$N \rightarrow \Delta(1232)$ $E2$ transition and Siegert's theorem

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Until now, calculations of the $E2$ amplitude in the $N \rightarrow \Delta$ transition using the spatial current operator have differed considerably from those using the one-body charge operator and Siegert's theorem. We show that this difference is almost entirely explained by spatial two-body exchange currents consistent with the chosen quark-quark potentials. These were not explicitly included in previous comparisons. Furthermore, using Siegert's theorem we find that the main contribution to the $E2$ amplitude comes from a two-quark spin-flip mechanism originating from pion and gluon degrees of freedom in the charge operator. Our results provide insight into the microscopic origin of the intrinsic deformation of the nucleon. [S0556-2813(98)00710-9]

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I. INTRODUCTION

The electromagnetic $\gamma + N \rightarrow \Delta$ transition form factors have received considerable attention during recent years. High-precision pion production experiments with real and virtual photons in the Δ-resonance region have recently been carried out and more are planned [1–4]. From these data, information on the transverse magnetic dipole $(M1)$, the transverse electric quadrupole $(E2)$, and the longitudinal charge quadrupole $(C2)$ transition form factors can be extracted. At small photon momentum transfers $|\mathbf{q}|$, the total $\gamma N \rightarrow \Delta$ amplitude is almost completely determined by the $M1$ spin-flip transition, which is the quark model analog of the 21 cm line due to hyperfine interaction between the proton and electron spin in atomic hydrogen. At zero photon momentum transfers $(|\mathbf{q}|=0)$, the strength of this $M1$ transition is (approximately) given by the Beg-Lee-Pais relation $\mu_{p\rightarrow\Delta^+}=(2\sqrt{2}/3)\mu_p$ [5], which connects the $N \rightarrow \Delta$ transition magnetic moment with the proton magnetic moment. In addition, there are also small $E2(C2)$ contributions [6]. A recent dispersion theoretical analysis [7] partly based on the data of Ref. [1] gives for the quadrupole transition form factor $G_{E2}(\mathbf{q})$ at the real K-matrix pole $G_{E2}=0.108$, while the data of Ref. [2] yield $G_{E2}^{\rm exp}=0.133(20)$. These quadrupole strengths are crucial observables because they are a measure of the intrinsic deformation of the nucleon and the Δ.

Various explanations for the microscopic origin of this deformation have been suggested. In the framework of the constituent quark model (CQM) the intrinsic deformation of the nucleon has been interpreted as arising from effective tensor forces between quarks and the ensuing D-wave admixtures in the nucleon wave function [8–11]. However, for realistic bag radii ($\sim$0.6 fm) and D-state admixtures ($\sim$0.5%) the calculated $E2$ transition strength is a factor of 5–10 smaller than experiment [8–11]. On the other hand, in the Skyrme model [12], the linear σ-model [13], the Nambu–Jona-Lasinio model [14], and some effective Lagrangian models [15], the $E2$ part of the amplitude is dominated by pion and/or sea-quark degrees of freedom, and usually $E2$ strengths compatible with experiment have been obtained. This suggests that some important dynamical feature has been missing in previous CQM calculations.

Previous CQM calculations with tensor forces due to one-gluon exchange predicted $E2$ transition strengths $G_{E2}(0)$ ranging between $G_{E2}(\mathbf{J})=0.0025$ and $G_{E2}(\rho)=0.065$ [11], depending on whether the calculation was performed with the spatial current or, using Siegert's theorem, with the charge density, respectively. This came as a surprise because in a gauge-invariant theory both methods of calculating the $E2$ strength should give the same answer. Thus the standard CQM explanation of G_{E2} based on tensor-force-induced D states in the nucleon and $\Delta(1232)$ wave function is incomplete, both from a phenomenological and conceptual point of view.

The main purpose of this work is to investigate the origin of the severe violation of gauge invariance observed in Ref. [11] and subsequently corroborated by several authors [16–18]. In a calculation which includes both D-wave admixtures *and* explicit spatial exchange currents, we show that the strong gauge dependence of G_{E2} observed in Refs. [11,16–18] is mainly due to the omission of explicit spatial two-body currents in previous comparisons. However, even after including spatial exchange currents, and establishing gauge invariance in leading order, the numerical results underestimate the data for G_{E2} by a factor of 2 or more. Another purpose of this paper is to show that the pion and gluon degrees of freedom in the *charge* operator lead via Siegert's theorem to an $E2$ amplitude in $\gamma N \rightarrow \Delta$ that is in better agreement with recent experiments than previous CQM results.

II. HAMILTONIAN AND WAVE FUNCTION

The Hamiltonian of the CQM consists of the kinetic energy, a quadratic confinement potential, and residual interactions $V^{\rm res}$ (see Fig. 1) which model the relevant properties of QCD:

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$$H=\sum_{i=1}^{3}\left(m_q+\frac{\mathbf{p}_i^2}{2m_q}\right)-\frac{\mathbf{P}^2}{6m_q}+\sum_{i<j}^{3}V^{\rm conf}(\mathbf{r}_i,\mathbf{r}_j)+\sum_{i<j}^{3}V^{\rm res}(\mathbf{r}_i,\mathbf{r}_j), \tag{1}$$

where $\mathbf{r}_i$, $\mathbf{p}_i$ are the spatial and momentum coordinates of the ith quark, respectively, and $\mathbf{P}$ is the center-of-mass momentum of the baryon. Color confinement is modeled by a linear or quadratic two-body quark-quark potential. Here, we take a two-body harmonic-oscillator confinement potential

$$V^{\rm conf}(\mathbf{r}_i,\mathbf{r}_j)=-a_c\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j(\mathbf{r}_i-\mathbf{r}_j)^2, \tag{2}$$

where $\boldsymbol{\lambda}_i$ is the color matrix of the ith quark.

The residual strong interaction between two spin-1/2 fermions, can quite generally be parametrized in terms of five relativistic bilinear invariants, namely scalar, pseudoscalar, vector, pseudovector, and tensor combinations that can be formed out of the Dirac spinors and γ matrices. The chiral quark model (see Fig. 1) emphasizes *effective* vector (gluon), pseudoscalar (pion), and scalar (sigma) exchange interactions

$$V^{\rm res}(\mathbf{r}_i,\mathbf{r}_j)=V^{\rm OGEP}(\mathbf{r}_i,\mathbf{r}_j)+V^{\rm OPEP}(\mathbf{r}_i,\mathbf{r}_j)+V^{\rm OSEP}(\mathbf{r}_i,\mathbf{r}_j), \tag{3}$$

which are motivated by the symmetries and dynamical properties of QCD. These are: asymptotic freedom at short distances, chiral symmetry and its dynamical breaking, which has important consequences for the form of the interaction at intermediate and larger distances.

A. Effective quark-quark interaction

Asymptotic freedom is modeled in the nonrelativistic quark model (NRQM) by the one-gluon exchange potential, $V^{\rm OGEP}$, which was introduced by De Rujula, Glashow, and Georgi in 1975 [19]. In an expansion up to third order in $1/m_q$ (the leading-order color Coulomb potential is counted as $1/m_q$), without retardation corrections, and for equal quark masses it reads

$$V^{\rm OGEP}(\mathbf{r}_i,\mathbf{r}_j)=\frac{\alpha_s}{4}\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j\Bigg\{\frac{1}{r}-\frac{\pi}{m_q^2}\left(1+\frac{2}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right)\delta(\mathbf{r})-\frac{1}{4m_q^2}\frac{1}{r^3}(3\,\boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)$$
$$-\frac{1}{2m_q^2}\frac{1}{r^3}\left[3\left(\mathbf{r}\times\frac{1}{2}(\mathbf{p}_i-\mathbf{p}_j)\right)\cdot\frac{1}{2}(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)-\left(\mathbf{r}\times\frac{1}{2}(\mathbf{p}_i+\mathbf{p}_j)\right)\cdot\frac{1}{2}(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_j)\right]\Bigg\}, \tag{4}$$

where $\mathbf{r}=\mathbf{r}_i-\mathbf{r}_j$; $\boldsymbol{\sigma}_i$ is the usual Pauli spin matrix, and $\boldsymbol{\lambda}_i$ is the color operator of the ith quark. Equation (4) contains a spin-independent central part (color Coulomb), a spin-dependent central part (color-magnetic interaction), a Galilean-invariant spin-orbit term, a Galilean-noninvariant spin-orbit term, and a tensor term [20]. With the exception of the color Coulomb term, all terms in Eq. (4) are higher-order relativistic corrections. However, these are the *lowest order nonvanishing terms* in a nonrelativistic reduction of the Feynman diagrams of Fig. 1(a), and for instance, the color-magnetic term $\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j$ of order $\mathcal{O}(m_q^{-3})$, is crucial in explaining why the $\Delta(1232)$ with spin 3/2 is heavier than the $N(939)$ with spin 1/2.

