigma; \omega, l) \\ \psi_3(v, I, I_z, j, \sigma; \omega, l, \omega_{12}, I_{12}, j_{12}, l_{12}) \end{pmatrix}. \tag{21}$$

The full inner product on this space is

$$\langle \Phi | \Psi \rangle = (1 + |v|^2)^{1/2} \delta^3(v - v') \times \delta_{j,j'} \delta_{\sigma,\sigma'} (\langle \varphi_2 | \psi_2 \rangle + \langle \varphi_3 | \psi_3 \rangle),$$

where the variables v, j, σ are extracted from the remainder of the inner product because of relativistic invariance; the reduced inner product is then

$$\langle \varphi_2 | \psi_2 \rangle = \sum_{I, I_z} \sum_{l} \int_{m_{12}}^{\infty} d\omega \, \frac{\omega^2 k(\omega)}{4} \times \varphi_2^*(I, I_z, j; \omega, l) \psi_2(I, I_z, j; \omega, l). \tag{22}$$

$$\langle \varphi_3 | \psi_3 \rangle = \sum_{I, I_z, I_{12}} \sum_{l} \sum_{l_{12}} \int_{m_{123}}^{\infty} d\omega \int_{m_{12}}^{\omega - m_3} d\omega_{12} \times \frac{\omega^2 k_3(\omega, \omega_{12}) k_1^{12}(\omega_{12})}{8} \varphi_3^* \times (I, I_z, j; \omega, l, I_{12}, j_{12}, \omega_{12}, l_{12}) \times \psi_3(I, I_z, j, \omega, l, I_{12}, j_{12}, \omega_{12}, l_{12}). \tag{23}$$

The normalization and phase space conventions used above are derived in the Appendix.

If the mass operator is also invariant with respect to isospin, the n-body to m-body kernel must satisfy

$$\langle v', I', I_z', j', \sigma', \xi_n' | M | v, I, I_z, j, \sigma, \xi_m \rangle = (1 + |v|^2)^{1/2} \delta^3(v - v') \delta_{I,I'} \delta_{I_z, I_z'} \delta_{j,j'} \delta_{\sigma,\sigma'} \langle \xi_n' | M_{I_j} | \xi_m \rangle, \tag{24}$$

where ξ_n' and ξ_m are the remaining ''internal'' variables for the n- and m-body systems, respectively. We denote the coupled-channel mass operator by

$$(M_{I_j})_{nm} := \langle \xi_n' | M_{I_j} | \xi_m \rangle. \tag{25}$$

For our models this can be written in obvious notation as

$$M_{I_j} = \begin{pmatrix} \langle N\pi | M_{I_j} | N\pi \rangle & \langle N\pi | M_{I_j} | N\pi\pi \rangle \\ \langle N\pi\pi | M_{I_j} | N\pi \rangle & \langle N\pi\pi | M_{I_j} | N\pi\pi \rangle \end{pmatrix}. \tag{26}$$

Using Eq. (14), the free mass operator in a velocity state basis becomes

$$M_{\mathrm{fr}} = \begin{pmatrix} \sum_{i=1}^{2} \sqrt{|\mathbf{k}_i|^2 + m_i^2} & 0 \\ 0 & \sum_{i=1}^{3} \sqrt{|\mathbf{k}_i|^2 + m_i^2} \end{pmatrix}. \tag{27}$$

Writing the mass operator as a sum of free and interacting parts, Eq. (7), and having extracted the four-velocity, the time-dependent relativistic Schrödinger equation, Eq. (1), becomes the time-independent Schrödinger equation on the reduced space

$$(M_{\mathrm{fr}} + V) | \Psi \rangle = M | \Psi \rangle = \omega | \Psi \rangle, \tag{28}$$

where ω is the eigenvalue of the mass operator for the coupled-channel system. The Lippmann-Schwinger equation is

$$| \Psi_{\pm} \rangle = | \Phi \rangle + G^{(\pm)} V | \Psi_{\pm} \rangle, \tag{29}$$

where

$$(\omega - M_{\mathrm{fr}}) | \Phi \rangle = 0, \tag{30}$$

and

$$G_{nm}^{(\pm)} := \delta_{n,m} [\omega - (M_{\mathrm{fr}})_n \pm i\varepsilon]^{-1}. \tag{31}$$

III. THE SEPARABLE POTENTIAL

The isobar model assumes production partial-wave amplitudes of the form

$$\mathcal{A}_{Ij}^{\pi N \to \pi\pi N} = \sum_{\sigma_\beta} \mathcal{A}_{j,\sigma_\beta}^{\pi N \to \pi\beta} \mathcal{A}_{j_\beta,\sigma_\beta}^{\beta \to \pi N}, \tag{32}$$

where $\mathcal{A}_{j,\sigma_\beta}^{\pi N \to \pi\beta}$ is the amplitude for the production of the quasi-two-body state $\pi\beta$, with β some hadronic resonance, and $\mathcal{A}_{j_\beta,\sigma_\beta}^{\beta \to \pi N}$ the decay amplitude for β. In this paper, for simplicity, we only consider one isobar as shown in Fig. 1. It is possible to model such amplitudes with the following separable potential:

$$V:=\begin{pmatrix} \sum_{\alpha} \lambda_{\alpha}|u_2^{\alpha}\rangle\langle u_2^{\alpha}| & \sum_{\alpha} \chi_{\alpha}|u_2^{\alpha}\rangle\langle u_3^{\alpha}| \\ \sum_{\alpha} \chi_{\alpha}^{*}|u_3^{\alpha}\rangle\langle u_2^{\alpha}| & 0 \end{pmatrix}. \quad (33)$$

The index α is related to the number of s-channel resonances which in this study will be the $\Delta(1232)$ and $\Delta(1660)$ resonances. The final-state resonance, in our case is also the Δ resonance. $|u_2^{\alpha}\rangle$ and $|u_3^{\alpha}\rangle$ are input functions whose form is discussed in the following paragraphs, and λ_{α} and χ_{α} are parameters that measure the strengths of the various resonances.

