Fermion-boson interactions in a solvable model

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The interplay between fermionic and bosonic degrees of freedom, in the context of many-body systems and with reference to applications in nuclear and elementary particle physics, is revisited. Starting from the formulation due to Geyer, Hahne, and co-workers, bifermion operators which obey bosonic transformation rules are introduced. Fermion-boson coupling terms are included in the original Hamiltonian. Exact solutions to this Hamiltonian are compared to approximate ones. The occurrence of fermion and boson condensates is discussed. [S0556-2813(98)03411-6]

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I. INTRODUCTION

The study of nuclear many body systems, i.e. finite nuclei, its constituents and the interactions between them at the baryon-meson scale as well as at the quark scale, is currently the subject of theoretical efforts [1–3].

The starting point in the treatment of these systems is closely related to the identification of effective degrees of freedom [4]. The advantages posed by this approach are obvious, since it manages to handle in a simplified form the otherwise unaffordable complexity of the interactions between constituents.

The identification of elementary excitations in highly degenerate fermionic systems, like the atomic nucleus, leads to the description of effective bosonic excitations [5–7].

In practice, the effective boson degrees of freedom representing fermionic interactions can be labeled by their fermionic contents or by the number of fermions which are interacting with them at lowest order in their couplings [6].

Among the models which have been proposed to describe fermion-fermion and fermion-boson interactions as well as the transformations between pairs of fermions and bosons, the model of Geyer and Hahne [4] features a system where fermions can be mapped onto a boson representation which includes three different bosons. The model has been analyzed in a series of papers by Geyer and co-workers [8–14].

The basic elements of the model are the interactions among fermions and the transformation of pairs of fermions (bi-fermions) onto particle-hole bosons (phonons), particle-particle bosons (pairons) and hole-hole bosons (holons). The parameters of the model are the strengths of the interaction in the particle-hole, particle-particle, and hole-hole channels and the fermion unperturbed energies. The Hamiltonian of Ref. [4] has been solved by applying group theoretical methods [15] and boson mapping techniques [6]. In the following we shall comment briefly on these results.

The Dyson-Maleev boson expansion method, extended to transform the generators of the Sp(4) algebra, has been applied in the work of [4]. As shown in Ref. [4] physical excitations can be described by ideal bosons representing bi-fermion operators. In principle, bosons can be assigned to each class of bi-fermion operators (particle-hole, particle-particle, and hole-hole).

The relationship between the ideal boson mapping of bi-fermions and seniority representations, of the model of [4], was discussed in [8] in connection with the collectivity associated to the bosons. Coupling terms between bosons and fermions were not considered in [8].

Besides purely algebraic aspects of the model, physically interesting scenarios have been devised to accommodate bosons which can carry bi-fermion numbers. Representations of this sort have been used to treat quadrupole interactions [12], generalized Hamiltonians belonging to the SO(8) representation [9,10] and realizations of physical bosons [14]. The appearance of spurious states, which originates in the transformation of bi-fermions onto bosons, was studied in [11,13].

The interest in physical situations represented by a Hamiltonian which is linear in bi-fermion operators, like the Hamiltonian of [4], has been renewed by the use of these degrees of freedom in nonperturbative QCD [1–3,16,17]. In a fashion similar to the one of standard nuclear structure models [18], the authors of [2] have introduced diquarks which can evolve into a Bose-condensate of pairs. As a toy model it has similarities with the bi-fermion model of [4]. In Ref. [1] similar considerations are raised in the discussion of diquarks correlations. There, scalar (particle-hole) and vector (particle-particle) pairs of quarks are treated as the elementary degrees of freedom of the approximation. The physics of diquarks-condensates, as pointed out in [1,2,19], may be the link between QCD inspired models [16,17] and models based on nucleons and mesons. In this context the model of Geyer and Hahne may be very useful [16,17].

In the present work we have extended the original Hamiltonian of [4] to accommodate the couplings between bi-fermions and bosons, at lowest order in the interactions, and we have used the methods of [20] to construct algebraic solutions. The solutions obtained in this way can be characterized by the structure of different condensates which may result from the dominance of a given channel among the class of particle-hole, particle-particle, and hole-hole excitations. In this respect the already known solutions of the model due to Schütte and Da Providencia [21], which contains a single external boson interacting with particle-hole bi-fermion excitations, will be taken as the natural limit of the present model. To illustrate the utility of the algebraic approach we have performed the comparison between exact and approximate results which, as for the case of Ref. [21], have been obtained by using the random phase approximation (RPA) [5]. In presenting numerical results we have discussed the structure of the effective excitations in the space of bi-fermion and boson configurations in order to establish a correspondence between these configurations and the occurrence of fermion or boson condensates. In the present approach we shall also treat bi-fermion and fermion-boson configurations, which are the proper degrees of freedom in the formalism of Geyer and Hahne [4] and Schütte and Da Providencia [21], respectively. Concerning bi-fermions, we shall investigate the permanence of the fermion and boson condensates [21] in presence of fermion couplings as in the model of [4]. By extending the treatment of bi-fermion excitations, as in the model of Ref. [4], we shall investigate BCS type of solutions to the present Hamiltonian.

In the following section, Sec. II, we shall introduce the basic elements of the formalism by presenting the Hamiltonian and the supporting algebra. The applications of the formalism, mainly to determine the dominance of fermion or boson condensates, will be shown in Sec. III, where the comparison between exact (algebraic) and approximate (RPA) solutions of the Hamiltonian will be discussed. Conclusions are drawn in Sec. IV. Details of the algebra are presented in the Appendix.

