Two center light cone calculation of pair production induced by ultrarelativistic heavy ions

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An exact solution of the two center time-dependent Dirac equation for pair production induced by ultrarelativistic heavy ion collisions is presented. Cross sections to specific final states approach those of perturbation theory. Multiplicity rates are reduced from perturbation theory. [S0556-2813(98)03409-8]

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I. INTRODUCTION

In this paper, we shall compute the production of electron-positron pairs in the central region for highly relativistic charged ions. We base our calculations on an exact solution of the time dependent Dirac equation in the ultrarelativistic limit. We show that except for possible cutoff effects the exact cross section for any specific final electron positron state equals the perturbation theory result. On the other hand, we argue that the rate for processes correlated with low impact parameters (such as pair multiplicity) are reduced from the perturbation theory rate.

Our notation will vary at times between that of high energy physics (light cone variables and Dirac γ matrices) and that of atomic physic (Dirac $\boldsymbol{\alpha}$ and β matrices and the usual four momentum and space time variables) as convenience and the connection to previous work dictates. The relation between different notations should be clear.

To properly define what we mean by central region and highly relativistic, it is useful to define light cone coordinates,

$$p^{\pm}=\frac{1}{\sqrt{2}}(p_0\pm p_z), \quad x^{\pm}=\frac{1}{\sqrt{2}}(t\pm z), \tag{1}$$

where the z direction is the beam direction. In this coordinate system, the invariant dot product of momenta is $p\cdot q=p^+q^-+p^-q^+-p_T\cdot q_T$. We will take the nucleus propagating along the positive z axis to have a large light cone momentum P^+ and that along the negative z axis to have large light cone momenta Q^-. By highly relativistic we mean, for example, the case of RHIC, colliding ions each with $\gamma(=1/\sqrt{1-v^2})$ of 100. Note that the Compton wavelength of the electron is large compared to the radius of either nucleus and thus the nucleus can be considered a point charge αZ.

To define what we mean by centrally produced, we define the lightcone momentum fractions of an electron or positron to be

$$x=p^+/P^+ \tag{2}$$

and

$$y=p^-/Q^-. \tag{3}$$

Central production will mean electrons and positrons which have $x,y\ll 1$. This can be satisfied for some range of longitudinal momenta of the electron or positron so long as the condition on the Compton wavelength is satisfied.

When the electron or positron in the central region sees the moving nuclei, it sees two oppositely Lorentz boosted Coulomb fields. These are the Liénard-Wiechert potentials

$$V(\boldsymbol{\rho},z,t)=\frac{\alpha Z(1-v\,\alpha_z)}{\sqrt{[(\mathbf{b}/2-\boldsymbol{\rho})/\gamma]^2+(z-vt)^2}}+\frac{\alpha Z(1+v\,\alpha_z)}{\sqrt{[(\mathbf{b}/2+\boldsymbol{\rho})/\gamma]^2+(z+vt)^2}}. \tag{4}$$

$\mathbf{b}$ is the impact parameter, perpendicular to the z axis along which the ions travel, $\boldsymbol{\rho}$, z, and t are the coordinates of the potential, α_z is the Dirac matrix, and Z,v and γ are the charge, velocity, and γ factor of the oppositely moving ions. We have specialized to the case of equal Z ions; unequal Z would only require a trivial change in what follows. If one makes the gauge transformation on the wave function

$$\psi=e^{-i\chi(\mathbf{r},t)}\psi', \tag{5}$$

where

$$\chi(\mathbf{r},t)=\frac{\alpha Z}{v}\ln[\gamma(z-vt)+\sqrt{b^2/4+\gamma^2(z-vt)^2}]-\frac{\alpha Z}{v}\ln[\gamma(z+vt)+\sqrt{b^2/4+\gamma^2(z+vt)^2}], \tag{6}$$

the interaction $V(\rho,z,t)$ is gauge transformed to [1]

$$V(\boldsymbol{\rho},z,t)=\frac{\alpha Z(1-v\,\alpha_z)}{\sqrt{[(\mathbf{b}/2-\boldsymbol{\rho})/\gamma]^2+(z-vt)^2}}-\frac{\alpha Z(1-(1/v)\,\alpha_z)}{\sqrt{b^2/4\gamma^2+(z-vt)^2}}+\frac{\alpha Z(1+v\,\alpha_z)}{\sqrt{[(\mathbf{b}/2+\boldsymbol{\rho})/\gamma]^2+(z+vt)^2}}-\frac{\alpha Z(1+(1/v)\,\alpha_z)}{\sqrt{b^2/4\gamma^2+(z+vt)^2}}. \tag{7}$$

