Radiative energy loss of high energy partons traversing an expanding QCD plasma

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We study analytically the medium-induced energy loss of a high energy parton passing through a finite size QCD plasma, which is expanding longitudinally according to Bjorken's model. We extend the Baier-Dokshitzer-Mueller-Peigné-Schiff formalism already applied to static media to the case of a quark which hits successive layers of matter of decreasing temperature, and we show that the resulting radiative energy loss can be as large as 6 times the corresponding one in a static plasma at the reference temperature $T=T(L)$, which is reached after the quark propagates a distance L. [S0556-2813(98)02109-8]

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I. INTRODUCTION

Recent work [1–8] on medium-stimulated gluon radiation from fast partons traversing (hot and cold) QCD matter starts from the assumption that the properties of the medium and its interactions with the energetic quark or gluon projectile do not change with time, i.e., the basic parameter μ, which is the typical transverse momentum given to the parton by a single scattering in the medium, and the parton's mean free path λ are kept constant in time. This also means, in particular, that the temperature T remains constant during the time the parton is passing through the QCD plasma.

In this paper we study analytically the propagation of a quark, of high energy E, traversing an expanding hot QCD medium, i.e., we investigate jet broadening, induced gluon radiation, and the resulting radiative energy loss of the quark. Thereby we extend the analysis of Ref. [1] to the case of time-dependent parameters μ and λ. We follow Baier-Dokshitzer-Mueller-Peigné-Schiff (BDMPS) [2,3] and we take into account our recent work [4], in which we also show the equivalence of our approach with Zakharov's [7,8] formulation of the Landau-Pomeranchuk-Migdal effect [9] for QCD.

For simplicity we consider a high energy quark entering and passing through a hot QCD medium. We may imagine the medium to be a quark-gluon plasma produced in a relativistic central $A-A$ collision, which occurs at $\tau=0$. At time τ_0 the quark enters the homogeneous plasma at high temperature T_0, which expands longitudinally with respect to the collision axis. We may consider τ_0 to be the thermalization time. For most of our results the limit $\tau_0 \rightarrow 0$ can be taken with impunity. We shall also state results for the more realistic situation where the quark is produced at $\tau_0=0$ *in* the (not yet thermalized) medium. The quark, for simplicity, is assumed to propagate in the transverse direction with vanishing longitudinal momentum, such that its energy is equal to its transverse momentum. On its way through the plasma the quark hits layers of matter which are cooled down due to the longitudinal expansion. We also assume that the plasma lives long enough so that the quark is able to propagate a given distance L within the quark-gluon phase of matter.

The properties of the expanding plasma are described by the hydrodynamical model proposed by Bjorken [10]. The parameters μ and λ depend on temperature, and therefore on time. The main relation is the scaling law

$$T^3 \tau^\alpha = \text{const}, \tag{1}$$

where τ is the proper time of the expanding medium; at rapidity $y=0$ it coincides with the distance (time) the quark has propagated through the plasma. The power α, which we approximate by a constant, may take values between $\alpha=0$ and $\alpha=1$ for an ideal fluid.

Let us state our main results for an expanding medium with $\alpha<1$. As for the static medium the transverse momentum broadening of the jet follows the random walk behavior, namely, the characteristic width $p_{\perp W}^2$ is proportional to the path length L. The radiative energy loss per unit distance $-dE/dz$ can be as large as 6 (2) times the corresponding loss in a static plasma at temperature $T=T(L)$. The number 6 (2) corresponds to the situation where the quark enters the expanding plasma from outside (is produced inside the plasma).

One expects indeed that the energy loss in an expanding medium would be larger than in the static case for the same final temperature, since the parton passes through hotter lay-

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ers during the early phase of the expansion. Perhaps the surprising feature is that there is no dependence of the enhancement factor on the initial temperature. This result has to be associated to the coherence pattern of the medium induced radiation. Gluons contributing to the energy loss require finite time for their emission, and therefore effects of the early stages of the quark-gluon plasma expansion are reduced.

This paper is organized as follows. In Sec. II we treat jet broadening due to multiple scattering in the case of an expanding plasma (with $\alpha<1$) and we estimate the characteristic width $p_{\perp W}^2$. Section III deals with the induced gluon radiation. In Sec. IV we derive the energy loss of a quark and relate it to $p_{\perp W}^2$. Following Bjorken [10] we review the main characteristics of an expanding plasma in Appendix A. The Green function of the Schrödinger-like equation with the time-dependent ''potential'' is studied in Appendix B. Integrals which are necessary in calculating the energy loss are presented in Appendix C.

II. JET BROADENING IN AN EXPANDING MEDIUM

In this section we consider a high energy parton propagating through an expanding QCD medium. By multiple scattering a transverse momentum is given to the parton. In Ref. [3] we summarized the derivation of the resulting transverse momentum broadening for the case of a static uniform medium. In the following we generalize this derivation taking into account the space-time development of the medium. As described in Appendix A we assume longitudinal expansion.