II. FORMALISM

We have considered a system of N fermions moving in two levels (hereafter denoted by the subindexes 1 and 2). Each level has 2Ω substates which are labeled by the quantum number k. The energy-difference between the energy of the lower level (level 1) and the upper one (level 2) is fixed by the scale ω_f. The creation and annihilation operators of particles belonging to level 2 are denoted by $a^\dagger_{2k}$ and a_{2k}, respectively, while for holes in level 1 the creation and annihilation operators are denoted by $b^\dagger_{1k}$ and b_{1k}. The fermions are coupled to three different bosons. These three independent bosons are created by the operators $B^\dagger$, $B^\dagger_p$, and $B^\dagger_h$ and their energies are ω_b, ω_p, and ω_h, respectively. The Hamiltonian is defined by

$$H=H_0+H_{ph}+H_{pp}+H_{hh}, \tag{1}$$

where

$$H_0=H_{0f}+H_{0b},$$

$$H_{0f}=\frac{\omega_f}{2}(\nu+\overline{\nu}),$$

$$H_{0b}=\omega_b B^\dagger B+\omega_p B^\dagger_p B_p+\omega_h B^\dagger_h B_h, \tag{2}$$

are the terms defined in [4]. To this Hamiltonian we have added the couplings between fermions and bosons

$$H_{ph}=G_1(T_+B^\dagger+T_-B),$$

$$H_{pp}=G_2(L_+B^\dagger_h+L_-B_h),$$

$$H_{hh}=G_3(K_+B^\dagger_p+K_-B_p). \tag{3}$$

G_1, G_2, and G_3 are the strengths of the interactions in the particle-hole, particle-particle, and hole-hole channels, respectively. The operators

$$\nu=\sum_k a^\dagger_{2k}a_{2k}, \quad \overline{\nu}=\sum_k b^\dagger_{1k}b_{1k}, \tag{4}$$

are the particle- and hole-number operators, respectively, and the operators $T_\pm$, $L_\pm$, $K_\pm$, $S_\pm$, L_0, and K_0 are the generators of the algebra of the simplectic group Sp(4) [20]. In terms of bi-linear combinations of fermion operators these generators read

$$L_+=\sum_{k>0} a^\dagger_{2k}a^\dagger_{2-k}, \quad L_-=(L_+)^\dagger,$$

$$L_0=\frac{1}{2}(\nu-\Omega),$$

$$K_+=\sum_{k>0} b^\dagger_{1k}b^\dagger_{1-k}, \quad K_-=(K_+)^\dagger,$$

$$K_0=\frac{1}{2}(\overline{\nu}-\Omega),$$

$$T_+=\sum_k a^\dagger_{2k}b^\dagger_{1k}, \quad T_-=(T_+)^\dagger,$$

$$T_0=L_0+K_0,$$

$$S_+=\sum_k sg(k)a^\dagger_{2-k}b_{1k}, \quad S_-=(S_+)^\dagger,$$

$$S_0=L_0-K_0. \tag{5}$$

Each of the four subgroups, consisting of the operators T, L, K, and S, belongs to a representation of the SU(2) quasispin algebra. The commutation rules of the Sp(4) and SU(2) groups are listed in [15]. In the above equations we have used a notation which is slightly different from the one used in [4] but it will allow us to compare the present formalism with the one of Ref. [21], where fermions are interacting with a single external boson. The results of [21] are particularly useful, for such a comparison, because they exhibit a phase transition of the type of a Bose condensate. This condensate is described in [21] as a superposition of particle-hole pairs. This particle-hole condensate (fermion condensate) can evolve into a condensate of bosons if the energy ω_b is comparable to the energy spacing between fermions. We shall see, by performing a similar analysis, that these features persist for the case of Hamiltonian (1).

The Hilbert space of the model is defined by the vectors

$$|klmnpq\rangle=B_p^{+q}K_+^pB_h^{+n}L_+^mB^{+l}O_+^k|\phi\rangle. \tag{6}$$

The operator O_+ has the form

$$O_+=K_+(L_+T_-+S_+(2L_0-1))+(L_+S_--T_+(2L_0-1))(2K_0-1), \tag{7}$$

the values of k, m, and p are given by

$$0\leqslant k\leqslant\Omega, \quad l\geqslant 0, \quad 0\leqslant m\leqslant\Omega-k,$$

$$n\geqslant 0, \quad 0\leqslant p\leqslant\Omega-k, \quad q\geqslant 0. \tag{8}$$

$|\phi\rangle$ is a pure fermion state of minimum weight which obeys $L_-|\phi\rangle=0$, $K_-|\phi\rangle=0$, $L_0|\phi\rangle=-\frac{1}{2}\Omega|\phi\rangle$, and $K_0|\phi\rangle=-\frac{1}{2}\Omega|\phi\rangle$. Both the definition of the basis and the relationship between the exponents of the operators in Eq. (6) have been taken from the work of Hecht [20], where the elements of the algebra of the Sp(4) group and its representations are given. The reader is kindly referred to [20] for further details.

The Hamiltonian H of Eq. (1) commutes with the symmetry operators

$$P=B^{\dagger}B+B_p^{\dagger}B_p+B_h^{\dagger}B_h-\frac{1}{2}\,(\nu+\overline{\nu}), \tag{9}$$

and

$$R=B_h^{\dagger}B_h-B_p^{\dagger}B_p-\frac{1}{2}\,(\nu-\overline{\nu}). \tag{10}$$

At this point we shall mention that these operators can be used to label the states, in a way which is similar to the one of [21]. For a single external boson, the symmetry operator P reduces to $P=B^{\dagger}B-1/2(\nu+\overline{\nu})$ and the eigenstates are labeled by the symmetry number L [21]. When three external bosons are considered the eigenvalues of P and R are defined by

$$P|klmnpq\rangle=\alpha|klmnpq\rangle, \tag{11}$$

with $\alpha=l-k+n-m+q-p$ and

$$R|klmnpq\rangle=\beta|klmnpq\rangle, \tag{12}$$

with $\beta=n-m-(q-p)$. Therefore we can construct a basis of states labeled by α and β

$$|\Phi^{\alpha\beta}_{NLkm}\rangle=N^{\alpha\beta}_{NLkm}\times|k,k+L;m,m+(\alpha+\beta-L)/2;\;\Omega-\frac{N}{2}+m,\Omega-\frac{N}{2}+m+(\alpha-\beta-L)/2\rangle, \tag{13}$$

with the normalization constant $N^{\alpha\beta}_{NLkm}$

$$(N^{\alpha\beta}_{NLkm})^2=\frac{1}{(k+L)!\left(m+\frac{\alpha+\beta-L}{2}\right)!}\times\frac{1}{\left(\Omega-\frac{N}{2}+m+\frac{\alpha-\beta-L}{2}\right)!}\times\frac{(\Omega-k-m)!}{m!(\Omega-k)!}\,\frac{\left(\frac{N}{2}-k-m\right)!}{\left(\Omega-\frac{N}{2}+m\right)!(\Omega-k)!}\times\frac{(\Omega-k)!}{\Omega!k!}\,\frac{(2\Omega-k+2)!}{(2\Omega+2)!}\left(\frac{(\Omega-k+1)!}{(\Omega+1)!}\right)^3. \tag{14}$$

