# Point form relativistic quantum mechanics and electromagnetic form factors

W. H. Klink
*Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242*
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A relativistic quantum mechanics of constituents is formulated in which particles are bound states of a mass operator. The point form of relativistic dynamics is used, in which Lorentz transformations are kinematic, and the four-momentum operator carries all the interactions. A general covariant expression for matrix elements of the electromagnetic current operator is given in which the invariant form factors are reduced matrix elements of the Poincaré group. A point form relativistic impulse approximation is formulated, in which invariant form factors of particles are given in terms of their underlying constituents. [S0556-2813(98)05512-5]

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## I. INTRODUCTION

The goal of this and succeeding papers is to formulate a relativistic quantum mechanics of constituent particles, the bound states of which are the observed hadrons, and whose scattering states should account for such complicated phenomenon as multiparticle production reactions. The term constituent particle is used to mean objects that transform as irreducible representations of the Poincaré group with definite mass, spin, and internal symmetry quantum numbers. Thus, constituent could mean the protons and neutrons that make up nuclei or it could mean the quarks that make up the hadrons. The many-particle Hilbert space of the constituents is the tensor product of the individual constituent Hilbert spaces; in this paper the number of constituents is fixed, while in following papers (for example, Ref. [1]) the number will be variable.

As is already evident in the language being used, the emphasis in these papers is on quantum mechanics, as opposed to quantum field theory. The goal is to formulate a relativistic quantum mechanics which is in spirit similar to nonrelativistic quantum mechanics, where a Hamiltonian acting on a suitable Hilbert space specifies the bound states and *S* matrix of the system. In point form relativistic quantum mechanics the four-momentum operator replaces the Hamiltonian and as will be shown in Sec. II, suitably defined relativistic states have properties very similar to their nonrelativistic multiparticle counterparts.

As is well known there are a number of ways of formulating a relativistic quantum mechanics, three of which Dirac [2] called the instant, front, and point forms. Calculations of spectra and form factors have been carried out using both instant [3] and front form dynamics [4]. Moreover quantum field theory is usually formulated as an instant form of dynamics, in which the interaction Lagrangian is integrated over a time constant surface, resulting in interactions in the energy and boost generators. More recently front form versions of quantum field theory have been developed [5].

There has been no analogous development of point form dynamics. The first use of the point form seems to have been made by Sokolov in the 1970s [6], to prove cluster properties for relativistic systems with a finite number of degrees of freedom. Lev [7] has compared some of the features of the point form with the instant and front forms, particularly with regard to the cluster properties of electromagnetic current operators. Keister and Polyzou [8] discuss the point form along with the other forms in their review article; however, there has been no investigation into the detailed properties of the point form, both as regards comparing calculations of such quantities as form factors and scattering amplitudes with data or comparing with results of other few-body calculations.

The point form has a number of features that distinguish it from the other forms of dynamics. First those operators that contain all the dynamics—namely, the four-momentum operators—commute with one another, and can be simultaneously diagonalized. Since the Lorentz generators do not contain any interaction terms, the theory is manifestly covariant. This means there is a more direct connection with models motivated by quantum field theory. In fact a point form quantum field theory could be developed by integrating the interaction Lagrangian over the forward hyperboloid, resulting in four-momentum operators with interactions in all four components. In Sec. III this construction will be used to couple the electromagnetic current operator to the photon field.

Second, electromagnetic current operators at an arbitrary space-time point are related to the electromagnetic current operator at a special space-time point (usually the space-time origin, hence the name "point form") by translating from the origin with the interaction dependent four-momentum operators. The resulting electromagnetic current operators automatically have the correct Poincaré transformation properties [see Eq. (3.3)], from which it follows—as will be shown in Sec. III—that there are the correct number of independent form factors for particles with spin. Moreover timelike and spacelike momentum transfers are handled on an equal footing. The current operator at the space-time origin is not uniquely determined, since the only constraints it must satisfy are Lorentz covariance and current conservation.

Finally, there is a close relationship with nonrelativistic quantum mechanics. In the contraction limit, when the speed of light is much larger than any particle velocities, the interacting three-momentum operator goes to zero. Spin and orbital angular momentum can be coupled together exactly as is done nonrelativistically, yet the theory is Lorentz covariant, and the spin is given by the relativistic Pauli-Lubanski operator.

 

There are two different Poincaré group actions on the constituent Hilbert space. One is the action inherited from the single-particle spaces, since the relevant Hilbert space is the tensor product of single-particle spaces. This action is called the free (or noninteracting) action, and results in operators representing free Lorentz transformations and space-time translations. The infinitesimal space-time transformations generate the free four-momentum operator $P^\mu_{\rm fr}$ and the free mass operator $M^2_{\rm fr}=P_{\rm fr}P_{\rm fr}$. The four-velocity operator $V^\mu$ is defined as $P^\mu_{\rm fr}M^{-1}_{\rm fr}$ and the relativistic multiparticle states mentioned previously can be chosen to be eigenstates of the four-velocity operator.

The second Poincaré action comes from the total four-momentum operator $P^\mu$, which is the sum of free and interacting four-momentum operators $P^\mu=P^\mu_{\rm fr}+P^\mu_I$. Since the Lorentz generators are not modified in the point form, the fundamental operator equations that must be satisfied can be written as

$$[P^\mu,P^\nu]=0,$$

$$U_\Lambda P^\mu, U_\Lambda^{-1}=\Lambda^{-1\mu}{}_\nu P^\nu,$$

where $U_\Lambda$ is the unitary operator representing Lorentz transformations on the constituent Hilbert space.

The main problem in point form dynamics is to construct interacting four-momentum operators satisfying the above equations. There are two known ways to do this; first one can use quantum field theory as a guide to construct interacting four-momentum operators; in particular for a given interaction Lagrangian the interacting four-momentum operator is constructed by integrating over the forward hyperboloid. For the electromagnetic interaction, needed in this paper to analyze form factors, this construction results in the electromagnetic four-momentum operator having the form

$$P^\mu_{\rm em}=\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)x^\mu J^\nu(x)A_\nu(x),$$

where $J^\nu(x)$ is the electromagnetic current operator and $A_\nu(x)$ the photon field operator. Appendix C shows that $P^\mu:=P^\mu_{\rm fr}+P^\mu_{\rm em}$ satisfies the Poincaré commutation relations if $J^\nu(x)$ and $A_\nu(x)$ are local. The integration defining $P^\mu_{\rm em}$ over the forward hyperboloid is specified by the proper time $\tau$.

A second way of constructing interacting four-momentum operators, called the Bakamjian-Thomas construction [9], is to write $P^\mu:=MV^\mu$, where, if the total mass operator $M$ commutes with the velocity operator $V^\mu$ and Lorentz transformations, the Poincaré commutation relations given above are satisfied. Further, if the mass operator is written as the sum of a free and interacting mass operator $M=M_{\rm fr}+M_{\rm int}$, then point form relativistic quantum mechanics has a structure very similar to nonrelativistic quantum mechanics. In particular, velocity states defined in Sec. II and analyzed in great detail in Ref. [10] have the property that orbital and spin angular momentum can be coupled together exactly as is done nonrelativistically. Examples of the Bakamjian-Thomas construction are given in the following paper [1], where a separable interacting mass operator is introduced to analyze $\pi$-nucleon production reactions.

The goal of the following series of papers is to make use of the distinctive features of the point form to explore the properties of few-body relativistic nuclear systems. This includes using a relativistic Lippman-Schwinger equation to analyze scattering and resonance phenomenon (see Ref. [1]) as well as computing form factors, initially of such simple systems as pions [11] and deuterons.

The goal of this paper is to develop the formalism needed to compute form factors in the point form. This will be done in Sec. III by rewriting the electromagnetic current operator as an irreducible tensor operator under the interacting Poincaré group. Using the Wigner-Eckhart theorem will then lead to the definition of form factors as reduced matrix elements, for particles of arbitrary mass and spin, and with the correct number of independent form factors. Section IV will use the covariant representation of current matrix elements to define a point form impulse approximation, so that form factors can be calculated from one-body current operators. The relevant Poincaré group Clebsch-Gordan coefficients needed for the Wigner-Eckhard theorem are derived in Appendix A, while the general covariance, including parity, of the current matrix elements is shown in Appendix B.

## II. RELATIVISTIC KINEMATICS AND DYNAMICS

### A. Kinematics

The positive mass, positive energy representation spaces of the Poincaré group $L(\mathbb{R}^3)\times V^j$ form the Hilbert space for constituent particles of mass $m$ and spin $j$; the Poincaré group transforms points $x$ in Minkowski space to $x'=\Lambda x+a$, where $x$, $x'$, and $a$ are four vectors, $\Lambda$ is a Lorentz transformation, and the inner product $x\cdot x$ on Minkowski space is given by

$$x\cdot x:=x^Tgx,\quad x=\begin{pmatrix}x_0\\ \mathbf{x}\end{pmatrix},\quad g=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$$

$$\Lambda\in \mathrm{SO}(1,3):=\{\Lambda\in GL(4,\mathbb{R})|\Lambda^Tg\Lambda=g\}.\tag{2.1}$$

Momentum states $|pj\sigma\rangle$ transform under Lorentz transformations and translations as

$$U_\Lambda|pj\sigma\rangle=\sum_\sigma D^j_{\sigma'\sigma}(R_w)|\Lambda pj\sigma'\rangle,$$

$$U_a|pj\sigma\rangle=e^{ip\cdot a}|pj\sigma\rangle.\tag{2.2}$$

Here $p$ is a four-momentum vector satisfying $p\cdot p=m^2$, where $m$ is the mass of the constituent. $V^j$ is the $(2j+1)$-dimensional representation space of the rotation group SO(3) and $R_w$ is a Wigner rotation defined by $R_w(p,\Lambda):=B^{-1}(\Lambda v)\Lambda B(v)$, with $B(v)$ a boost, a coset representative of SO(1,3)/SO(3); in particular, $p=B(v)p(\text{rest})$, with $p(\text{rest})$ the rest frame four-vector $(m,0,0,0)$ and $v:=p/m$. Various types of boosts and their properties are discussed in detail in Ref. [12].

The infinitesimal transformations of Eq. (2.2) generate the operators $P^{\mu}_{\rm fr}$ and $J^{\alpha\beta}$, out of which the free mass operator and spin operators are formed:

$$P_{\rm fr}\cdot P_{\rm fr}=M^2_{\rm fr},\quad V^{\mu}:=\frac{P^{\mu}_{\rm fr}}{M_{\rm fr}},$$

$$W_{\mu}=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}J^{\alpha\beta}P^{\gamma}_{\rm fr}\quad\text{(Pauli-Lubanski operator)},$$

$$\widetilde{W}_{\mu}=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}J^{\alpha\beta}V^{\gamma}\quad\text{(modified Pauli-Lubanski operator)}.\tag{2.3}$$

$V^{\mu}$ is the four-velocity operator, which, acting on a momentum eigenstate, has eigenvalue $v$. The labels $p$, $j$, and $\sigma$ appearing in the state, Eq. (2.2), are eigenvalues of the operators $P^{\mu}_{\rm fr}$, $\widetilde{W}\cdot\widetilde{W}$, and $n\cdot\widetilde{W}$ (see Ref. [12] and the following paper [10]). The Casimir invariant $\widetilde{W}\cdot\widetilde{W}$ has the eigenvalues $j(j+1)$, in contrast to the usual Pauli-Lubanski operator, which has an additional mass factor; $n$ is the momentum-dependent four vector $B(v)^{\mu}{}_3$ (see Ref. [10]).

Many-particle states are defined as products of single-particle states:

$$|p_1j_1\sigma_1,\ldots,p_nj_n\sigma_n\rangle:=|p_1j_1\sigma_1\rangle\cdots p_nj_n\sigma_n\rangle$$

$$U_{\Lambda}|p_1j_1\sigma_1,\ldots,p_nj_n\sigma_n\rangle=\prod_{k=1}^{n}D^{j_k}_{\sigma'_k\sigma_k}(R_{w_k})|(\Lambda p_1)j_1\sigma'_1,\ldots,(\Lambda p_n)j_n\sigma'_n\rangle$$

$$U_a|p_1j_1\sigma_1,\ldots,p_nj_n\sigma_n\rangle=e^{-i\Sigma_{k=1}^{n}p_k\cdot a}|p_1j_1\sigma_1,\ldots,p_nj_n\sigma_n\rangle.\tag{2.4}$$

To develop a relativistic dynamics it is useful to have multiparticle states with labels describing the $n$-particle system as a whole, and labels describing the internal configuration of the $n$-particle system. In Ref. [12] simultaneously coupled states were introduced that have such internal and external variables. However, the internal variables do not include the orbital angular momentum of the $n$-particle system. In this paper we want to work with relativistic states related as closely as possible to nonrelativistic states, where orbital angular momentum is a possible internal variable. To that end define a velocity state as

$$|v,\mathbf{k}_i,\mu_i\rangle:=U_{B(v)}|k_1j_1\mu_1,\ldots,k_nj_n\mu_n\rangle,$$

$$\sum\mathbf{k}_i=\mathbf{0},\tag{2.5}$$

$$\begin{aligned}U_{\Lambda}|v,\mathbf{k}_i\mu_i\rangle&=U_{\Lambda}U_{B(v)}|k_1j_1\mu_1,\ldots,k_nj_n\mu_n\rangle\\&=U_{B(\Lambda v)}U_{R_w}|k_1j_1\mu_1,\ldots,k_nj_n\mu_n\rangle\\&=\sum_{\mu'_i}\prod_{i=1}^{n}D^{j_i}_{\mu'_i\mu_i}[R_w(k_i,R_w)]|\Lambda v,R_w\mathbf{k}_i,\mu'_i\rangle.\end{aligned}\tag{2.6}$$

$R_w$ is the Wigner rotation $R_w(v,\Lambda)$ and $R_w(k_i,R_w)$ the Wigner rotation of a Wigner rotation. If the boost $B(k_i)$ is chosen to be a canonical boost, then $R_w(k_i,R_w)=R_w$ [see Eq. (2.7), Ref. [12]] and Eq. (2.6) becomes

$$U_{\Lambda}|v,\mathbf{k}_i,\mu_i\rangle=\sum_{\mu'_i}\prod_{i=1}^{n}D^{j_i}_{\mu'_i\mu_i}(R_w)|\Lambda v,R_w\mathbf{k}_i,\mu'_i\rangle.\tag{2.7}$$

Since all of the rotations are the same, the spins can be coupled together to give an overall spin state, as is done nonrelativistically. If a boost other than canonical spin is used, it is easy to modify Eq. (2.5) so that under Lorentz transformations the velocity states still transform as in Eq. (2.7). The steps needed to construct relativistic orbital angular momentum states from the velocity states, Eq. (2.5) is carried out in detail in Ref. [10]. Under space-time translations, velocity states transform as

$$\begin{aligned}U_a|v,\mathbf{k}_i\mu_i\rangle&=U_aU_{B(v)}|k_1j_1\mu_1,\ldots,k_nj_n\mu_n\rangle\\&=U_{B(v)}U_{B^{-1}(v)a}|k_1j_1\mu_1,\ldots,k_nj_n\mu_n\rangle\\&=e^{-im_nv\cdot a}|v,\mathbf{k}_i\mu_i\rangle,\end{aligned}\tag{2.8}$$

where $m_n:=\Sigma\sqrt{m_i^2+\mathbf{k}_i\mathbf{k}_i}$ is the free $n$-particle mass.