Because of the evolution of the medium the parton propagating in the transverse direction, z is affected by the position-dependent density of the plasma $\rho(z)$ and the parton cross section $d\sigma/d^2\vec{q}_\perp(\vec{q}_\perp,z)$. Based on the probabilistic interpretation[1] the master equation for the probability $f(q_\perp^2,z)$ for a quark to have transverse momentum $\vec{q}_\perp$ (orthogonal to its direction of motion) at position z is

$$\frac{\partial f(q_\perp^2,z)}{\partial z}=-\int f(q_\perp^2,z)\rho(z)\frac{d\sigma}{d^2\vec{q}_\perp'}(\vec{q}_\perp-\vec{q}_\perp',z)d^2\vec{q}_\perp' + \int f(q_\perp'^2,z)\rho(z)\frac{d\sigma}{d^2\vec{q}_\perp'}(\vec{q}_\perp'-\vec{q}_\perp,z)d^2\vec{q}_\perp' . \tag{2}$$

The first term (loss term) accounts for partons which are scattered out of the direction $\vec{q}_\perp, \vec{q}_\perp\to\vec{q}_\perp'$, and the second one (gain term) counts those partons which are scattered into the direction $\vec{q}_\perp$ from all other directions $\vec{q}_\perp'$, $\vec{q}_\perp'\to\vec{q}_\perp$. The result given in Ref. [3] is reproduced with $1/\sigma d\sigma/d^2\vec{q}_\perp$ and the mean free path $\lambda=1/\rho\sigma$ independent of z, where $\sigma=\int d^2\vec{q}_\perp d\sigma/d^2\vec{q}_\perp$. With a z-dependent mean free path

$$\lambda(z)=[\rho(z)\sigma(z)]^{-1}, \tag{3}$$

Eq. (2) can be written as

$$\lambda(z)\frac{\partial f(q_\perp^2,z)}{\partial z}=-f(q_\perp^2,z)+\int\frac{1}{\sigma}\frac{d\sigma}{d^2\vec{q}_\perp'}(\vec{q}_\perp',z)\times f[(\vec{q}_\perp'-\vec{q}_\perp)^2,z]d^2\vec{q}_\perp' , \tag{4}$$

which can be diagonalized by defining

$$\tilde{f}(b^2,z)=\int d^2\vec{q}_\perp e^{-i\vec{b}\cdot\vec{q}_\perp}f(q_\perp^2,z), \tag{5}$$

and

$$\tilde{V}(b^2,z)=\int d^2\vec{q}_\perp e^{-i\vec{b}\cdot\vec{q}_\perp}\frac{1}{\sigma}\frac{d\sigma}{d^2\vec{q}_\perp}(\vec{q}_\perp,z). \tag{6}$$

The resulting equation becomes

$$\lambda(z)\frac{\partial\tilde{f}(b^2,z)}{\partial z}=-\frac{1}{4}\vec{b}^2\tilde{v}(b^2,z)\tilde{f}(b^2,z), \tag{7}$$

where

$$\tilde{v}(b^2,z)=\frac{4}{\vec{b}^2}[1-\tilde{V}(b^2,z)] \tag{8}$$

and $\tilde{V}(0,z)=1$. As discussed in Ref. [2] in QCD $\tilde{v}(b^2,z)$ has no finite limit for $b^2\to 0$, nevertheless, Eq. (7) may be solved in a logarithmic approximation

$$\tilde{v}(b^2,z)\simeq\mu^2(z)\tilde{v}, \tag{9}$$

independent of $\vec{b}$. As in Refs. [2,3] we introduce the scale μ^2, with $\mu(z)$ representing a typical momentum transfer to the parton in a parton-medium collision, evaluated at position z. An explicit model for the scattering cross section is given by the screened ''potential'' [5]

$$V(\vec{q}_\perp^2)=\frac{1}{\sigma}\frac{d\sigma}{d^2\vec{q}_\perp}=\frac{\mu^2}{\pi(\vec{q}_\perp^2+\mu^2)^2}. \tag{10}$$

For $\vec{b}^2\simeq 0$ we obtain, using Eq. (9),

$$\frac{\partial\tilde{f}(b^2,z)}{\partial z}\simeq-\frac{\vec{b}^2}{4}\hat{q}(z)\tilde{f}(b^2,z), \tag{11}$$

with the (transport) coefficient [3] defined by

$$\hat{q}(z)\equiv\frac{\mu^2(z)}{\lambda(z)}\tilde{v}\simeq\rho(z)\int_0^{1/b^2}d^2\vec{q}_\perp\vec{q}_\perp^2\frac{d\sigma}{d^2\vec{q}_\perp}. \tag{12}$$

The solution of Eq. (11) is

$$\tilde{f}(b^2,z)=\tilde{f}_0(b^2,z_0)\exp\left\{-\frac{b^2}{4}\int_{z_0}^z dz'\hat{q}(z')\right\}, \tag{13}$$

[1] The main difference from the static case is the expression for the absorption of the parton along its path between points z_0 and z: $\exp[-(z-z_0)/\lambda]$ for the static and $\exp[-\int_{z_0}^z dz'\rho(z')\sigma(z')]$ for the expanding plasma, respectively.

from which the characteristic width of the distribution $f(q_\perp^2, z)$ is deduced:

$$p_{\perp W}^2(z) = \langle q_\perp^2(z) \rangle \equiv \int_{z_0}^{z} dz' \hat{q}(z'). \tag{14}$$

For a hot (massless) medium the z dependence of $\hat{q}(z)$ may be determined from the temperature dependence of the expanding fluid, $T = T(z)$. The leading term of the high-temperature expansion for $\hat{q}(z)$ in Eq. (12) is determined by the T dependence of the density $\rho(z)$ of the medium

$$\hat{q}(z) = \hat{q}(z_0)(T/T_0)^3. \tag{15}$$

This implies that the medium undergoes cooling from T_0 to T when the parton propagates from z_0 to z. Using Bjorken's model [10] summarized in Appendix A, we may write [see Eq. (A6)]

$$\hat{q}(z) = \hat{q}(z_0)\left(\frac{z_0}{z}\right)^{\alpha}. \tag{16}$$

Let us consider the interesting case of an interacting and expanding plasma, i.e., the case $\alpha < 1$. Inserting Eq. (16) into Eq. (14) the integration (with $\alpha = \text{const}$) gives for $z = L$ in the limit $z_0 \to 0$ the random walk behavior

$$p_{\perp W}^2(L) = \frac{\hat{q}(L)L}{(1-\alpha)}. \tag{17}$$

In general the relationship is

$$p_{\perp W}^2(L) = \hat{q}(L)L\,\frac{1-(z_0/L)^{1-\alpha}}{1-\alpha}, \tag{18}$$

which shows the delicacy of taking the limits $z_0 \to 0$, $\alpha \to 1$.