This gauge transformation reduces the range in $(z\pm vt)$ to more closely map the **B** and **E** fields (which have the denominator to the $\frac{3}{2}$ power rather than the $\frac{1}{2}$ power of the untransformed Lorentz gauge). In the ultrarelativistic limit (ignoring correction terms in $[(\mathbf{b}/2\pm\boldsymbol{\rho})/\gamma]^2$) Eq. (7) takes the form [2]

$$V(\boldsymbol{\rho},z,t)=\delta(z-t)(1-\alpha_z)\Lambda^-(\boldsymbol{\rho})+\delta(z+t)(1+\alpha_z)\Lambda^+(\boldsymbol{\rho}) \tag{8}$$

where

$$\Lambda^{\pm}(\boldsymbol{\rho})=-Z\alpha\ln\frac{(\boldsymbol{\rho}\pm\mathbf{b}/2)^2}{(b/2)^2}. \tag{9}$$

The potential as written here will be referred to as the singular gauge solution. In this gauge the field vanishes everywhere except along the lightcone $x^{\pm}=0$. We can gauge transform from this field to the less singular light cone gauge by again utilizing $\psi=e^{-i\chi(\mathbf{r},t)}\psi'$ where

$$\chi(\mathbf{r},t)=\theta(t-z)\Lambda^-(\boldsymbol{\rho})+\theta(t+z)\Lambda^+(\boldsymbol{\rho}). \tag{10}$$

This leads to added gauge terms in the transformed potential

$$\begin{aligned}-\frac{\partial\chi(\mathbf{r},t)}{\partial t}-\boldsymbol{\alpha}\cdot\nabla=&-\delta(z-t)(1-\alpha_z)\Lambda^-(\boldsymbol{\rho})\\&-\delta(z+t)(1+\alpha_z)\Lambda^+(\boldsymbol{\rho})\\&-\theta(t-z)\boldsymbol{\alpha}\cdot\nabla\Lambda^-(\boldsymbol{\rho})\\&-\theta(t+z)\boldsymbol{\alpha}\cdot\nabla\Lambda^+(\boldsymbol{\rho})\end{aligned} \tag{11}$$

and we thus obtain the light cone gauge

$$V(\boldsymbol{\rho},z,t)=-\theta(t-z)\boldsymbol{\alpha}\cdot\nabla\Lambda^-(\boldsymbol{\rho})-\theta(t+z)\boldsymbol{\alpha}\cdot\nabla\Lambda^+(\boldsymbol{\rho}), \tag{12}$$

with $\boldsymbol{\alpha}$ the Dirac matrix. This construction fits in well with the similar non-Abelian treatment for QCD [3].

The reason why we expect that we can exactly compute pair production in the central region should be obvious. In either of the gauges above, the propagation of the electron or positron is essentially trivial. Except at $x^{\pm}=0$, the electron or positron propagates as a free particle. Therefore, to construct the propagator for the electron, one needs to solve a boundary value problem with free propagation everywhere except at the surfaces of discontinuity at the lightcone $x^{\pm}=0$.

In fact to solve the problem of pairs production is arguably a little simpler than constructing the propagator. What we will do is assume we begin with a negative energy solution of the Dirac equation in the initial state. This will correspond to a positive energy positron. We will then let it propagate forward in time. At late times we will compute the amplitude that this state is a positive energy electron.

The organization of this paper is as follows. In the second section, we compute the amplitude for electron positron pair production in the singular (δ function) gauge. The technique used will closely follow that previously used in the exact calculation of bound electron positron pair production [4], and will use conventional Green's functions methods. In the third section we work in light cone gauge. We show that the result for the pair production amplitude agrees between the two computations. In the fourth section, we discuss the cross sections and their relation to perturbation theory. In Appendix A, we write down our conventions for light cone coordinates and projection operators in the Dirac equation. In Appendix B, we evaluate the transverse integral of the Coulomb field which is necessary for the solution in both gauges. In Appendix C, we show how to solve the Dirac equation across a boundary corresponding to a charged nucleus in light cone gauge. We show how this result maps into the similar result computed in singular gauge.

II. SOLUTION IN SINGULAR GAUGE FOR PAIR PRODUCTION

A strategy to find the exact semiclassical solution for pair production in a two-center model is to first find the exact wave function at the point of interaction (in terms of the appropriate Green's function) and then to construct the exact amplitude (incorporating the initial exact wave function, the interaction, and the final plane wave function).