Chiral symmetry is the other important property of QCD. The spontaneous breaking of this symmetry by the physical vacuum is responsible for the constituent quark mass generation, as well as for the appearence of pseudoscalar Goldstone bosons that couple to the constituent quarks. This aspect of QCD is modeled by a one-pion exchange potential $V^{\rm OPEP}$ between constituent quarks

$$V^{\rm OPEP}(\mathbf{r}_i,\mathbf{r}_j)=\frac{g_{\pi q}^2}{4\pi(4m_q^2)}\frac{1}{3}\frac{\Lambda^2}{\Lambda^2-\mu^2}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\left[\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\mu^2\frac{e^{-\mu r}}{r}+(3\,\boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)\mu^2\frac{e^{-\mu r}}{r}-(\mu\leftrightarrow\Lambda)\right], \tag{5}$$

where $r=|\mathbf{r}|$; μ is the pion mass, and Λ the chiral cutoff. Here, $\boldsymbol{\tau}_i$ denotes the isospin of the ith quark. The pion-quark coupling constant, $g_{\pi q}^2/(4\pi)$, is related to the well-known πN coupling constant $f_{\pi N}^2/(4\pi)$. In addition to the pion, its chiral partner, namely a massive scalar-isoscalar sigma-meson, and a σ-exchange potential $V^{\rm OSEP}$ is introduced [21]. Quite generally, the scalar-isoscalar potential between two spin-1/2 fermions, including relativistic corrections to order $\mathcal{O}(m_q^{-3})$ is given as [22]:

$$V^S=V_0^S-\frac{1}{4m_q^2}\Bigg(\{(\boldsymbol{\sigma}_i\cdot\mathbf{p}_i)^2,V_0^S\}+\{(\boldsymbol{\sigma}_j\cdot\mathbf{p}_j)^2,V_0^S\}+\frac{1}{2}(\nabla_1^2+\nabla_2^2)V_0^S$$
$$+\frac{1}{r}\frac{dV_0^S}{dr}\left[\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}\times(\mathbf{p}_1-\mathbf{p}_2)+\frac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{r}\times(\mathbf{p}_1+\mathbf{p}_2)\right]\Bigg), \tag{6}$$

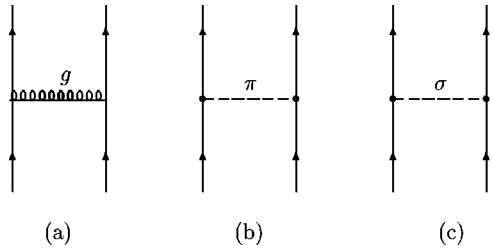

FIG. 1. Residual (a) one-gluon, (b) one-pion, and (c) one-σ exchange potentials between constituent quarks. The hadronic size of the constituent quarks is indicated by small dots.

where

$$V_0^S = V^{\mathrm{OSEP}} = -\frac{g_{\sigma q}^2}{4\pi}\frac{\Lambda^2}{\Lambda^2 - m_\sigma^2}[\exp(-m_\sigma r)/r - (m_\sigma \leftrightarrow \Lambda)].$$

The parameters of this one-σ exchange potential are fixed by the requirements of chiral symmetry as [21]

$$\frac{g_{\sigma q}^2}{4\pi} = \frac{g_{\pi q}^2}{4\pi} = \frac{f_{\pi q}^2}{4\pi}\left(\frac{2m_q}{\mu}\right)^2, \quad m_\sigma^2 \approx (2m_q)^2 + \mu^2,$$

$$\Lambda_\pi = \Lambda_\sigma = \Lambda, \tag{7}$$

where $f_{\pi q}^2/(4\pi) = (3/5)^2 f_{\pi N}^2/(4\pi)$. If one takes $V_0^S = V^{\mathrm{conf}}$ one obtains a generalized spin-dependent confinement potential including relativistic corrections [27].

The potential model outlined above should be seen as an effective description that models the symmetries and main properties of the underlying quantum field theory. Despite its obvious shortcomings, the constituent quark model with its *lowest nonvanishing order terms*, has been very successful in describing a wide spectrum of experimental data, ranging from the properties of single baryons to the properties of nuclear few-body systems. The phenomenological success of the potential model coupled with the scarcity of parameters employed indicates that the notion of constituent quarks interacting via two-body potentials provides a viable description of low-energy QCD.

B. Wave functions

As usual we employ harmonic-oscillator wave functions to diagonalize the Hamiltonian. For three quarks in the $(0s)^3$ harmonic-oscillator ground state, the total baryon wave function $\Phi_{N(\Delta)}$ is an inner product of the orbital, spin-isospin, and color wave function and given by

$$|\Phi_{N(\Delta)}\rangle = (1/\sqrt{3}\pi b^2)^{3/2}\exp(-(\boldsymbol{\rho}^2/4b^2 + \boldsymbol{\lambda}^2/3b^2))|ST\rangle^{N(\Delta)} \times |[111]\rangle_{\mathrm{color}}^{N(\Delta)}, \tag{8}$$

where the Jacobi coordinates $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ are defined as $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$ and $\boldsymbol{\lambda} = \mathbf{r}_3 - (\mathbf{r}_1 + \mathbf{r}_2)/2$; b is the harmonic-oscillator constant (quark core radius) which describes the size of the baryon. Due to the residual interactions between the constituent quarks, they can be scattered from their unperturbed $(0s)^3$ ground-state into higher oscillator shells. This process, usually referred to as configuration mixing, admixes excited states to the unperturbed ground state wave functions [23].

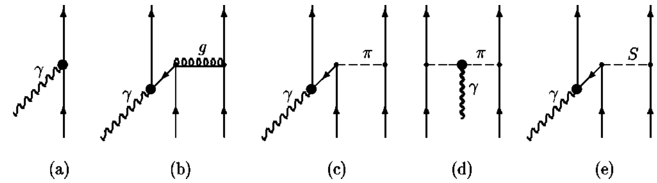

FIG. 2. One-body and two-body exchange currents between quarks: (a) impulse, (b) gluon pair, (c) pion pair, (d) pionic, (e) scalar pair. The finite electromagnetic size of the constituent quarks and the pion is indicated by the filled circles.

If, as usual, we restrict ourselves to $2\hbar\omega$ excitations, we have four excited states ($\Phi_{S'_S}^N$, $\Phi_{S_M}^N$, $\Phi_{D_M}^N$, $\Phi_{P_A}^N$) for the N and three excited states ($\Phi_{S'_S}^\Delta$, $\Phi_{D_S}^\Delta$, $\Phi_{D_M}^\Delta$) for the Δ. The subscripts L_{sym} describe the orbital angular momentum (L) and the symmetry (sym) of the orbital wave function under particle exchange. Here, S denotes symmetric, M mixed symmetric, and A antisymmetric orbital wave functions. The N and Δ wave functions are then given by

$$\Phi_N = a_{S_S}\Phi_{S_S}^N + a_{S'_S}\Phi_{S'_S}^N + a_{S_M}\Phi_{S_M}^N + a_{D_M}\Phi_{D_M}^N + a_{P_A}\Phi_{P_A}^N,$$

$$\Phi_\Delta = b_{S_S}\Phi_{S_S}^\Delta + b_{S'_S}\Phi_{S'_S}^\Delta + b_{D_S}\Phi_{D_S}^\Delta + b_{D_M}\Phi_{D_M}^\Delta. \tag{9}$$

Typical D-state probabilities range between 0.16% [8] and 0.8% [11]. A detailed description of these wave functions can be found in Refs. [23,24].