If $a$ becomes infinitesimal, the free four-momentum operator is seen to be

$$P^{\mu}_{\rm fr}|v,\mathbf{k}_i,\mu_i\rangle=m_nv^{\mu}|v,\mathbf{k}_i,\mu_i\rangle\tag{2.9}$$

so that the free mass operator $M_{\rm fr}:=\sqrt{P_{\rm fr}\cdot P_{\rm fr}}$ acting on $|v,\mathbf{k}_i,\mu_i\rangle$ gives

$$M_{\rm fr}|v,\mathbf{k}_i,\mu_i\rangle=m_n|v,\mathbf{k}_i\mu_i\rangle,\tag{2.10}$$

while the free four-velocity operator $V^{\mu}:=P^{\mu}_{\rm fr}/M_{\rm fr}$ gives

$$V^{\mu}|v,\mathbf{k}_i,\mu_i\rangle=v^{\mu}|v,\mathbf{k}_i,\mu_i\rangle,\quad v\cdot v=+1.\tag{2.11}$$

The connection between velocity states and single-particle states is given by

$$\begin{aligned}|v,\mathbf{k}_i,\mu_i\rangle&=U_{B(v)}|k_1j_1\mu_1,\ldots,k_nj_n\mu_n\rangle\\&=\sum_{\sigma_i}\prod_{i=1}^{n}D^{j_i}_{\sigma_i\mu_i}[k_i,B(v)]|p_1j_1\sigma_1,\ldots,p_nj_n\sigma_n\rangle;\end{aligned}\tag{2.12}$$

$\langle p_ij_i\sigma_i|v\mathbf{k}_i\mu_i\rangle=\Pi_{i=1}^{n}D^{j_i}_{\sigma_i\mu_i}[R_w(k_i,B(v))]$ with $p_i=B(v)k_i$; $p:=\Sigma p_i=B(v)\binom{m_n}{\mathbf{0}}=vm_n$, so $v=p/m_n$ as expected.

### B. Dynamics

Given the multiparticle Hilbert space, we want to introduce a relativistic dynamics by perturbing the free four-momentum operator $P^{\mu}_{\rm fr}$. There will then be interactions in the spatial part of the four-momentum operator, as well as the time component of the four-momentum operator $P^0$, which is the Hamiltonian. Dirac was the first to observe that a relativistic dynamics could take various forms, three of which he called the point, instant, and front forms [2]. The best known form of relativistic dynamics is the instant form, in which the three-momentum operator is not perturbed, but instead interactions are put in the boost generators of Lorentz transformations. Thus in the instant form Lorentz transformations are not kinematic, but the momentum operators are. In contrast in the point form, all Lorentz transformations are kinematic and all four momentum operators are dynamic.

That introducing interactions in relativistic quantum mechanics is more complicated than in nonrelativistic quantum mechanics can be seen from the Poincaré commutator

$$[K_i, P_j] = \delta_{ij} P^0, \tag{2.13}$$

where $K_i$ is a generator of Lorentz transformations along the $i$th axis, $P_j$ is the momentum generator along the $j$th axis, $i,j=1,2,3$, and $P^0$ is the Hamiltonian. In the absence of interactions these generators, obtained from Eq. (2.2), satisfy Eq. (2.13). If interactions are added to $P^\mu_{\mathrm{fr}}$, $P^0_{\mathrm{fr}} \to P^0 = P^0_{\mathrm{fr}} + V$, then in order that the commutation relation be satisfied, either $K_i$ must be modified (instant form) or $P_j$ must be modified (point form); the front form modifies pieces of both $P_j$ and $K_i$ as seen in Ref. [4]. In nonrelativistic quantum mechanics, the right-hand side of Eq. (2.13) contracts to the identity operator, so that when the free Hamiltonian is modified to include interactions, it is not necessary to modify $K_i$ (which contracts to the nonrelativistic position operator) or $P_i$.

A feature of the point form is that it is manifestly Lorentz covariant. If the free four-momentum operator is perturbed, $P^\mu_{\mathrm{fr}} \to P^\mu := P^\mu_{\mathrm{fr}} + P^\mu_{\mathrm{int}}$, then the total four-momentum operator $P^\mu$ must satisfy the covariant Poincaré commutation relations

$$[P^\mu, P^\nu] = 0,$$

$$U_\Lambda P^\mu U_\Lambda^{-1} = \Lambda^{-1\mu}{}_\nu P^\nu, \tag{2.14}$$

where, since Lorentz transformations are kinematic, $U_\Lambda$ is given in Eq. (2.7).

Bound and scattering states are eigenvectors of the mass operator, defined as

$$M := \sqrt{P \cdot P}; \tag{2.15}$$

a fundamental requirement on $M$ is that its spectrum be positive. The associated spin spectrum for a given $P^\mu$ is obtained from the square of the Pauli-Lubanski operator, defined in Eq. (2.3).

A key issue in point form relativistic quantum mechanics is constructing four-momentum operators that satisfy Eq. (2.14). In this paper we assume that the strongly interacting four-momentum operator $P^\mu_{\mathrm{st}}$ is given; the goal is to analyze the properties of electromagnetic current operators that have definite transformation properties with respect to $P^\mu_{\mathrm{st}}$ [see Eq. (C3)] and in particular find a covariant expression for current matrix elements, in which the states are eigenstates of $P^\mu_{\mathrm{st}}$. In this analysis, the specific form of $P^\mu_{\mathrm{st}}$ is not needed; to calculate actual form factors requires of course that a specific choice of $P^\mu_{\mathrm{st}}$ be made (see, for example, Ref. [11]).

## III. ELECTROMAGNETIC CURRENT OPERATORS AND FORM FACTORS

In this section we consider a hadronic four-momentum operator which is the sum of strong, photon, and electromagnetic four-momentum operators:

$$P^\mu = P^\mu_{\mathrm{st}} + P^\mu_\gamma + P^\mu_{\mathrm{em}}, \tag{3.1}$$

where $P^\mu_{\mathrm{st}}$ is the strongly interacting four-momentum operator, $P^\mu_\gamma$ is the free photon four-momentum operator given in terms of photon creation and annihilation operators, and $P^\mu_{\mathrm{em}}$ the electromagnetic four-momentum operator; in the point form $P^\mu_{\mathrm{em}}$ is obtained by integrating $J^\nu(x) A_\nu(x)$ over the forward hyperboloid

$$P^\mu_{\mathrm{em}} = \int d^4x\, \delta(x\cdot x - \tau^2)\, \theta(x^0) x^\mu J^\nu(x) A_\nu(x), \tag{3.2}$$

with $J^\nu(x)$ the hadronic current operator and $A_\nu(x)$ the photon field. The goal of this section is to find a covariant expression for matrix elements of the current operator, in which the invariant form factors are reduced matrix elements of the Poincaré group. As shown in Appendix C, the hadronic current operator must satisfy the following conditions:

$$\frac{\partial J^\mu(x)}{\partial x^\mu} = 0 \quad \text{(current conservation)}, \tag{3.3a}$$

$$U_\Lambda J^\mu(x) U_\Lambda^{-1} = \Lambda^{-1\mu}{}_\nu J^\nu(\Lambda x) \quad \text{(Lorentz covariance)}, \tag{3.3b}$$

$$[P^\nu_{\mathrm{st}}, J^\mu(x)] = i\, \frac{\partial J^\mu}{\partial x_\nu} \quad \text{(space-time covariance)}. \tag{3.3c}$$

We want to write the operator $J^\mu(x)$ in such a way that it transforms as an irreducible tensor operator under the strongly interacting Poincaré group, for the matrix elements of such an operator $J_b(Q)$ can be reduced to Clebsch-Gordan coefficients times reduced matrix elements, which are the invariant form factors. Write

$$J^\mu(x) := \sum_b \int d^4Q\, e^{-iQ\cdot x} D[B(Q)]^\mu{}_b J_b(Q), \tag{3.4}$$

where

$$b = 1,2,3 \quad \text{for} \quad Q^2 > 0 \quad \text{(timelike)},$$

$$b = 1,2 \quad \text{for} \quad Q^2 = 0 \quad \text{(lightlike)},$$

$$b = 0,1,2 \quad \text{for} \quad Q^2 < 0 \quad \text{(spacelike)}, \tag{3.5}$$

and

$$D(\Lambda)^\mu{}_\nu := \Lambda^\mu{}_\nu g_{\nu\nu}, \quad \text{with} \quad g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1),$$

the Minkowski metric. Note there is no sum on $g_{\nu\nu}$ in the definition of $D(\Lambda)$. $B(Q)$ is a boost defined below. Since the Poincaré group properties of spacelike representations are not as well known as the timelike and lightlike representations, and further, since the applications of the representation of current matrix elements is to spacelike momentum transfers, only the case $Q^2<0$ will be analyzed in detail in this paper.

For $Q^2<0$, write $Q(\mathrm{st}):=(0,0,0,q)$ and $\tilde{R}\in\mathrm{SO}(1,2)$ with

$$\mathrm{SO}(1,2):=\{\tilde{R}\in GL(3,\mathbb{R})|\tilde{R}\tilde{g}\tilde{R}^T=\tilde{g},\quad \tilde{g}=\mathrm{diag}(1,-1,-1)\}. \tag{3.6}$$

$\tilde{R}\in\mathrm{SO}(1,2)$ leaves $Q(\mathrm{st})$ invariant, $\left(\begin{smallmatrix}\tilde{R} & 0\\ 0 & 1\end{smallmatrix}\right)Q(\mathrm{st})=Q(\mathrm{st})$. Choose boosts $B(Q)$ [that is, coset representatives of SO(1,3) with respect to SO(1,2)] such that $\Lambda=B(Q)\tilde{R}$, and $Q=B(Q)Q(\mathrm{st})$. Then

$$\Lambda B(Q)=B(\Lambda Q)\tilde{R}_w, \tag{3.7}$$

where $\tilde{R}_w$ is a spacelike Wigner "rotation" defined by

$$\tilde{R}_w:=B^{-1}(\Lambda Q)\Lambda B(Q)\in\mathrm{SO}(1,2). \tag{3.8}$$

If the operator $J_b(Q)$ of Eq. (3.4) transforms as a spacelike representation of the Lorentz subgroup

$$U_\Lambda J_b(Q)U_\Lambda^{-1}=\sum_{b'=0}^{2}(\tilde{R}_w)_{b'b}J_{b'}(\Lambda Q), \tag{3.9}$$

then $J^\mu(x)$ will transform as Eq. (3.3b):

$$\begin{aligned}U_\Lambda J^\mu(x)U_\Lambda^{-1}&=\sum_{b',b}\int d^4Qe^{-iQ\cdot x}B(Q)^\mu_b g_{bb}(\tilde{R}_w)_{b'b}J_{b'}(\Lambda Q)\\&=\sum_{b'}\int d^4Qe^{-iQ\cdot x}D[B(Q)(\tilde{R}_w)^{-1}]^\mu_{b'}J_{b'}(\Lambda Q)\\&=\Lambda^{-1\mu}{}_\nu\sum_{b'}\int d^4Q'e^{-iQ'\cdot\Lambda x}D[B(Q')]^\mu_{b'}J_{b'}(Q')\\&=\Lambda^{-1\mu}J^\nu(\Lambda x).\end{aligned} \tag{3.10}$$

Further, if

$$[P^\nu,J_b(Q)]=Q^\nu J_b(Q), \tag{3.11}$$

then

$$\begin{aligned}[P^\nu,J^\mu(x)]&=\sum_b\int d^4Qe^{-iQ\cdot x}D[B(Q)]^\mu{}_b[P^\nu,J_b(Q)]\\&=i\,\frac{\partial J^\mu(x)}{\partial x_\nu}.\end{aligned} \tag{3.12}$$

This means that if $J_b(Q)$ transforms as an irreducible tensor operator under the interacting Poincaré group, then $J^\mu(x)$ will automatically satisfy the covariance properties Eqs. (3.3b) and (3.3c).