In the high temperature phase of QCD matter, we note that $\alpha = 1/(1+\Delta_1/3)$ and, since $\Delta_1 = O(\alpha_s^2)$ [cf. Eq. (A7) in Appendix A],

$$1-\alpha \simeq \Delta_1/3 = O(\alpha_s^2). \tag{19}$$

III. GLUON RADIATION SPECTRUM IN AN EXPANDING MEDIUM

Here we generalize the derivation of the soft gluon emission spectrum [2] to the case of an expanding hot medium. As described in the Introduction we assume that the fast quark is produced by a hard collision outside the medium. Let us first start with the key equations—valid in the static case—of Sec. 4 in Ref. [2], which are reexamined in Ref. [4].

Because of the Landau-Pomeranchuk-Migdal phenomenon [9], the induced spectrum is determined by an interference, essentially by the gluon emission amplitude at t_1, $\vec{f}(\vec{b}, t_2 - t_1)$, evolved in time to $t_2 > t_1$, and the complex conjugate Born amplitude $\vec{f}_0{}^*(\vec{b})$ for emission at t_2. We keep all the variables unscaled, as we did in the previous section. The Born $\vec{b}$-space amplitude for gluon emission is given by

$$\vec{f}_0(\vec{b}) = -4\pi i[1-\tilde{V}(b^2)]\frac{\vec{b}}{b^2} \simeq -i\pi\mu^2\tilde{v}\vec{b}, \tag{20}$$

where we work in the logarithmic approximation [cf. Eq. (9)]. The two-dimensional vector structure of $\vec{f}_0$ and $\vec{f}$ takes into account the two polarizations of the emitted gluon.

The induced gluon radiation spectrum (per unit length), in the limit of soft gluon energy ω and in the large N_c limit (cf. Eq. (4.24) in Ref. [2]), is given by

$$\frac{\omega dI}{d\omega dz} = \frac{\alpha_s N_c}{2\pi^2}\frac{1}{L} 2\,\mathrm{Re}\left\{ \int_0^L \frac{dt_2}{\lambda}\int_0^{t_2}\frac{dt_1}{\lambda} \times \int \frac{d^2\vec{b}}{(2\pi)^2}\vec{f}(\vec{b}, t_2 - t_1)\cdot \vec{f}_0{}^*(\vec{b}) \Bigg|_{\kappa}^{\kappa=0} \right\}. \tag{21}$$

Instead of the variable z used in the equations for $p_\perp$ broadening it is equivalent to use the time variable t. In the large N_c limit the coupling of the quark emitting a gluon is given by $\alpha_s C_F/\pi^2 \simeq \alpha_s N_c/2\pi^2$.

The factor $dt/\lambda = \rho\sigma dt$ counts the number of scatterers in the medium. The factor $1/L$ appears in Eq. (21) because the spectrum is given per unit length. λ is the mean free path of the quark and $\kappa = \lambda\mu^2/2\omega$. The κ limits indicated in Eq. (21) eliminate the medium-independent factorization contribution.

These characteristic properties are now taken into account to allow the natural generalization to the expanding medium. By properly specifying the time dependences we rewrite Eq. (21) as

$$\frac{\omega dI}{d\omega dz} = \frac{\alpha_s N_c}{2\pi^2}\frac{1}{L} 2\,\mathrm{Re}\left\{ \int_{\tau_0}^{\tau_0+L} \frac{dt_2}{\lambda(t_2)}\int_{\tau_0}^{t_2}\frac{dt_1}{\lambda(t_1)} \times \int \frac{d^2\vec{b}}{(2\pi)^2}\vec{f}(\vec{b}; t_2, t_1)\cdot \vec{f}_0{}^*(\vec{b}, t_2) \Bigg|_{\kappa}^{\omega=\infty(\kappa=0)} \right\}, \tag{22}$$

where we assume that the quark hits the medium at time τ_0 and travels a path length L, as discussed in the Introduction. The gluon propagation from $t_1 \to t_2$ is controlled by a Green function

$$\vec{f}(\vec{b}; t_2, t_1) = \int d^2\vec{b}'\, G(\vec{b}, t_2; \vec{b}', t_1)\vec{f}(\vec{b}'; t_1, t_1). \tag{23}$$

The initial condition is

$$\vec{f}(\vec{b}; t_1, t_1) = \vec{f}_0(\vec{b}, t_1), \tag{24}$$

which is given by Eq. (20), where now $\mu = \mu(t_1)$. With the definition of the coefficient $\hat{q}(t)$ given in Eq. (12) the emission spectrum is expressed in a rather symmetric form with respect to t_1 and t_2, namely, by

$$\frac{\omega dI}{d\omega dz}=\alpha_s N_c \frac{1}{L}\mathrm{Re}\left\{\int_{\tau_0}^{\tau_0+L} dt_2 \int_{\tau_0}^{t_2} dt_1 \hat{q}(t_2)\hat{q}(t_1) \times \int \frac{d^2\vec{b}}{2\pi}\int \frac{d^2\vec{b}'}{2\pi}\vec{b}\cdot\vec{b}' G(\vec{b},t_2;\vec{b}',t_1)\Bigg|_{\omega}^{\omega=\infty}\right\}. \tag{25}$$