III. ELECTROMAGNETIC CURRENTS AND GAUGE INVARIANCE

In most applications of the CQM [8–11,17,18], the total electromagnetic current has been approximated by a sum of free single-quark currents [see Fig. 2(a)]:

$$\rho_{[1]}^{\mathrm{imp}}(\mathbf{r}_i, \mathbf{q}) = e_i e^{i\mathbf{q}\cdot\mathbf{r}_i};$$

$$\mathbf{J}_{[1]}^{\mathrm{imp}}(\mathbf{r}_i, \mathbf{q}) = \frac{e_i}{2m_q}(i[\boldsymbol{\sigma}_i \times \mathbf{p}_i, e^{i\mathbf{q}\cdot\mathbf{r}_i}] + \{\mathbf{p}_i, e^{i\mathbf{q}\cdot\mathbf{r}_i}\}). \tag{10}$$

This approximation is called impulse approximation. Equation (10) corresponds to the operator of lowest nonvanishing order in a nonrelativistic expansion of the relativistic one-body quark current. In Eq. (10), $e_i = \frac{1}{6}e(1 + 3\boldsymbol{\tau}_{i3})$ is the quark charge operator; $\boldsymbol{\tau}_{i3}$ is the third component of the isospin of the ith quark, and $\mathbf{q}$ is the three-momentum of the photon.

Constituent quarks are quasiparticles, i.e., current quarks surrounded by a cloud of $q\bar{q}$ pairs and gluons. The internal structure of the constituent quarks as seen by the electromagnetic probe is usually described by a monopole form factor

$$F_{\gamma q}(\mathbf{q}^2) = \frac{1}{1 + (1/6)\mathbf{q}^2 r_{\gamma q}^2}, \tag{11}$$

where $r_{\gamma q}^2$ is the charge radius of the constituent quark. In order to take the internal electromagnetic structure of the constituent quarks into account, the charge and current operators used in this work must be multiplied by the form factor of Eq. (11). While the mass and size of the constituent quarks are different from those of the current quarks, the anomalous magnetic moment of constituent quarks is small [25] and therefore neglected here.

Before discussing two-body operators, we give our expressions for the matrix elements of the one-body charge and current operators of Eq. (10) evaluated between mixed wave functions. We use the definition of the $E2$ transition form factor G_{E2} at $\mathbf{q}=0$ as in Ref. [24]

$$G_{E2}(\rho_{[1]}^{imp}) = b^2 N\Bigg((a_{S_S} b_{D_M} - a_{D_M} b_{S_S}) + \frac{2}{\sqrt{3}}(a_{S'_S} b_{D_M} - a_{D_M} b_{S'_S}) + \frac{7}{\sqrt{30}} a_{D_M} b_{D_S}\Bigg),$$

$$G_{E2}(\mathbf{J}_{[1]}^{imp}) = \frac{1}{\sqrt{5}}(a_{S_S} b_{D_M} + a_{D_M} b_{S_S}), \tag{12}$$

where $N = M_N(M_\Delta - M_N)/(2\sqrt{45})$. Our expression for $G_{E2}(\rho_{[1]}^{\text{imp}})$ differs from the ones used in Refs. [9,11,24], which contain an additional $4a_{S'_M} b_{D_M}/\sqrt{6}$ term. Such a term is absent in Eq. (12) for the following reason: In order for the orbital-spin-isospin wave function to be fully symmetric, there must be a relative minus sign between the two terms in the mixed symmetric nucleon wave function [see, e.g., Eq. (3.12) in Ref. [24]]. If nucleon wave functions with the correct permutational symmetry are used, the term proportional to $a_{S'_M} b_{D_M}$ drops from the final result. Note that this term is also absent in Ref. [16].

The numerical difference between the two ways of calculating the electric quadrupole transition in Eq. (12) has been repeatedly discussed in the literature. However, the importance of exchange currents for a correct explanation of this difference has not been recognized.

The current of Eq. (10) is not conserved in the presence of interactions between the quarks. In a bound system of quarks the electromagnetic current operator is not simply a sum of free quark currents as in Eq. (10), but necessarily contains various exchange currents for the total electromagnetic current to be conserved [26]. We construct the four-vector exchange current operators from the Feynman diagrams of Figs. 2(b)–2(e) and subsequent nonrelativistic expansion up to lowest nonvanishing order. The spatial parts of the nonrelativistically reduced exchange currents are closely related to the quark-quark potentials from which they can be derived by minimal substitution [27].

The total four vector current density consists of a one-body operator, and several two-body operators (three-body operators are neglected):

$$\rho(\mathbf{q}) = \sum_{i=1}^{3} \rho_{[1]}^{\text{imp}}(\mathbf{r}_i) + \sum_{i<j}^{3} [\rho_{[2]}^{gq\bar{q}}(\mathbf{r}_i,\mathbf{r}_j) + \rho_{[2]}^{\pi q\bar{q}}(\mathbf{r}_i,\mathbf{r}_j) + \rho_{[2]}^{\sigma q\bar{q}}(\mathbf{r}_i,\mathbf{r}_j) + \rho_{[2]}^{\text{conf}}(\mathbf{r}_i,\mathbf{r}_j)] = \rho_{[1]} + \rho_{[2]}, \tag{13}$$

TABLE I. Lowest nonvanishing order of each charge and current operator according to the $\mathcal{O}(m_q)^{-n}$ classification scheme [28].

	Imp	Pion	Gluon	Scalar
ρ	m_q^0	m_q^{-2}	m_q^{-4}	m_q^{-4}
$\mathbf{J}$	m_q^{-1}	m_q^{-1}	m_q^{-3}	m_q^{-3}

$$\mathbf{J}(\mathbf{q}) = \sum_{i=1}^{3} \mathbf{J}_{[1]}^{\text{imp}}(\mathbf{r}_i) + \sum_{i<j}^{3} [\mathbf{J}_{[2]}^{gq\bar{q}}(\mathbf{r}_i,\mathbf{r}_j) + \mathbf{J}_{[2]}^{\gamma\pi\pi}(\mathbf{r}_i,\mathbf{r}_j) + \mathbf{J}_{[2]}^{\pi q\bar{q}}(\mathbf{r}_i,\mathbf{r}_j) + \mathbf{J}_{[2]}^{\sigma q\bar{q}}(\mathbf{r}_i,\mathbf{r}_j) + \mathbf{J}_{[2]}^{\text{conf}}(\mathbf{r}_i,\mathbf{r}_j)] = \mathbf{J}_{[1]} + \mathbf{J}_{[2]}. \tag{14}$$

On the right-hand side of Eqs. (13),(14) the dependence of each operator on the photon momentum $\mathbf{q}$ is suppressed.

A. Exchange charge and current operators

In the following, we list the two-body charge and current operators employed in this work. These operators have been derived by a nonrelativistic reduction of the Feynman diagrams of Fig. 2, keeping only the lowest nonvanishing order terms in a $1/m_q$ expansion for each process. As usual the static limit is taken and nonlocal terms are discarded [29]. For classifying the relativistic corrections, we use the scheme of Friar [28] in which the leading-order potential of each diagram in Fig. 1 is counted as being of order $\mathcal{O}(m_q^{-1})$. The relativistic orders of the charge and current operators used in this work are listed in Table I.