The states are specified by the following combinations of indexes: (i) $N\leqslant 2\Omega$, $0\leqslant k\leqslant N/2$, $0\leqslant m\leqslant N/2-k$, and (ii) $N>2\Omega$, $0\leqslant k\leqslant 2\Omega-N/2$, $N/2-\Omega\leqslant m\leqslant\Omega-k$, with $-\Omega\leqslant\alpha$, $-(N/2+\alpha)\leqslant\beta\leqslant(\alpha+2\Omega-N/2)$, and $-k\leqslant L\leqslant\min\{\alpha+\beta+2m,\alpha-\beta+2m+2\Omega-N\}$.

In this basis H has the nonvanishing matrix elements given in the Appendix. After diagonalization the eigenvectors $\Psi^{\alpha\beta}_n$ are written as linear combinations

$$|\Psi^{\alpha\beta}_n\rangle=\sum_{NLkm}C^{\alpha\beta}_{n,NLkm}|\Phi^{\alpha\beta}_{N,L,k,m}\rangle, \tag{15}$$

of the states of the basis Eq. (13), with eigenvalues $E^n_{\alpha\beta}$. As a consequence of the symmetries represented by P and R the spectrum of H can be ordered into bands with energies $E^n_{\alpha\beta}$. The structure of the minima can change, as a function of the adopted values of the coupling constants, since the actual values of α and β are affected by changes in G_i. A similar dependence of the ground state wave function is observed in the model of Ref. [21].

III. RESULTS AND DISCUSSION

In analogy with the numerical search of minima performed in Ref. [21] the dependence of the ground-state energy, for the system described by H [Eq. (1)] can be explored as a function of the symmetry numbers α and β and for different values of the scaled coupling constants $x_i(i=b,p,h)$. Following the notation of [21] these parameters are defined by $x_b=G_1\sqrt{2\Omega/\omega_f\omega_b}$, $x_p=G_2\sqrt{\Omega/\omega_f\omega_h}$, and $x_h=G_3\sqrt{\Omega/\omega_f\omega_p}$.

We shall consider two extreme cases for the ratios between the energies of the bosons, namely (a) $\omega_p=\omega_h=\omega_b$ and (b) $\omega_b>\omega_p=\omega_h$ or $\omega_b<\omega_p=\omega_h$. These values will be given in units of ω_f. By setting the couplings $x_p=x_h$, fixing the scale $\omega_f=1$ and varying x_b the evolution of the ground state energy, as a function of α and β, can be determined. Results corresponding to this search of minima are shown in Fig. 1 and they are equivalent to the results shown in [21]. Figure 1(a) shows the result of case (a) ($\omega_p=\omega_h=\omega_b$) for a purely boson condensate and the insets (b), (c), and (d) of Fig. 1 show cases where the structure of the condensate changes from a fermionic to a bosonic structure, a fact which is demonstrated by the position of the minima around $\alpha=0$ (the normal phase of [21]) and around $\alpha=\pm 4$ (the deformed phase of Ref. [21]). The normal phase is attained by values of the coupling constants $x_i<1$. The correlated ground-state corresponds to $\alpha=0$ (which has the meaning of the quantity L of [21]) and it shows the same symmetry as the unperturbed ground state.

For $x_i>1$ the absolute ground state corresponds to values of $\alpha>0$ (boson-condensate). The values of $E^0_{\alpha\beta}$, in this regime, are shown in Fig. 1(a). In this region the ground-state wave function is mainly given by the states $(B^{\dagger})^L\phi$, $(B_p^{\dagger})^{(\alpha-L)/2}\phi$ and $(B_h^{\dagger})^{(\alpha-L)/2}\phi$, where ϕ is the ground-state of the fermion sector in the absence of bosons.

For larger values of $\omega_{(b,p,h)}>\omega_f$ and for $x_i>1$ it exists a regime for which the absolute ground state yields values of $\alpha<0$. We shall call this regime the phase of fermionic condensation. Results corresponding to this phase are shown in Figs. 1(b), 1(c), and 1(d), as mentioned before. The corresponding wave function is dominated by one of the following configurations: $(T^{\dagger})^{-L}\phi$, $(K_+)^{-(\alpha-L)/2}\phi$ or

$(L_+)^{-(\alpha-L)/2}\phi$. For larger values of x_i the position of the minima indicates a boson condensate.

In these calculations the number of particle-particle pairs and hole-hole pairs in the ground state, at the minima, is the same, i.e., $\beta=0$.

The normal phase

As it is shown in the previous section, the Hamiltonian of Eq. (1) can be solved exactly and from the analysis of the solutions it becomes evident that the correlations described by them belong to different phases. In the so-called normal-phase the fermionic excitations and the external bosons are coupled and from these couplings one can define effective bosonic degrees of freedom. The RPA [5] is a suitable method to describe, at leading order, these effective degrees of freedom. In the following we shall compare the structure of the exact solution of the model with the one obtained by using the RPA method.

In analogy with [21], one can define three sets of RPA bosons, namely:

$$\Gamma^\dagger_{ph}=Xt_+-Yt_-+ZB^\dagger-WB,$$

$$\Gamma^\dagger_a=X_a l_+-Y_a k_-+Z_a B^\dagger_p-W_a B_h,$$

$$\Gamma^\dagger_r=X_r k_+-Y_r l_-+Z_r B^\dagger_h-W_r B_p, \qquad (16)$$

with $t_+=T_+/\sqrt{2\Omega}$, $l_+=L_+/\sqrt{\Omega}$ and $k_+=K_+/\sqrt{\Omega}$.