Finally, Eq. (3.4) automatically incorporates current conservation, $\partial J^\mu/\partial x^\mu=0$; note that $J_b(Q)$ has only three components in $b$ for $Q^2<0$, namely, $b=0,1,2$. Current conservation follows from the fact that $Q_\mu D[B(Q)]^\mu{}_b=0$, for then

$$\frac{\partial J^\mu(x)}{\partial x^\mu}=-i\sum_b\int d^4Qe^{-iQ\cdot x}Q_\mu D[B(Q)]^\mu{}_bJ_b(Q)=0. \tag{3.13}$$

To show that $Q_\mu D[B(Q)]^\mu{}_b=0$, notice that $B^{-1}(Q)Q=Q(\mathrm{st})=0$ for $b=0,1,2$. But

$$\begin{aligned}Q\cdot B(Q)&=Q^TgB(Q)=Q^TB^{-1T}(Q)g=[B^{-1}(Q)Q]^Tg\\&=Q(\mathrm{st})^Tg=0\end{aligned} \tag{3.14}$$

for $b=0,1,2$, which implies current conservation.

Since $J_b(Q)$ transforms as a tensor operator, all $J_b(Q)$'s can be obtained from one standard, say $J_0[Q(\mathrm{st})]$; that is,

$$U_{B(Q)}J_0[Q(\mathrm{st})]U_{B(Q)}^{-1}=J_0[B(Q)Q(\mathrm{st})]=J_0(Q)$$

$$U_{\tilde{R}}J_0[Q(\mathrm{st})]U_{\tilde{R}}^{-1}=\sum_b\tilde{R}_{b0}J_b[Q(\mathrm{st})],\quad \tilde{R}\in\mathrm{SO}(1,2). \tag{3.15}$$

Though this analysis has been carried out only for spacelike $Q$, it is clear that a similar analysis can be carried out for $Q$ timelike or lightlike.

Consider now states $|pj\sigma\rangle$ which are eigenstates of $P^\mu_{\mathrm{st}}$, $\tilde{W}\cdot\tilde{W}$, and $n\cdot\tilde{W}$, the four-momentum, spin, and spin component. In perturbation theory the scattering amplitude for electrons scattering off a bound state of constituents is given by

$$\begin{aligned}\langle p'j'\sigma';k_e'|S-I|pj\sigma;k_e\rangle&=\frac{-ie^2}{(2\pi)^3}\int d^4Q\,\frac{1}{Q^2}\,\delta^4(p'-p-Q)\delta^4(k_e'-k_e+Q)\\&\quad\times\langle p'j'\sigma'|J^\mu(0)|pj\sigma\rangle\langle k_e'|J_e^\nu(0)|k_e\rangle g_{\mu\nu},\end{aligned} \tag{3.16}$$

where $\langle k_e'|J_e^\nu(0)|k_e\rangle$ is the electron current matrix element, with $k_e$, $k_e'$ the initial and final electron four momenta (electron spin labels have been suppressed).

But

$$\langle p'j'\sigma'|[P^\mu_{\mathrm{st}},J_b(Q)]|pj\sigma\rangle=Q^\mu\langle p'j'\sigma'|J_b(Q)|pj\sigma\rangle$$

$$(p'^\mu-p^\mu)\langle p'j'\sigma'|J_b(Q)|pj\sigma\rangle=Q^\mu\langle p'j'\sigma'|J_b(Q)|pj\sigma\rangle \tag{3.17}$$

so that either

(a) $p'-p=Q$, with $\langle p'j'\sigma'|J_b(Q)|pj\sigma\rangle$ nonzero, or

(b) $p'-p\neq Q$, in which case $\langle p'j'\sigma'|J_b(Q)|pj\sigma\rangle=0$.

It thus follows that the hadronic current matrix element can be written as

$$\langle p'j'\sigma'|J^\mu(0)|pj\sigma\rangle=\sum_b D[B(Q)]^\mu{}_b\langle p'j'\sigma'|J_b(Q)|pj\sigma\rangle; \tag{3.18}$$

the matrix element of $J_b(Q)$ is zero unless $Q=p'-p$ which is just the condition that a bound state of four-momentum $p=mv$ produce a (possibly new) state of four-momentum $p'=m'v'=p+Q$, with $Q$ the four-momentum transfer of the photon.

Using the fact that $J_b(Q)$ transforms as an irreducible tensor operator under the (interacting) Poincaré group, the matrix element, Eq. (3.18), can be written as a product of Clebsch-Gordan coefficients and reduced matrix elements:

$$\langle p'j'\sigma'|J_\nu(Q)|pj\sigma\rangle = \sum_{r',r} \langle p'j'\sigma'|Q\nu,r'r;pj\sigma\rangle \times\langle m'j'r'\|Q^2\|mjr\rangle, \quad (3.19)$$

where the sum over $r'$ and $r$ is between $-j'\leqq r'\leqq j'$, $-j\leqq r\leqq j$ such that $r'-r=0,\pm 1$ and $\nu=0,\pm 1$. [See Eq. (A1).] $\langle p'j'\sigma'|Q\nu,r'r;pj\sigma\rangle$ is a Poincaré group Clebsch-Gordan coefficient coupling a time-like $p$ to a spacelike $Q$ to produce a timelike $p'$. As shown in Appendix A it is the product of three $D$ functions times a four-dimensional delta function $\delta^4(p'-p-Q)$, so that Eq. (3.19) indeed satisfies the commutator conditions for translational covariance as given in Eq. (3.17). The quantities

$$\langle m'j'r'\|Q^2\|mjr\rangle, \quad (3.20)$$

are reduced matrix elements, which, as will be shown, can be related to the definition of more usual invariant form factors. Equation (3.19) is a consequence of the Wigner-Eckhart theorem for the Poincaré group, in which the spin projection labels $r'$ and $r$ are degeneracy parameters (they are actually eigenvalues of $\tilde{W}'\cdot Q$ and $\tilde{W}\cdot Q$, see Ref. [12], p. 40).

The connection with form factors defined as current matrix elements evaluated in a standard frame is obtained by choosing a standard frame

$$p'(\mathrm{st}) = \begin{pmatrix} \sqrt{m'^2+p_z'^2} \\ 0 \\ 0 \\ p_z' \end{pmatrix}, \quad p(\mathrm{st}) = \begin{pmatrix} \sqrt{m^2+p_z^2} \\ 0 \\ 0 \\ p_z \end{pmatrix},$$

$$Q(\mathrm{st}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ q \end{pmatrix}; \quad (3.21)$$

$p_z'$ and $p_z$ are related to the invariants $m$, $m'$ and $Q^2$ in Eq. (A7). Then using the Clebsch-Gordan coefficients, Eq. (A10) in the standard frame,

$$\langle p'(\mathrm{st})j'r'|J_\nu[Q(\mathrm{st})]|p(\mathrm{st})jr\rangle = \delta_{\nu,r'-r}\langle m'j'r'\|Q^2\|mjr\rangle. \quad (3.22)$$

For $\nu=0$, $r'=r$ and the change form factor is defined as

$$F_r^0 := \langle p'(\mathrm{st})j'r|J_{\nu=0}[Q(\mathrm{st})]|p(\mathrm{st})jr\rangle = \langle m'j',r'=r\|Q^2\|mjr\rangle. \quad (3.23)$$

For $\nu=\pm 1$, $r'-r=\pm 1$ and the current form factor is defined as

$$F_r^{\pm} := \langle p'(\mathrm{st})j'r'|J^x[Q(\mathrm{st})]\pm iJ^y[Q(\mathrm{st})]|p(\mathrm{st})jr\rangle = \langle m'j',r'=r\pm 1\|Q^2\|mjr\rangle. \quad (3.24)$$

To convert these invariant form factors to Cartesian components, needed in Eq. (3.18), we write

$$F_{r'r}^{b=1}(=F_{r'r}^x) = \frac{1}{2}(F_r^+ + F_r^-),$$

$$F_{r'r}^{b=2}(=F_{r'r}^y) = \frac{1}{2i}(F_r^+ - F_r^-). \quad (3.25)$$

Then the current matrix element, Eq. (3.18), can be written as

$$\begin{aligned}\langle p'j'\sigma'|J^\mu(x)|pj\sigma\rangle &= \sum_b \int d^4Q e^{-iQ\cdot x} D[B(Q)]^\mu{}_b \langle p'j'\sigma'|J_b(Q)|pj\sigma\rangle \\ &= \sum_b \int d^4Q e^{-iQ\cdot x}\delta^4(p'-p-Q)\sum_{r'r}\Lambda^\mu{}_b(p,Q)D^{j'}_{\sigma'r'}(R_w')D^{j*}_{\sigma r}(R_w)F^b_{r'r} \\ &= e^{i(p'-p)\cdot x}\sum_{b,r'r}\Lambda^\mu{}_b(p,Q)D^{j'}_{\sigma'r'}(R_w')F^b_{r'r}D^j_{r\sigma}(R_w^{-1}),\end{aligned} \quad (3.26)$$

where the Lorentz transformation $\Lambda^\mu{}_b(p,Q)$ [coset representative of SO(1,3) with respect to SO(2)] is given by

$$\Lambda(p,Q) = \left(w_0, w_1, w_2, \frac{Q}{q}\right), \quad (3.27)$$

with $\tilde{m}w_0^\mu = p^\mu - (p\cdot Q)/(Q^2)Q^\mu$, $\tilde{m} = \sqrt{m^2+(p\cdot Q)/q^2} = E(\mathrm{st})$, $Q=-q^2$, $w_0^2=1$, $w_0\cdot Q=0$. $w_1$ and $w_2$ are two four vectors in the Lorentz transformation $\Lambda(p,Q)$, satisfying the relations $w_i\cdot w_0 = w_i\cdot Q = 0$, $w_i^2=-1$, $i=1,2$, and $w_1\cdot w_2=0$; they are given explicitly in Eq. (A18). If the masses of the initial and final states are the same, $m'=m$, then $w_0=(4m^2+q^2)^{-1/2}(p+p')$.

The Wigner rotations $R_w$, $R_w'$ in Eq. (3.26) are defined by

$$R_w = B^{-1}(p)\Lambda(p,Q)B[p(\mathrm{st})],$$

$$R_w' = B^{-1}(p')\Lambda(p,Q)B[p'(\mathrm{st})]; \quad (3.28)$$

their explicit form is given in Eq. (A21) for canonical spin boosts and Eq. (A22) for helicity boosts and are functions of

$p,p'$ only, independent of the coset choice made for $\Lambda(p,Q)$. Notice that if $p=p(\mathrm{st})$ and $p'=p'(\mathrm{st})$, then $R_w=R'_w=e$, the identity rotation.

The translational and Lorentz covariance of the current matrix element representation, Eq. (3.26), can now be readily demonstrated. If $a$ is an arbitrary space-time translation, it follows that

$$\begin{aligned}\langle p'j'\sigma'|U_a^{-1}U_aJ^\mu(x)U_a^{-1}U_a|pj\sigma\rangle &= e^{-i(p'-p)\cdot a}\langle p'j'\sigma'|J^\mu(x+a)|pj\sigma\rangle \\ &= e^{-i(p'-p)\cdot a}e^{i(p'-p)\cdot x+a} \\ &\quad\times\sum_{r'r,b}\Lambda^\mu{}_b(p,Q)D^{j'}_{\sigma'r'}(R'_w)D^{j*}_{\sigma r}(R_w)F^b_{r'r} \\ &= \langle p'j'\sigma'|J^\mu(x)|pj\sigma\rangle. \end{aligned} \tag{3.29}$$

This means that it suffices to evaluate the matrix element for $J^\mu(0)$; translational covariance, Eq. (3.3c) with $J^\mu(x):=U_xJ^\mu(0)U_x^{-1}$ is then automatically satisfied. Checking the Lorentz covariance of Eq. (3.21) is a little more complicated and is carried out in Appendix B (including parity covariance).

By virtue of current conservation, the third component of the standard current matrix element $\langle p'(\mathrm{st})j'\sigma'|J^3(0)|p(\mathrm{st})j\sigma\rangle$ is zero. More generally current conservation is the condition that

$$(p'_\mu-p_\mu)\langle p'j'\sigma'|J^\mu(0)|pj\sigma\rangle=0; \tag{3.30}$$

but in the standard frame $p'_\mu(\mathrm{st})-p_\mu(\mathrm{st})=Q_\mu(\mathrm{st})$ has only a $z$ component which means the matrix element of $J^3(0)$ is zero. Conversely, if $\langle p'(\mathrm{st})j'r'|J^3(0)|p(\mathrm{st})jr\rangle$ is zero for all $r',r$, then by boosting with $\Lambda(p,Q)$, Eq. (3.30) follows.