We begin with the gluon-exchange current of Fig. 2(b)

$$\rho_{[2]}^{gq\bar{q}}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q}) = -i\frac{\alpha_s}{16 m_q^3}\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j\{e_i e^{i\mathbf{q}\cdot\mathbf{r}_i}[\mathbf{q}\cdot\mathbf{r} + (\boldsymbol{\sigma}_i\times\mathbf{q})\cdot(\boldsymbol{\sigma}_j\times\mathbf{r})] + (i\leftrightarrow j)\}\frac{1}{r^3},$$

$$\mathbf{J}_{[2]}^{gq\bar{q}}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q}) = -\frac{\alpha_s}{4m_q^2}\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j\left\{e_i e^{i\mathbf{q}\cdot\mathbf{r}_i}\frac{1}{2}(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\times\mathbf{r} + (i\leftrightarrow j)\right\}\frac{1}{r^3}. \tag{15}$$

The above coordinate space expressions describe a quark-antiquark pair creation process induced by the external photon, or the coupling of the external photon to a quark-antiquark pair in the nucleon. These gluon-pair currents are of relativistic origin as reflected by the higher powers of $1/m_q$ as compared to the nonrelativistic impulse current of Eq. (10).

In addition to the gluon exchange currents, we include pion pair exchange currents [see Fig. 2(c)]. For the $N\rightarrow\Delta$ transition only the isovector pion-pair current contributes

$$\rho_{[2]}^{\pi q\bar{q}}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q})=\frac{ie}{2}\,\frac{g_{\pi q}^2}{4\pi(4m_q^2)}\,\frac{\Lambda^2}{\Lambda^2-\mu^2}\,\frac{1}{m_q}\times\{\boldsymbol{\tau}_{j3}e^{i\mathbf{q}\cdot\mathbf{r}_i}\boldsymbol{\sigma}_i\cdot\mathbf{q}\boldsymbol{\sigma}_j\cdot\nabla_{\mathbf{r}}+(i\leftrightarrow j)\}\times\left(\frac{e^{-\mu r}}{r}-\frac{e^{-\Lambda r}}{r}\right),$$

$$\mathbf{J}_{[2]}^{\pi q\bar{q}}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q})=e\,\frac{g_{\pi q}^2}{4\pi(4m_q^2)}\,\frac{\Lambda^2}{\Lambda^2-\mu^2}\times\{(\boldsymbol{\tau}_i\times\boldsymbol{\tau}_j)_3e^{i\mathbf{q}\cdot\mathbf{r}_i}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\cdot\nabla_{\mathbf{r}}+(i\leftrightarrow j)\}\times\left(\frac{e^{-\mu r}}{r}-\frac{e^{-\Lambda r}}{r}\right). \tag{16}$$

The pionic current of Fig. 2(d) describes a process where the photon couples to the pion directly. In the static limit Fig. 2(d) does not contribute to the charge density but only to the spatial current

$$\rho_{[2]}^{\gamma\pi\pi}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q})\approx 0,$$

$$\mathbf{J}_{[2]}^{\gamma\pi\pi}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q})=e\,\frac{g_{\pi q}^2}{4\pi(4m_q^2)}\,\frac{\Lambda^2}{\Lambda^2-\mu^2}(\boldsymbol{\tau}_i\times\boldsymbol{\tau}_j)_3\times\boldsymbol{\sigma}_i\cdot\boldsymbol{\nabla}_i\boldsymbol{\sigma}_j\cdot\boldsymbol{\nabla}_j\int_{-1/2}^{1/2}dv\,e^{i\mathbf{q}\cdot(\mathbf{R}-\mathbf{r}v)}\times\left(\mathbf{z}_\mu\frac{e^{-L_\mu r}}{L_\mu r}-\mathbf{z}_\Lambda\frac{e^{-L_\Lambda r}}{L_\Lambda r}\right). \tag{17}$$

In the pionic exchange current we have used the following abbreviations: $\mathbf{R}=(\mathbf{r}_i+\mathbf{r}_j)/2$, $\mathbf{z}_m(\mathbf{q},\mathbf{r})=L_m\mathbf{r}+iv r\mathbf{q}$, and $L_m(q,v)=[\frac{1}{4}q^2(1-4v^2)+m^2]^{1/2}$.

Next, we construct scalar exchange currents corresponding to the generalized scalar σ-meson exchange and confinement potentials. As in the one-gluon exchange potential, we have kept relativistic corrections up to order $\mathcal{O}(m_q^{-3})$ in the scalar exchange potential. Although scalar exchange currents do not contribute to the $N\to\Delta$ transition at $\mathbf{q}=0$ we list them for completeness. Note that the spatial part of the scalar exchange current has the same spin-isospin structure as the single-quark current:

$$\rho_{[2]}^{S}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q})=\frac{1}{(2m_q)^3}\left\{e_ie^{i\mathbf{q}\cdot\mathbf{r}_i}\left(\frac{3}{2}\mathbf{q}^2-i\mathbf{q}\cdot\nabla_r+\frac{1}{2}\nabla_r^2\right)\times V_0^S(\mathbf{r}_i,\mathbf{r}_j)+(i\leftrightarrow j)\right\},$$

$$\mathbf{J}_{[2]}^{S}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{q})=-\frac{i}{2m_q^2}\{e_ie^{i\mathbf{q}\cdot\mathbf{r}_i}\boldsymbol{\sigma}_i\times\mathbf{q}V_0^S(\mathbf{r}_i,\mathbf{r}_j)+(i\leftrightarrow j)\}. \tag{18}$$

Equation (18) is used to calculate both the confinement- and σ-meson-exchange currents.

With the exception of the isovector pion-pair current of Eq. (16) and the isovector pionic current of Eq. (17), which are of the same relativistic order $\mathcal{O}(m_q^{-1})$ as the one-body current, all other two-body currents are relativistic corrections. This is shown in Table I. We point out that some of these operators have a *spin-isospin structure* not present in the lowest order one-body or two-body currents. Thus, although being formally of higher order, they can induce, e.g., transitions to states not accessible by the lowest order one-body or two-body currents. We will come back to this point.

B. Gauge invariance and current conservation

In this section we investigate to what extent the electromagnetic charge and current operators of Sec. III satisfy the continuity equation

$$\mathbf{q}\cdot\mathbf{J}(\mathbf{q})=[H,\rho(\mathbf{q})], \tag{19}$$

where $\rho(\mathbf{q})$ and $\mathbf{J}(\mathbf{q})$ are, respectively, the total charge density and current density for an interacting three-quark system. Verification of Eq. (19) in approximate theories is usually a complicated problem. In this work we cannot do more than take a few steps in the right direction, and remove two approximations that are usually employed.

We start with a decomposition of all operators in Eq. (19) into one- and two-body operators and denote them by subscript $[1]$ and $[2]$, respectively,

$$\rho(\mathbf{q})=\rho_{[1]}(\mathbf{q})+\rho_{[2]}(\mathbf{q}),$$
$$\mathbf{J}(\mathbf{q})=\mathbf{J}_{[1]}(\mathbf{q})+\mathbf{J}_{[2]}(\mathbf{q}),$$
$$H=T_{[1]}+V_{[2]}. \tag{20}$$

Proper symmetrization of the two-body operators is understood. In principle, this decomposition could be extended to three-body operators. Then we perform a nonrelativistic reduction of each diagram in Fig. 2 and group terms of the same order $\mathcal{O}(m_q^{-n})$ with $n=0,1,2,\ldots$ denoted by the superscripts in parentheses (n)

$$\rho_{[1]}(\mathbf{q})=\rho_{[1]}^{(0)}(\mathbf{q})+\rho_{[1]}^{(2)}(\mathbf{q}),$$
$$\rho_{[2]}(\mathbf{q})=\rho_{[2]}^{(2)}(\mathbf{q})+\rho_{[2]}^{(4)}(\mathbf{q}),$$
$$\mathbf{J}_{[1]}(\mathbf{q})=\mathbf{J}_{[1]}^{(1)}(\mathbf{q})+\mathbf{J}_{[1]}^{(3)}(\mathbf{q}),$$
$$\mathbf{J}_{[2]}(\mathbf{q})=\mathbf{J}_{[2]}^{(1)}(\mathbf{q})+\mathbf{J}_{[2]}^{(3)}(\mathbf{q}). \tag{21}$$