The phonon operator $\Gamma^\dagger_{ph}$ describes the particle-hole excitations of the system, $\Gamma^\dagger_a$ corresponds to the addition (particle-particle) mode and $\Gamma^\dagger_r$ corresponds to the removal (hole-hole) mode. The treatment of H [Eq. (1)] in the harmonic approximation leads to the commutators

$$[H,\Gamma^\dagger_{ph}]=\omega_{ph}\Gamma^\dagger_{ph}, \quad [H,\Gamma^\dagger_a]=\omega_a\Gamma^\dagger_a, \quad [H,\Gamma^\dagger_r]=\omega_r\Gamma^\dagger_r. \qquad (17)$$

The RPA eigenvalues ω_{ph}, ω_a, and ω_r are determined by solving dispersion relations and they are written as

$$\omega_{ph}=-\frac{1}{2}(\omega_b-\omega_f)\pm\frac{1}{2}(\omega_b+\omega_f)\sqrt{1-\left(\frac{2G_1\sqrt{2\Omega}}{(\omega_b+\omega_f)}\right)^2},$$

$$\omega_{a1}=-\frac{1}{2}(\omega_p-\omega_f)\pm\frac{1}{2}(\omega_p+\omega_f)\sqrt{1-\left(\frac{2G_2\sqrt{\Omega}}{(\omega_p+\omega_f)}\right)^2},$$

$$\omega_{a2}=-\frac{1}{2}(\omega_f-\omega_h)\pm\frac{1}{2}(\omega_h+\omega_f)\sqrt{1-\left(\frac{2G_2\sqrt{\Omega}}{(\omega_h+\omega_f)}\right)^2},$$

$$\omega_{r1}=-\frac{1}{2}(\omega_p-\omega_f)\pm\frac{1}{2}(\omega_p+\omega_f)\sqrt{1-\left(\frac{2G_3\sqrt{\Omega}}{(\omega_p+\omega_f)}\right)^2},$$

$$\omega_{r2}=-\frac{1}{2}(\omega_f-\omega_h)\pm\frac{1}{2}(\omega_h+\omega_f)\sqrt{1-\left(\frac{2G_3\sqrt{\Omega}}{(\omega_h+\omega_f)}\right)^2}. \qquad (18)$$

The comparison with the exact solutions shows that the RPA treatment yields good results for small values of the coupling constants. It also shows that the RPA becomes critical and that it breaks-down when the particle-particle (pp) and hole-hole (hh) channels are weighted by renormalized coupling constants of the order of unity. However, the particle-hole (ph) channel of the RPA solution also becomes critical and the comparison with the exact result gets worse beyond $x_b=0.75$. A compilation of RPA results, for the energy of the first excited state in the normal phase, is shown in Fig. 2. The first case, Fig. 2(a), shows results which coincide, naturally, with the ones of Ref. [21]. As seen from these results, the renormalization of the pp and hh channels does not have much influence upon the RPA energy but the critical behavior is dominated by the coupling in the ph channel [Fig. 2(b)]. The same is true for the RPA solutions obtained by the renormalization of the ph channel [Figs. 2(c) and 2(d)]. Exact and RPA results tend to differ for values of x_i larger than 0.75, however, the RPA does a good job in reproducing the exact solution for small values of x_i even in the presence of pp and hh channels. In conclusion, the addition of the two extra bosons does not affect the RPA results and it does not prevent the collapse of the first RPA eigenvalue.

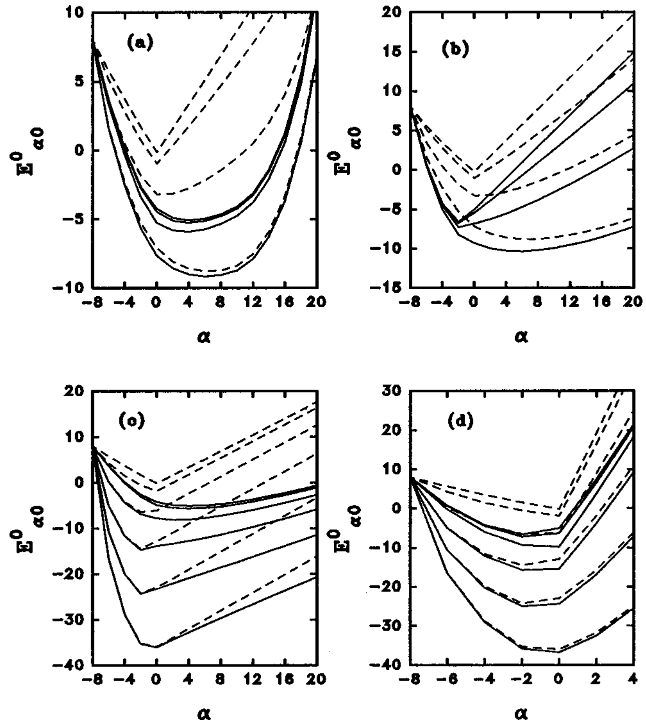

FIG. 1. (a) Exact ground-state energies $E^0_{\alpha\beta}$ for $\omega_f=\omega_b=\omega_p=\omega_h=1$ and $N=8$, $\Omega=4$. The reduced couplings are fixed at the values $x_b=0.5$, and $x_p=x_h=0$, 1, 2, and 3 (long-dashed lines, from top to bottom) and $x_b=2$ and $x_p=x_h=0$, 1, 2, and 3 (solid lines, from top to bottom). (b) Exact ground state energy with the same parametrization of (a) but with $\omega_b=10$. (c) Exact ground-state energies for $\omega_f=\omega_b=1$ and $\omega_p=\omega_h=10$. The curves correspond to $x_b=0.5$, and $x_p=x_h=0$, 1, 2, 3, 4, and 5 (long-dashed lines, from top to bottom) and $x_b=2$ and $x_p=x_h=0$, 1, 2, 3, 4, and 5 (solid lines, from top to bottom). (d) Same as (c) for $\omega_b=10$.