It will be shown that the relevant amplitudes obtained with this potential are of the form

$$\mathcal{A}_{Ij}^{\pi N\to\pi N}=\sum_{\alpha,\alpha'} u_{2Ij}^{\alpha}(\omega,l)\mathbf{T}_{22}^{\alpha,\alpha'}(\omega,I,j)u_{2Ij}^{\alpha'*}(\omega,l), \quad (34)$$

and

$$\mathcal{A}_{Ij}^{\pi N\to\pi\pi N}=\sum_{\alpha,\alpha'} u_{3Ij}^{\alpha}(\omega,l,\omega_{12},I_{12},j_{12},l_{12}) \times\mathbf{T}_{32}^{\alpha,\alpha'}(\omega,I,j)u_{2Ij}^{\alpha'*}(\omega,l), \quad (35)$$

where the functions $u_{2Ij}^{\alpha*}(\omega,l)$ contain the initial state variables and the functions $u_{2Ij}^{\alpha}(\omega,l)$ and $u_{3Ij}^{\alpha}(\omega,l,\omega_{12},I_{(12)},j_{12},l_{12})$ the final-state variables. Then, the potential for πN scattering in the P_{33} channel will be nonzero only for $I=\frac{3}{2}$, $j=\frac{3}{2}$ and $I_{12}=\frac{3}{2}$, $j_{12}=\frac{3}{2}$. The isobar amplitude for producing the $\Delta\pi$ quasi-two-body state mixes the spin projections σ_{β} among the partial-wave channels in such a way that the $l=1$ orbital angular momentum channel is the only one that conserves parity.

Given the initial state

$$|\Phi^{\text{in}}\rangle=\begin{pmatrix} |\phi_2^{\text{in}}\rangle \\ 0 \end{pmatrix}, \quad (36)$$

and interacting state $|\Psi\rangle$ denoted by

$$|\Psi\rangle=\begin{pmatrix} |\psi_2\rangle \\ |\psi_3\rangle \end{pmatrix}, \quad (37)$$

inserting the potential (33) into the Lippmann-Schwinger equation, Eq. (29) results in the following pair of coupled equations:

$$|\psi_2\rangle=|\phi_2^{\text{in}}\rangle+\sum_{\alpha'=1}^{n} G_2\lambda_{\alpha'}|u_2^{\alpha'}\rangle\langle u_2^{\alpha'}|\psi_2\rangle + \sum_{\alpha=1}^{n} G_2\chi_{\alpha'}|u_2^{\alpha'}\rangle\langle u_3^{\alpha'}|\psi_3\rangle, \quad (38)$$

$$|\psi_3\rangle=\sum_{\alpha'=1}^{n} G_3\chi_{\alpha}|u_3^{\alpha'}\rangle\langle u_2^{\alpha'}|\psi_2\rangle. \quad (39)$$

Multiplying both sides by $\langle u_2^{\alpha}|$ gives

$$\langle u_2^{\alpha}|\psi_2\rangle=\langle u_2^{\alpha}|\phi_2^{\text{in}}\rangle+\sum_{\alpha'=1}^{n} g_2^{\alpha,\alpha'}(\omega)\lambda_{\alpha'}\langle u_2^{\alpha'}|\psi_2\rangle + \sum_{\alpha'=1}^{n} g_2^{\alpha,\alpha'}(\omega)\chi_{\alpha'}\langle u_3^{\alpha''}|\psi_3\rangle, \quad (40)$$

$$\langle u_3^{\alpha}|\psi_3\rangle=\sum_{\alpha'=1}^{n} g_3^{\alpha,\alpha'}(\omega)\chi_{\alpha'}\langle u_2^{\alpha'}|\psi_2\rangle, \quad (41)$$

where

$$g_2^{\alpha,\alpha'}(\omega):=\langle u_2^{\alpha}|G_2|u_2^{\alpha'}\rangle = \sum_{l=1,2}\int_{m_{12}}^{\infty} d\omega' \frac{\omega'^2 k(\omega')}{4}\frac{u_{2Ij}^{\alpha*}(\omega',l)u_{2Ij}^{\alpha'}(\omega',l)}{\omega-\omega'+i\varepsilon}, \quad (42)$$

$$g_3^{\alpha,\alpha'}(\omega):=\langle u_3^{\alpha}|G_3|u_3^{\alpha'}\rangle=\sum_{l=0,1}\sum_{l_{12}=1,2}\int_{m_{123}}^{\infty} d\omega'\int_{m_{12}}^{\omega-m_3} d\omega_{12}\frac{\omega^2 k_3(\omega',\omega_{12})k_1^{12}(\omega_{12})}{8} \times\frac{u_{3Ij}^{\alpha*}(\omega',l,I_{12},j_{12},\omega_{12},l_{12})u_{3Ij}^{\alpha'\alpha'}(\omega',l,I_{12},j_{12},\omega_{12},l_{12})}{\omega-\omega'+i\varepsilon}. \quad (43)$$

Inserting Eq. (41) into (40), one finds

$$\langle u_2^\alpha|\psi_2\rangle=\langle u_2^\alpha|\phi_2^{\rm in}\rangle+\sum_{\alpha'=1}^{n} g_2^{\alpha,\alpha'}(\omega)\lambda_{\alpha'}\langle u_2^{\alpha'}|\psi_2\rangle + \sum_{\alpha'=1}^{n}\sum_{\alpha''=1}^{n} g_{2Ij}^{\alpha,\alpha'}(\omega)\chi_{\alpha'}g_3^{\alpha',\alpha''}(\omega)\chi_{\alpha''}\langle u_2^{\alpha''}|\psi_2\rangle \tag{44}$$

which can be written in matrix notation as

$$[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]\mathbf{u}_2=\mathbf{u}_2^{\rm in}, \tag{45}$$

where

$$\mathbf{u}_2=\langle u_2^\alpha,\psi_2\rangle,$$

$$\mathbf{u}_2^{\rm in}=\langle u_2^\alpha,\phi_2^{\rm in}\rangle,$$

$$\mathbf{\Lambda}=\boldsymbol{\lambda}+\chi\mathbf{g}_3\chi,$$

$$(\boldsymbol{\lambda})_{\alpha\alpha'}:=\delta_{\alpha\alpha'}\lambda_\alpha,$$

$$(\chi)_{\alpha\alpha'}:=\delta_{\alpha\alpha'}\chi_\alpha. \tag{46}$$

The hermiticity of the potential (33) requires that all the λ be real. Equation (45) can be rewritten as

$$\mathbf{u}_2=[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]^{-1}\mathbf{u}_2^{\rm in}, \tag{47}$$