In particular, one would like to solve the two center time-dependent Dirac equation,

$$[\boldsymbol{\alpha}\mathbf{p}+\beta-V(\boldsymbol{\rho},z,t)-E]\Psi(\mathbf{r},t)=0, \tag{13}$$

in terms of a Dirac Green's function based on the plane wave solutions,

$$[\boldsymbol{\alpha}\mathbf{p}+\beta-E]\phi(\mathbf{r},t)=0. \tag{14}$$

One obtains

$$\begin{aligned}[\boldsymbol{\alpha}\mathbf{p}+\beta-E]\Psi(\mathbf{r},t)=&V(\boldsymbol{\rho},z,t)\Psi(\mathbf{r},t)\\&+[\boldsymbol{\alpha}\mathbf{p}+\beta-E]\phi(\mathbf{r},t).\end{aligned} \tag{15}$$

If one acts on both sides with the Dirac plane wave Green's function

$$G_0=[\boldsymbol{\alpha}\mathbf{p}+\beta-E]^{-1} \tag{16}$$

one obtains the usual form

$$\Psi=\phi+G_0V\Psi. \tag{17}$$

The Green's function G_0 obeys the equation

$$\begin{aligned}&[\boldsymbol{\alpha}\mathbf{p}+\beta-E]G_0(\mathbf{r},t;\mathbf{r}',t')\\&\quad=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\\&\quad=\frac{1}{(2\pi)^4}\int d^3k\,d\omega e^{ik(\mathbf{r}-\mathbf{r}')-i\omega(t-t')},\end{aligned} \tag{18}$$

and the solution is

$$G_0(\mathbf{r},t;\mathbf{r}',t')=\frac{1}{(2\pi)^4}\int d^3k\,d\omega \times[\boldsymbol{\alpha}\mathbf{k}+\beta+\omega]\frac{e^{ik(\mathbf{r}-\mathbf{r}')-i\omega(t-t')}}{k^2+1-\omega^2}. \quad (19)$$

Note the positive sign of ω in the brackets. We will get the boundary conditions later.

We now make use of the usual light cone coordinates, x^+,x^-,q^+, and q^-, as in Eq. (1) in place of t,z,q_0, and q_z. Begin with a plane wave ϕ representing one of the electrons in the filled negative energy Dirac sea. Our convention will be that q_0 is positive, corresponding to the positron energy we eventually are interested in, and likewise the three momentum is opposite in sign to that of the negative electron

$$\phi=v(q,s_i)e^{-iqx}=v(q,s_i)e^{-iq_\perp x_\perp+iq^+x^-+iq^-x^+}. \quad (20)$$

The function $v(q,s_i)$ is the usual spinor of a positron. The Green's function can likewise be written in light cone coordinates

$$G_0(x;x')=\frac{1}{(2\pi)^4}\int dk^+dk^-d^2k_\perp[\boldsymbol{\alpha}\mathbf{k}+\beta+\omega] \times\frac{e^{ik_\perp(x_\perp-x'_\perp)-ik^+(x^--x'^-)-ik^-(x^+-x'^+)}}{-2k^+k^-+k_\perp^2+1}, \quad (21)$$

and in light cone coordinates the two center time-dependent potential Eq. (8) takes the form

$$V(x)=\frac{1}{\sqrt{2}}\delta(x^-)(1-\alpha_z)\Lambda^-(x_\perp)+\frac{1}{\sqrt{2}}\delta(x^+)(1+\alpha_z)\Lambda^+(x_\perp). \quad (22)$$

A space-time diagram is presented in Fig. 1. Obviously $V(x)$ only acts on the boundaries between regions I, II, III, and IV.

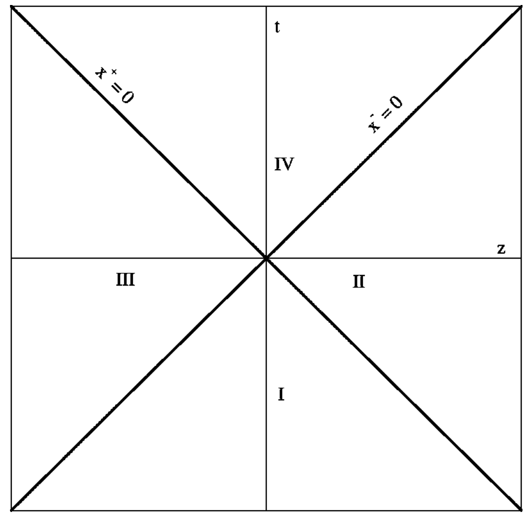

FIG. 1. Light cone boundaries of the four regions of the z, t plane for an ultrarelativistic collision.