In the logarithmic approximation the amplitude $\vec{f}(\vec{b};t_2,t_1)$, and therefore the Green function, satisfies a Schrödinger equation for the two-dimensional harmonic oscillator (actually with imaginary potential) [2,8]. For fixed t_1 the equation reads

$$i\frac{\partial}{\partial t_2}\vec{f}(\vec{b};t_2,t_1)=\left[+\frac{1}{2\omega}\vec{\nabla}_b{}^2-\frac{1}{2}\omega\omega_0^2(t_2)\vec{b}^2\right]\vec{f}(\vec{b};t_2,t_1), \tag{26}$$

with $\omega_0^2(t)\equiv i\hat{q}(t)/\omega$. For an expanding medium the frequency of the oscillator $\omega_0(t)$ is time dependent. In the Bjorken model [10] the temperature of the hot medium scales with time, as given in Eq. (1), which translates to

$$\omega_0^2(t)=\omega_0^2(\tau_0)\left(\frac{\tau_0}{t}\right)^{\alpha}. \tag{27}$$

The explicit expression for the Green function is derived in Appendix B [12], and given by Eq. (B13). In order to perform the $\vec{b}$-space integrations in Eq. (25) it is convenient to change variables

$$z_{i,f}=2i\nu\omega_0(\tau_0)\tau_0\left(\frac{t_{1,2}}{\tau_0}\right)^{1/2\nu}, \tag{28}$$

with the index $\nu=1/(2-\alpha)$, such that $\frac{1}{2}\leqslant\nu<1$. The $\vec{b}$-space integral is given by

$$\begin{aligned} I&\equiv\int \frac{d^2\vec{b}}{2\pi}\int \frac{d^2\vec{b}'}{2\pi}\vec{b}\cdot\vec{b}' G(\vec{b},t_2;\vec{b}',t_1)\\ &=-\frac{1}{\pi}\left[\frac{2\nu\tau_0}{\omega(2i\nu\omega_0(\tau_0)\tau_0)^{2\nu}}\right]^2\\ &\quad\times\frac{(z_iz_f)^{2\nu-2}}{[I_{\nu-1}(z_i)K_{\nu-1}(z_f)-I_{\nu-1}(z_f)K_{\nu-1}(z_i)]^2} \end{aligned} \tag{29}$$

in terms of modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ [13]. Inserting I and the time dependence of the coefficient $\hat{q}(t)$ as specified in Eq. (16) into the spectrum (25) a rather simple expression is obtained:

$$\frac{\omega dI}{d\omega dz}=\frac{\alpha_s N_c}{\pi}\frac{1}{L}\frac{1}{4\nu^2}\mathrm{Re}\left\{\int_{\tau_0}^{\tau_0+L}\frac{dt_2}{t_2}\int_{\tau_0}^{t_2}\frac{dt_1}{t_1} \times\frac{1}{[I_{\nu-1}(z_i)K_{\nu-1}(z_f)-I_{\nu-1}(z_f)K_{\nu-1}(z_i)]^2}\Bigg|_{\omega}^{\omega=\infty}\right\}. \tag{30}$$

If we set

$$x_{i,f}=\tau_0\left(\frac{t_{1,2}}{\tau_0}\right)^{1/2\nu}, \tag{31}$$

and express the function $K_\nu(z)$ in terms of $I_{\pm\nu}(z)$ (excluding the case $\nu=1$), we arrive at

$$\frac{\omega dI}{d\omega dz}=\frac{\alpha_s N_c}{\pi}\frac{1}{L}\left[\frac{2\sin\pi(\nu-1)}{\pi}\right]^2\mathrm{Re}\left\{\int_{\tau_0}^{\hat{\tau}_0}\frac{dx_i}{x_i}\int_{\tau_0}^{x_i}\frac{dx_f}{x_f} \times\frac{1}{[I_{\nu-1}(2i\nu\omega_0x_i)I_{1-\nu}(2i\nu\omega_0x_f)-(x_i\leftrightarrow x_f)]^2}\Bigg|_{\omega}^{\omega=\infty}\right\}, \tag{32}$$

where we put $\omega_0\equiv\omega_0(\tau_0)$ and $\hat{\tau}_0\equiv\tau_0(1+L/\tau_0)^{1/2\nu}$ for shorter notation.

In order to compare with our previous result for the nonexpanding plasma [2] we take $\nu=1/2$. The induced spectrum (32) then becomes

$$\frac{\omega dI}{d\omega dz}=\frac{\alpha_s N_c}{\pi L}\mathrm{Re}\int_{\tau_0}^{L+\tau_0}\frac{dx_i}{x_i}\int_{\tau_0}^{x_i}\frac{dx_f}{x_f} \times\frac{4}{\pi^2[I_{1/2}(i\omega_0x_f)I_{-1/2}(i\omega_0x_i)-(x_i\leftrightarrow x_f)]^2}\Bigg|_{\omega}^{\omega=\infty}. \tag{33}$$