which yields the amplitude

$$\begin{aligned}\mathcal{A}_{Ij}^{\pi N\to\pi N}(\omega)&=\langle\phi_2^{\rm out}|V|\Psi\rangle=\sum_{\alpha=1}^{n}\left(\lambda_\alpha\langle\phi_2^{\rm out}|u_2^\alpha\rangle\langle u_2^\alpha|\psi_2\rangle+\chi_\alpha\langle\phi_2^{\rm out}|u_2^\alpha\rangle\langle u_3^\alpha|\psi_3\rangle\right)\\ &=\sum_{\alpha=1}^{n}\left(\lambda_\alpha\langle\phi_2^{\rm out}|u_2^\alpha\rangle\langle u_2^\alpha|\psi_2\rangle++\langle\phi_2^{\rm out}|u_2^\alpha\rangle\sum_{\alpha'=1}^{n}\chi_\alpha g_3^{\alpha,\alpha'}(\omega)\chi_{\alpha'}\langle u_2^{\alpha'}|\psi_2\rangle\right)\\ &=\sum_{\alpha=1}^{n}\langle\phi_2^{\rm out}|u_2^\alpha\rangle\left(\delta_{\alpha,\alpha'}\lambda_{\alpha'}+\sum_{\alpha'=1}^{n}\chi_\alpha g_3^{\alpha,\alpha'}(\omega)\chi_{\alpha'}\right)\langle u_2^{\alpha'}|\psi_2\rangle=\mathbf{u}_2^{\rm out\dagger}\cdot\mathbf{\Lambda}[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]^{-1}\cdot\mathbf{u}_2^{\rm in}=\mathbf{u}_2^{\rm out\dagger}T_{22}\mathbf{u}_2^{\rm in},\end{aligned} \tag{48}$$

where

$$\mathbf{T}_{22}:=\mathbf{\Lambda}[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]^{-1}=[\mathbf{I}-\mathbf{\Lambda}\mathbf{g}_2(\omega)]^{-1}\mathbf{\Lambda}. \tag{49}$$

This matrix solution has the usual structure of a Lippmann-Schwinger equation, although $\mathbf{\Lambda}$ contains the inelasticity due to the presence of the open three-particle channel. The $\pi N\to\pi\pi N$ amplitude is

$$\begin{aligned}\mathcal{A}^{\pi N\to\pi\pi N}(\omega)&=\langle\phi_3^{\rm out}|V|\Psi\rangle=\sum_\alpha\langle\phi_3^{\rm out}|u_3^\alpha\rangle\chi_\alpha\langle u_2^\alpha|\psi_2\rangle\\ &=\mathbf{u}_3^{\rm out}\boldsymbol{\chi}[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]^{-1}\mathbf{u}_2^{\rm in}=\mathbf{u}_3^{\rm out}\mathbf{T}_{32}\mathbf{u}_2^{\rm in}\end{aligned} \tag{50}$$

with

$$\mathbf{T}_{32}:=\boldsymbol{\chi}[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]^{-1}\mathbf{u}_2^{\rm in}. \tag{51}$$

We must now make choices for $u_{2Ij}(\omega,l)$ and $u_{3Ij}(\omega lI_{12}\omega_{12}j_{12}l_{12})$ as input functions for the partial wave amplitudes. A simple choice for u_2 which is square integrable and had the correct threshold behavior can be given in terms of the basic input function used in this paper

$$u_K(k,l):=\frac{k^l}{(k^2+K^2)^{n(l)}},$$

$$u_{2Ij}^\alpha(\omega,l)=\frac{k^*(\omega)^l}{(k^*(\omega)^2+K_{\alpha Ij}^2)^{n(l)}}=u_{K_\alpha}(k^*,l), \tag{52}$$

where $k^*(\omega)$ is the relativistic center of mass momentum of the two-particle system, and K is an adjustable parameter used for fitting data. Since we are fitting partial-wave data, I and j are fixed for each channel and K then depends on α only. $n(l)$ is an integer depending on l; it is chosen to be the smallest integer that makes u_2 square integrable in the reduced two-body space, namely, $n(l)=(l+3)/2$.

Since the Δ resonance decays into πN, u_3 is chosen to be of the form

$$\begin{aligned}&u_{3Ij}^\alpha(\omega lI_{12}\omega_{12}j_{12}l_{12})\\ &=\frac{k_3^*(\omega,\omega_{12})^l}{[k_3^*(\omega,\omega_{12})^2+K_{\alpha Ij}^2]^{n(l)}}\times\frac{\Gamma_\Delta}{2(\omega_{12}-\omega_\Delta)}\frac{k_1^{12}(\omega_{12})^{l_{12}}}{[k_1^{12}(\omega_{12})^2+K_{0I_{12}j_{12}}^2]^{n(l_{12})}}\\ &=u_{K_\alpha}(k_3^*,l)\,\frac{\Gamma_\Delta}{2(\omega_{12}-\omega_\Delta)}\,u_{K_0}(k_1^{12},l_{12}),\end{aligned} \tag{53}$$

where $k_3^*(\omega,\omega_{12})$ is the magnitude of the momentum of the outgoing (nonresonant) π in the overall center of mass, and k_1^{12} is the magnitude of the momentum of the outgoing

nucleon in the $N\pi$ center of mass [which is the (12) center of mass]. The values of the parameters Γ_Δ and ω_Δ in the Breit-Wigner terms are taken from the Particle Data Book, namely, $\omega_\Delta = 1232$ MeV and $\Gamma_\Delta = 120$ MeV.

As seen in Eq. (53), for a fixed I,j, u_3 depends on parameters K_α and K_0. Since the numerical calculation of the three-body Greens' function, $g_{3Ij}^{\alpha\alpha'}(\omega)$, Eq. (43), is relatively complicated, to make the fitting procedure simpler we have chosen all the K_α to have the same value, namely, K_0, which is fixed to be 800 MeV. Then $g_{3Ij}^{\alpha\alpha'}(\omega)$ has the same value for all α, α', and the matrix $\mathbf{\Lambda}$, Eq. (46) simplifies to

$$\mathbf{\Lambda} = \boldsymbol{\lambda} + g_{3Ij}(\omega)\boldsymbol{\chi}\cdot\boldsymbol{\chi}. \tag{54}$$

The significance of $g_{3Ij}(\omega)$ and $\boldsymbol{\chi}$ are as follows: For purely elastic scattering, with $\boldsymbol{\chi}=0$, $\mathbf{\Lambda}$ equals the real diagonal matrix $\boldsymbol{\lambda}$. Using the form of u_{3Ij}, Eq. (53) in the definition of $g_{3Ij}(\omega)$, Eq. (43), shows $g_{3Ij}(\omega)$ to be a nonpositive, real function of ω below the production threshold, which becomes complex above threshold. The $2\rightarrow 2$ amplitudes of Eq. (48) lie on the unitarity circle, that is, $\mathcal{A}^{\pi N\rightarrow\pi N}(\omega) = \sin\delta e^{i\delta}$ when $\mathrm{Im}[g_3(\omega)]=0$, but take the form

$$\mathcal{A}_{Ij}^{\pi N-\pi N}(\omega) = \eta\ \sin\ \delta e^{i\delta} \tag{55}$$

with the inelasticity parameter $\eta\leqslant 1$ when $\mathbf{\Lambda}$ has a nonzero imaginary component. This implies that $1-\eta(\omega) \sim \mathrm{Im}\, g_3(\omega)$; hence the shape of $\mathrm{Im}\, g_3(\omega)$ controls the rate at which the inelasticity changes with energy while χ controls the degree of inelasticity (production) which may occur.