Thus we have shown that the current matrix elements can be written as

$$\begin{aligned}\langle p'j'\sigma'I'|J^\mu(0)|pj\sigma I\rangle &= \sum_{r',r,b}\Lambda^\mu{}_b(p,Q)D^{j'}_{\sigma'r'}(R'_w) \\ &\quad\times F^b_{r'r}[(p'-p)^2,I',I]D^j_{r\sigma}(R_w^{-1}),\\ \langle p'j'\sigma'I'|J^\mu_{\mathrm{elec}}(0)|pj\sigma I\rangle &= \sum_{r',r}w^\mu_0(p,Q)D^{j'}_{\sigma'r'}(R'_w) \\ &\quad\times F^0_{r'r}[(p'-p)^2,I',I]D^j_{r\sigma}(R_w^{-1}),\\ \langle p'j'\sigma'I'|J^\mu_{\mathrm{mag}}(0)|pj\sigma I\rangle &= \sum_{\substack{b=1,2\\ r',r}}w^\mu{}_b(p,Q)D^{j'}_{\sigma'r'}(R'_w) \\ &\quad\times F^b_{r'r}[(p'-p)^2,I',I]D^j_{r\sigma}(R_w^{-1}),\end{aligned} \tag{3.31}$$

a manifestly covariant form valid for particles of arbitrary masses $m'$, $m$ and spins $j',j$. The labels $I'$ and $I$ have been included in the initial and final states to allow for multiparticle states where $I'$ and $I$ might include subenergies, spins of subsystems or any other invariant labels needed to specify a multiparticle system [10,12]. Further, the reduced matrix elements $F^b_{r'r}$ become the invariant form factors given by Yennie *et al.* [13] when the spin and mass of the final particle is the same as that of the initial particle. Notice that although the representation for the current matrix element, Eq. (3.31), is covariant, all the spin dependence resides in the Wigner $D^j$ and $D^{j'}$ functions. There are no spinor labels, $\gamma$ matrices or doubling of states for half-integer spin to include parity. Moreover, in contrast to form factors defined using spinor variables (see, for example, Ref. [14]), the reduced matrix elements $F^b_{r'r}$ give the correct number of independent form factors. Although the arguments of the $D$ functions are a bit complicated [see Eqs. (A21) and (A22) for the exact expressions] the representation, Eq. (3.31), is basically the same for all matrix elements, including higher spin transitions. This can be seen from the following examples, in which it is assumed that parity is conserved ($\eta,\eta'$ are the intrinsic parities):

$$(1)\quad j'=j=\tfrac{1}{2},\quad F^0_r=\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix},\quad F^{b=2}_{r'r}=\begin{pmatrix}0 & B\\ B & 0\end{pmatrix}.$$

There is one electric and one magnetic form factor, related, as discussed by Yennie *et al.* to the Dirac form factors by

$$A=e\left[F_1-\kappa\frac{Q^2}{4m^2}F_2\right],$$

$$B=2e\left[\frac{|Q^2|}{2m}(F_1+\kappa F_2)\right]\quad[\text{their Eq. (A-22)}].$$

$\kappa$ is the anomalous magnetic moment.

$$(2)\quad j'=j=1,\quad \eta'=\eta,\quad F^0_r=\begin{pmatrix}A_1 & & \\ & A_2 & \\ & & A_1\end{pmatrix},$$

$$F^{b=2}_{r'r}=\begin{pmatrix}0 & B_1 & 0\\ B_2 & 0 & B_2\\ 0 & B_1 & 0\end{pmatrix}.$$

There are two electric and two magnetic form factors. If the initial and final particles are the same, then hermiticity [Eq. (B16)] implies $B_2=B_1$ and there is only one magnetic form factor. The form factor is often written covariantly as

$$\begin{aligned}\langle p'1\sigma'|J^\mu(0)|p1\sigma\rangle &= \frac{p^\mu+p'^\mu}{2}\left\{F_1(Q^2)[B^T(p')gB(p)]_{\sigma'\sigma}\right.\\ &\quad\left.+\frac{F_2(Q^2)}{2m^2}[(Q^TgB(p')]_{\sigma'}[Q^TgB(p)]_\sigma\right\}\\ &\quad-\frac{G_1(Q^2)}{2}\{[Q^TgB(p')]_{\sigma'}B(p)^\mu{}_\sigma\\ &\quad-B(p')^\mu{}_\sigma[Q^TgB(p)]_\sigma\}.\end{aligned}$$

$$(3) \quad j'=j=\tfrac{3}{2}, \quad \eta'=\eta, \quad F_r^0 0=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & A_2 & \\ & & & A_1 \end{pmatrix},$$

$$F_{r'r}^{b=2}=\begin{pmatrix} 0 & B_1 & & \\ B_3 & 0 & B_2 & \\ & B_2 & 0 & B_3 \\ & & B_1 & 0 \end{pmatrix}.$$

There are two electric and three magnetic form factors. If the initial and final particles are the same, then hermiticity implies $B_3=B_1$ and there are two magnetic form factors.

(4)

$$j'>\tfrac{1}{2}, \; j=\tfrac{1}{2}, \quad F_r^0=\left(\bigcirc \; \overbrace{\begin{matrix} A & 0 \\ 0 & A \end{matrix}}^{2j'+1} \; \bigcirc\right)$$

$$F_{r'r}^{b=2}=\left(\bigcirc \; \overbrace{\begin{matrix} \pm B_2 & 0 & B_1 & 0 \\ 0 & \pm B_1 & 0 & B_2 \end{matrix}}^{2j'+1} \; \bigcirc\right).$$

There is one electric and two magnetic form factors for all $N^{1/2}\rightarrow N^{j'}$ transitions. The signs in $F_{r'r}^{b=2}$ depend on the value of $\eta'\eta(-1)^{j'+j+2r+1}$ [see Eq. (B14)].

## IV. THE POINT FORM RELATIVISTIC IMPULSE APPROXIMATION

The general idea behind the impulse approximation is that the electromagnetic properties of composite particles should be determined by the electromagnetic properties of their constituents. In particular the electromagnetic properties of hadrons thought of as bound states of constituents should be determined by the electromagnetic properties of the constituents. In practice this means approximating the electromagnetic current operator $J^\mu(0)$ by one-body operators. The hope is that the matrix elements of the many-body current operator are small in comparison with the matrix elements of the one-body operator. How this approximation is to be made in the point form of relativistic quantum mechanics is the subject of this section. We will consider the general case of a bound state of $n$ constituents in this section, and in succeeding papers use the results to calculate deuteron as well as hadronic form factors [11].

Consider then a particle as a bound state of $n$ constituents of mass $m_i$ and spin $j_i=\frac{1}{2}$, $i=1\ldots n$. Assume there is a mass operator $M$ defined on the constituent Hilbert space $\mathcal{H}_n$ whose discrete spectrum includes a particle of mass $m$, spin $j$, and spin projection $\sigma$:

$$M\psi_{pj\sigma}=m\psi_{pj\sigma}, \quad p^2=m^2; \tag{4.1}$$

$j$ is the eigenvalue of $\tilde{W}\cdot\tilde{W}$, and $\sigma$ the eigenvalue of $n\cdot\tilde{W}$, as discussed in Sec. II. Assume also that the constituents have known invariant form factors so that for the $i$th constituent

$$\langle p_i'\sigma_i'|J_{(i)}^\mu(0)|p_i\sigma_i\rangle=\frac{e_i}{(2\pi)^3}\,\overline{u}(p_i'\sigma_i')[\gamma^\mu(F_1^{(i)}+2m_iF_2^{(i)}) -(p_i'^{\mu}+p_i^\mu)F_2^{(i)}]u(p_i\sigma_i), \tag{4.2}$$

where $F_1^{(i)},F_2^{(i)}$ are the invariant Dirac form factors for the $i$th constituent. The goal is to compute the invariant form factor of the particle in terms of the invariant form factors of the underlying constituents. To this end write

$$J^\mu(0)=J_{\rm fr}^\mu(0)+J_>^\mu(0), \tag{4.3}$$

where $J_{\rm fr}^\mu(0)$ is the ''free'' or one-body operator and is of the form

$$J_{\rm fr}^\mu(0)=\sum_{i=1}^n J_i^\mu(0), \tag{4.4}$$

with $J_i^\mu(0)$ the current operator of the $i$th constituent [Eq. (4.2)]; $J_>^\mu(0)$ is the many-body current operator.

As shown in Sec. III the invariant form factor is the matrix element of the current operator in the standard frame. That is,

$$F_r^0=\langle p'({\rm st})j'r|J^0(0)|p({\rm st})jr\rangle$$

$$F_r^\pm=\langle p'({\rm st})j'r\pm 1|J^\pm(0)|p({\rm st})jr\rangle, \tag{4.5}$$

where $F_r^0$ and $F_r^\pm$ are the electric and the magnetic form factors. Thus, what must be computed is the current matrix element in the standard frame. In terms of one-body and many-body operators this is

$$\langle p'({\rm st})j'r'|J^\mu(0)|p({\rm st})jr\rangle=\langle p'({\rm st})j'r'|J_{\rm fr}^\mu(0)|p({\rm st})jr\rangle +\langle p'({\rm st})j'r'|J_>^\mu(0)|p({\rm st})jr\rangle. \tag{4.6}$$

We define the point form impulse approximation to be

$$\langle p'({\rm st})j'r'|J^\mu(0)|p({\rm st})jr\rangle\cong\langle p'({\rm st})j'r'|J_{\rm fr}^\mu(0)|p({\rm st})jr\rangle \tag{4.7}$$

for $\mu=0,\pm 1$. Notice that for current conservation to be valid, the $\mu=3$ component must include many-body current operators. That is, for $\mu=3$ the left-hand side of Eq. (4.7) is zero [see Eq. (3.30)], yet if the right-hand side of Eq. (4.7) were evaluated for $\mu=3$, it would not in general be zero. It is the addition of the many-body current matrix element that makes the sum zero. More generally any operator, including two or more body operators, can be used in Eq. (4.7) without violating Poincaré covariance or current conservation.

The goal now is to evaluate the right-hand side of Eq. (4.7) for $\mu=b=0,\pm 1$. Writing out the matrix element gives

$$F^b_{r'r} = \langle p'(\mathrm{st})j'r'|J^b_{\mathrm{fr}}(0)|p(\mathrm{st})jr\rangle$$
$$= \sum\!\!\!\!\!\!\int \psi^*_{p'(\mathrm{st})j'r'}(1'\cdots n')[\langle 1'|J^b_1(0)|1\rangle \times \delta(2'-2)\cdots\delta(n'-n)+\cdots+\delta(1'-1)\cdots \times \delta[(n'-1)-(n-1)]\langle n'|J^b_n(0)|n\rangle]\psi_{p(\mathrm{st})jr}(1\cdots n). \tag{4.8}$$

The bound-state wave functions must be symmetric or antisymmetric under particle interchange. Since the impulse current matrix elements are symmetric under interchange of particles, the matrix element, Eq. (4.8), is $n$ times the matrix element for particle 1 the struck particle.

The free particle variables used in the above wave functions are not very convenient for carrying out the integrations. Choosing particle 1 as the struck particle, the remaining free particle labels can be coupled together to give a multiparticle with four-momentum $p_{n-1}$, mass $m_{n-1} := \sqrt{p_{n-1}\cdot p_{n-1}}$, spin $j_{n-1}$, spin component $\sigma_{n-1}$, and other internal invariant labels $I_{n-1}$ [10,12]. Then the state $|1\cdots n\rangle$ is written $|p_1\sigma_1, p_{n-1}j_{n-1}\sigma_{n-1}I_{n-1}\rangle$ and the matrix element, Eq. (4.8), becomes

$$F^b_{r'r} = n\sum_{\substack{j_{n-1}\\ \sigma,\sigma'_1}} \int \frac{d^3p_1}{2E_1}\frac{d^3p'_1}{2E'_1}\, d^4p_{n-1}d\mu(I_{n-1})\psi^*_{p'(\mathrm{st})j'r'} \times(p'_1\sigma'_1, p_{n-1}j_{n-1}\sigma_{n-1}I_{n-1})\langle p'_1\sigma'_1|J^b_1(0)|p_1\sigma_1\rangle \times \psi_{p(\mathrm{st})jr}(p_1\sigma_1, p_{n-1}j_{n-1}\sigma_{n-1}I_{n-1}), \tag{4.9}$$

where $d\mu(I_{n-1})$ is the measure for the internal variables of the multiparticle and depends on the choice of variables. Examples for $n=2$ and 3 will be given subsequently.

The ‘‘multiparticle’’ can be coupled to particle 1 to give the velocity state $|v\mathbf{k}\mu_1 m_{n-1}j_{n-1}\mu_{n-1}I_{n-1}\rangle$ for both initial and final states. For the Bakamjian-Thomas construction of the four-momentum operator, the velocity label then becomes the velocity of the initial or final state, $v_{\mathrm{in}}$ or $v_f$, and the matrix element of Eq. (4.9) takes the simpler form

$$F^b_{r'r} = n\sum_{\substack{j_{n-1}\\ \mu_1\mu'_1\mu_{n-1}\mu'_{n-1}}} \int \frac{d\mathbf{k}d\mathbf{k}'}{\omega_1\omega_{n-1}\omega'_1}\, dm_{n-1}d\mu(I_{n-1})\psi^*_{m'j'r'}(\mathbf{k}'\mu'_1\mu'_{n-1}j_{n-1}I_{n-1})\,\delta^3[k'-B^{-1}(v_f)B(v_{\mathrm{in}})k] \times\langle p'_1\sigma'_1|J^b_1(0)|p_1\sigma_1\rangle D^{1/2*}_{\mu'_1\sigma'_1}\{R_w[k'_1,B(v_f)]\}D^{j_{n-1}}_{\mu'_{n-1}\mu_{n-1}}\{R_w[k,B^{-1}(v_f)B(v_{\mathrm{in}})]\}D^{1/2}_{\sigma_1\mu_1}\{R_w[k_1,B(v_{\mathrm{in}})]\} \times\psi_{mjr}(\mathbf{k}\mu_1\mu_{n-1}j_{n-1}I_{n-1}), \tag{4.10}$$

where $p_{n-1}=B(v_{\mathrm{in}})k$, $k=(\sqrt{m^2_{n-1}+\mathbf{k}^2},\mathbf{k})$, $p_1=B(v_{\mathrm{in}})k_1$, $k_1=(\sqrt{m^2_1+\mathbf{k}^2},-\mathbf{k})$, with similar notation for the primed variables. Now from Eqs. (A6) and (A7)

$$p(\mathrm{st}) = \begin{pmatrix}\sqrt{m^2+p_z^2}\\0\\0\\p_z\end{pmatrix} = m\begin{pmatrix}\cosh\alpha\\0\\0\\\sinh\alpha\end{pmatrix},$$

$$\sinh\alpha = \frac{p_z}{m} = \frac{m^2-q^2-m'^2}{2qm},$$

$$p'(\mathrm{st}) = \begin{pmatrix}\sqrt{m'^2+p'^2_z}\\0\\0\\p'_z\end{pmatrix} = m'\begin{pmatrix}\cosh\alpha'\\0\\0\\\sinh\alpha'\end{pmatrix},$$

$$\sinh\alpha' = \frac{p'_z}{m'} = \frac{m^2+q^2-m'^2}{2qm'}. \tag{4.11}$$

Since, in the standard frame, $v_f$ and $v_{\mathrm{in}}$ have no $x$ or $y$ components,

$$B^{-1}(v_f)B(v_{\mathrm{in}}) = \begin{pmatrix}\cosh\Delta & 0 & 0 & \sinh\Delta\\ 0&1&0&0\\ 0&0&1&0\\ \sinh\Delta & 0&0&\cosh\Delta\end{pmatrix}, \tag{4.12}$$

where

$$\Delta = \alpha-\alpha' = \sinh^{-1}\left(\frac{m^2-q^2-m'^2}{2qm}\right) - \sinh^{-1}\left(\frac{m^2+q^2-m'^2}{2qm}\right)$$

depends only on the invariant masses and momentum transfer. The delta function in Eq. (4.10) can then be written as

$$\sqrt{m^2_{n-1}+k'^2} = (\cosh\Delta)\sqrt{m^2_{n-1}+k^2}+(\sinh\Delta)k\cos\theta,$$

$$k'\sin\theta'\cos\varphi' = k\sin\theta\cos\varphi,$$

$$k'\sin\theta'\sin\varphi' = k\sin\theta\sin\varphi,$$

$$k'\cos\theta' = (\sinh\Delta)\sqrt{m^2_{n-1}+k^2}+(\cosh\Delta)k\cos\theta. \tag{4.13}$$