Using $I_{1/2}(z)=\sqrt{2/\pi}\sinh z/\sqrt{z}$ and $I_{-1/2}(z)=\sqrt{2/\pi}\cosh z/\sqrt{z}$ [13] gives

$$\frac{\omega dI}{d\omega dz}=\frac{\alpha_s N_c}{\pi L}\mathrm{Re}\int_0^L dx_i\int_0^{x_i}dx_f\left[\frac{i\omega_0}{\sinh[i\omega_0(x_f-x_i)]}\right]^2\Bigg|_{\omega}^{\omega=\infty}, \tag{34}$$

where we have put $\tau_0=0$. The remaining integrals can be performed explicitly,

$$\begin{aligned}\frac{\omega dI}{d\omega dz}&=\frac{\alpha_s N_c}{\pi L}\mathrm{Re}\int_0^L\frac{dx_i}{x_i}\left[\frac{i\omega_0x_i}{\tanh(i\omega_0x_i)}-1\right]\\ &=\frac{\alpha_s N_c}{\pi L}\mathrm{Re}\left\{\ln\left(\frac{\sinh(i\omega_0L)}{i\omega_0L}\right)\right\}\\ &=\frac{\alpha_s N_c}{\pi L}\ln\left|\frac{\sin(\omega_0L)}{\omega_0L}\right|.\end{aligned} \tag{35}$$

This is the radiation spectrum in the $N_c\to\infty$ limit, derived and discussed in Refs. [2,4] for a hard quark entering the static QCD medium and radiating a soft gluon.

On can easily go beyond the large N_c limit and the soft gluon approximation in Eqs. (32) and (35). For a particle in an arbitrary color representation R, ω_0^2 should be replaced by

$$\omega_0^2\frac{N_c}{2C_R}\left(1-x+\frac{C_R}{N_c}x^2\right), \tag{36}$$

where x is the gluon energy fraction $x=\omega/E$. In addition the right-hand side of Eqs. (32) and (35) should be multiplied by

$$\frac{2C_R}{N_c}\left(1-x+\frac{x^2}{2}\right) \quad (37)$$

for a spin $\frac{1}{2}$ fermion, and by

$$\frac{2C_R}{N_c}\frac{[1+x^4+(1-x)^4]}{2(1-x)} \quad (38)$$

for a spin 1 particle (e.g., gluon, $C_R=N_c$) [4].

IV. ENERGY LOSS IN AN EXPANDING MEDIUM

Next we integrate the radiation spectrum Eq. (32) with respect to the gluon energy ω in order to obtain the energy loss per unit length

$$-\frac{dE}{dz}=\int_0^E d\omega\frac{\omega dI}{d\omega dz}, \quad (39)$$

where we extend the limit $E\to\infty$. In analogy with the static case [2] we introduce new integration variables x and $\hat{z}$

$$2i\nu\omega_0(\tau_0)x_i=i(1+i)x\equiv\hat{x},\quad \hat{z}=\frac{x_f}{x_i}, \quad (40)$$

leading to

$$\omega=\frac{2\nu^2\hat{q}(\tau_0)}{x^2}x_i^2. \quad (41)$$

Taking $\tau_0=0$ and performing the x_i integration

$$\int_0^{\tau_0(L/\tau_0)^{1/2\nu}}dx_i x_i\hat{q}(\tau_0)=\frac{1}{2}\tau_0^2\hat{q}(\tau_0)\left(\frac{L}{\tau_0}\right)\equiv\frac{1}{2}\hat{q}(L)L^2, \quad (42)$$

the energy loss can be written as

$$-\frac{dE}{dz}=\frac{2\alpha_s N_c}{\pi}\left[\frac{2\nu\sin\pi(\nu-1)}{\pi}\right]^2\hat{q}(L)L\,\mathrm{Re}\int_0^1\frac{d\hat{z}}{\hat{z}}\int_0^\infty\frac{dx}{x^3}\times\frac{1}{[I_{\nu-1}(\hat{x})I_{1-\nu}(\hat{x}\hat{z})-I_{\nu-1}(\hat{x}\hat{z})I_{1-\nu}(\hat{x})]^2}\Bigg|_x^{x=0}. \quad (43)$$

In order to obtain the subtraction term we expand the modified Bessel functions around $x=0$ [13], $I_\nu(z)\simeq(\frac{1}{2}z)^\nu/\Gamma(\nu+1)$. This enables us to write Eq. (43) as

$$-\frac{dE}{dz}=\frac{2\alpha_s N_c}{\pi}\hat{q}(L)L\left[\frac{2\Gamma(\nu+1)\Gamma(2-\nu)\sin\pi(\nu-1)}{\pi}\right]^2\times\int_0^1\frac{d\hat{z}}{\hat{z}[\hat{z}^{\nu-1}-\hat{z}^{1-\nu}]^2}I(\nu,\hat{z}), \quad (44)$$

where the function $I(\nu,\hat{z})$ is defined in Eq. (C1) and evaluated in Appendix C. We integrate over the $\hat{z}$ variable (see Appendix C) and obtain the analytic expression for the energy loss

$$-\frac{dE}{dz}=\frac{\alpha_s N_c}{2}\hat{q}(L)L\left[\frac{2\Gamma(\nu+1)\Gamma(2-\nu)\sin\pi(\nu-1)}{\pi}\right]^2 I(\nu), \quad (45)$$

where the function

$$I(\nu)=\frac{1}{4(1-\nu)^2(2-\nu)}, \quad (46)$$

for $\frac{1}{2}\leqslant\nu<1$ is derived in Eqs. (C4)–(C10). Notice that for $\nu=1/2$ one recovers the energy loss for a quark traversing a static medium of size L, as discussed in Refs. [2–4]

$$-\frac{dE}{dz}\bigg|_{\mathrm{static}}=\frac{\alpha_s N_c}{12}\hat{q}(L)L. \quad (47)$$

Equations (45) and (47) require that the ω integration in Eq. (39) be dominated by small x gluons. These formulas remain true beyond the large N_c limit. The color properties of the traversing particle are contained in the (transport) coefficient $\hat{q}(L)$ given in Eq. (12).