In region I we have the initial plane wave ϕ. We would like to begin by constructing the solutions in regions II and III. We will work out region III explicitly and then region II will follow by symmetry. We have from Eq. (17)

$$\Psi_{III}(x)=\phi(x)+\int d^4zG_0(x,z)V(z)\Psi(z). \quad (23)$$

But we have on the boundary between I and III [4]

$$V(z)\Psi(z)=\frac{1}{\sqrt{2}}\delta(z^-)(1-\alpha_z)\Lambda^-(z_\perp)e^{i\theta(z^-)\Lambda^-(z_\perp)}\phi(z), \quad (24)$$

which can be expressed equivalently

$$V(z)\Psi(z)=\frac{-i}{\sqrt{2}}\delta(z^-)(1-\alpha_z)(e^{i\Lambda^-(z_\perp)}-1)\phi(z). \quad (25)$$

We now obtain

$$\Psi_{III}(x)=\phi(x)-\int dz^+dz^-d^2z_\perp\frac{1}{(2\pi)^4}\int dk^+dk^-d^2k_\perp\left[\alpha_\perp k_\perp+\beta+\frac{1}{\sqrt{2}}(1-\alpha_z)k^-+\frac{1}{\sqrt{2}}(1+\alpha_z)k^+\right] \times\frac{e^{ik_\perp(x_\perp-z_\perp)-ik^+(x^--z^-)-ik^-(x^+-z^+)}}{-2k^+k^-+k_\perp^2+1}\frac{i}{\sqrt{2}}\delta(z^-)(1-\alpha_z)(e^{i\Lambda^-(z_\perp)}-1)v(q,s_i)e^{-iq_\perp z_\perp+iq^+z^-+iq^-z^+}. \quad (26)$$

The term in $(1+\alpha_z)$ obviously vanishes. Now integrate over z^-,z^+, and k^- to obtain

$$\Psi_{III}(x)=\phi(x)-\int\frac{dk^+d^2k_\perp}{(2\pi)^3}\left[\alpha_\perp k_\perp+\beta-\frac{1}{\sqrt{2}}(1-\alpha_z)q^-\right]\frac{e^{ik_\perp x_\perp-ik^+x^-+iq^-x^+}}{2k^+q^-+k_\perp^2+1}\frac{i}{\sqrt{2}}(1-\alpha_z) \times\int d^2z_\perp e^{-i(q_\perp+k_\perp)z_\perp}(e^{i\Lambda^-(z_\perp)}-1)v(q,s_i). \quad (27)$$

Now consider the integration over k^+. We did not previously specify the boundary condition on G_0 and we must do so now. We want the G_0 term in Eq. (27) to be nonvanishing for $x^- > 0$ and to vanish for $x^- < 0$. If we move the singularity just below the real axis by adding $i\epsilon$ to the denominator then this physical boundary condition will be fulfilled. We have

$$\int_{-\infty}^{\infty} \frac{dk^+ e^{-ik^+x^-}}{k^+ + (k_\perp^2 + 1/2q^-) + i\epsilon} = -2\pi i e^{i[(k_\perp^2+1)/(2q^-)]x^-} \tag{28}$$

and our expression for the wave function in region III now becomes

$$\Psi_{III}(x) = -\int \frac{d^2k_\perp}{(2\pi)^2}\left[\frac{1}{\sqrt{2}}(\alpha_\perp k_\perp + \beta)(1-\alpha_z) - (1-\alpha_z)q^-\right] \times \frac{e^{ik_\perp x_\perp + i[(k_\perp^2+1)/(2q^-)]x^- + iq^-x^+}}{2q^-}\int d^2z_\perp e^{-i(q_\perp+k_\perp)z_\perp} e^{i\Lambda^-(z_\perp)} v(q,s_i). \tag{29}$$