Set $\sinh\gamma' := (k'/m_{n-1})$, $\sinh\gamma = (k/m_{n-1})$, $\gamma,\gamma' \geqslant 0$. Then Eq. (4.13) becomes

$$\cosh\gamma' = \cosh\Delta\cosh\gamma + \sinh\Delta\sinh\gamma\cos\theta,$$

$$\tan\varphi' = \tan\varphi,$$

$$\sinh\gamma'\cos\theta' = \sinh\Delta\cosh\gamma + \cosh\Delta\sinh\gamma\cos\theta. \tag{4.14}$$

With the change of variables $\mathbf{k}\rightarrow|\mathbf{k}|\hat{k}\rightarrow\gamma$, $\cos\theta,\varphi$, the integration can be split into wave function and current matrix element integrations. When $\theta=0$ or $\pi$,

$$\cosh\gamma' = \cosh\Delta\cosh\gamma \pm \sinh\Delta\sinh\gamma,$$

$$\gamma' = \gamma+\Delta, \theta=0,$$

$$\gamma' = \gamma-\Delta, \theta=\pi, \gamma\geqslant\Delta,$$

$$\gamma' = -\gamma+\Delta, \theta=\pi, \gamma<\Delta, \tag{4.15}$$

and the integration over the independent variables $\gamma,\gamma'$ is between $\gamma'=\gamma-\Delta$ and $\gamma'=\gamma+\Delta$. Further Eq. (4.14) can be used to solve for the angles:

$$\cos\theta = \frac{\cosh\gamma' - \cosh\Delta\cosh\gamma}{\sinh\Delta\sinh\gamma}, \quad \Delta\neq 0$$

$$\cos\theta' = \frac{\cosh\gamma - \cosh\Delta\cosh\gamma'}{\sinh\Delta\sinh\gamma'},$$

$$\varphi' = \varphi, \tag{4.16}$$

so that finally the form factor integral becomes

$$F^b_{r'r} = n \sum_{\substack{j_{n-1} \\ l' m_l l m_l \\ \mu_1\mu_1'\mu_{n-1}'\mu_{n-1}}} \int m_{n-1}^2 dm_{n-1} d\mu(I_{n-1}) \int \frac{\sinh^2\gamma d\gamma}{\sqrt{(m_1/m_{n-1})^2+\sinh^2\gamma}} \frac{\sinh^2\gamma' d\gamma'}{\sqrt{(m_1/m_{n-1})^2+\sinh^2\gamma'}}$$

$$\times \psi^*_{m'j'r'}(\gamma' l' m_l' \mu_1' \mu_{n-1}' j_{n-1} m_{n-1} I_{n-1}) J^b(\gamma'\gamma m_{n-1} j_{n-1}\Delta\mu_1'\mu_1\mu_{n-1}'\mu_{n-1}) \psi_{mjr}(\gamma l m_l \mu_1 \mu_{n-1} j_{n-1} m_{n-1} I_{n-1}), \tag{4.17}$$

where

$$J^b(\gamma'\gamma m_{n-1} j_{n-1}\Delta\mu_1'\mu_1\mu_{n-1}'\mu_{n-1}) := P_{l'm_l'}(\cos\theta') P_{lm_l}(\cos\theta) \int d\varphi e^{i(m_l'-m_l)\varphi} D^{j_{n-1}}_{\mu_{n-1}'\mu_{n-1}}\{R_w[k,B^{-1}(v_f)B(v_{\rm in})]\}$$

$$\times \sum_{\sigma_1'\sigma_1} D^{1/2}_{\mu_1'\sigma_1'}\{R_w^{-1}[k_1',B(v_f)]\}\langle p_1'\sigma_1'|J_1^b(0)|p_1\sigma_1\rangle D^{1/2}_{\sigma_1\mu_1} R_w[k_1,B(v_{\rm in})]. \tag{4.18}$$

$l$ is the relative orbital angular momentum of the $n$-particle bound state; $m_l$ is the projection. The form factor calculation has been split into a part involving only the internal $n-1$ variables of the unstruck particles, and a part linking the struck particle with the current matrix element. These latter variables include the mass of the collective unstruck particles $m_{n-1}$ and $\gamma,\gamma'$, related to the energies of the struck particle. The sums over the remaining discrete variables are limited by their coupling to $j$ and $j'$, the initial and final angular momenta. Examples will be given in the following paragraphs. The arguments of the Legendre polynomials, $P_{lm_l}(\cos\theta)$ and $P_{l'm_l'}(\cos\theta')$ are given in Eq. (4.16).

The integration over $\gamma'$ and $\gamma$ can be further simplified by setting $\gamma_\pm = \gamma' \pm \gamma$; then

$$\int d\gamma' d\gamma = \frac{1}{2}\int_\Delta^\infty d\gamma_+ \int_{-\Delta}^{+\Delta} d\gamma_-. \tag{4.19}$$

Finally, the arguments of the Wigner $D$ functions must be evaluated. Now the argument of the $D^{j_{n-1}}$ function in Eq. (4.18) is the Wigner rotation

$$R_w[k,B^{-1}(v_f)B(v_{\rm in})] = B^{-1}(k')B^{-1}(v_f)B(v_{\rm in})B(k), \tag{4.20}$$

which, after a somewhat tedious calculation, can be written as the SU(2) element

$$\begin{pmatrix} u & v \\ -v^*, & u^* \end{pmatrix}, |u|^2+|v|^2=1:$$

$$u=\frac{(\omega_{n-1}+m_{n-1})\cosh(\Delta/2)+k\cos\theta\sinh(\Delta/2)}{\sqrt{(\omega_{n-1}+m_{n-1})(\omega'_{n-1}+m_{n-1})}}=\frac{(1+\cosh\gamma)\cosh(\Delta/2)+\sinh\gamma\cos\theta\sin(\Delta/2)}{\sqrt{(1+\cosh\gamma)(1+\cosh\gamma')}}$$
$$=\sqrt{\frac{1+\cosh\gamma}{1+\cosh\gamma'}}\cosh\frac{\Delta}{2}+\frac{\cosh\gamma'-\cosh\Delta\cosh\gamma}{\sinh\Delta}\sinh\frac{\Delta}{2},$$
$$v=\frac{k\sin\theta e^{-i\varphi}\sinh(\Delta/2)}{\sqrt{(\omega_{n-1}+m_{n-1})(\omega'_{n-1}+m_{n-1})}}=\frac{e^{-i\varphi}}{2\cosh(\Delta/2)}\sqrt{\frac{1-\cosh^2\gamma-\cosh^2\Delta-\cosh^2\gamma'+2\cosh\gamma'\cosh\Delta\cosh\gamma}{(1+\cosh\gamma)(1+\cosh\gamma')}}, \tag{4.21}$$

which gives $u$ and $v$ as functions of $\gamma$, $\gamma'$, $\Delta$ and $\varphi$ only.

The remaining two Wigner $D^{1/2}$ functions can be absorbed into the struck particle current matrix element. To see this consider the term multiplying the Dirac spinor $u(p,\sigma)$:

$$\sum_{\sigma_1} u_\alpha(p_1\sigma_1)D^{1/2}_{\sigma_1\mu_1}\{R_w[k_1,B(v_{\rm in})]\}=\sum_{\sigma_1} S_{\alpha\sigma_1}[B(p_1)]S_{\sigma_1\mu_1}[B^{-1}(p_1)B(v_{\rm in})B(k_1)]$$
$$=S_{\alpha\mu_1}[B(v_{\rm in})B(k_1)]=\sum_\beta S_{\alpha\beta}[B(v_{\rm in})]u_\beta(k_1\mu_1); \tag{4.22}$$

here use has been made of the intertwining properties of matrix elements $S(\Lambda)$ of the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group with respect to the Dirac spinors $u(p,\sigma)$.

The relevant expression in Eq. (4.18) then becomes

$$\sum_{\sigma'_1\sigma_1} D^{1/2}_{\mu'_1\sigma'_1}\{R_w^{-1}[k'_1,B(v_f)]\}\langle p'_1\sigma'_1|J_1^b(0)|p_1\sigma_1\rangle D^{1/2}_{\sigma_1\mu_1}\{R_w[k_1,B(v_{\rm in})]\}$$
$$=D^{1/2}_{\mu'_1\sigma'_1}[B^{-1}(k'_1)B^{-1}(v_f)B(p'_1)]\overline{u}(p'_1\sigma'_1)[\gamma^b(F_1+2mF_2)-(p_1^b+p_1'^b)F_2]u(p_1\sigma_1)D^{1/2}_{\sigma_1\mu_1}[B^{-1}(p_1)B(v_{\rm in})B(k_1)]$$
$$=\overline{u}(k'_1\mu'_1)S[B^{-1}(v_f)][\gamma^b(F_1+2m_1F_2)-(p_1^b+p_1'^b)F_2]S[B(v_{\rm in})]u(k_1\mu_1)=B(v_f)^b{}_\nu\overline{u}(k'_1\mu'_1)\{\gamma^\nu(F_1+2m_1F_2)$$
$$-[k'_1+B^{-1}(v_f)B(v_{\rm in})k_1]^\nu F_2\}S[B^{-1}(v_f)B(v_{\rm in})]u(k_1\mu_1). \tag{4.23}$$

$B^{-1}(v_f)B(v_{\rm in})$ is given in Eq. (4.12) and is a pure $z$ axis Lorentz boost depending on $m$, $m'$, and $Q^2$ invariants. The $z$ axis Lorentz transformation $B(v_f)^b{}_\nu$ also depends only on invariants and can be taken outside the form factor integral. Thus the entire integral depends on momentum transfer only through the variable $\Delta$, defined in Eq. (4.12).

To conclude this section we give two examples that are of particular interest in form factor calculations, namely two-body and three-body bound states. For a two-body wave function $m_{n-1}=m_2$, the mass of the unstruck particle, and $j_{n-1}=j_2$, the spin of the unstruck particle. There are no additional $I_{n-1}$ variables. The wave function is

$$\psi_{mjr}(\gamma l m_l \mu_1\mu_2),$$

and if the spins of the two particles are coupled to $l$, the relative orbital angular momentum, to give $j$, the spin of the compound, then $l$ is limited by Clebsch-Gordan coefficients.

For three-particle bound states with particle 1 the struck particle, particles 2 and 3 can be coupled to give a multiparticle described by variables $p_{23}$, $j_{23}$, $\sigma_{23}$, $l_{23}$, and $s_{23}$, where $l_{23}$ and $s_{23}$ are the orbital and spin angular momentum of the 2-3 system. (The justification for being able to do these stepwise couplings for relativistic systems is given in a following paper, Ref. [10].) Then the wave function variables are given by

$$\psi_{mjr}(\gamma l m_l \mu_1\mu_{23}m_{23}j_{23}l_{23}s_{23}),$$

and the sums over $l_{23}$ and $s_{23}$ in Eq. (4.17) are independent of the current matrix kernel, Eq. (4.18).

## V. CONCLUSION

Though the point form of relativistic quantum mechanics may seem further removed from nonrelativistic quantum mechanics than the instant form, there are a number of ways in which it resembles nonrelativistic quantum mechanics; as shown in Sec. II, multiparticle velocity states have the property that under Lorentz transformations, the internal momenta and spins transform like their nonrelativistic counterparts. In this case the overall four velocity of the state replaces the total momentum nonrelativistically, and in contrast to the instant form, the boosts which generate the velocity states are entirely kinematic. Further, unlike the instant form, the point form Hamiltonian is additive in the mass operators, meaning that various decompositions of the total Hamiltonian needed in the interaction picture of time evolution are readily carried out. Also unlike both the instant and front forms, the point form is manifestly covariant under arbitrary Lorentz transformations.