Using Eqs. (45) and (47) one finds

$$-\frac{dE}{dz}=\frac{6\nu^2}{2-\nu}\left(-\frac{dE}{dz}\bigg|_{\mathrm{static}}\right)=\frac{6}{(2-\alpha)(3-2\alpha)}\left(-\frac{dE}{dz}\bigg|_{\mathrm{static}}\right),\quad \alpha=2-\frac{1}{\nu}. \quad (48)$$

In the case where the quark is produced in the medium rather than outside,

$$-\frac{dE}{dz}=2\nu\left(-\frac{dE}{dz}\bigg|_{\mathrm{static}}\right)=\frac{2}{2-\alpha}\left(-\frac{dE}{dz}\bigg|_{\mathrm{static}}\right) \quad (49)$$

replaces Eq. (48), where $-dE/dz|_{\mathrm{static}}$ also corresponds to a quark produced in the medium and is 3 times the expression given in Eq. (47) [4].

We notice that the limit $\nu=1(\alpha=1)$ for an expanding ideal relativistic plasma can be taken. In this limit the maximal loss is achieved. It is bigger by a factor 6 for a quark produced outside (2 for inside) than in the corresponding static case. In this comparison the temperature is taken after the expansion. The coefficient $\hat{q}(L)=\hat{q}[T(L)]$ has to be evaluated at the temperature $T(L)$ the quark finally ''feels'' after having passed the distance L through the medium which during this propagation cools down to $T(L)$. Obviously in the expanding medium the energy loss should be larger than in the case of constant temperature, since for the same final temperature the quark has passed hotter, i.e., denser layers of the plasma. It is remarkable that the initial temperature T_0 of the hot medium does not enter the formulas (48) and (49), although T_0 is actually diverging in the limit $\tau_0\to0$.

As a consequence it is straightforward to generalize the relationship between energy loss and $p_\perp$ broadening derived in Ref. [3] for static nuclear matter to the case of an expanding plasma. We derive the relationships for a quark approaching the medium

$$-\frac{dE}{dz}=\frac{\alpha_s N_c}{2}\frac{1}{(2-\alpha)(3-2\alpha)}L\frac{\partial}{\partial L}p_{\perp W}^2(L) \qquad (50)$$

and for a quark produced in the medium

$$-\frac{dE}{dz}=\frac{\alpha_s N_c}{2}\frac{1}{2-\alpha}L\frac{\partial}{\partial L}p_{\perp W}^2(L), \qquad (51)$$

relating the energy loss per unit distance in a hot expanding medium with the typical transverse momentum squared (14) a jet receives in traversing a length L of a longitudinally expanding plasma. For $\alpha=0$ the results of Ref. [4] are reproduced.

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APPENDIX A: PROPERTIES OF AN EXPANDING PLASMA

Here we recall and briefly summarize the main properties of the space-time evolution of a hadronic fluid, which is produced by highly relativistic nucleus-nucleus collisions. We consider a hydrodynamical model and follow Bjorken [10] in assuming one-dimensional longitudinal expansion.

In order to obtain the dependence of the fluid's temperature $T=T(\tau)$ on the proper-time τ we use the conservation law for the entropy density s,

$$ds/d\tau+s/\tau=0. \qquad (A1)$$

We take into account the thermodynamic equation for the pressure

$$dp/dT=s[T(\tau)] \qquad (A2)$$

and express p in terms of a monotonically increasing function of temperature $n(T)$,

$$p=\frac{\pi^2}{90}n(T)T^4. \qquad (A3)$$

Defining the parameter

$$\Delta_1\equiv\frac{T}{n(T)}\frac{dn(T)}{dT}, \qquad (A4)$$

which we assume to be temperature independent.[2] It follows from Eqs. (A1)–(A4)

$$\frac{dT(\tau)}{d\tau}=-v_s^2\frac{T}{\tau}, \quad v_s^2=(3+\Delta_1)^{-1/2}, \qquad (A5)$$

with v_s the sound velocity. In the approximation of $v_s=\text{const}$ Eq. (A5) gives

$$(T/T_0)^3=(\tau_0/\tau)^{\alpha}, \quad \text{with} \quad \alpha\equiv 3v_s^2. \qquad (A6)$$

The parameter α is bounded by $0\leqslant\alpha\leqslant 1$, where $\alpha=0$ means constant temperature and a static medium. $\alpha=1$ is an ideal relativistic plasma.

In perturbative thermal QCD [11] the parameter Δ_1 turns out to be small, indicating small deviations from ideal gas behavior. For the case of a gluon gas

$$\Delta_1=\frac{165}{8}\left(\frac{\alpha_s}{\pi}\right)^2[1+O(\sqrt{\alpha_s})], \qquad (A7)$$

in terms of the QCD coupling constant α_s, which at very high temperatures should be evaluated at the scale T.