The deformed phase

To describe the deformed phase we can introduce the quasiparticle transformation

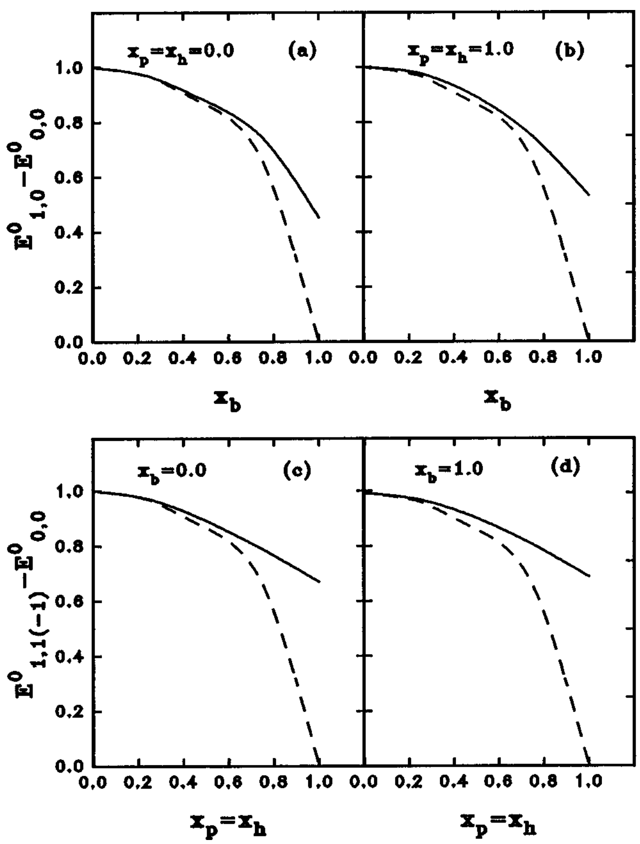

FIG. 2. Comparison between the RPA (long-dashed lines) and the exact solution (solid lines) for the excitations of the system in the normal phase. The RPA values of the particle-hole and particle-particle (hole-hole) correlated energies are compared with the exact results $E^0_{1,0}-E^0_{0,0}$ [(a) and (b)] and $E^0_{1,\pm1}-E^0_{0,0}$ [(c) and (d)], respectively. The parameters of the Hamiltonian are defined in the caption to Fig. 1(a). The actual values of the reduced coupling constants are indicated in each inset of the figure.

$$\begin{pmatrix} \alpha^\dagger_{1m} \\ \alpha^\dagger_{2m} \\ \alpha_{1-m} \\ \alpha_{2-m} \end{pmatrix} = \begin{pmatrix} u^*_{11} & u^*_{12} & v_{11} & v_{12} \\ u^*_{21} & u^*_{22} & v_{21} & v_{22} \\ -v^*_{11} & -v^*_{12} & u_{11} & u_{12} \\ -v^*_{21} & -v^*_{22} & u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} a^\dagger_{1m} \\ a^\dagger_{2m} \\ a_{1-m} \\ a_{2-m} \end{pmatrix} \tag{19}$$

for fermions and shift the boson operators

$$\beta^\dagger = B^\dagger - \sqrt{2\Omega}\, b_0, \quad \beta^\dagger_h = B^\dagger_h - \sqrt{\Omega}\, b_{0h},$$
$$\beta^\dagger_p = B^\dagger_p - \sqrt{\Omega}\, b_{0p}. \tag{20}$$

The amplitudes u and v of Eq. (19) are determined from the anticommutation and normalization conditions applied to the fermion operators α_{km} and $\alpha^\dagger_{km}$.

The quasiparticle energy-spectrum E_i is determined by the commutator

$$[H,\alpha^\dagger_i] = E_i \alpha^\dagger_i, \tag{21}$$

which leads to the matrix equation

$$\begin{pmatrix} \epsilon_1 & g_1 & -g_3 & 0 \\ g^*_1 & \epsilon_2 & 0 & g^*_2 \\ -g^*_3 & 0 & -\epsilon_1 & -g^*_1 \\ 0 & g_2 & -g_1 & -\epsilon_2 \end{pmatrix} \begin{pmatrix} u_{i1} \\ u_{i2} \\ v_{i1} \\ v_{i2} \end{pmatrix} = E_i \begin{pmatrix} u_{i1} \\ u_{i2} \\ v_{i1} \\ v_{i2} \end{pmatrix} \tag{22}$$

with $g_1 = G_1 b_0 \sqrt{2\Omega}$, $g_2 = G_2 b_{0h}\sqrt{\Omega}$ and $g_3 = G_3 b_{0p}\sqrt{\Omega}$.

The RPA commutators, for shifted boson operators, yield the additional conditions

$$[H,\beta^\dagger] = \omega_b \beta^\dagger \rightarrow \omega_b b_0 + g_1(v_{11}v^*_{12} + v_{21}v^*_{22}) = 0,$$
$$[H,\beta^\dagger_h] = \omega_h \beta^\dagger_h \rightarrow \omega_h b_{0h} - g_2(u^*_{12}v^*_{12} + u^*_{22}v^*_{22}) = 0,$$
$$[H,\beta^\dagger_p] = \omega_p \beta^\dagger_p \rightarrow \omega_p b_{0p} + g_3(u^*_{21}v^*_{21} + u^*_{11}v^*_{11}) = 0. \tag{23}$$

The Hamiltonian [Eq. (1)] can be written in terms of these quasiparticle and boson operators by using the definitions

$$n_1 = \sum_m \alpha^\dagger_{1m}\alpha_{1m}, \quad n_2 = \sum_m \alpha^\dagger_{2m}\alpha_{2m},$$
$$\mathcal{L}_+ = \sum_{m>0} \alpha^\dagger_{2m}\alpha^\dagger_{2-m}, \quad \mathcal{K}_+ = \sum_{m>0} \alpha^\dagger_{1m}\alpha^\dagger_{1-m},$$
$$\mathcal{T}_+ = \sum_{m>0} (\alpha^\dagger_{2m}\alpha^\dagger_{1-m} + \alpha^\dagger_{1m}\alpha^\dagger_{2-m}),$$
$$\mathcal{S}_+ = \sum_{m>0} (\alpha^\dagger_{2m}\alpha_{1m} + \alpha^\dagger_{2-m}\alpha_{1-m}). \tag{24}$$

Next, the generators (5) are transformed accordingly. After a cumbersome but otherwise straightforward calculation one gets for H the expression

$$H' = H - \lambda N,$$
$$= H_{00} + H_{11} + H_{20} + H_{res},$$
$$H_{res} = H_{40} + H_{22} + H_{31}. \tag{25}$$

The notation used to indicate the number of creation and annihilation quasiparticle operators appearing at each term is the standard one. The explicit expression of each term of Eq. (25) is rather lengthy and it will be omitted, for convenience. The conditions $H_{20}=0$ and $\langle N\rangle = 2\Omega(|v_{11}|^2 + |v_{12}|^2 + |v_{21}|^2 + |v_{22}|^2)$ determine the values of b_0, b_{0h}, b_{0p}, and λ.