Where the point form differs most strikingly from nonrelativistic quantum mechanics is in the momentum operator, which carries interactions, and thus is not equal to the free momentum operator. But in the limit when $c \rightarrow \infty$ and the Poincaré group contracts to the Galilei group, the interacting part of the momentum operator goes to zero leaving only the free momentum operator.

The main goal of this paper has been to examine properties of electromagnetic current operators in the point form. The hadronic electromagnetic current operator $J^{\mu}(x)$ must satisfy a number of constraints due to Poincaré covariance and current conservation. By defining a new current operator $J_b(Q)$ which transforms as an irreducible tensor operator under the Poincaré group, the constraints on $J^{\mu}(x)$ are automatically satisfied. Moreover, in contrast to the difficulties encountered with the front form in dealing with timelike and lightlike momentum transfers, the point form operator $J_b(Q)$ deals with all these cases on an equal footing. Only the spacelike case has been discussed in this paper because the spacelike representations are the least well known of the Poincaré group representations, and it is spacelike momentum transfers that are of most interest in applications. Utilizing the tensor transformation properties of $J_b(Q)$ allows one to give a group theoretical derivation of form factors, as seen in Eq. (3.20); when supplemented with parity and time reversal invariance, the electric and magnetic form factors are directly related to current matrix elements in Eq. (3.31). Relativistic invariance including parity also gives the correct number of independent form factors, with no relations between form factors, as is the case with front form calculations [4].

Moreover all current matrix elements have the same general form. All the spin information for the initial and final states resides in the two Wigner $D$ functions; nevertheless, as shown in Appendix B, under arbitrary Lorentz transformations the expression for the current matrix element, Eq. (3.31), is generally covariant. The distinction between electric and magnetic form factors is not the usual one in that the four-vector multiplying the electric form factors [$w_0^{\mu}$ in Eq. (3.31)] is orthogonal to the two four-vectors [$w_1^{\mu}, w_2^{\mu}$ in Eq. (3.31)] multiplying the magnetic form factors. As seen in example 1 following Eq. (3.31), for current matrix elements of spin-$\frac{1}{2}$ systems, the electric and magnetic form factors are linear combinations of the usual Dirac and Pauli form factors.

Form factors have been defined as reduced matrix elements in the Poincaré group Wigner-Eckhard theorem. But an examination of the Poincaré group Clebsch-Gordan coefficients [see Eq. (3.19)] shows that these reduced matrix elements are the current matrix elements in a standard frame, namely, for $Q^2$ spacelike, that frame where $Q(\mathrm{st})$ has only a $z$ component. Since all Lorentz transformations are kinematic, magnetic current matrix elements in any other frame connected by a Lorentz transformation along the $z$ direction to the standard frame, do not change. That is, the magnetic form factors are the same in any frame connected to the standard frame by a $z$-axis Lorentz transformation. The electric form factors, however, differ from those of the standard frame by a cosh factor arising from the Lorentz transformation. Therefore, form factors, defined as current matrix elements with final and initial momenta along the $z$ axis only, are very simply related to the form factors defined in this paper. In particular, form factors in the rest frame of one of the particles differs by a factor depending on invariants only, from our standard form factors.

Although canonical spin boosts were chosen in Sec. II [Eq. (2.7)] to define velocity states, it is straightforward to generalize the definition of velocity states to include any boost. This generalization is given in the appendix of Ref. [10]. In particular if helicity boosts are used in computing the Wigner rotations occurring in the expression for the current matrix elements, the invariant form factors then coincide with the so-called helicity form factors [15].

Our second main result concerns the existence of a point form relativistic impulse approximation. If $J_b(Q)$ is written as the sum of one-body and many-body operators, where the one-body operators are the current operators of the underlying constituents, we have shown that hadronic form factors can be written entirely in terms of the form factors of the underlying constituents; nevertheless, there must be many-body currents as demonstrated in Ref. [16]. But as seen in Eq. (4.7) there are no effects of the many-body current matrix elements on the hadronic form factors, while still preserving Poincaré covariance and current conservation.

The invariant form factors are integrals over bound state wave functions in the standard frame and the struck constituent current matrix elements. We have shown how to choose wave function variables in which the $(n-1)$ unstruck constituents are coupled together to form a multiparticle, which is then coupled to the struck constituent. Using these variables the form factor integral can be written in such a way that all of the momentum transfer information [given through the variable $\Delta$, see Eq. (4.12)] resides in the limits of integration or in the matrix element of the struck particle. By splitting the wave function variables into ''struck'' and ''unstruck'' variables it should be possible to separate the effects of wave function and struck constituent current matrix elements on the momentum transfer dependence of the invariant form factors. These variables are being used to calculate deuteron form factors for both elastic and quasielastic channels.

## APPENDIX A: CLEBSCH-GORDAN COEFFICIENTS FOR SPACELIKE MOMENTUM TRANSFER

Because the current operator $J_b(Q)$ [Eq. (3.4)] transforms as an irreducible tensor operator under the (interacting) Poincaré group, the matrix elements of the current operator can be reduced to Clebsch-Gordan coefficients times a reduced matrix element. The goal of this appendix is to compute the relevant Clebsch-Gordan coefficients for $Q^2$ spacelike, namely, $\langle p'j'\sigma'|Qbr'r;pj\sigma\rangle$; here $|pj\sigma\rangle$ is a timelike state with four momentum $p$, spin $j$, and spin projection $\sigma$, with transformation properties under the Poincaré group given by Eq. (2.2).

$|Qb\rangle$ is a spacelike state with momentum transfer $Q^2<0$; its Lorentz transformation properties are given in Eq. (3.9). For calculating Clebsch-Gordan coefficients it is convenient to switch from ''Cartesian'' coordinates, $b=0,1,2$, to spherical coordinates $\nu=0,\pm 1$. Then under Poincaré transformations $|Q\nu\rangle$ transforms as

$$U_\Lambda|Q\nu\rangle=\sum_{\nu'=0,\pm1} D_{\nu'\nu}(\tilde{R}_w)|\Lambda Q,\nu'\rangle,$$
$$U_a|Q\nu\rangle=e^{-iQ\cdot a}|Q,\nu\rangle, \tag{A1}$$

where $\tilde{R}_w\in \mathrm{SO}(1,2)$ is the Wigner ''rotation'' defined in Eq. (3.8). $D_{\nu'\nu}(\tilde{R}_w)$ is $\tilde{R}_w$ written in spherical coordinates.

To get the Clebsch-Gordan coefficients coupling a spacelike four momentum $Q$ with ''spin'' 1 to a timelike four-momentum $p$ with spin $j$ resulting in a timelike four momentum $p'$ with spin $j'$, it is actually more convenient to couple $p'$ with spin $j'$ to a negative energy $\overline{p}=-p$ with spin $j$ to get a spacelike $Q$; this is done by writing

$$|Q\nu r'r\rangle := U_{B(Q)}\int_{\mathrm{SO}(1,2)} d\tilde{R} \times\tilde{g}_{\nu\nu}D_{\nu\overline{\nu}}(\tilde{R})U_{\tilde{R}}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle, \tag{A2}$$

where $p'(\mathrm{st})$ and $\overline{p}(\mathrm{st})$ are standard four-vectors to be specified, $\tilde{g}_{\nu\nu}$ is the SO(1,2) metric defined in Eq. (3.6), and $r'r$ are degeneracy parameters that are related to the independent invariant form factors. Then

$$\begin{aligned} U_\Lambda|Q\nu r'r\rangle &= U_\Lambda U_{B(Q)}\int_{\mathrm{SO}(1,2)} d\tilde{R}\tilde{g}_{\nu\nu}D_{\nu\overline{\nu}}(\tilde{R})U_{\tilde{R}}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle \\ &= U_{B(\Lambda Q)}\int_{\mathrm{SO}(1,2)} d\tilde{R}\tilde{g}_{\nu\nu}D_{\nu\overline{\nu}}(\tilde{R})U_{\tilde{R}_w\tilde{R}}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle \\ &= U_{B(\Lambda Q)}\int_{\mathrm{SO}(1,2)} d\tilde{R}\tilde{g}_{\nu\nu}D_{\nu\overline{\nu}}(\tilde{R}_w^{-1}\tilde{R})U_{\tilde{R}}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle=\sum_{\nu'} D_{\nu'\nu}(\tilde{R}_w)|\Lambda Q,\nu'r'r\rangle, \end{aligned} \tag{A3}$$

so that the state defined in Eq. (A2) transforms correctly under Lorentz transformations as required by Eq. (A1). In going from the second to the third line of Eq. (A3) the invariance of the Haar measure of SO(1,2) was used.

Further,

$$\begin{aligned} U_a|Q\nu r'r\rangle &= U_{B(Q)}\int_{\mathrm{SO}(1,2)} d\tilde{R}\tilde{g}_{\nu\nu}D_{\nu\overline{\nu}}(\tilde{R})U_{\tilde{R}}U_{\tilde{R}^{-1}B^{-1}(Q)a}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle \\ &= U_{B(Q)}\int_{\mathrm{SO}(1,2)} d\tilde{R}\tilde{g}_{\nu\nu}D_{\nu\overline{\nu}}(\tilde{R})e^{-iQ(\mathrm{st})\cdot\tilde{R}^{-1}B^{-1}(Q)a}U_{\tilde{R}}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle, \end{aligned} \tag{A4}$$

where $Q(\mathrm{st})$ is defined to be

$$Q(\mathrm{st}):=p'(\mathrm{st})+\overline{p}(\mathrm{st})=p'(\mathrm{st})-p(\mathrm{st}). \tag{A5}$$

If $Q(\mathrm{st})$ is chosen (following Yennie *et al.* [13]) to have the form $(0,0,0,q)$, with $Q^2=-q^2<0$, $q>0$, then

$$Q(\mathrm{st})\cdot\tilde{R}^{-1}B^{-1}(Q)a=Q\cdot a,$$

as required for the correct space-time translation properties given in Eq. (A1).

Set

$$p'(\mathrm{st})=[E'(\mathrm{st}),0,0,p_z']=(\sqrt{m'^2+p_z},0,0,p_z'),$$

$$p(\mathrm{st})=[E(\mathrm{st}),0,0,p_z]=(\sqrt{m^2+p_z},0,0,p_z),\quad E'(\mathrm{st})=E(\mathrm{st}), \tag{A6}$$

with $m'$ and $m$ the masses of the four vectors $p'(\mathrm{st})$ and $p(\mathrm{st})$, respectively. Substituting the choices for $p'(\mathrm{st})$ and $p(\mathrm{st})$ given in Eq. (A6) into Eq. (A5) gives

$$p_z'=\frac{m^2+q^2-m'^2}{2q},$$

$$p_z=\frac{m^2-q^2-m'^2}{2q}. \tag{A7}$$

Notice that if $m'=m$, the expressions for $p_z'$ and $p_z$ reduce to the Yennie *et al.* result [their Eq. (A-7)] [13].

Since $p'(\mathrm{st})$ and $p(\mathrm{st})$ have zero $x$ and $y$ components, they remain unchanged under a rotation about the $z(=3)$ axis or the $t(=0)$ axis. If SO(1,2) is decomposed into cosets with respect to a rotation about the $t$ axis, the Haar measure $d\tilde{R}$ can be written as $d\tilde{R}_c d\varphi$, where $\tilde{R}_c$ is some choice of coset representative of SO(1,2) with respect to SO(2). Then

$$|Q\nu r'r\rangle = U_{B(Q)}\int_{SO(1,2)/SO(2)} d\tilde{R}_c \int_{SO(2)} d\varphi \tilde{g}_{\nu\nu} D_{\nu\bar{\nu}}(\tilde{R}_c)$$
$$\times e^{i\bar{\nu}\varphi} U_{\tilde{R}_c} U_{\varphi}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle$$
$$= U_{B(Q)}\int_{SO(1,2)/SO(2)} d\tilde{R}_c \tilde{g}_{\nu\nu} D_{\nu,r'-r}(\tilde{R}_c)$$
$$\times U_{\tilde{R}_c}|p'(\mathrm{st})j'r'\rangle|\overline{p}(\mathrm{st})jr\rangle, \quad \text{(A8)}$$

where use has been made of the fact that the Wigner rotation of $p'(\mathrm{st})$ with $R_z(\varphi)$ is just $R_z(\varphi)$. Thus, $\bar{\nu}=r'-r$ follows from integrating the resulting exponentials over $\varphi$.