APPENDIX B: GREEN FUNCTION

In order to discuss the solution of Eq. (26) we make a logarithmic approximation and assume that $\ln(1/\vec{b}^2)$ is slowly varying for small $\vec{b}^2$. We then have to solve the Schrödinger equation for a two-dimensional harmonic oscillator with time-dependent frequency. Using variables familiar from quantum mechanics the equation is

$$i\frac{\partial}{\partial t}\vec{f}(\vec{b},t)=\left[-\frac{1}{2m}\vec{\nabla}_b^2+\frac{1}{2}m\omega_0^2(t)\vec{b}^2\right]\vec{f}(\vec{b},t), \qquad (B1)$$

where the mass of the oscillator is identified with the energy of the emitted gluon, $m\equiv-\omega$, and the time dependence of the frequency is given by the power behavior

$$\omega_0^2(t)=\omega_0^2(t_0)(t_0/t)^{\alpha}, \qquad (B2)$$

where the parameter α was discussed in Appendix A and

$$\omega_0^2(t_0)=\frac{i\hat{q}(t_0)}{\omega}. \qquad (B3)$$

The Green function of the Schrödinger equation (B1) can be written in the form [12]

$$G(\vec{b},t;\vec{b}',t')=\frac{m}{2\pi i D(t,t')}\exp\{iS_{\text{cl}}(\vec{b},t;\vec{b}',t')\}, \qquad (B4)$$

where the fluctuation determinant satisfies the homogeneous differential equation

$$\frac{d^2}{dt^2}D(t,t')+\omega_0^2(t)D(t,t')=0, \qquad (B5)$$

with the initial conditions

$$D(t',t')=0, \quad \frac{d}{dt}D(t,t')|_{t=t'}=1. \qquad (B6)$$

[2] Possible T dependences of Δ_1 are sketched in Ref. [10].

The classical action $S_{\rm cl}$ in Eq. (B4) is determined by the classical path $\vec{b}_{\rm cl}(t)$ obeying

$$\frac{d^2}{dt^2}\vec{b}_{\rm cl}(t)+\omega_0^2(t)\vec{b}_{\rm cl}(t)=0, \tag{B7}$$

with

$$\vec{b}_{\rm cl}(t)=\vec{b} \quad \text{and} \quad \vec{b}_{\rm cl}(t')=\vec{b}'. \tag{B8}$$

It follows that

$$S_{\rm cl}(\vec{b},t;\vec{b}',t')=\frac{m}{2}\left[\vec{b}_{\rm cl}(t)\cdot\frac{d}{dt}\vec{b}_{cl}(t)\right]\Bigg|_{t'}^{t}. \tag{B9}$$

The explicit solution of Eq. (B5) with Eq. (B2) is found in terms of modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ [13] to be

$$\begin{aligned}D(t,t')=&\frac{2\nu t_0}{[2i\nu\omega_0(t_0)t_0]^{2\nu}}(zz')^{\nu}[I_\nu(z)K_\nu(z')\\&-K_\nu(z)I_\nu(z')],\end{aligned} \tag{B10}$$

where we introduce the variables

$$z=z(t)\equiv 2i\nu\omega_0(t_0)t_0(t/t_0)^{1/2\nu}, \quad z'=z(t'), \tag{B11}$$

with

$$\nu=1/(2-\alpha), \tag{B12}$$

such that $1/2\leqslant\nu\leqslant 1$.

Using the solution to Eqs. (B7) and (B8) in Eq. (B9) gives the Green function (B4) as

$$\begin{aligned}G(\vec{b},t;\vec{b}',t')=&\frac{i\omega}{2\pi D(t,t')}\\&\times\exp\left\{\frac{-i\omega}{2D(t,t')}[c_1\vec{b}^2+c_2\vec{b}'^2-2\vec{b}\cdot\vec{b}']\right\},\end{aligned} \tag{B13}$$

with the coefficients

$$\begin{aligned}c_1&=z(z'/z)^{\nu}[I_{\nu-1}(z)K_\nu(z')+K_{\nu-1}(z)I_\nu(z')],\\c_2&=z'(z/z')^{\nu}[K_\nu(z)I_{\nu-1}(z')+I_\nu(z)K_{\nu-1}(z')].\end{aligned} \tag{B14}$$

The case $\nu=1/2$ is especially easy to handle, and allows a direct comparison with the results already obtained in Refs. [2,4]. The variables given in Eq. (B11) become $z(z')=i\omega_0 t(t')$ with $\omega_0\equiv\sqrt{i\mu^2\tilde{v}/\lambda\omega}$. The functions $I_{1/2}(z)$ and $K_{1/2}(z)$ are expressed in terms of hyperbolic functions [13], so that the determinant (B10) simplifies to

$$D(t,t')=\frac{1}{i\omega_0}\sinh(z-z')=\frac{1}{\omega_0}\sin\omega_0(t-t').$$

We note that the Green function (B13) is time-translational invariant for $\nu=1/2$. It correctly reproduces the result of Eq. (5.6) in Ref. [2].

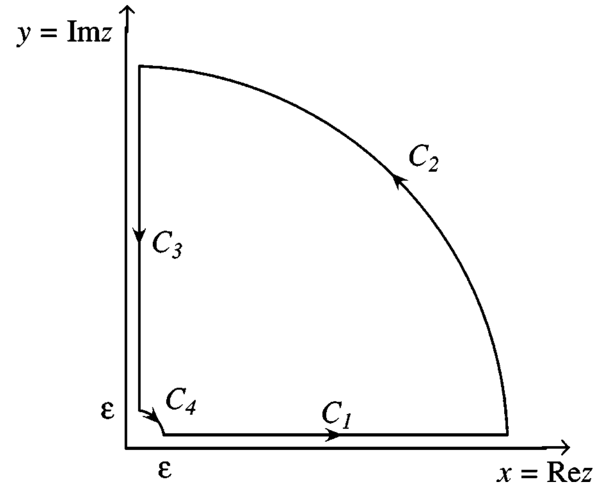

FIG. 1. Integration contour for the integral $I(\nu,\hat{z})$.