As a check on the validity of our expression we note that if we project with $(1-\alpha_z)$ the expected boundary condition holds for $x^- = +\epsilon$:

$$(1-\alpha_z)\Psi_{III} = (1-\alpha_z)e^{i\Lambda^-(z_\perp)}\phi. \tag{30}$$

Now the exact amplitude takes the form

$$M = i\int dt \langle\phi|V|\Psi\rangle. \tag{31}$$

Begin by constructing V times the exact wave function on the boundary between regions III and IV. In analogy to Eq. (25) we have

$$V(x)\Psi(x) = \frac{-i}{\sqrt{2}}\delta(x^+)(1+\alpha_z)(e^{i\Lambda^+(x_\perp)} - 1)\Psi_{III}(x). \tag{32}$$

Note that we need the projection $(1+\alpha_z)$ for this boundary due to the opposite direction of motion of the ion producing the V as compared to the V on the boundary between I and III. Thus we have at $x^+ = 0, x^- > 0$

$$V(x)\Psi(x) = \frac{i}{\sqrt{2}}\delta(x^+)(1+\alpha_z)e^{i\Lambda^+(x_\perp)}\int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{\sqrt{2}}(\alpha_\perp k_\perp + \beta)(1-\alpha_z) \times \frac{e^{ik_\perp x_\perp + i[(k_\perp^2+1)/(2q^-)]x^- + iq^-x^+}}{2q^-}\int d^2z_\perp e^{-i(q_\perp+k_\perp)z_\perp} e^{i\Lambda^-(z_\perp)} v(q,s_i). \tag{33}$$

Likewise at $x^- = 0, x^+ > 0$

$$V(x)\Psi(x) = \frac{i}{\sqrt{2}}\delta(x^-)(1-\alpha_z)e^{i\Lambda^-(x_\perp)}\int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{\sqrt{2}}(\alpha_\perp k_\perp + \beta)(1+\alpha_z) \times \frac{e^{ik_\perp x_\perp + i[(k_\perp^2+1)/(2q^+)]x^+ + iq^+x^-}}{2q^+}\int d^2z_\perp e^{-i(q_\perp+k_\perp)z_\perp} e^{i\Lambda^+(z_\perp)} v(q,s_i). \tag{34}$$

In constructing the transition amplitude one makes use of the fact that only two interaction (two photon) terms have a net contribution. (Single interaction terms integrated over the four boundaries give a null contribution.) The amplitude then has two pieces corresponding to the boundary of region IV with regions II and III

$$M(p,q) = M(p,q)_{IV,III} + M(p,q)_{IV,II}. \tag{35}$$

The final state is a positive energy electron

$$\phi = u(p,s_f)e^{ipx} = u(p,s_f)e^{ip_\perp x_\perp - ip^+x^- - ip^-x^+}. \tag{36}$$

Then we have

$$M(p,q)_{IV,III}=i\int_0^\infty dx^-\int dx^+dx_\perp\overline{u}(p,s_f)\beta e^{-ip_\perp x_\perp+ip^+x^-+ip^-x^+}\frac{i}{\sqrt{2}}\delta(x^+)(1+\alpha_z)(e^{i\Lambda^+(x_\perp)}-1)\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{\sqrt{2}}$$
$$\times(\alpha_\perp k_\perp+\beta)(1-\alpha_z)\frac{e^{ik_\perp x_\perp+i[(k_\perp^2+1)/(2q^-)]x^-+iq^-x^+}}{2q^-}\int d^2z_\perp e^{-i(q_\perp+k_\perp)z_\perp}e^{i\Lambda^-(z_\perp)}v(q,s_i). \tag{37}$$

Now integrate over x^+ and x^- to obtain

$$M(p,q)_{IV,III}=-\int\frac{d^2k_\perp}{(2\pi)^2}\frac{\overline{u}(p,s_f)\beta(1+\alpha_z)(\alpha_\perp k_\perp+\beta)v(q,s_i)}{2p^+q^-+k_\perp^2+1}\int d^2x_\perp e^{-i(p_\perp-k_\perp)x_\perp}e^{i\Lambda^+(x_\perp)}\int d^2z_\perp e^{-i(q_\perp+k_\perp)z_\perp}e^{i\Lambda^-(z_\perp)}. \tag{38}$$

The transverse spatial integrals can be done in closed form as is shown in Appendix B. We obtain

$$\int d^2y_\perp e^{-ik_\perp y_\perp}e^{i\Lambda^\pm(y_\perp)}=-e^{\mp ibk_\perp/2}\left(\frac{b^2}{16}\right)^{i\eta}\frac{4\pi}{k_\perp^{2-2i\eta}}\frac{\Gamma(1-i\eta)}{\Gamma(i\eta)}, \tag{39}$$

where $\eta=+Z\alpha$.