In leading nonrelativistic order $\mathcal{O}(m_q^0)$, the charge density is not modified by exchange current effects, i.e., one has $\rho\approx\rho_{[1]}^{(0)}$. This can be seen from Table I and the explicit expressions for the two-body charge densities. On the other hand, the spatial current density is already affected by exchange currents in leading nonrelativistic order $\mathcal{O}(m_q^{-1})$. According to Eq. (21) one has

$$\mathbf{q}\cdot(\mathbf{J}_{[1]}(\mathbf{q})+\mathbf{J}_{[2]}(\mathbf{q}))=[H,\rho_{[1]}^{(0)}(\mathbf{q})]. \tag{22}$$

In this work, we ignore the higher relativistic orders in the kinetic energy and the one-body charge and current density. For the two-body operators we keep for each degree of freedom the lowest nonvanishing order, even if it is a higher-order correction. The reasons for this apparently asymmetrical treatment are discussed in Sec. III C. Equation (22) can be further decomposed. Let us first consider the standard nonrelativistic one-body current of the quarks of Fig. 2(a). It is straightforward to show that the one-body current of Eq. (10) and the kinetic energy term in Eq. (1) satisfy an independent continuity equation:

$$\mathbf{q}\cdot\mathbf{J}^{(1)}_{[1]}(\mathbf{q})=[T_{[1]},\rho^{(0)}_{[1]}(\mathbf{q})], \tag{23}$$

where $\rho^{(0)}_{[1]}$ is the nonrelativistic one-body charge density of order $\mathcal{O}(m_q^0)$ in Eq. (10). Therefore, the two-body spatial current in Eq. (22) and the potential energy term in Eq. (1) satisfy the following continuity equation:

$$\mathbf{q}\cdot\mathbf{J}_{[2]}(\mathbf{q})=[V_{[2]},\rho^{(0)}_{[1]}(\mathbf{q})]. \tag{24}$$

Equation (24) provides a connection between the potentials used to calculate the excitation spectrum of the nucleon and the electromagnetic currents of bound quarks that are responsible for its electromagnetic properties. There are two sources which contribute to the commutator on the right-hand side of Eq. (24): (i) the momentum dependence and (ii) the isospin-dependence of the potential. For example, the spatial gluon exchange current of Eq. (15) originates from the momentum dependence of the one-gluon exchange potential, while the spatial pion exchange currents of Eqs. (16) and (17) are a consequence of the isospin dependence of the one-pion exchange potential. Equation (24) is shown to be satisfied for each potential term in Eq. (1) individually [27].

C. Higher-order relativistic corrections

The reader may ask why we did not include relativistic corrections to the one-body operators in the Hamiltonian, charge, and spatial current operators. At first sight it seems to be inconsistent to use relativistic corrections in two-body operators but to ignore them in one-body operators. Recall that in our approach we include for each Feynman diagram and for each nonrelativistic invariant only *the lowest nonvanishing order*. We have previously shown that the inclusion of next-to-leading order terms in the one-body current destroys the successful constituent quark model predictions of baryon magnetic moments [27]. We have argued that use of these next-to-leading orders in one-body operators contradicts the CQM paradigm, which says that the bulk of relativistic corrections is already taken care of by the choice of the constituent quark (quasiparticle) mass, and the size parameters b and $r_{\gamma q}$.

Can one justify the neglect of relativistic corrections in one-body currents? For a heuristic argument in favor of our approach consider the nonrelativistic expansion of the relativistic kinetic energy of a single quark

$$\sqrt{m_q^2+\mathbf{p}^2}=m_q+\frac{\mathbf{p}^2}{2m_q}-\frac{\mathbf{p}^4}{8m_q^3}+\frac{\mathbf{p}^6}{16m_q^5}-\cdots.$$

An estimate for p based on the uncertainty relation shows that $p\approx 1/b\approx m_q$, where b is the quark core radius. This means the series converges only very slowly or does not converge at all, if b is significantly smaller than 0.6 fm. However, if we truncate the series after the $p^2/(2m_q)$ term and use $p\approx m_q$, the numerical value for the relativistic kinetic energy is for the left-hand side $\sqrt{2}m_q$ while for the right-hand side it is $1.5m_q$; a 6% deviation (see also the review by Lucha *et al.* [30]). Thus, despite its nonrelativistic appearance, the kinetic energy on the right-hand side contains a considerable amount of relativistic corrections. This is due to the actual values of the constituent quark mass m_q and the quark core radius b. For another estimate consider the matrix elements of the nonrelativistically expanded kinetic energy between the wave function of Eq. (8)

$$\langle\Phi_N|T|\Phi_N\rangle=3m_q+\frac{3}{2m_qb^2}-\frac{5}{8m_q^3b^4}+\frac{35}{48m_q^5b^6}+\cdots. \tag{25}$$

We observe that for our choice $m_q\approx b^{-1}$ the latter two corrections largely cancel each other, and we are again left with the usual nonrelativistic expression, i.e., the first two terms in Eq. (25).

It appears that due to our choice of $m_q=313$ MeV and $b\approx 0.6$ fm, the lowest-order terms in the kinetic energy are sufficient to account for the bulk of relativistic effects. Therefore, in the framework of the CQM, it seems to be legitimate to ignore next-to-leading order corrections in *all* one-body operators (kinetic energy, one-body charge and current density) consistently. The reason for this is Eq. (23). One should not use next-to-leading order relativistic corrections in the one-body charge and current operators if one ignores them in the kinetic energy. It seems that a proper choice of quark model parameters such as m_q, b, and $r_{\gamma q}$ in the kinetic energy, wave function, and electromagnetic one-body current operator is a better way of including relativistic effects than insisting on formal consistency of a series expansion [up to a certain order $\mathcal{O}(m_q^{-n})$] [27]. We therefore propose that relativistic corrections to one-body operators be consistently neglected.

The same reasoning does not apply to two-body operators. For the two-body currents it is important to include terms of order $\mathcal{O}(m_q^{-2})$ and higher, because in many cases these *are the lowest nonvanishing orders*. For example, there is no pion contribution to the charge operator in leading order $\mathcal{O}(m_q^0)$. The pion contribution to the charge operator enters only in order $\mathcal{O}(m_q^{-2})$ (see Table I). It is important to keep such terms because they contribute isospin-spin structures, not present in the lowest order charge operator [31]. In nuclear physics, the use of these lowest nonvanishing pion exchange corrections in the charge operator is common practice [32] even though the consistency with the lowest order one-pion exchange potential is broken. For further details on relativistic corrections in one- and two-body operators in the nonrelativistic quark model see Ref. [33].

Summarizing this section, it is clear that an effective operator expansion in powers of $1/m_q$ converges only very slowly or does not converge at all for constituent quarks with $p\approx m_q$, the problem being already apparent at the level of the Hamiltonian. Despite the asymmetrical treatment of one- and two-body operators (the kinetic energy is expanded up to order $\mathcal{O}(m_q^{-1})$, whereas the two-body potentials may contain terms up to order $\mathcal{O}(m_q^{-3})$ [19]), the constituent quark model provides a vast amount of results in agreement with experiment [23]. In this work we extend this approach to the charge and current operator which enter the theory in the presence of an external electromagnetic field. We believe that by including the lowest nonvanishing order pion-, gluon-, and scalar exchange charges and currents, one gains deeper insight into the problem which dynamical processes govern the electromagnetic properties of baryons, than by ignoring two-body currents altogether. Without exchange currents there is no gauge invariance at all as we will see in the next section.

IV. THE $N \to \Delta$ QUADRUPOLE TRANSITION AND SIEGERT'S THEOREM

Some time ago it was pointed out [11] that a calculation of the transversal $N \to \Delta$ quadrupole transition form factor, $G_{E2}(\mathbf{q})$, evaluated at the pseudothreshold $\mathbf{q=0}$ gives very different results depending on whether the transverse current density or the charge density of Eq. (10) is used. This seems to constitute a violation of the gauge invariance condition

$$q_\mu J^\mu = \omega\rho - \mathbf{q}\cdot\mathbf{J} = 0, \tag{26}$$

according to which both calculations should give the same answer because the corresponding operators are related according to Eq. (26).