In terms of these parameters the expectation values of P and R are given by

$$\langle P\rangle = \Omega(2|b_0|^2 + |b_{0h}|^2 + |b_{0p}|^2 - 1 - |v_{12}|^2 - |v_{22}|^2 + |v_{11}|^2 + |v_{21}|^2)$$
$$\langle R\rangle = \Omega(|b_{0h}|^2 - |b_{0p}|^2 + 1 - |v_{11}|^2 - |v_{21}|^2 - |v_{12}|^2 - |v_{22}|^2). \tag{26}$$

As in the case of the normal regime we shall use the RPA formalism to describe the excited states of the system. The RPA Hamiltonian is given by the terms

$$H_{RPA}=H_{11}+H_{22}+H_{40} \tag{27}$$

and the RPA bosons are defined by

$$\begin{aligned}\Gamma^{\dagger}=&Xt_{+}-Yt_{-}+Z\beta^{\dagger}-W\beta\\&+X_{p}l_{+}-Y_{p}l_{-}+Z_{p}\beta_{p}^{\dagger}-W_{p}\beta_{p}\\&+X_{h}k_{+}-Y_{h}k_{-}+Z_{h}\beta_{h}^{\dagger}-W_{h}\beta_{h}\\&+X_{1}s_{+}-Y_{1}s_{-}\end{aligned} \tag{28}$$

with

$$t_{+}=\mathcal{T}_{+}/\sqrt{2\Omega},\quad s_{+}=\mathcal{S}_{+}/\sqrt{2\Omega},$$

$$l_{+}=\mathcal{L}_{+}/\sqrt{\Omega},\quad k_{+}=\mathcal{K}_{+}/\sqrt{\Omega}. \tag{29}$$

The amplitudes *X, Y, Z,* and *W* are obtained from the RPA commutator

$$[H_{RPA},\Gamma^{\dagger}]=\omega\Gamma^{\dagger}. \tag{30}$$

As an example of the ability of the RPA to reproduce exact results in the deformed region we show, in Fig. 3, the fragmentation of the first RPA-energy induced by increasing values of the particle-particle and hole-hole couplings. From the results shown in this figure one can also see that the spreading of the RPA eigenvalues is very small compared with the exact one.

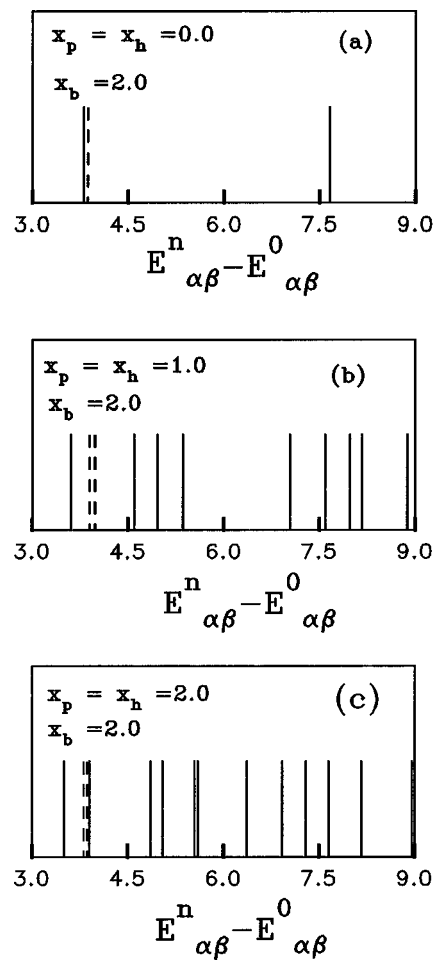

FIG. 3. Distribution of exact (solid lines) and RPA (long-dashed) energies in the deformed phase of the system. The values used for the reduced couplings x_i are given in each inset. The parameters of *H* are given in the caption to Fig. 1(a).

Residual bifermion interactions

As said above, the present model is a natural extension of the fermion-boson coupling model of [21]. The study of [21] shows that a condensate, either of fermionic or bosonic type, can be obtained depending upon the values of the coupling constants and for a given symmetry. The permanence of this feature upon the inclusion of bi-fermion interactions may be relevant for the study of the mechanism leading to fermionic condensates in hadronic physics [1–3]. In the context of the present discussion this question can be answered by adding bi-fermion interactions to the fermion-boson coupling Hamiltonian. For the sake of convenience we shall limit the discussion to the case of fermions interacting with a particle-hole like boson, as in the model of [21]. To the corresponding Hamiltonian, which is a limit of Hamiltonian (1) for only one type of bosons, we have added a bi-fermion interaction of the Lipkin type [6], with an arbitrary coupling *V*,

$$H=-V(T_{+}^{2}+T_{-}^{2}). \tag{31}$$

Then, we have searched for the minimum of the energy, as a function of the bi-fermion coupling. The results of this search are shown in Fig. 4. It is evident from the results shown in this figure that the inclusion of bi-fermion interactions can change the structure of the minima and produce the breaking of a given symmetry (*L*-value), either of the bosonic or fermionic type. The general trend shown by the curves of Fig. 4 is consistent with the breakdown of the boson condensate and with the appearance of a fermion condensate for larger values of the bi-fermion couplings.