The quality $\mathrm{Re}[g_3(\omega)]$ can take appreciable values below threshold. This does *not* result in any deviation from the unitarity circle below threshold (only the imaginary part does this and that is zero below threshold) but it *does* produce a phase shift relative to the elastic amplitude with the same $\boldsymbol{\lambda}$. Thus, it is necessary to take account of the χ_i's when adjusting the parameters to fit resonances below the production threshold.

Consider now parametrizing the resonances in the elastic case ($\mathbf{\Lambda}=\boldsymbol{\lambda}$). For a single resonance it is sufficient to use a single u_2 function but there is not a one-to-one relation between resonances and u_2 functions as shown by the following considerations: for a single u_2,

$$[\mathbf{I}-\mathbf{\Lambda}\mathbf{g}_{2Ij}(\omega)]^{-1}\rightarrow\frac{1}{1-\lambda g_2(\omega)}.$$

Simple square integrable u_2's enforce certain generic properties in $g_2(\omega)$. Consider

$$g_2(\omega) = \sum_{l=1,2}\int_{m_{12}}^{\infty} d\omega'\ \frac{\omega^2 k(\omega')}{4}\frac{|u_{2jI}(\omega',l)|^2}{\omega-\omega'+i\varepsilon}$$
$$= P\int_{m_{12}}^{\infty} d\omega'\ \frac{F(\omega')}{\omega-\omega'} - i\pi F(\omega), \tag{56}$$

where

$$F(\omega) := \sum_l\ \frac{\omega^2 k(\omega)}{4}\,|u_{2jI}(\omega,l)|^2. \tag{57}$$

Now, the numerator $F(\omega')$ is a positive definite function of ω'. For a given value of ω above threshold, the denominator has an integrable singularity and changes sign from negative to positive at $\omega=\omega'$. Near threshold, i.e., $\omega\rightarrow m_{12}$, the principal value part approaches a negative-definite constant value while the singular part goes to zero. As $\omega\rightarrow\infty$, the principal value part approaches zero from above while the singular part approaches zero from below.

A resonance occurs when the real part of the denominator vanishes. That is, when

$$0 = \mathrm{Re}\{[\mathbf{I}-\mathbf{\Lambda}\mathbf{g}_{2Ij}(\omega)]\} = 1 - P\int_{m_{12}}^{\infty} d\omega'\ \frac{F(\omega')}{\omega-\omega'}.$$

There is only one u_2 here and for most reasonable choices of u_2 the principal value integral will not significantly oscillate in passing from its threshold value to its asymptotic limit at ∞. For the input functions used in this paper, the types of behavior are exhausted by the following two cases. The first case has $\mathrm{Re}[1-\lambda g_2(\omega)]$ starting out negative, passing through zero once and asymptotically approaching one. This behavior produces one resonance and the phase shift goes to π as ω goes to ∞. In the second case, $\mathrm{Re}[1-\lambda g_2(\omega)]$ starts out positive, passes through zero from above and back through zero from below to again approach one asymptotically. The first zero produces an antiresonance followed by a resonance and the phase shift is negative until after the antiresonance.

For two input functions u_2^1 and u_2^2 the same possibilities occur as for a single u_2 function but with the following additional possibilities. There may now be a triple of zeroes of $\mathrm{Re}\{[\mathbf{I}-\mathbf{\Lambda}\mathbf{g}_{2Ij}(\omega)]\}$ giving rise to a combination of a resonance followed by an antiresonance followed by another resonance. It is also possible to have four zeroes making up a succession of two resonance-antiresonance pairs. Finally, there is a third possibility, in which $\mathrm{Re}\{[\mathbf{I}-\mathbf{\Lambda}\mathbf{g}_{2Ij}(\omega)]\}$ only has two zeroes but there is a zero of $\mathrm{Im}\{[\mathbf{I}-\mathbf{\Lambda}\mathbf{g}_{2Ij}(\omega)]\}$ occurring between them ($\delta_l=\pi$). There are two resonances with a zero of the amplitude occurring between them, analogous to the Ramsauer-Townsend effect [12] in electron atom scattering.

Evidently the number of input functions is related to the number of allowed resonances but not quite in a simple one-to-one fashion. The objective in this paper is to fit two or three resonances so we need at least two input functions. However, with only two input functions of the form above, the resonance masses and widths predicted by the model are very tightly constrained by nonlinear relations so it may not in general be possible to fit the data. Using three input functions makes it possible to get the masses and widths right for two resonances along with a possible third. In addition, it is advantageous to model the two resonances with a Ramsauer-Townsend–like zero between them because if there is an antiresonance between, say, the $\Delta(1232)$ and the $\Delta(1600)$, the potential is never sufficiently attractive to give reasonable widths and requires so much inelasticity to bring the antiresonance down that the amplitudes are highly unstable and oscillatory in the parameter space.

Setting

$$D(\omega) := |[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]| \tag{58}$$

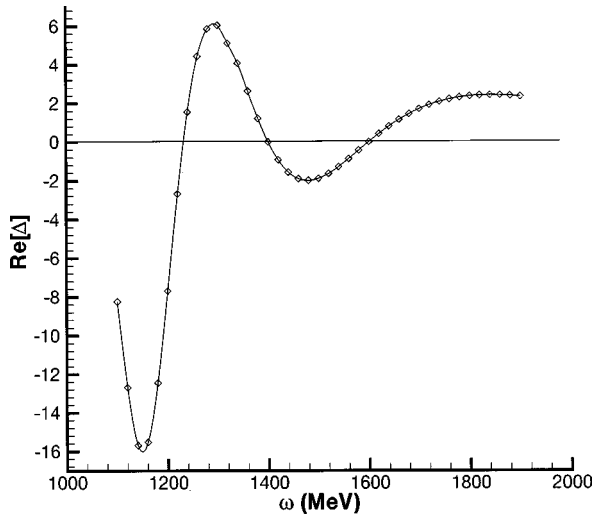

FIG. 2. Re[$D(\omega)$] with Im[$D(\omega)$]>0, $\chi_0=0$.