APPENDIX C: THE INTEGRALS $I(\nu,\hat{z})$ AND $I(\nu)$

Here we evaluate the integral

$$\begin{aligned}I(\nu,\hat{z})\equiv&\mathrm{Re}\int_0^\infty\frac{dx}{x^3}\\&\times\left\{1-\frac{[\hat{z}^{\nu-1}-\hat{z}^{1-\nu}]^2[\Gamma(\nu)\Gamma(2-\nu)]^{-2}}{[I_{1-\nu}(\hat{x})I_{\nu-1}(\hat{x}\hat{z})-I_{1-\nu}(\hat{x}\hat{z})I_{\nu-1}(\hat{x})]^2}\right\},\end{aligned} \tag{C1}$$

with $\hat{x}\equiv i(1+i)x$, by using the integration contour $C_{1,\ldots,4}$ in the complex $z=(x+iy)$ plane, which we already introduced in Ref. [2] (see Appendix D), and which for convenience is reproduced here in Fig. 1.

Following the detailed discussion given in Ref. [2] the integral is performed by calculating the residue of the pole at $x=0$. Since the contribution along C_2 vanishes, we find the result for $I(\nu,\hat{z})$ by adding the contributions from the paths C_1 and C_3, i.e.,

$$\begin{aligned}I(\nu,\hat{z})&=\frac{1}{2}[I(\nu,\hat{z})]_{C_1+C_3}\\&=-\frac{1}{2}[I(\nu,\hat{z})]_{C_4}\\&=+\frac{2\pi i}{8}\ \text{Residue}\left[\frac{1}{x^3}\{\cdot/.\}\right]\Bigg|_{x=0},\end{aligned} \tag{C2}$$

leading to

$$I(\nu,\hat{z})=\frac{\pi\hat{z}[[1/(2-\nu)](\hat{z}^{\nu-2}-\hat{z}^{2-\nu})+(1/\nu)(\hat{z}^{\nu}-\hat{z}^{-\nu})]}{4(\hat{z}^{\nu-1}-\hat{z}^{1-\nu})}. \tag{C3}$$

For the reader who may not be convinced by the above arguments we note that we have evaluated the integral (C1) numerically using the program MATHEMATICA [14]. Stable results were obtained agreeing with Eq. (C3) for a large domain of z and ν, $0.1\leqslant z\leqslant 0.8$, and $\frac{1}{2}\leqslant\nu\leqslant 0.95$.

In Eq. (45) we stated that the energy loss $-dE/dz$ is proportional to the integral $I(\nu)$ defined by

$$I(\nu)\equiv\frac{4}{\pi}\int_0^1\frac{dz}{z}\,\frac{I(\nu,z)}{[z^{\nu-1}-z^{1-\nu}]^2}$$
$$=\int_0^1 dz\,\frac{[1/(2-\nu)][z^{\nu-2}-z^{2-\nu}]+(1/\nu)[z^{\nu}-z^{-\nu}]}{[z^{\nu-1}-z^{1-\nu}]^3}. \tag{C4}$$

For the special case $\nu=1/2$

$$I(1/2)=\frac{2}{3}\int_0^1 dz=2/3. \tag{C5}$$

For ν in the interval $\frac{1}{2}\leqslant\nu<1$ we evaluate $I(\nu)$ as follows. We change the integration variable to

$$t=z^{2(1-\nu)}. \tag{C6}$$

We regularize the integrand near $t=1$ by $(1-t)^{-3}\to(1-t)^{-3+\varepsilon}$, $\varepsilon>0$, and arrive at

$$I_\varepsilon(\nu)\equiv\frac{1}{2\nu(1-\nu)(2-\nu)}\int_0^1 dt\times\frac{\nu[1-t^{1/(1-\nu)+1}]+(2-\nu)[t^{1/(1-\nu)}-t]}{(1-t)^{3-\varepsilon}}, \tag{C7}$$

where the limit $\varepsilon\to 0$ is to be taken after the integration. Using the Euler β function [13] gives

$$I_\varepsilon(\nu)=\frac{1}{2\nu(1-\nu)(2-\nu)}\Bigg\{\frac{\nu}{\varepsilon-2}-\frac{2-\nu}{(\varepsilon-1)(\varepsilon-2)}-\frac{\nu(2-\nu)\Gamma(\varepsilon+1)}{(1-\nu)^2\varepsilon(\varepsilon-1)(\varepsilon-2)}\bigg[\frac{\Gamma[1/(1-\nu)-1]}{\Gamma[\varepsilon+1/(1-\nu)-1]}-\frac{\Gamma[1/(1-\nu)]}{\Gamma[\varepsilon+1/(1-\nu)]}\bigg]\Bigg\}. \tag{C8}$$

One can easily check that $I_\varepsilon(\nu)$ is regular at $\varepsilon=0$. Using [13]

$$\Gamma(z)/\Gamma(\epsilon+z)\xrightarrow[\varepsilon\to 0]{}1-\varepsilon\psi(z)+O(\varepsilon^2) \tag{C9}$$

in terms of the digamma function $\psi(z)$, and with the recurrence formula $\psi(z+1)=\psi(z)+1/z$, we finally obtain

$$I_\varepsilon(\nu)\xrightarrow[\varepsilon\to 0]{}I(\nu)=\frac{1}{4(1-\nu)^2(2-\nu)},\quad \frac{1}{2}\leqslant\nu<1. \tag{C10}$$

This is in agreement with Eq. (C5) for $\nu=1/2$.

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