The amplitude now becomes

$$M(p,q)_{IV,III}=-4\frac{\Gamma^2(1-i\eta)}{\Gamma^2(i\eta)}\int d^2k_\perp\frac{\overline{u}(p,s_f)\beta(1+\alpha_z)(\alpha_\perp k_\perp+\beta)v(q,s_i)}{2p^+q^-+k_\perp^2+1}e^{-ib(p_\perp/2-q_\perp/2-k_\perp)}[(p_\perp-k_\perp)^2(k_\perp+q_\perp)^2]^{i\eta-1}, \tag{40}$$

where trivial b dependent and constant phases have been removed.

The other piece of the amplitude coming from the boundary between region II and region IV has the corresponding form

$$M(p,q)_{IV,II}=-4\frac{\Gamma^2(1-i\eta)}{\Gamma^2(i\eta)}\int d^2k_\perp\frac{\overline{u}(p,s_f)\beta(1-\alpha_z)(\alpha_\perp k_\perp+\beta)v(q,s_i)}{2p^-q^++k_\perp^2+1}e^{+ib(p_\perp/2-q_\perp/2-k_\perp)}[(p_\perp-k_\perp)^2(k_\perp+q_\perp)^2]^{i\eta-1}$$
$$=-4\frac{\Gamma^2(1-i\eta)}{\Gamma^2(i\eta)}\int d^2k_\perp\frac{\overline{u}(p,s_f)\beta(1-\alpha_z)[\alpha_\perp(p_\perp-q_\perp-k_\perp)+\beta]v(q,s_i)}{2p^-q^++(p_\perp-q_\perp-k_\perp)^2+1}$$
$$\times e^{-ib(p_\perp/2-q_\perp/2-k_\perp)}[(p_\perp-k_\perp)^2(k_\perp+q_\perp)^2]^{i\eta-1}, \tag{41}$$

and the total amplitude takes the form

$$M(p,q)=4\eta^2\int d^2k_\perp e^{ibk_\perp}[(p_\perp-k_\perp)^2(k_\perp+q_\perp)^2]^{i\eta-1}\left(\frac{\overline{u}(p,s_f)\beta(1+\alpha_z)(\alpha_\perp k_\perp+\beta)v(q,s_i)}{2p^+q^-+k_\perp^2+1}\right.$$
$$\left.+\frac{\overline{u}(p,s_f)\beta(1-\alpha_z)[\alpha_\perp(p_\perp-q_\perp-k_\perp)+\beta]v(q,s_i)}{2p^-q^++(p_\perp-q_\perp-k_\perp)^2+1}\right), \tag{42}$$

where trivial phases depending on η and on initial and final momenta have been removed.

Rewriting with the mass of the electron explicit, and making use of more modern notation we have

$$M(p,q)=4\eta^2\int d^2k_\perp e^{ibk_\perp}[(p_\perp-k_\perp)^2(k_\perp+q_\perp)^2]^{i\eta-1}$$
$$\times\left(\frac{\overline{u}(p,s_f)(1-\alpha_z)(-\not{k}_\perp+m)v(q,s_i)}{2p^+q^-+k_\perp^2+m^2}\right.$$
$$\left.+\frac{\overline{u}(p,s_f)(1+\alpha_z)(-\not{p}_\perp+\not{q}_\perp+\not{k}_\perp+m)v(q,s_i)}{2p^-q^++(p_\perp-q_\perp-k_\perp)^2+m^2}\right). \tag{43}$$

At this point one notices that the infinite range of the transverse potential provides us with a result lacking an infrared cutoff. In Appendix B, we introduce a convergence parameter ϵ to regularize this infrared singular behavior. Strictly speaking we were not allowed to let the convergence parameter ϵ take on the value zero. In the previous use of the singular interaction for calculating bound electron positron pair production, the transverse integrals were cut off by the finite range of the bound state wave function [4]. With plane waves there is no such natural cutoff. This suggests that we attach a physical interpretation to the convergence factor ϵ in Appendix B. In fact if we set ϵ equal to ω/γ, where ω is the energy of the produced electron or positron, then we obtain the expected spatial cutoff in a heavy ion electromagnetic interaction, and a result that can naturally be compared with the corresponding perturbation theory result of Bottcher and Strayer [5]. We discuss this further in Sec. IV. But first we shall come to the same solution in the light cone gauge.