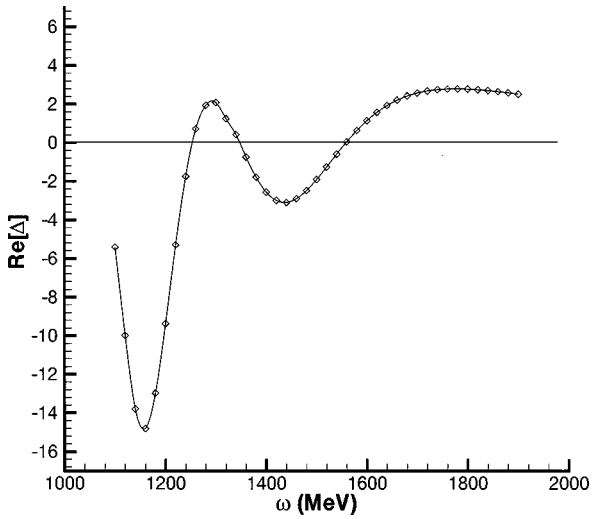

FIG. 4. Re[$D(\omega)$] with Im[$D(\omega)$]>0, $\chi_0=0.8$.

means that the general condition for the position of a resonance is Re $D(\omega)=0$. Expanding the partial wave amplitude $\mathcal{A}_{Ij}^{\pi N\to\pi N}(\omega)$ [Eq. (48)] about the resonance frequency then gives the resonance width as

$$\frac{\Gamma}{2}=\frac{\text{Im } D(\omega)}{(d/d\omega)\text{Re } D(\omega)|_{\omega=\omega_R}}, \tag{59}$$

where ω_R is the position of the resonance Re $D(\omega_R)=0$.

IV. FITTING DATA

We have shown that a separable potential of the form given in Eq. (33) very naturally models isobar amplitudes. The direct channel resonances are given by the zeroes of the determinant Re $D(\omega)$, Eq. (58), while the final-state resonances in the production amplitudes are given through the u_3 functions, Eq. (53). Nevertheless, though the relativistic Lippmann-Schwinger equation with a separable potential reduces to a linear matrix equation, the expressions for the amplitudes in terms of the parameters of typical input functions are in general, highly nonlinear. When production channels are open, the nonlinearity is particularly evident, as seen in the oscillatory behavior of attempted fits to the data. As a result it is extremely unlikely that a straightforward attempt at a nonlinear fit will converge to a good fit unless one starts with parameters that are already very close to best-fit parameters.

Since the parameters of the final-state resonance are fixed by the Δ resonance parameters [see Eq. (53)], the goal of this section is to fit the (multiple) resonances in the direct channel of the $\pi N\to\pi N$ reaction, as well as the inelasticity due to the $\pi N\to\pi\pi N$ channel. With the three-particle Green's function parameters all fixed, the remaining parameters are λ_α and χ_α from the separable potential, Eq. (33), and the K_α from the u_2 functions, Eq. (51). For $\alpha=1,2,3$ there are nine parameters to fit two resonances [Δ(1232) and Δ(1600)] and antiresonance (at 1420 MeV), their widths, and their strengths.

Because of the highly nonlinear relation between the resonance position and widths and the K_α appearing in the two-body Green's functions, our fitting strategy is to begin with purely elastic scattering, $\chi_\alpha=0$, fit the resonance (and antiresonance) positions and widths, and then slowly turn on the

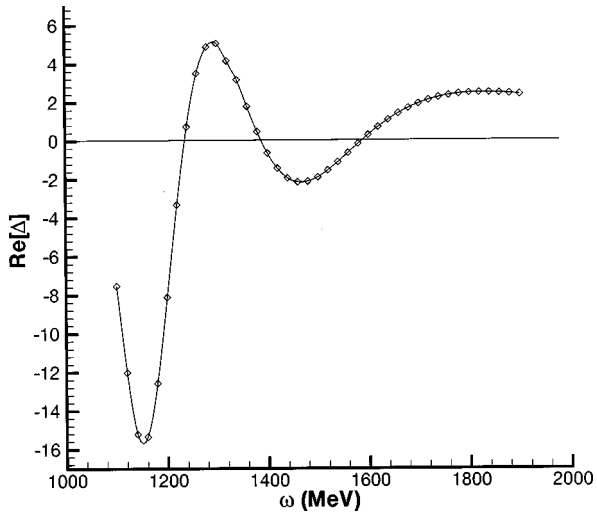

FIG. 3. Re[$D(\omega)$] with Im[$D(\omega)$]>0, $\chi_0=0.4$.

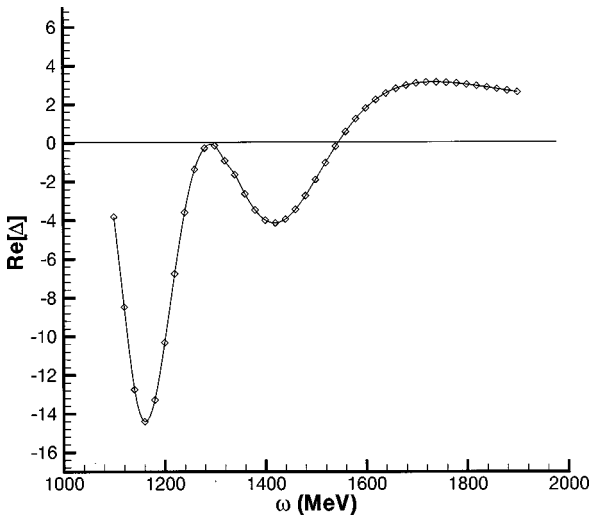

FIG. 5. Re[$D(\omega)$] and Im[$D(\omega)$] with $\chi_0=1.2$.

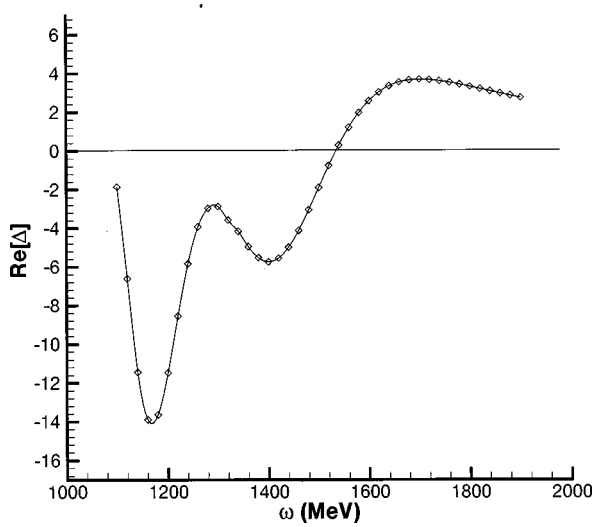

FIG. 6. Re[$D(\omega)$] and Im[$D(\omega)$] with $\chi_0 = 1.4$.

inelasticity, at each step readjusting the λ_α and K_α to fit the resonances. When increasing the inelasticity (all χ_α are chosen at first to have the same χ_0 value), resonances may not only change positions or widths, they may actually disappear, reflecting the fact that the number of solutions to the resonance conditions, Eq. (58) have changed. This is shown in Figs. 2–6, in which χ_0 is changed in increments of 0.4; in these figures $K_1 = 405$ MeV, $K_2 = 420$ MeV, and $K_3 = 460$ MeV, while $\lambda_1 = -5.89 \times 10^4$, $\lambda_2 = 1.25 \times 10^5$, and $\lambda_3 = -8.91 \times 10^4$ (in MeV).