Numerically, the violation of gauge invariance is huge. For example, a calculation based on the one-body spatial current density $\mathbf{J}_{[1]}$, the admixture coefficients of Ref. [34], and the expression for $G_{E2}(0)$ given in Ref. [11], yields an $E2$ transition moment $G_{E2}(\mathbf{J}_{[1]}) = 0.0076$. This differs by almost an order of magnitude from the corresponding result based on the one-body charge density $G_{E2}(\rho_{[1]}) = 0.065$ [11]. In Ref. [11] it is stated that a calculation of the $E2$ transition strength via the one-body charge density is to be preferred, because it is less sensitive to the restriction to a finite number of oscillator shells in the expansion of the true wave function (truncation of configuration space).

The main findings of the pioneering study of Ref. [11] were corroborated by Weyrauch and Weber [16] and by Bourdeau and Mukhopadhyay [17], who extended the calculation to finite momentum transfers, and also investigated the role of pions. Later Capstick and Karl [18] enlarged the model space to $6\hbar\omega$ in order to test the conclusions of Refs. [11,16,17]. They pointed out that a larger wave function basis and the use of relativized wave functions does not remove the lack of current conservation and suggested investigating the electromagnetic transition *operator* in more detail.

Thus, although there is general agreement that the truncation of configuration space is partly responsible for the strong gauge dependence of previous results for G_{E2}, the main source for this discrepancy has not been identified in previous CQM calculations. In this work, we show that the main reason for the large difference between $G_{E2}(\rho_{[1]})$ and $G_{E2}(\mathbf{J}_{[1]})$ is that the former includes the lowest order (in a relativistic expansion) spatial exchange currents $\mathbf{J}_{[2]}$. This is a consequence of the continuity equation used in the derivation of Siegert's theorem.

A. Siegert's theorem

Siegert's theorem [35] states that the matrix elements of the transverse electric multipole operators (T^{EJ}) as derived from the spatial current density, $\mathbf{J}$, can be calculated in the low-momentum transfer limit from the corresponding matrix elements of the Coulomb multipole (T^{CJ}) operators which are based on the charge density ρ:

$$\langle f | T^{EJ}(|\mathbf{q}|\to 0)|i\rangle = -\frac{\omega}{|\mathbf{q}|}\sqrt{\frac{J+1}{J}}\langle f|T^{CJ}(|\mathbf{q}|\to 0)|i\rangle. \tag{27}$$

Here, ω is the energy difference between initial and final states and equals the energy transfer of the photon, and J is the total angular momentum of the photon. We use the definition of the electromagnetic multipole operators as in Ref. [36]. We will not rederive Siegert's theorem here but merely state the necessary requirements for its derivation [36]:

(i) The low-momentum transfer limit $\mathbf{q}\to 0$.

(ii) The use of exact eigenstates $|i\rangle$ and $|f\rangle$ of the Hamiltonian H.

(iii) The continuity equation for the electromagnetic current (gauge invariance), which relates the total charge density ρ, the total current density $\mathbf{J}$ and the total Hamiltonian of the system $\mathbf{q}\cdot\mathbf{J}(\mathbf{q}) = [H, \rho(\mathbf{q})]$.

All previous quark model calculations of the $N\to\Delta$ quadrupole transition violate the latter two requirements for Siegert's theorem to some extent. In shell-model calculations, which use a finite number of harmonic oscillator states (finite configuration space), one always deals with approximate eigenstates. In addition, there is a violation of the continuity equation if, in the presence of momentum and/or isospin-dependent two-body interactions only spatial one-body currents are employed [35]. As in Refs. [11,16,17], we use only the limited $2\hbar\omega$ configuration space given by Eq. (9) for the N and Δ wave functions. It has been previously shown [18] that an increase in the number of oscillator shells does not remove the violation of gauge invariance. Therefore, we will focus on the implications of the continuity equation.

The continuity equation for the electromagnetic current involves the total charge, the total current and the total Hamiltonian of the interacting quark system. However, in most applications of Siegert's theorem, the continuity equation is used to replace the divergence of the spatial *one-body* current density by the commutator of the *total* Hamiltonian with the one-body charge density [11,16–18]. One thereby implicitly includes just those two-body exchange currents $\mathbf{J}_{[2]}$ that are minimally required by the two-body exchange potential $V_{[2]}$.

In lowest nonvanishing order one usually writes

$$[H, \rho^{(0)}_{[1]}(\mathbf{q})] = \mathbf{q}\cdot(\mathbf{J}_{[1]}(\mathbf{q}) + \mathbf{J}_{[2]}(\mathbf{q})), \tag{28}$$

where $\rho^{(0)}_{[1]}$ is the nonrelativistic one-body charge operator in Eq. (13). Taking matrix elements on both sides of Eq. (28) and assuming that the states $|N\rangle$ and $|\Delta\rangle$ are exact eigenstates of H with eigenvalues E_N and E_Δ, respectively, one obtains

$$\begin{aligned}\langle\Delta|[H,\rho^{(0)}_{[1]}(\mathbf{q})]|N\rangle &= (E_\Delta - E_N)\langle\Delta|\rho^{(0)}_{[1]}(\mathbf{q})|N\rangle\\ &= \mathbf{q}\cdot\langle\Delta|(\mathbf{J}_{[1]}(\mathbf{q}) + \mathbf{J}_{[2]}(\mathbf{q}))|N\rangle.\end{aligned} \tag{29}$$

It is evident from Eq. (29) that an evaluation of the matrix elements of the one-body charge operator $\rho_{[1]}$ implicitly includes the effect of spatial two-body exchange currents $\mathbf{J}_{[2]}$. This fact has long been known [35], and it is widely recognized that Siegert's theorem provides a convenient way of including the spatial two-body exchange currents connected with the two-body potentials when calculating electric multipole transitions.

B. Siegert's theorem including exchange charges

The second purpose of this work is to explain the dynamical origin (in our approach) for the large $E2$ amplitude in the $N\to\Delta$ transition. In Eq. (29), we have approximated the total charge density by the nonrelativistic one-body charge density. This is the usual approximation made in nuclear physics and corresponds to Siegert's original derivation. It is sometimes called Siegert's hypothesis [37] in order to distinguish it from Siegert's theorem, which is more general. Siegert's hypothesis is sufficient for establishing consistency between each potential term in Eq. (3) and the corresponding spatial exchange currents in Sec. III A. It is also the underlying assumption of previous quark model work on the $N\to\Delta$ transition. However, Siegert's theorem of Eq. (27) is more general, and it is not necessary to use Siegert's hypothesis in its derivation.

Consider the following continuity equation with two-body terms in the charge operator:

$$\mathbf{q}\cdot(\mathbf{J}_{[1]}(\mathbf{q})+\tilde{\mathbf{J}}_{[2]}(\mathbf{q})+\mathbf{J}_{[3]})=[H,\rho_{[1]}(\mathbf{q})+\rho_{[2]}(\mathbf{q})]. \tag{30}$$

Here, $\tilde{\mathbf{J}}_{[2]}$ is a spatial two-body current, which is more general than the one in Sec. III A, and $\mathbf{J}_{[3]}$ is a heretofore unexplored three-body current.

It is interesting to study the implications of Siegert's theorem based on the more general continuity equation (30) in more detail. The Coulomb quadrupole operator entering Eq. (27) is defined [39] as

$$T^{C2}(|\mathbf{q}|)=-\frac{1}{4\pi}\int d\Omega_q\rho(\mathbf{q})Y^{[2]}_0(\hat{\mathbf{q}}), \tag{31}$$

where $\rho(\mathbf{q})$ is the total charge operator of Eq. (13) containing one- and two-body terms. Obviously, in order for the Coulomb quadrupole operator to be nonzero, $\rho(\mathbf{q})$ must be proportional to $Y^{[2]}(\hat{\mathbf{q}})$. Next we investigate the one-body and the two-body terms in the charge-density operator of Eq. (13) entering Eq. (30) separately.