The present model has several points of contact with the bi-fermion model of Geyer and Hahne [4]. To illustrate this aspect of the present formalism we shall show, in the following, the interplay between bi-fermion excitations and particle-particle (hole-hole) type of bosons. To show it we shall perform first a mean field treatment of the model Hamiltonian (1) enlarged to include bi-fermion interactions. The result of such a treatment will be expressed in terms quasiparticle degrees of freedom. Then, we shall construct the excitation spectrum by applying the random phase approximation to the remaining part of the Hamiltonian.

The bi-fermion interactions introduced in [4] can be written

$$H_{bi-f}=-\lambda L_{+}L_{-}-\kappa K_{+}K_{-}, \tag{32}$$

in the notation of Sec. II. We have added these terms to the Hamiltonian (1) and applied the transformations given by Eqs. (19) and (20) of Sec. III. The transformed Hamiltonian can be written

$$H=H_{00}+H_{11}+H_{22}+H_{40}, \tag{33}$$

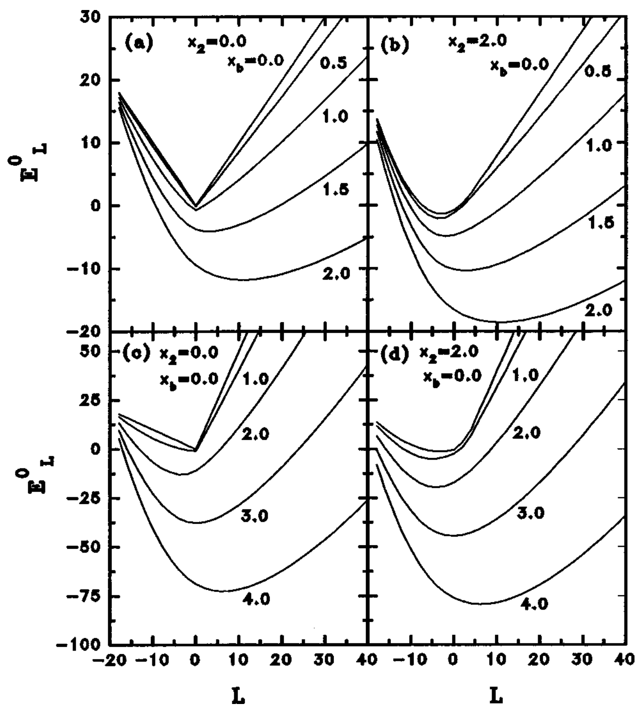

FIG. 4. Ground state energy E_L^0 as a function of the symmetry number L for different values of the adimensional coupling constants x_b (fermion-boson coupling) and $x_2=4\Omega V/\omega_f$ (bifermion coupling). Insets (a) and (b) show results for $\omega_f=\omega_b=1$ and $N=20$. Insets (c) and (d) correspond to $\omega_f=1$, $\omega_b=5$, for the same number of particles.

where

$$\begin{aligned}
H_{00}&=-2\Omega\epsilon_1+2\Omega(\epsilon_1 v_{11}^2+\epsilon_2 v_{22}^2)\\
&\quad-\Omega\,\frac{\Delta_2^2}{g_2^2/\omega_h+\lambda\Omega}-\Omega\,\frac{\Delta_1^2}{g_3^2/\omega_p+\kappa\Omega},\\
H_{11}&=\omega_p\beta_p^\dagger\beta_p+\omega_h\beta_h^\dagger\beta_h+E_1n_1+E_2n_2,\\
H_{22}&=-g_2v_{22}^2(l_+\beta_h+\beta_h^\dagger l_-)-g_3u_{11}^2(k_+\beta_p+\beta_p^\dagger k_-)\\
&\quad-\lambda\Omega(u_{22}^4+v_{22}^4)l_+l_--\kappa\Omega(u_{11}^4+v_{11}^4)k_+k_-,\\
H_{40}&=g_2u_{22}^2(l_+\beta_h^\dagger+\beta_h^\dagger l_+)+g_3v_{11}^2(k_+\beta_p^\dagger+\beta_p^\dagger k_+)\\
&\quad+\lambda\Omega u_{22}^2v_{22}^2(l_+^2+l_-^2)+\kappa\Omega u_{11}^2v_{11}^2(k_+^2+k_-^2).
\end{aligned} \tag{34}$$

In the above equations we have defined

$$\epsilon_1=-\omega_f/2-\lambda_F,\quad \epsilon_2=\omega_f/2-\lambda_F. \tag{35}$$

The quasiparticle spectrum corresponding to this Hamiltonian has the values

$$E_1=\sqrt{\epsilon_1^2+\Delta_1^2},\quad E_2=\sqrt{\epsilon_2^2+\Delta_2^2}, \tag{36}$$

for the quasiparticle energies, with pairing gaps given by

$$\Delta_1=(g_3^2/\omega_p+\kappa\Omega)u_{11}v_{11},\quad \Delta_2=(g_2^2/\omega_h+\lambda\Omega)u_{22}v_{22}. \tag{37}$$

These quantities show the interplay between fermion-boson interactions, with couplings g_2 and g_3, and bi-fermion interactions, with couplings λ,κ [Eq. (32)].

The Lagrange multiplier λ_F is given by

$$\lambda_F=\frac{w_f}{2}\,\frac{E_1-E_2}{E_1+E_2}, \tag{38}$$

and it enforces the constraint $N=2\Omega$. The occupancies u and v, of Eq. (22), satisfy the conditions

$$u_{11}=v_{22},\quad v_{11}=u_{22}, \tag{39}$$

in correspondence with the gap equation

$$1=\frac{1}{4}\left(\frac{g_3^2}{\omega_p}+\kappa\Omega\right)\frac{1}{E_1}+\frac{1}{4}\left(\frac{g_2^2}{\omega_h}+\lambda\Omega\right)\frac{1}{E_2}. \tag{40}$$

These equations show that the model can display a superconductive behavior which can be due both to bi-fermion excitations and to fermion-boson interactions. The condensate can, therefore, be produced by boson-fermion interactions, as in the model of [21] or by bi-fermion interactions, an aspect of the model introduced in [4].