Finally, the operations $B(Q)\tilde{R}_c$ on the tensor product states can be carried out. Now $B(Q)\tilde{R}_c$ is a coset representative of SO(1,3) with respect to SO(2) and is related to $p$, $p'$, and $Q$ by

$$p = B(Q)\tilde{R}_c p(\mathrm{st}),$$
$$p' = B(Q)\tilde{R}_c p'(\mathrm{st}),$$
$$Q = B(Q)\tilde{R}_c Q(\mathrm{st}) = B(Q)Q(\mathrm{st}). \quad \text{(A9)}$$

Carrying out the action of $B(Q)$ and $\tilde{R}_c$ in Eq. (A8) then gives the desired Clebsch-Gordan coefficients:

$$\langle p'j'\sigma'|Q\nu,r'r;pj\sigma\rangle$$
$$= \langle p'j'\sigma';\overline{pj}\sigma|Q\nu,r'r\rangle$$
$$= \delta^4(p'-p-Q)\tilde{g}_{\nu\nu}D_{\nu,r'-r}(\tilde{R}_c)$$
$$\times D^{j'}_{\sigma'r'}(R'_w)D^{j*}_{\sigma r}(R_w). \quad \text{(A10)}$$

There are many different ways of choosing coset representatives $\Lambda(p,Q)$ of SO(1,3) with respect to SO(2). Instead of writing $\Lambda(p,Q)=B(Q)\tilde{R}_c$, which emphasizes the spacelike momentum transfer $Q$, it is more convenient to write

$$\Lambda(p,Q) = B(w_0)R, \quad \text{(A11)}$$

which emphasizes the timelike four-vector $\tilde{m}w_0^{\mu}=p^{\mu}-(p\cdot Q)/(Q^2)Q^{\mu}$, introduced in Eq. (3.27), and satisfying $w_0^2=1$, $w_0\cdot Q=0$. Then

$$p = \Lambda(p,Q)p(\mathrm{st}) = B(w_0)Rp(\mathrm{st}),$$
$$p' = \Lambda(p,Q)p'(\mathrm{st}) = B(w_0)Rp'(\mathrm{st}),$$
$$Q = \Lambda(p,Q)Q(\mathrm{st}) = B(w_0)RQ(\mathrm{st}),$$
$$w_0 = \Lambda(p,Q)e_0 = B(w_0)e_0, \quad \text{(A12)}$$

where $e_0=(1,0,0,0)$. If $B(w_0)$ is chosen to be a canonical boost,

$$B_c(w) = \begin{pmatrix} w^{\mu} & \mathbf{w}^T \\ & I+\dfrac{\mathbf{w}\otimes\mathbf{w}^T}{1+w^0} \end{pmatrix}, \quad \text{(A13)}$$

then the coset representative $\Lambda(p,Q)=(w_0,w_1,w_2,Q/q)$ can be given explicitly. In particular,

$$\frac{Q}{q} = B(w_0)Re_3,$$
$$\frac{Q^0}{q} = \mathbf{w}_0\cdot R\hat{z},$$
$$\frac{\mathbf{Q}}{q} = R\hat{z} + \frac{\mathbf{w}_0}{1+w_0^0}\,\mathbf{w}_0\cdot R\hat{z};$$
$$R\hat{z} := R_z(\varphi)R_y(\theta)\hat{z} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \frac{\mathbf{Q}}{q} - \frac{\mathbf{w}_0}{1+w_0^0}\frac{Q^0}{q} \quad \text{(A14)}$$

fixes the polar and azimuthal angles of $R$ in terms of $Q$ and $w_0$ (or equivalently, in terms of $p$ and $p'$).

Then

$$w_1 := B_c(w_0)Re_1,$$
$$w_1^0 = \mathbf{w}_0\cdot R\hat{x},$$
$$\mathbf{w}_1 = R\hat{x} + \frac{\mathbf{w}_0}{1+w_0^0}\,w_1^0 \quad \text{(A15)}$$

and

$$w_2 := B_c(w_0)Re_2,$$
$$w_2^0 = \mathbf{w}_0\cdot R\hat{y},$$
$$\mathbf{w}_2 = R\hat{y} + \frac{\mathbf{w}_0}{1+w_0^0}\,w_2^0. \quad \text{(A16)}$$

$R_w := R_w[p(\mathrm{st}),\Lambda(p,Q)] = B^{-1}(p)\Lambda(p,Q)B[p(\mathrm{st})]$, and $R'_w := R_w[p'(\mathrm{st}),\Lambda(p,Q)] = B^{-1}(p')\Lambda(p,Q)B[p'(\mathrm{st})]$. These Wigner rotations can be computed explicitly as functions of $v=p/m$, $v'=p'/m'$ by writing

$$R_wB^{-1}[p(\mathrm{st})] = B^{-1}(p)\Lambda(p,Q),$$
$$R_wB^{-1}[p(\mathrm{st})]p'(\mathrm{st}) = B^{-1}(p)\Lambda(p,Q)p'(\mathrm{st}) = B^{-1}(p)p'. \quad \text{(A17)}$$

Now the $x$ and $y$ components of $B^{-1}[p(\mathrm{st})]p(\mathrm{st})$ are zero, so that $R_w$ is specified by the unit vector extracted from $B^{-1}(p)p'$. For canonical spin boosts

$$B_c^{-1}(p)p' = m'\begin{pmatrix} v_0 & -\mathbf{v}^T \\ -\mathbf{v} & I+\dfrac{\mathbf{v}\otimes\mathbf{v}^T}{1+v_0} \end{pmatrix}\begin{pmatrix} v'_0 \\ \mathbf{v}' \end{pmatrix}$$
$$= m'\begin{bmatrix} v\cdot v' \\ \mathbf{v}'+\mathbf{v}\left(\dfrac{\mathbf{v}\cdot\mathbf{v}'}{1+v_0}-v'_0\right) \end{bmatrix}. \quad \text{(A18)}$$

The vector magnitude of $B_c^{-1}(p)p'$ is $m'\sqrt{(v\cdot v')^2-1}$, so that the two angles in the Wigner rotation can be specified by the unit vector

$$\hat{n}_c(\mathbf{v},\mathbf{v}')=\frac{\mathbf{v}'+\mathbf{v}[\mathbf{v}\cdot\mathbf{v}'/(1+v_0)-v_0']}{\sqrt{(v\cdot v')^2-1}}. \tag{A19}$$

For $R_w'$ it is simply necessary to interchange $v$ and $v'$. Note that Eq. (A17) already shows that $R_w$ and $R_w'$ do not depend on the choice of the coset representative $\Lambda(p,Q)$.

For helicity boosts, with

$$B_H(v)=R(\hat{v})\Lambda_z(|\mathbf{v}|)=(v,u_1,u_2,u_3),$$

$$v=\begin{pmatrix}v_0\\ \mathbf{v}\end{pmatrix}=\begin{pmatrix}v_0\\ |\mathbf{v}|\hat{v}\end{pmatrix},\quad u_1=\begin{pmatrix}0\\ \cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix},$$

$$u_2=\begin{pmatrix}0\\ -\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix},\quad u_3=\begin{pmatrix}|\mathbf{v}|\\ v_0\hat{v}\end{pmatrix},\quad \hat{v}=\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}, \tag{A20}$$

the Wigner rotations are obtained from

$$B_H^{-1}(v)p'=m'gB^T(v)gv'=m'g\begin{pmatrix}v\cdot v'\\ u_1\cdot v'\\ u_2\cdot v'\\ u_3\cdot v'\end{pmatrix}=m'\begin{pmatrix}v\cdot v'\\ -u_1\cdot v'\\ -u_2\cdot v'\\ -u_3\cdot v'\end{pmatrix}; \tag{A21}$$

then the Wigner rotation for a helicity boost is given by

$$\hat{n}_H(\mathbf{v},\mathbf{v}')=-\begin{pmatrix}u_1\cdot v'\\ u_2\cdot v'\\ u_3\cdot v'\end{pmatrix}\Big/\sqrt{(v\cdot v')^2-1}=\begin{pmatrix}\mathbf{u}_1\cdot\mathbf{v}'\\ \mathbf{u}_2\cdot\mathbf{v}'\\ v_0\hat{v}\cdot\mathbf{v}'-|\mathbf{v}|v_0'\end{pmatrix}\Big/\sqrt{(v\cdot v')^2-1}. \tag{A22}$$

## APPENDIX B: COVARIANCE OF THE CURRENT MATRIX ELEMENT REPRESENTATION UNDER LORENTZ TRANSFORMATIONS

To demonstrate the covariance of the representation of the current matrix element [Eq. (3.26)] under Lorentz transformations use will be made of the Lorentz transformation properties of states, Eq. (2.2), and the transformation of current operators, Eq. (3.3b). Inserting the identity operator $U_\Lambda^{-1}U_\Lambda$, $\Lambda\in \mathrm{SO}(1,3)$, into the current matrix element gives

$$\begin{aligned}
&\langle p'j'\sigma'|U_\Lambda^{-1}U_\Lambda J^\mu(0)U_\Lambda^{-1}U_\Lambda|pj\sigma\rangle\\
&=\Lambda^{-1\mu}{}_\nu D^{j'*}_{\overline{\sigma'}\sigma'}[R_w(p',\Lambda)]\langle\Lambda p'j'\overline{\sigma'}|J^\nu(0)|\Lambda p,j\bar{\sigma}\rangle D^j_{\bar{\sigma}\sigma}[R_w(p,\Lambda)]\\
&=\Lambda^{-1\mu}{}_\nu\sum_{b,r',r}\Lambda^\nu(\Lambda p,\Lambda Q)D^{j'}_{\sigma'\overline{\sigma'}}[R_w^{-1}(p',\Lambda)]D^{j'}_{\overline{\sigma'}r'}\{R_w[p'(\mathrm{st}),\Lambda(\Lambda p,\Lambda Q)]\}F^b_{r'r}\\
&\quad\times D^j_{r\bar{\sigma}}\{R_w^{-1}[p(\mathrm{st}),\Lambda(\Lambda p,\Lambda Q)]\}D^j_{\bar{\sigma}\sigma}[R_w(p,\Lambda)]\\
&=\sum_{b,r',r}\Lambda^{-1}\Lambda(\Lambda p,\Lambda Q)^\mu{}_b D^{j'}_{\sigma'r'}\{R_w^{-1}(p',\Lambda)R_w[p'(\mathrm{st}),\Lambda(\Lambda p,\Lambda Q)]\}F^b_{r'r}D^j_{r\sigma}\{R_w^{-1}[p(\mathrm{st}),\Lambda(\Lambda p,\Lambda Q)]R_w(p,\Lambda)\}.
\end{aligned} \tag{B1}$$

We must show that the last line of Eq. (B1) is equal to $\langle p'j'\sigma'|J^\mu(0)|pj\sigma\rangle$.

Now $\Lambda(p,Q)$ is a coset representative of SO(1,3) with respect to SO(2); that means for any element $\Lambda\in$ SO(1,3) one can write

$$\Lambda=\Lambda(p,Q)R_z,\quad R_z\in \mathrm{SO}(2). \tag{B2}$$

$\Lambda(p,Q)$ is defined by the relations

$$p=\Lambda(p,Q)p(\mathrm{st}), \tag{B3}$$

$$Q=\Lambda(p,Q)Q(\mathrm{st}).$$

Once the coset representative $\Lambda(p,Q)$ has been determined, $R_z=\Lambda^{-1}(p,Q)\Lambda$. Since $\Lambda\Lambda(p,Q)$ is also an element of SO(1,3), it too can be decomposed into cosets, as in Eq. (B2), but with different arguments for the coset representative, say $\bar{p}$ and $\bar{Q}$. Then

$$\Lambda\Lambda(p,Q)=\Lambda(\bar{p},\bar{Q})R_z,$$

$$\Lambda\Lambda(p,Q)p(\mathrm{st})=\Lambda(\bar{p},\bar{Q})R_zp(\mathrm{st}),$$

$$\Lambda p=\overline{p},$$
$$\Lambda\Lambda(p,Q)Q(\mathrm{st})=\Lambda(\overline{p},\overline{Q})R_zQ(\mathrm{st}),$$
$$\Lambda Q=\overline{Q}; \tag{B4}$$

that is $\overline{p}=\Lambda p$, $\overline{Q}=\Lambda Q$, so that for any Lorentz transformation $\Lambda$,

$$\Lambda\Lambda(p,Q)=\Lambda(\Lambda p,\Lambda Q)R_z, \tag{B5}$$

and as with Wigner rotations, $R_z=\Lambda^{-1}(\Lambda p,\Lambda Q)\Lambda\Lambda(p,Q)$, but this will not be needed in the following discussion.

Using these results the Wigner rotations that appear in the $D$ functions in Eq. (B1) can be written as

$$\begin{aligned}R_w^{-1}&\{[p(\mathrm{st}),\Lambda(\Lambda p,\Lambda Q)]R_w(p,\Lambda)\}\\&=B^{-1}[p(\mathrm{st})]\Lambda^{-1}(\Lambda p,\Lambda Q)B(\Lambda p)B^{-1}(\Lambda p)\Lambda B(p)\\&=B^{-1}[p(\mathrm{st})]R_z\Lambda^{-1}(p,Q)B(p)\\&=R_zR_w^{-1}[p(\mathrm{st}),\Lambda(p,Q)];\end{aligned}$$

$$D^j_{r\sigma}\{R_w^{-1}[p(\mathrm{st}),\Lambda(\Lambda p,\Lambda Q)]R_w(p,\Lambda)\}=e^{-ir\varphi}D^j_{r\sigma}\{R_w^{-1}[p(\mathrm{st}),\Lambda(p,Q)]\}. \tag{B6}$$

A similar result holds for $D^{j'}_{\sigma' r'}()$ in Eq. (B1). Substituting Eqs. (B6) and (B5) into Eq. (B1), then gives $\langle p'j'\sigma'|J^\mu(0)|pj\sigma\rangle$, which was to be shown.