In Figs. 7–10, the real part of $D(\omega)$ is positive below threshold and then goes to zero at 1232 MeV with a negative slope. However, as seen in Eq. (59), since the imaginary part of $D(\omega)$ is negative here, a resonance rather than an antiresonance results. At $\omega = 1400$ MeV, Im $D(\omega) = 0$, resulting in the Ramsauer-Townsend–like effect, mentioned in the previous section. Figure 7 shows the case of no inelasticity ($\chi_0 = 0$), while Figs. 8–10 show shifts with increasing inelasticity χ_0. In Fig. 10 the point at which Im $D(\omega) = 0$ has shifted sufficiently to produce an antiresonance at about 1400 MeV. Here $\lambda_1 = -6.07 \times 10^4$, $\lambda_2 = -7.99 \times 10^4$, and $\lambda_3 = -1.68 \times 10^5$ (in MeV).

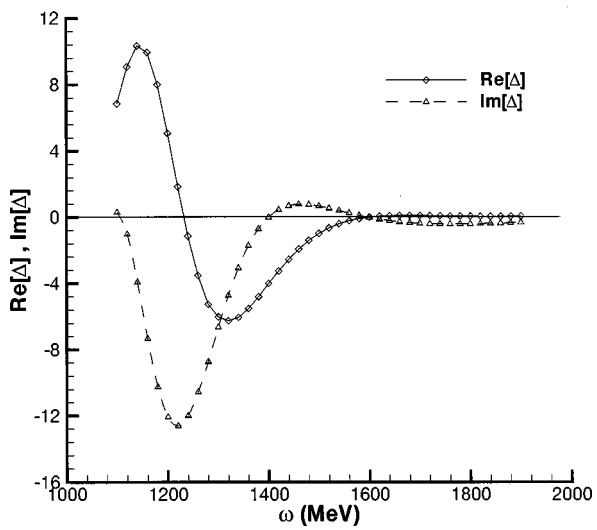

FIG. 7. Re[$D(\omega)$] and Im[$D(\omega)$] with $\chi_0 = 0$.

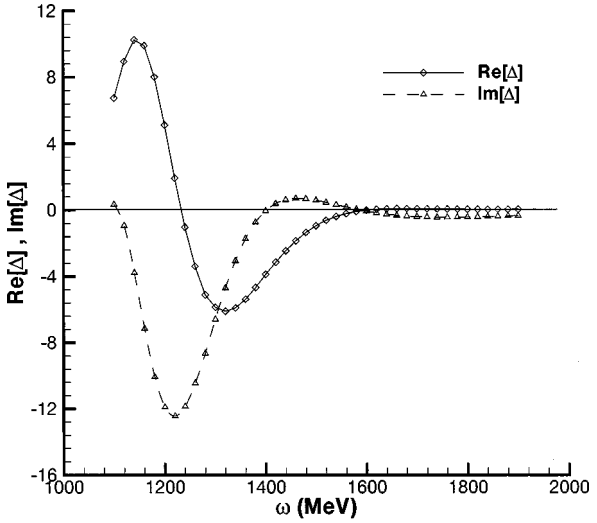

FIG. 8. Re[$D(\omega)$] and Im[$D(\omega)$] with $\chi_0 = 0.4$.

To fit the P_{33} data, we have varied χ_0 for a rough fit. Then by varying the χ_α separately and making small adjustments in K_α, we have obtained the following fits to the partial cross section data (Fig. 11 for P_{33}) and Argand plot (Fig. 12 for P_{33}). Table I shows the parameters giving the P_{33} fit, while Table II gives the P_{33} parameters. The orders of magnitude of $\lambda_i (\sim 10^7)$ and $\chi_i (\sim 10^4)$ in Table II came about mainly from the normalization of the u_2 and u_3 input functions given in Eqs. (52) and (53).

V. CONCLUSION

We have shown how to use point form relativistic quantum mechanics to construct a dynamical isobar model that couples production to elastic channels with the correct production thresholds and inelastic unitarity. In this paper we have taken the model Hilbert space to be the direct sum of a pion-nucleon and two-pion nucleon space, though the formalism is easily extended to other channels and more than three particle spaces. The point form has interactions in all four of the momentum operators; nonetheless it is possible to

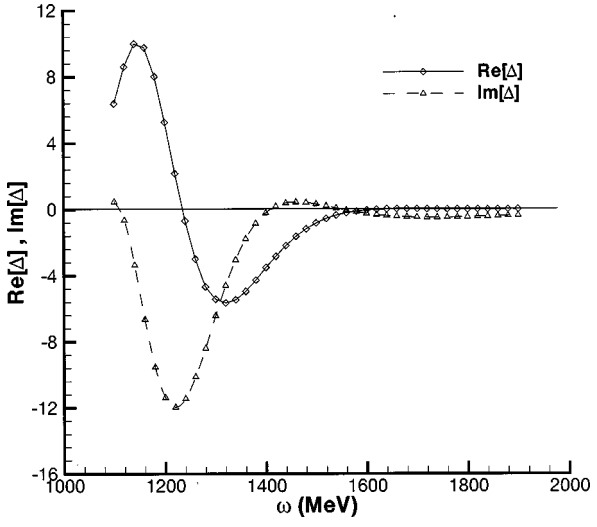

FIG. 9. Re[$D(\omega)$] and Im[$D(\omega)$] with $\chi_0 = 0.8$.

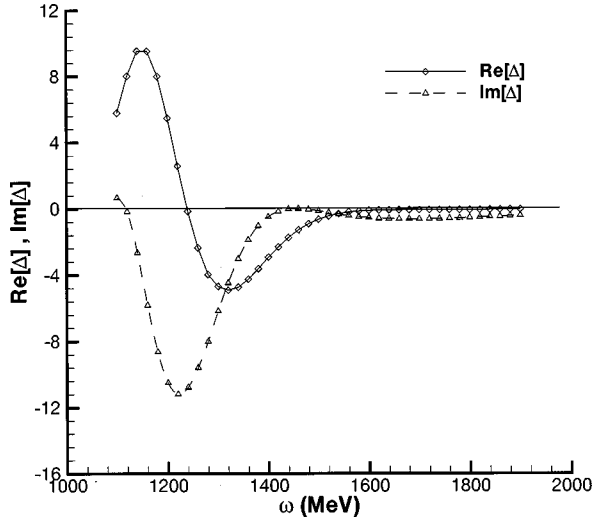

FIG. 10. Re[$D(\omega)$] and Im[$D(\omega)$] with $\chi_0 = 1.0$.