III. THE SOLUTION TO THE TWO CHARGE PROBLEM IN LIGHT CONE GAUGE

In this section, we present an alternative derivation to the results presented above in singular gauge. This serves as a check on the results of the previous section. Also the technique which we present here appears to generalize directly to the non-Abelian problem.

We begin by considering a negative energy plane wave. Here we will take $q^{\pm}<0$. This negative energy incoming plane wave will eventually be interpreted as an incoming positron with momentum $q\rightarrow -q$. So we begin with an incoming wave

$$\psi_I(x)=e^{iqx}v_\lambda(-q). \tag{44}$$

Here v_λ is a positron spinor. The label I on the solution refers to the fact that this is a solution in region I as in Fig. 1, and as in Sec. II.

Recall that the background field in light cone gauge is of the form

$$A^i(x)=-\theta(x^-)\nabla^i\Lambda^-(x_T)-\theta(x^+)\nabla^i\Lambda^+(x_T) \tag{45}$$

as was discussed in the Introduction.

Let us first construct the solution in region II. This involves constructing the solution in region across the boundary at $x^+=0$ from region I. Everywhere in region II, $x^-<0$. Using the results of Appendix C, a general solution is of the form

$$\psi_{II}(x)=\frac{1}{\sqrt{2}}e^{-i\Lambda^+(x_T)}\int\frac{dp^+d^2p_T}{(2\pi)^3}e^{\{-ip^+x^-+ip_Tx_T-i(p_T^2+m^2)x^+/2(p^++i\epsilon)\}}\left\{1+\frac{\alpha_T\cdot p_T+m}{\sqrt{2}(p^++i\epsilon)}\right\}\alpha^+v_\lambda(-q)$$
$$\times\left\{\frac{1}{i}\frac{1}{p^+-q^+-i\epsilon}\int d^2z_Te^{i(q_t-p_T)z_T}e^{i\Lambda^+(z_T)}+H(p_T,p^+,q)\right\}. \tag{46}$$

In this equation, H has all its singularities in the negative half p^+ plane. Therefore at $x^+=0$, this contribution gives nothing. In fact throughout region II, there is no contribution so that H can be dropped. The choice of singularity for p^+ in the exponential is so that there is convergence at large positive x^+. Notice that for $x^-<0$ corresponding to region II, the term which does not involve H may be evaluated by closing in the upper half p^+ plane. Here the solution satisfies the correct boundary condition at $x^+=0$ and solves the Dirac equation in region II. It is unique.

In exactly the same way, we have in region III, that the solution is

$$\psi_{III}(x)=\frac{1}{\sqrt{2}}e^{-i\Lambda^-(x_T)}\int\frac{dp^-d^2p_T}{(2\pi)^3}e^{\{-ip^-x^++ip_Tx_T-i(p_T^2+m^2)x^-/2(p^-+i\epsilon)\}}\left\{1+\frac{\alpha_T\cdot p_T+m}{\sqrt{2}(p^-+i\epsilon)}\right\}\alpha^-v_\lambda(-q)$$
$$\times\left\{\frac{1}{i}\frac{1}{p^--q^--i\epsilon}\int d^2z_Te^{i(q_t-p_T)z_T}e^{i\Lambda^-(z_T)}\right\}. \tag{47}$$

We now write down an ansatz for the solution in region IV. After writing down the solution, we will show that it solves the Dirac equation and the boundary conditions at $x^{\pm}=0$ when $x^{\mp}>0$. Consider

$$\psi_{IV}(x)=\frac{i}{2}e^{-i\Lambda^+(x_T)-i\Lambda^-(x_T)}\int\frac{d^2p_Td^2z_T}{(2\pi)^2}\frac{d^2p_T'd^2z_T'}{(2\pi)^2}\left\{\int\frac{dp^+}{2\pi}\frac{1}{p^+-(p_T^2+m^2)/2q^-+i\epsilon}\right.$$
$$\times e^{-ip^+x^-+ip_T'x_T-i(p_T'^2+m^2)x^+/2(p^++i\epsilon)}e^{iz_T'(p_T-p_T')+iz_T(q_T-p_T)}e^{i\Lambda^+(z_T)+i\Lambda^-(z_T')}$$
$$\times\left(1+\frac{\alpha_T\cdot p_T'+\beta m}{\sqrt{2}(p^++i\epsilon)}\right)\alpha^+\frac{\alpha_T\cdot p_T+\beta m}{\sqrt{2}q^-}\alpha^-v_\lambda(-q)$$
$$+\int\frac{dp^-}{2\pi}\frac{1}{p^--(p_T^2+m^2)/2q^++i\epsilon}e^{-ip^-x^++ip_T'x_T-i(p_T'^2+m^2)x^+/2(p^-+i\epsilon)}e^{iz_T'(p_T-p_T')+iz_T(q_T-p_T)}e^{i\Lambda^-(z_T)+i\Lambda^+(z_T')}$$
$$\left.\times\left(1+\frac{\alpha_T\cdot p_T'+\beta m}{\sqrt{2}(p^++i\epsilon)}\right)\alpha^-\frac{\alpha_T\cdot p_T+\beta m}{\sqrt{2}q^+}\alpha^+v_\lambda(-q)\right\}. \tag{48}$$