The current matrix element representation is also covariant with respect to parity. Parity is the operation that takes **x** to $-$**x**, leaving the time component unchanged. As a matrix it is equal to the metric $g=\mathrm{diag}(1,-1,-1,-1)$. The action of parity, $\mathcal{P}$, on timelike states is given by

$$\begin{aligned}U_{\mathcal{P}}|pj\sigma\rangle&=U_{\mathcal{P}}U_{B(p)}|p(\mathrm{rest})j\sigma\rangle=U_{\mathcal{P}B(p)\mathcal{P}}U\mathcal{P}|p(\mathrm{rest})j\sigma\rangle\\&=\eta U_{B(\mathcal{P}p)}U_{R_{\mathcal{P}}(p)}|p(\mathrm{rest})j\sigma\rangle\\&=\eta\sum_{\overline{\sigma}}|\mathcal{P}p,j\overline{\sigma}\rangle D^j_{\overline{\sigma}\sigma}[R_{\mathcal{P}}(p)],\end{aligned} \tag{B7}$$

where use has been made of the fact that $\mathcal{P}B(p)\mathcal{P}$ is a proper Lorentz transformation, so it can be decomposed into cosets with respect to SO(3):

$$R_{\mathcal{P}}(p):=B^{-1}(\mathcal{P}p)\mathcal{P}B(p)\mathcal{P}, \tag{B8}$$

with $\eta$ the intrinsic parity. Similarly for the current operator

$$U_{\mathcal{P}}J^0(0)U_{\mathcal{P}}^{-1}=J^0(0),$$
$$U_{\mathcal{P}}J^i(0)U_p^{-1}=-J^i(0),\quad i=1,2,3. \tag{B9}$$

Now since covariance has been demonstrated for arbitrary continuous Lorentz transformations, it suffices to investigate the effects of parity on standard states, which can then be boosted to arbitrary states. Following Yennie *et al.* it is more convenient to use $\mathcal{P}_2:=\mathcal{P}R_y(\pi)$, which only changes the 2 component, for then $\mathcal{P}_2p(\mathrm{st})=p(\mathrm{st})$. It follows that

$$\begin{aligned}U_{\mathcal{P}_2}|p(\mathrm{st})jr\rangle&=U_{R_y(\pi)}U_{\mathcal{P}}|p(\mathrm{st})jr\rangle\\&=\eta U_{R_y(\pi)}|\mathcal{P}p(\mathrm{st})j\overline{r}\rangle D^j_{\overline{r}r}\{R_{\mathcal{P}}[p(\mathrm{st})]\}\\&=\eta|p(\mathrm{st})j\bar{\bar{r}}\rangle D^j_{\bar{\bar{r}}r}\{R_w[\mathcal{P}p(\mathrm{st}),R_y(\pi)]R_{\mathcal{P}}[p(\mathrm{st})]\}.\end{aligned} \tag{B10}$$

But

$$\begin{aligned}R_w&[\mathcal{P}p(\mathrm{st}),R_y(\pi)]R_{\mathcal{P}}[p(\mathrm{st})]\\&=B^{-1}[p(\mathrm{st})]R_y(\pi)B[\mathcal{P}p(\mathrm{st})]B^{-1}[\mathcal{P}p(\mathrm{st})]\mathcal{P}B[p(\mathrm{st})]\mathcal{P}\\&=B^{-1}[p(\mathrm{st})]\mathcal{P}_2B[p(\mathrm{st})]\mathcal{P}_2R_y(\pi)=R_y(\pi).\end{aligned} \tag{B11}$$

Therefore,

$$\begin{aligned}U_{\mathcal{P}_2}|p(\mathrm{st})jr\rangle&=\eta|p(\mathrm{st})j\overline{r}\rangle D^j_{\overline{r}r}[R_y(\pi)]\\&=\eta|p(\mathrm{st})j,-r\rangle(-1)^{j-r},\end{aligned} \tag{B12}$$

for all boosts, including canonical and helicity boosts, that have pure $z$ axis Lorentz transformations as elements.

Then

$$\begin{aligned}F^b_{r'r}&=\langle p'(\mathrm{st})j'r'|J^b(0)|p(\mathrm{st})jr\rangle\\&=\langle p'(\mathrm{st})j'r'|U_{\mathcal{P}_2}U_{\mathcal{P}_2}J^b(0)U_{\mathcal{P}_2}U_{\mathcal{P}_2}|p(\mathrm{st})jr\rangle\\&=\eta\eta'(-1)^{j'-r'}(-1)^{j-r}\\&\quad\times\langle p'(\mathrm{st})j',\ -r'|\mathcal{P}_2J^b(0)|p(\mathrm{st})j,-r\rangle.\end{aligned} \tag{B13}$$

For the electric form factors $b=0$, $r'=r$ and

$$F^0_r=\eta'\eta(-1)^{j'+r}F^0_{-r}(-1)^{j+r},$$
$$F^0_{-r}=\eta'\eta(-1)^{j'+j+2r}F^0_r. \tag{B14}$$

If the initial and final particles are the same, then $\eta'=\eta$, $j'=j$, and $F^0_{-r}=F^0_r$ for any spin $j$. Further, time reversal invariance means that $F^0_r$ is real [13].

For the magnetic form factors it is most convenient to switch back to spherical coordinates [see Eq. (3.25)], for then

$$\begin{aligned}F^+_r&=\langle p'(\mathrm{st})j'r'|J_x+iJ_y|p(\mathrm{st})jr\rangle\\&=\langle p'(\mathrm{st})j'r'|U_{\mathcal{P}_2}U_{\mathcal{P}_2}(J_x+iJ_y)U_{\mathcal{P}_2}U_{\mathcal{P}_2}|p(\mathrm{st})jr\rangle\\&=(-1)^{j'+r'}\eta'\eta\langle p'(\mathrm{st})j_1-r'|J_x+iJ_y|p(\mathrm{st})j,-r\rangle\\&\quad\times(-1)^{j+r}\\&=\eta'\eta(-1)^{j'+j+2r+1}F^-_{-r},\quad r'=r+1,\end{aligned}$$

$$F^{b=1}_{r'r}=\frac{1}{2}[F^+_r+(-1)^{j+j'+2r+1}\eta'\eta F^+_{-r}],\quad r'=r+1,$$

$$F^{b=2}_{r'r}=\frac{1}{2i}[F^+_r-(-1)^{j+j'+2r+1}\eta'\eta F^+_{-r}]. \tag{B15}$$

Time reversal invariance now implies that the $F^+_r$ are pure imaginary, which when written as $iB_+(r)$, give

$$F^{b=1}_{r'r}=\frac{i}{2}\,[B_+(r)+(-1)^{j+j'+2r+1}B_+(-r)],$$

$$F^{b=2}_{r'r}=\frac{1}{2}\,[B_+(r)-(-1)^{j+j'+2r+1}B_+(-r)],\quad r'=r+1. \tag{B16}$$

Again if the initial and final particles are the same, $\eta'=\eta$, $j'=j$, and $(-1)^{2j+2r+1}=-1$, in agreement with Yennie *et al.* [their Eq. (A-12b)].

Finally if the initial and final particles are the same, the hermiticity of the current operator can be used to get further relations on the invariant form factors:

$$F^b_{r'r}=\langle p'(\mathrm{st})jr'|J^b(0)|p(\mathrm{st})jr\rangle$$
$$=\langle p(\mathrm{st})jr|J^b(0)|p'(\mathrm{st})jr'\rangle^*=F^{b*}_{rr'},\quad b=1,2. \tag{B17}$$

Note that the momenta of $p(\mathrm{st})$ and $p'(\mathrm{st})$ are equal and opposite for $m'=m$ [Eq. (A7)]. A Lorentz transformation that takes $p(\mathrm{st})=(\sqrt{m^2+p_z},0,0,-p_z)$ to $(\sqrt{m^2+p_z},0,0,+p_z)$, means, using the covariance proved earlier in this appendix, that $F^b_{r'r}$ is Hermitian in $r'$ and $r$.

## APPENDIX C: POINCARÉ COVARIANCE OF THE FOUR-MOMENTUM OPERATOR

In this appendix we find conditions under which the total four-momentum operator given in Eq. (3.1), $P^\mu=P^\mu_{\mathrm{st}}+P^\mu_\gamma+P^\mu_{\mathrm{em}}$, the sum of the strong, photon, and electromagnetic four-momentum operators, satisfies the Poincaré conditions given in Eq. (2.14), namely,

$$[P^\mu,P^\nu]=0,$$
$$U_\Lambda P^\mu U_\Lambda^{-1}=\Lambda^{-1\mu}{}_\nu P^\nu.$$

The proof that $[P^\mu,P^\nu]=0$ is carried out in several steps. Since by assumption

$$[P^\mu_{\mathrm{st}}+P^\mu_\gamma,P^\nu_{\mathrm{st}}+P^\nu_\gamma]=0, \tag{C1}$$

it suffices to show that $[P^\mu_{\mathrm{em}},P^\nu_{\mathrm{em}}]=0$ and $[P^\mu_{\mathrm{st}}+P^\mu_\gamma,P^\nu_{\mathrm{em}}]+[P^\mu_{\mathrm{em}},P^\nu_{\mathrm{st}}+P^\nu_\gamma]=0$. Consider first the commutator

$$[P^\mu_{\mathrm{em}},P^\nu_{\mathrm{em}}]$$
$$=\int d^4x d^4y\,\delta(x\cdot x-\tau^2)\,\delta(y\cdot y-\tau^2)\,\theta(x^0)\,\theta(y^0)x^\mu y^\nu$$
$$\times[J^\alpha(x)A_\alpha(x),J^\beta(y)A_\beta(y)]. \tag{C2}$$

Now the integration is over timelike $x$ and $y$, with $(x-y)^2=2\tau^2-2x\cdot y<0$, so that if $J^\alpha(x)$ is local, the commutator in Eq. (C2) is zero.

Further, under space-time translations, $U_a=e^{-i(P_{\mathrm{st}}+P_\gamma)\cdot a}$, we require that

$$U_aJ^\mu(x)U_a^{-1}=J^\mu(x+a),$$
$$U_aA^\mu(x)U_a^{-1}=A^\mu(x+a). \tag{C3}$$

Then

$$U_aP^\mu_{\mathrm{em}}U_a^{-1}=\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)x^\mu U_aJ^\alpha(x)A_\alpha(x)U_a^{-1}=\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)x^\mu J^\alpha(x+a)A_\alpha(x+a),$$

$$[P^\mu_{\mathrm{st}}+P^\mu_\gamma,P^\nu_{\mathrm{em}}]=\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)x^\nu\,\frac{\partial}{\partial x_\mu}\,J^\alpha(x)A_\alpha(x),$$

$$[P^\mu_{\mathrm{st}}+P^\mu_\gamma,P^\nu_{\mathrm{em}}]+[P^\mu_{\mathrm{em}},P^\nu_{\mathrm{st}}+P^\nu_\gamma]=[P^\mu_{\mathrm{st}}+P^\mu_\gamma,P^\nu_{\mathrm{em}}]-[P^\nu_{\mathrm{st}}+P^\nu_\gamma,P^\mu_{\mathrm{em}}]$$
$$=\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)\left(x^\nu\,\frac{\partial}{\partial x_\mu}-x^\mu\,\frac{\partial}{\partial x_\nu}\right)J^\alpha(x)A_\alpha(x)=0. \tag{C4}$$

The last line of Eq. (C4) follows from the fact that $J^\alpha(x)A_\alpha(x)$ is a scalar density, $U_\Lambda J^\alpha(x)A_\alpha(x)U_\Lambda^{-1}=J^\alpha(\Lambda x)A_\alpha(\Lambda x)$, for then

$$\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)U_\Lambda J^\alpha(x)A_\alpha(x)U_\Lambda^{-1}$$
$$=\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)J^\alpha(\Lambda x)A_\alpha(\Lambda x)$$
$$=\int d^4x\,\delta(x\cdot x-\tau^2)\,\theta(x^0)J^\alpha(x)A_\alpha(x). \tag{C5}$$

Since Eq. (C5) holds for all Lorentz transformations, for infinitesimal Lorentz transformations the integral in Eq. (C4) is zero.

Thus,

$$[P^\mu,P^\nu]=[P^\mu_{\mathrm{st}}+P^\mu_\gamma+P^\mu_{\mathrm{em}},P^\nu_{\mathrm{st}}+P^\nu_\gamma+P^\mu_{\mathrm{em}}]$$
$$=[P^\mu_{\mathrm{st}}+P^\mu_\gamma,P^\nu_{\mathrm{st}}+P^\nu_\gamma]+[P^\mu_{\mathrm{em}},P^\nu_{\mathrm{em}}]$$
$$+\{[P^\mu_{\mathrm{st}}+P^\mu_\gamma,P^\nu_{\mathrm{em}}]-[P^\nu_{\mathrm{st}}+P^\nu_\gamma,P^\mu_{\mathrm{em}}]\}$$
$$=0, \tag{C6}$$

since each of the three terms are separately zero. Finally,

$$U_\Lambda P^\mu_{\rm em} U^{-1}_\Lambda = \int d^4x\, \delta(x\cdot x - \tau^2)\, \theta(x^0) x^\mu J^\alpha(\Lambda x) A_\alpha(\Lambda x)$$
$$= \Lambda^{-1\mu}{}_\nu P^\nu_{\rm em}; \tag{C7}$$

from this it follows that $U_\Lambda P^\mu U^{-1}_\Lambda = \Lambda^{-1\mu}{}_\nu P^\nu$, and thus Eq. (2.14) is satisfied for $P^\mu$ defined in Eq. (3.2).

It remains to show that in the nonrelativistic limit, $\mathbf{P}_{\rm em} = 0$ and $P^0_{\rm em} = H_{\rm em}$ is the usual electromagnetic interaction. Group theoretically these limits correspond to contracting a representation of the Poincaré group to the Galilei group by letting $c$, the speed of light, go to infinity. Inserting factors of $c$ gives $x^0 = ct$ and

$$P^\mu_{\rm em} = \int d^4x\, \delta[x\cdot x - (c\tau)^2] x^\mu \theta(ct) J^\alpha(x) A_\alpha(x)$$
$$= \int \frac{d^3x}{\sqrt{(c\tau)^2 + \mathbf{x}\cdot\mathbf{x}}} \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}^\mu J^\alpha(x) A_\alpha(x);$$
$$\lim_{c\to\infty} \mathbf{P}_{\rm em} = \int \frac{d^3x}{c\tau} \mathbf{x} J^\alpha(x) A_\alpha(x) = 0$$
$$\lim_{c\to\infty} P^0_{\rm em} = \int \frac{d^3x}{c\tau} c\tau J^\alpha(x) A_\alpha(x) = \int d^3x J^\alpha(x) A_\alpha(x)$$
$$= H^{\rm nonrel}_{\rm em}. \tag{C8}$$


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