The quasiparticle excitations can now be used to define collective excitations (one-phonon states) of the system. One can, therefore, apply the random phase approximation described in Sec. III to construct the phonon creation operators

$$\begin{aligned}
\Gamma^\dagger&=X_1k_+-Y_1k_-+W_1\beta_p^\dagger-Z_1\beta_p\\
&\quad+X_2l_+-Y_2l_-+W_2\beta_h^\dagger-Z_2\beta_h,
\end{aligned} \tag{41}$$

which are the solutions of the equation of motion

$$[H,\Gamma^\dagger]=\omega\Gamma^\dagger. \tag{42}$$

The eigenfrequencies ω, which are solutions of this equation, are given by the expression

$$\begin{aligned}
\omega_1&=\sqrt{\frac{g_3^4}{2\omega_p^2}+\frac{\omega_p^2}{2}+g_3^2\left(\frac{\kappa\Omega}{2\omega_p}(2u_{11}v_{11})^2+(u_{11}^2-v_{11}^2)\right)},\\
\omega_2&=\sqrt{\frac{g_2^4}{2\omega_h^2}+\frac{\omega_h^2}{2}+g_2^2\left(\frac{\lambda\Omega}{2\omega_h}(2u_{22}v_{22})^2-(u_{22}^2-v_{22}^2)\right)},
\end{aligned} \tag{43}$$

and they are degenerate if $g_2=g_3$, $\omega_p=\omega_h$ and $\lambda=\kappa$. These solutions represent the energies of collective excitations associated to the addition (removal) of a pair of fermions. Both the coexistence of a bi-fermion and boson condensate and the collective structure of bi-fermion excitations indicate that the present model exhibits a variety of modes which correspond to the conditions imposed in the fermion-boson- [21] and bi-fermion- [4] coupling models.

IV. CONCLUSIONS

In this work we have presented an extension of the model of Ref. [4] which includes the coupling of fermions to bosons. We have obtained exact solutions of this model and we have also compared them with the solutions given by the RPA method. The limit of the proposed model corresponding to a single external boson was demonstrated. From the analysis of the results it is concluded that (i) the bosonic condensate, found in [21], persists after the inclusion of the pp and hh bosons. The coupling of fermions to these bosons affects the structure of the condensate but for some cases, which are characterized by the values of the coupling constants x_i, the bosonic contributions prevail; (ii) the model can describe purely fermionic and bosonic ''phases'' and also shows a regime of permanent deformation, meaning that one of these degrees of freedom dominates; (iii) the RPA results agree rather well with the exact solution, but the approximation fails for some ''critical'' values of the coupling constants.

The bosons which we have introduced can represent, at fermionic level, correlated bi-fermion excitations. The results of Sec. II and Sec. III show that the extension of Geyer and Hahne's model, to include the fermion-boson couplings, does preserve the algebra of the Sp(4) group thus allowing for the determination of exact solutions. This group structure is very rich and it can be used in other scenarios of physical interest, as for example in QCD-inspired models [22]. The translation of the present formalism at the fermion level into elementary quark degrees of freedom would allow for the algebraic interpretation of the diquark Bose condensates of [2]. Other applications of the present formalism are found in the treatment of quasiparticle-pair excitations in open shell nuclei [18], where the complexity of the correlations makes the use of the RPA mandatory.

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APPENDIX

In the following a list of the matrix elements of H in the basis (13) is given. The complete set of nonzero matrix elements is obtained by observing the selection rules $N,k,m \rightarrow N\pm 2,\ k\pm 1,\ m\pm 1$.

$$\langle \Phi^{\alpha\beta}_{N,L,k,m}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = \omega_f(m+\Omega-N/2+m+k)+\omega_b(L+k) + \omega_p(\Omega-N/2+m+(\alpha-\beta-L)/2) + \omega_h(M+(\alpha+\beta-L)/2),$$

$$\langle \Phi^{\alpha\beta}_{N+2,L,k,m+1}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = G_2\sqrt{(m+1)\left(m+1+\frac{\alpha+\beta-L}{2}\right)(\Omega-k-m)},$$

$$\langle \Phi^{\alpha\beta}_{N-2,L,k,m-1}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = G_2\sqrt{m\left(m+\frac{\alpha+\beta-L}{2}\right)(\Omega-k-m+1)},$$

$$\langle \Phi^{\alpha\beta}_{N+2,L,k,m}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = G_3\sqrt{\left(\Omega-\frac{N}{2}+m+1\right)} \times\sqrt{\left(\Omega-\frac{N}{2}+m+1+\frac{\alpha-\beta-L}{2}\right)\left(\frac{N}{2}-k-m\right)},$$

$$\langle \Phi^{\alpha\beta}_{N-2,L,k,m}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = G_3\sqrt{\left(\Omega-\frac{N}{2}+m\right)} \times\sqrt{\left(\Omega-\frac{N}{2}+m+\frac{\alpha-\beta-L}{2}\right)\left(\frac{N}{2}-k-m+1\right)},$$

$$\langle \Phi^{\alpha\beta}_{N,L,k+1,m}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = -G_1\sqrt{(k+1)(k+1+L)} \times\sqrt{\frac{(2\Omega+2-k)(\Omega-m-k)\left(\frac{N}{2}-m-k\right)}{(\Omega+1-k)(\Omega-k)}},$$

$$\langle \Phi^{\alpha\beta}_{N,L,k-1,m}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = -G_1\sqrt{k(k+L)} \times\sqrt{\frac{(2\Omega+3-k)(\Omega-m-k+1)\left(\frac{N}{2}-m-k+1\right)}{(\Omega+2-k)(\Omega+1-k)}},$$

$$\langle \Phi^{\alpha\beta}_{N,L+2,k-1,m+1}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = G_1\sqrt{k(m+1)} \times\sqrt{\frac{(\Omega-N/2+m+1)(L+k+1)(2\Omega+3-k)}{(\Omega-k+1)(\Omega+2-k)}},$$

$$\langle \Phi^{\alpha\beta}_{N,L-2,k+1,m-1}|H|\Phi^{\alpha\beta}_{N,L,k,m}\rangle = G_1\sqrt{(k+1)m} \times\sqrt{\frac{(\Omega-N/2+m)(L+k)(2\Omega+2-k)}{(\Omega-k)(\Omega+1-k)}}.$$

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