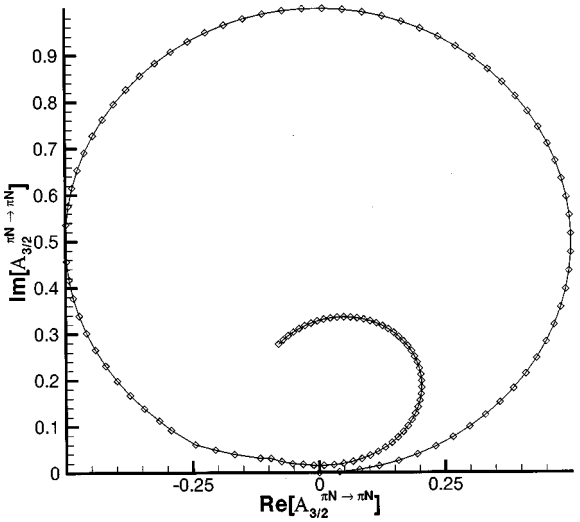

FIG. 12. P_{33} Argand plot.

specify the dynamics by a single mass operator [see Eq. (6)], analogous to the Hamiltonian in nonrelativistic quantum mechanics. If the mass operator is chosen to have a separable form [see Eq. (33)], the resulting Lippmann-Schwinger equation can be solved analytically. The elastic and production amplitudes then depend on certain input functions [Eqs. (52) and (53)] as well as the inverse of a matrix [Eq. (47)]. The zeros of the real part of the determinant of this matrix locate the positions of the various resonances in a given angular momentum channel, while the imaginary part is related to the widths of the resonances. By increasing the number of input functions, it is possible to generate more and more resonances in the same $L_{2I,2j}$ channel. In this paper we have focused on the P_{33} channel with resonances at 1232 and 1600 MeV.

In contrast, the final state resonances in the production channel are determined solely by the parameters of the u_{3Ij} input functions [Eq. (53)]. Thus the final state resonances are not related in any necessary way to the direct channel resonances. This corresponds experimentally to the fact that final state resonances are seen in distributions in subenergy variables (Dalitz plots), at a fixed total energy, while direct channel resonances are seen in total cross section data.

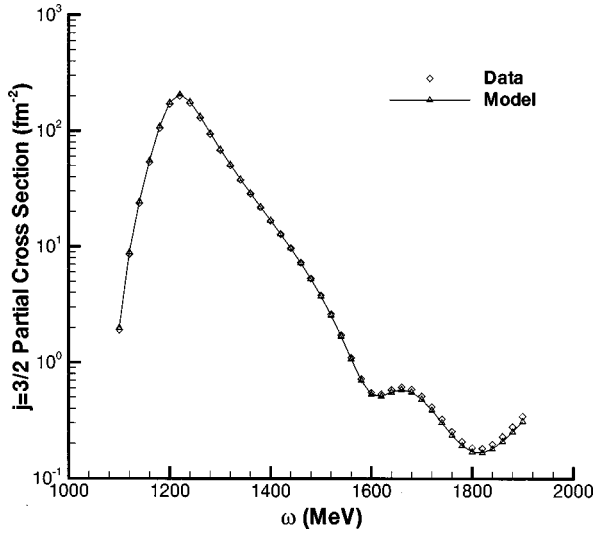

FIG. 11. P_{33} Partial cross section vs c.m.s. energy.

The isobar model developed in this paper is to be contrasted with other models, such as those given in Refs. [1–6] (see also Refs. [11,13,14] for *NN* models), where resonances such as the Δ resonance are treated as particles, with their own Hilbert spaces. In our model the only particles are pions and nucleons, and resonances such as the Δ resonance appear as poles in the partial wave scattering amplitudes. And for the final state resonances there is no $\Delta\pi$ channel, only a Δ resonance in the $\pi\pi N$ channel. Such a treatment of resonances is closely related to earlier attempts at fitting πN data, as seen for example in Refs. [15] and [16]. However, in contrast to our work, in these references a pure *S* matrix approach, with no Hamiltonian or mass operator, was used.

The biggest shortcoming in the model developed in this paper is that final state resonances other than the delta resonance have not been taken into account. Thus, the $N\rho$ state, which is a possible final state in the $N\pi\pi$ system, as well as other π-π resonances, have been ignored in the u_{3Ij} functions, Eq. (53). The reason is that the natural variables for describing such resonances include ω_{23} and j_{23}, the mass and spin of the π-π system. But these variables are incompatible with the 1-2 type variables used in the u_{3Ij} functions. What is needed are Racah coefficients that connect different stepwise coupling schemes. In a following paper [8] we derive the coefficients that connect any stepwise coupling scheme to a so-called simultaneously coupled scheme, in which the variables include the subenergies used in Dalitz plots. These Racah coefficients are products of Clebsch-Gordan coefficients and Wigner *D* functions and contain all the spin information about the final state resonances.

Though final state resonances other than Δ resonances are not included in the u_{3Ij} input functions, it is still possible to get good fits for the elastic partial cross sections and inelasticity data in some channels. In Figs. 11 and 12 we have obtained good fits to plots given by Manley *et al.* [4] for the P_{33} channel, from which the resonance parameters given in Tables I and II have been extracted. The P_{33} channel is, however, a particularly clean channel, with no other interfering resonances. In other channels such as the P_{11} channel it

TABLE I. $P33$ resonance parameters.

Resonance	M^* (MeV)	Γ^* (MeV)	$M_{\rm fit}$ (MeV)	$\Gamma_{\rm fit}$ (MeV)
$\Delta(1232)$	1232	$\approx$120	1234.9	126.1
$\Delta(1600)$	1600	$\approx$350	1638.1	378.0

will be necessary to take into account other resonances such as the ρ resonance mentioned in the previous paragraph to get good fits.

The main focus of this paper has been to develop an isobar model from a dynamical theory, in this case point form relativistic quantum mechanics. The separable potentials used in the mass operator allow for a great deal of freedom in the input functions used for fitting data. However, it is possible to use more realistic potentials in the elastic channel, while retaining the separable form for that part of the mass operator that mixes two and three particle spaces, thus simulating the effect of multiparticle channels on two-body channels: For example, in form factor calculations of the deuteron, both elastic and breakup, it is necessary to include the effects of pion production at sufficiently high energies. Using a realistic N-N potential for the deuteron along with the separable potential provides a simple way to include production in deuteron form factor calculations.