We start with the one-body charge operator $\rho_{[1]}$. After an expansion of the plane wave $\exp(i\mathbf{q}\cdot\mathbf{r}_i)$ into partial waves we see that its $Y^{[2]}(\hat{\mathbf{q}})$ component is proportional to $Y^{[2]}(\hat{\boldsymbol{\rho}})$

$$\rho_{[1]}\propto[Y^{[l]}(\hat{\mathbf{q}})\times Y^{[l]}(\hat{\boldsymbol{\rho}})]^{[0]}. \tag{32}$$

Thus, according to Eq. (31) $T^{C2}(\rho_{[1]})\propto Y^{[2]}(\hat{\boldsymbol{\rho}})$, and we get nonvanishing matrix elements of the $C2$ multipole operator only for the off-diagonal $S\to D$, $D\to S$, and the diagonal $D\to D$ transitions, as can be explictly seen from Eq. (12). Obviously all terms in Eq. (12) involve the small D-wave components in the nucleon and Δ.

On the other hand, the two-body gluon and pion charge densities $\rho_{[2]}$ contain a rank-2 tensor in spin space

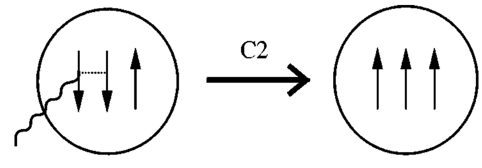

FIG. 3. Two-body gluon and pion exchange charges induce a double spin flip $N(S-wave)\to\Delta(S-wave)$ quadrupole ($E2$) transition not allowed in nonrelativistic impulse approximation. The strength of this $N\to\Delta$ quadrupole transition is given by the neutron charge radius $Q_{N\to\Delta}=r_n^2/\sqrt{2}$, or equivalently by the difference between proton and Δ^+ charge radii, $Q_{N\to\Delta}=(r_p^2-r_{\Delta^+}^2)/\sqrt{2}$ [39].

$$\rho_{[2]}\propto[[\boldsymbol{\sigma}_i^{[1]}\times\boldsymbol{\sigma}_j^{[1]}]^{[2]}\times[Y^{[0]}(\hat{\boldsymbol{\rho}})\times Y^{[2]}(\hat{\mathbf{q}})]^{[2]}]^{[0]}. \tag{33}$$

Therefore, $T^{C2}(\rho_{[2]})\propto Y^{[0]}(\hat{\boldsymbol{\rho}})$, and we get a nonvanishing matrix element of the $C2$ multipole operator also for an $S\to S$ transition, i.e., a transition involving only the dominant S wave in the nucleon and the dominant S wave in the Δ. This transition corresponds to a double spin-flip transition involving two quarks (see Fig. 3). With the one-body charge operator such a transition is impossible. That is why the two-body charge densities derived from Figs. 2(b) and 2(d) lead to a nonvanishing quadrupole moment, even if there are no D waves in the nucleon and/or Δ.

In anticipation of our numerical results, we point out that it is not justified to assume that the $\rho_{[2]}$ contribution is small, as in nuclear physics. On the contrary, there are observables in baryon physics for which the two-body pair terms in the charge-density operator are very important. Examples are the neutron charge radius [38] and the $N\to\Delta$ transition quadrupole moment [39]. Using Siegert's theorem we will see that the two-body charge operator $\rho_{[2]}$ provides also the largest contribution for the $N\to\Delta$ transition form factor G_{E2}.

V. RESULTS AND DISCUSSION

A. Role of spatial exchange currents

In Table II we use the wave functions of Refs. [8,9,11,16,38] to calculate (i) $G_{E2}(\rho_{[1]})$ using Siegert's theorem of Eq. (28), and (ii) $G_{E2}(\mathbf{J})$ using the total spatial current of Eq. (14). The latter calculation explicitly includes the two-body exchange currents $\mathbf{J}_{[2]}$ consistent with the two-body exchange potentials $V_{[2]}$ used in the calculation of the N and Δ wave functions. Note that only the pion and gluon exchange currents appear in Table II, and that the scalar σ and confinement exchange currents do not contribute in the limit $\mathbf{q}=0$. As is evident from Table II, both methods of calculating the $E2$ strength agree with each other within 20% for the models investigated. For the models of Refs. [8,9,38], the agreement between $G_{E2}(\rho_{[1]})$ and $G_{E2}(\mathbf{J})$ is better than 5%. For the model used in Ref. [11] the inital order of magnitude difference between $G_{E2}(\rho_{[1]})$ and $G_{E2}(\mathbf{J}_{[1]})$ shrinks to a 20% discrepancy once the exchange currents $\mathbf{J}_{[2]}$ consistent with the chosen Hamiltonian are included. Furthermore, we point out that the contribution of the spatial two-body current $\mathbf{J}_{[2]}$ is usually larger than the one of the spatial one-body current $\mathbf{J}_{[1]}$. Concerning the relative contribution of gluon and pion exchange currents we see from

TABLE II. The transverse electric quadrupole form factor $G_{E2}(\mathbf{q}^2=0)$ for the $\gamma+p\to\Delta^+$ transition calculated with (i) the one-body charge density $\rho_{[1]}^{(0)}$ using Siegert's theorem, (ii) with the spatial current density $\mathbf{J}_{[1]}+\mathbf{J}_{[2]}$. The difference between $G_{E2}(\rho_{[1]}^{(0)})$ and $G_{E2}(\mathbf{J}_{[1]})$ is almost entirely explained by the spatial two-body currents $\mathbf{J}_{[2]}$ needed to satisfy the continuity equation (24) with the corresponding potentials. A comparison of the entries in the *first and last rows* shows how well the continuity equation is satisfied in a truncated $(2\hbar\omega)$ configuration space for various quark-quark interaction models. The experimental range for $G_{E2}(0)$ is: $G_{E2}=0.133(20)$ [2], $G_{E2}=0.108(17)$ [1,7], $G_{E2}=0.095(16)$ [41,42], $G_{E2}=0.066(18)$ [40]. The remaining discrepancy between theory and recent experiments can be explained by the two-body gluon and pion charge densities $\rho_{[2]}$ [39] [see Eq. (36)].

	Ref. [8]	Ref. [9]	Ref. [11]	Ref. [16]	Ref. [38](π)
$G_{E2}(\rho_{[1]}^{(0)})$	0.0192	0.0203	0.0796	0.0177	0.0165
$G_{E2}(\mathbf{J}_{[1]}^{(1)})$	0.0118	0.0092	0.0076	0.0027	0.0058
$G_{E2}(\mathbf{J}_{[2]}^{gqq})$	0.0084	0.0114	0.0561	0.0044	0.0039
$G_{E2}(\mathbf{J}_{[2]}^{\pi qq})$	0.0000	0.0000	0.0000	0.0122	0.0103
$G_{E2}(\mathbf{J}_{[2]}^{\gamma\pi\pi})$	0.0000	0.0000	0.0000	-0.0039	-0.0037
$G_{E2}(\mathbf{J}_{[2]})$	0.0084	0.0114	0.0561	0.0127	0.0105
$G_{E2}(\mathbf{J}_{[1]}+\mathbf{J}_{[2]})$	0.0202	0.0206	0.0637	0.0154	0.0163

Table II that once pions are included (see the models of Refs. [16,38]) they tend to dominate.

We conclude that gauge invariance is approximately restored even in a truncated model space, provided that the spatial two-body exchange currents required by Eq. (28) are explicitly taken into account. The results in Table II clearly show that for the $N\to\Delta$ quadrupole transition, the violation of gauge invariance induced by the finite wave function basis is small compared to the error induced by neglecting spatial exchange currents. Thus the gauge dependence observed in Refs. [11,17,18] seems to be mainly due to the neglect of *explicit* spatial exchange currents, and has relatively little to do with the truncation of the model space. This lends further support to our assertion in Ref. [39] that spatial two-body exchange currents are crucial for a gauge-invariant calculation of the $E2$ transition form factor via the spatial current density. Having explicitly demonstrated that gauge invariance holds to good approximation in a $2\hbar\omega$ model space, one can take either side of the continuity equation to calculate G_{E2}. We obtain using Eqs. (12) and (27)

$$G_{E2}(\rho_{[1]})=0.0166, \tag{34}$$

whereas the most recent experimental result is $G_{E2}^{\exp}=0.133(20)$ [2].

B. The importance of two-body terms in the charge operator

Evidently, a large discrepancy remains between the recent experimental results $G_{E2}^{\exp}=0.133(20)$ [2] or $G_{E2}^{\exp}=0.108(17)$ [1,7], and the calculation with the one-body charge operator or the spatial current operator of Eq. (14) (see Table II). What is the origin of this discrepancy? From our discussion in Sec. IV B and Eq. (12) it is evident that the one-body charge operator or the leading-order spatial two-body currents contribute to the $E2$ transition only if there are D-wave components in the nucleon and/or Δ wave functions. The calculated D-wave probablities in the nucleon and Δ wave function are in the constituent quark model [9,8,10,38] much less than 1%, and therefore the contribution of the one-body charge operator, despite being of leading nonrelativistic order $\mathcal{O}(m_q^0)$ is very small.

In Ref. [39] we have shown that the two-body exchange corrections to the *charge* operator in Eq. (13) provide the major contribution to the $C2$ amplitude in the $N\to\Delta$ transition. By virtue of Siegert's theorem they will also lead to a large contribution to the transverse electric $E2$ transition amplitude as we have argued in Sec. IV B. Due to their particular spin and isospin structure, these pion and gluon exchange charge operators can induce transitions between the *dominant S waves* in the nucleon and Δ wave functions, and explain the $N\to\Delta$ quadrupole transition as a simultaneous spin-flip of two quarks (*double spin-flip transition*). Using our parameter-independent relation between the transition quadrupole moment and the neutron charge radius: $Q_{N\to\Delta}(\rho_{[2]})=r_n^2/\sqrt{2}$ [39], as well as Siegert's theorem we find

$$G_{E2}(\rho_{[2]})=-\frac{\omega M_N\sqrt{3}}{12}r_n^2=0.107, \tag{35}$$

for the corresponding $E2$ strength induced by the exchange terms in the charge operator. If we add this to the contribution of the one-body charge density in Eq. (34) we obtain

$$G_{E2}(\rho_{[1]})+G_{E2}(\rho_{[2]})=0.0166+0.107=0.124. \tag{36}$$

This has to be compared to the most recent experimental values $G_{E2}=0.133(20)$ [2], and $G_{E2}=0.108(17)$ [1,7].

Note that Eq. (35) is an approximate relation derived using pure S waves for the N and Δ wave functions. A more complete calculation using mixed wave functions for the evaluation of the two-body charge densities cannot qualitatively change our conclusion that $G_{E2}(\rho_{[2]})$ is the dominant term. Because the admixture coefficients of the other components in the N and Δ wave functions of Eq. (9) are small [8,9,16,38], we expect Eq. (35) to hold within some 30%.

As to the relative importance of $\rho_{[2]}^{\pi qq}$ and $\rho_{[2]}^{gqq}$ in Eq. (35), we have shown before that gluons dominate, if there is no configuration mixing [39,38], whereas pions and gluons give comparable contributions if configuration mixing is included [38]. However, the $N\rightarrow\Delta$ quadrupole transition is probably not a good observable to pin down the relative importance of pion and gluon degrees of freedom. Because pions do not couple to the strange quarks, the radiative $E2$ decays of the strange decuplet hyperons provide additional information on the importance of *effective* gluon degrees of freedom in nonperturbative QCD [43].

VI. CONCLUSION

In this work, we have calculated the $E2$ strength in the $N\rightarrow\Delta$ transition in two different ways. First, we have evaluated the spatial current density including exchange currents between mixed wave functions in a $2\hbar\omega$ configuration space. Second, using Siegert's theorem, we have calculated the $E2$ amplitude with the help of the one-body charge density and mixed wave functions. We have found that both calculations agree remarkably well. This conclusion is independent of the particular model considered. Thus, we have shown that the strong violation of gauge invariance, found in the pioneering study of Ref. [11] and also in subsequent works [16–18], was mainly due to the omission of *explicit* spatial two-body currents. The importance of two-body exchange currents for a correct explanation of the apparent violation of gauge invariance has not been recognized before.

The present calculation also shows that the contribution of the spatial current of Eq. (14) evaluated between mixed wave functions is far too small to explain the empirical $E2$ transition form factor. This is a consequence of the small D-state probabilities in the nucleon and Δ wave functions of less than 1%. We have previously shown that even if there are no D states in the nucleon and/or Δ, one obtains a $N\rightarrow\Delta$ charge quadrupole moment of the right sign and magnitude [39], if one includes two-body pion and gluon pair terms (Z graphs) in the charge operator. Siegert's theorem then implies that the $E2$ transition to the $\Delta(1232)$ can be attributed to these *two-body* terms in the charge-density operator. Despite being a higher-order relativistic correction, they provide numerically the largest contribution. The reason for this is simple. Unlike the one-body charge operator, which gives nonzero matrix elements only when D states are involved, the two-body charge operators can, due to their tensorial structure in spin space, connect the dominant S waves in the nucleon and Δ by a two-quark spin-flip transition [39].

Let us recapitulate the two steps that we have made towards a gauge-invariant calculation of the $E2$ strength in the quark potential model. We have used Siegert's theorem of Eq. (27) to calculate the transverse $E2$ amplitude. The basic ingredient in Siegert's theorem is the continuity equation for the electromagnetic current. In a first step, we have explicitly proven the equivalence of the matrix elements of both sides of the restricted continuity equation (28), in which the charge operator has been approximated by the leading order one-body contribution. This was sufficient to explain the *apparent* violation of gauge invariance found in previous quark model calculations that did not include spatial exchange currents. However, it was insufficient to explain the data. In a second step, we have investigated the role of the two-body terms in the charge operator when applying Siegert's theorem. Our total result of Eq. (36) for the $E2$ transition strength is based on the commutator side of the more complete continuity equation (30). The latter involves the total charge operator including the two-body exchange charge $\rho_{[2]}$. It should be stressed that both ways of calculating the $E2$ amplitude [via the total charge or via the total spatial current operator in Eq. (30)] will give approximately the same results, and that one can take either side of the continuity equation if one is only interested in transverse electric multipoles. This is the content of Siegert's theorem. Thus, irrespective of whether G_{E2} is calculated with the spatial current or the charge operator, exchange currents dominate the $N\rightarrow\Delta$ quadrupole $E2$ transition form factor. We consider this result as compelling evidence for the important role of two-body exchange currents in the $N\rightarrow\Delta$ quadrupole transition form factor.

It would be interesting to calculate the $M1$ transition strength including exchange currents. In this case, one has to *explicitly* calculate the spatial current operator of Eq. (30) or a generalization of it. This issue, and the explicit proof of current conservation when $\rho_{[2]}$ is taken into account deserve further study. However, for the present application to the electric quadrupole transition form factor this is not necessary. We stress that our total result for the $E2$ strength is based on the more complete charge operator of Eq. (13) and Siegert's theorem.

In conclusion, the present CQM calculation with exchange currents and tensor force induced D states clearly shows that the microscopic origin of the deformation lies in the pion and gluon degrees of freedom connected with the quark-antiquark pairs (in our language: pair exchange currents) and not with the D-wave motion of valence quarks as was heretofore assumed. The double spin-flip mechanism resulting from our theory with pair-exchange currents seems to be the physical mechanism that explains the empirical $N\rightarrow\Delta$ quadrupole transition in the constituent quark model.

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