ACKNOWLEDGMENT

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APPENDIX: RELATIVISTIC KINEMATICS

The four-velocity is defined in terms of the four-momentum by

$$p^\mu := \omega v^\mu,$$

where

$$\omega := \sqrt{\sum_{i=1}^{n} p_i^2}.$$

Here, the square denotes the four-vector dot product. This is the (positive) square root of the Mandlestam s variable. The invariant mass (or subenergy) of the 12 cluster is

$$\omega_{12} := \sqrt{p_1^2 + p_2^2}.$$

Frequent use is made of several special kinematic functions corresponding to free-particle momenta in certain Lorentz frames. These follow from the well-known formula for the center of mass momentum for two free particles with masses m_1 and m_2. Inverting the equation

$$\omega = \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2},$$

we find

$$k(\omega) = \frac{\sqrt{(\omega^2 + m_1^2 - m_2^2)^2 - 4\omega^2 m_1^2}}{2\omega}.$$

TABLE II. $P33$ fitting parameters.

Parameter	Value
K_0	800 MeV/c
K_1	400 MeV/c
K_2	410 MeV/c
K_3	465 MeV/c
λ_1	-3.92333×10^7 MeV
λ_2	-5.04267×10^7 MeV
λ_3	-2.58521×10^8 MeV
χ_1	2.7265×10^4 MeV
χ_2	2.1866×10^4 MeV
χ_3	4.1854×10^4 MeV

Extending this, the center of mass momentum of particle 3 of a three-body system in terms of the overall mass and the 12 cluster subenergy is

$$k_3(\omega, \omega_{12}) = \frac{\sqrt{(\omega^2 + \omega_{12}^2 - m_3^2)^2 - 4\omega^2\omega_{12}^2}}{2\omega}.$$

And finally, the momentum of particle 1 in the 12 center of momentum system is given by

$$k_1^{12}(\omega_{12}) = \frac{\sqrt{(\omega_{12}^2 + m_1 - m_2^2)^2 - 4\omega_{12}^2 m_1^2}}{2\omega_{12}}.$$

These exhaust the special momenta used since we only consider final states with clusters in particles 1 and 2 but it is clear how one extends the above notation to other cases.

For normalizations it is convenient to adopt the invariant measure

$$\frac{d^3p_i}{2E_i} := d^4p_i\,\theta(p_i^0)\,\delta(p_i^2 - m_i^2),$$

where $E_i = \sqrt{\mathbf{p}_i^2 + m_i^2}$ for states in momentum space. Then, the momentum states are normalized as follows:

$$\langle p_i, \sigma_i | p_i', \sigma_i' \rangle = 2E_i\,\delta^3(p_i - p_i'),$$

which implies that plane-wave states, in these units, have a density of $2E_i$ particles of type i per unit volume.

Using the above normalizations the differential cross section for n-body scattering is

$$d\sigma = \frac{1}{4\omega k^*(\omega)}\,(2\pi)^4\delta^4(p_f - p_i)\sum_\alpha\,|\langle f|T|i\rangle|^2\prod_{\alpha=1}^{n}\,\frac{d^3p_\alpha}{2E_\alpha},$$

where $\langle f|T|i\rangle$ is the scattering amplitude.

For converting to velocity states the respective measures are given by

$$\sum_{\sigma_1}\,\frac{d^3p_1}{2E_1}\,\frac{d^3p_2}{2E_2} = \sum_{\sigma_1}\,\frac{d^3v}{4(1+|v|^2)^{1/2}}\,d\omega\omega^2k^*(\omega)d\hat{p}_1$$

$$= \frac{d^3v}{4(1+|v|^2)^{1/2}}\,d\omega\omega^2k^*(\omega)\sum_{j\sigma l}$$

and

$$\sum_{\sigma_1} \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}$$
$$= \sum_{\sigma_1} \frac{d^3v}{8(1+|v|^2)^{1/2}}\, d\omega\omega^2 k_{12}^*(\omega,\omega_{12}) k_1^{12}(\omega_{12}) d\hat{p}_{12}^* d\hat{p}_1^{12}$$
$$= \frac{d^3v}{8(1+|v|^2)^{1/2}}\, d\omega\omega^2 d\omega_{12} k_{12}^*(\omega,\omega_{12}) k_1^{12}(\omega_{12}) \sum_{j\sigma l}\sum_{j_{12}l_{12}}.$$

Relations for determining masses and widths. The masses and widths for the resonances can be determined from the isobar amplitudes by expanding the amplitudes about the resonance energy and equating the first-order expression with a Breit-Wigner resonance form. The $N\pi \to N\pi$ amplitude has the form

$$\mathcal{A}_{Ij}^{\pi N\to\pi N}(\omega) = \langle \phi_2^{\rm out}|\mathbf{V}|\Psi\rangle = u_2^{\rm out\dagger}\cdot\mathbf{\Lambda}[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]^{-1}\cdot u_2^{\rm in}$$
$$= u_2^{\rm out\dagger} T_{22} u_2^{\rm in}, \qquad \text{(A1)}$$

where

$$\mathbf{T}_{22} = \mathbf{\Lambda}[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]^{-1} = [\mathbf{I}-\mathbf{\Lambda}\mathbf{g}_2(\omega)]^{-1}\mathbf{\Lambda} \qquad \text{(A2)}$$

and

$$\mathbf{\Lambda} = \lambda + \boldsymbol{\chi} g_3 \boldsymbol{\chi}.$$

Now, the energy dependent complex component of $\mathbf{\Lambda}$ due to inelasticity does not change the pole structure of the amplitude, and because it is second order in $\mathbf{g}_3$ does not significantly shift the phase for physically reasonable inelasticities. Thus, the condition for determining the position of a resonance with inelasticity is just that the real part of the determinant

$$D(\omega) := |\mathbf{\Lambda}[\mathbf{I}-\mathbf{g}_2(\omega)\mathbf{\Lambda}]|$$

vanishes. Therefore we determine the resonance masses ω by

$$\mathrm{Re}[D(\omega)] = 0.$$

Near resonance the amplitude has the approximate form

$$\mathcal{A}_{Ij}^{\pi N\to\pi N}(\omega) \approx \eta\ \mathrm{Im}[D(\omega)]\{\mathrm{Re}[D(\omega)] + i\ \mathrm{Im}[D(\omega)]\}^{-1},$$

where η is the usual inelasticity parameter at the resonance energy. Expanding the denominator about ω and recalling that the zeroth-order term vanishes at resonance results in a Breit-Wigner form for the amplitude where we identify the resonance width via

$$\frac{\Gamma}{2} = \frac{\mathrm{Im}[D(\omega)]}{d/d\omega\ \mathrm{Re}[D(\omega)]|_{\omega=\omega_R}}.$$

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