First it is straightforward to apply the Dirac equation to ψ_{IV}, and show that it solves the Dirac equation. What about the behavior on the boundaries $x^{\pm}=0$, $x^{\mp}>0$? First consider $x^{-}=0$ and $x^{+}>0$. We first evaluate ψ_{II} on this boundary. We require that the ψ^{-} be continuous. Using ψ_{II}, we see that

$$\psi^{-}_{II}|_{x^{-}=0,x^{+}>0}=\frac{1}{\sqrt{2}}e^{-i\Lambda^{+}(x_T)}\int\frac{d^2p_T}{(2\pi)^2}e^{ip_Tx_T-i(p_T^2+m^2)x^{+}/2q^{+}}\times\frac{\alpha_T\cdot p_T+\beta m}{\sqrt{2}q^{+}}\alpha^{+}v_{\lambda}(-q)\times\int d^2z_Te^{i(q_T-p_T)z_T}e^{i\Lambda^{+}(z_T)}. \tag{49}$$

Now, how does ψ^{-}_{IV} behave on this boundary? The first integral over p^{+} in the integral representation for ψ^{-}_{IV} vanishes at $x^{-}=0$ as all the singularities of the integration over p^{+} are in the same side of the real p^{+} axis in the complex p^{+} plane. In the second term which involves p^{-}, we must close in the lower half p^{-} planes when $x^{-}=0$ and $x^{+}>0$. Closing in this plane and doing the integrals over z'_T and p'_T shows that our formula solves the correct boundary condition.

By symmetry, we see that along the other boundary at $x^{+}=0$ for $x^{-}>0$ the boundary condition is solved for the plus component of the wave function. Therefore ψ_{IV} solves the Dirac equation and the boundary conditions in region IV.

Now we must extract the matrix element for pair production from this wave function. To do this, we observe that after a little algebra, and making the substitution $q\to -q$ that

$$\psi_{IV}(x)=-\frac{1}{2}e^{-i\Lambda^{+}(x_T)-i\Lambda^{-}(x_T)}\int\frac{d^4p'}{(2\pi)^4}\frac{d^2p_T}{(2\pi)^2}d^2z_Td^2z'_T\times\frac{m-p'\cdot\gamma}{p'^2+m^2-i\epsilon}e^{-ip'x}\times\left\{\frac{2p'^{+}}{p'^{+}+(p_T^2+m^2)/2q^{-}+i\epsilon}e^{iz'_T(p_T-p'_T)-iz_T(q_T+p_T)}\times e^{i\Lambda^{+}(z'_T)+i\Lambda^{-}(z_T)}\alpha^{-}\frac{m+p_T\cdot\gamma}{\sqrt{2}q^{-}p'^{+}}v_{\lambda}(q)\times\frac{2p'^{-}}{p'^{-}+(p_T^2+m^2)/2q^{+}+i\epsilon}e^{iz'_T(p_T-p'_T)-iz_T(q_T+p_T)}\times e^{i\Lambda^{-}(z'_T)+i\Lambda^{+}(z_T)}\alpha^{+}\frac{m+p_T\cdot\gamma}{\sqrt{2}q^{+}p'^{-}}v_{\lambda}(q).\right. \tag{50}$$

Amputating the external line which corresponds to the propagator for the electron, we find the matrix element for pair production to be

$$M_{\eta\lambda}=\int\frac{d^2p'_T}{(2\pi)^2}\left\{\kappa^{+}(p'_T-p_T)\kappa^{-}(p'_T+q_T)\frac{1}{2q^{-}p'^{+}+p_T'^2+m^2+i\epsilon}\overline{u}_{\eta}(p)\sqrt{2}\alpha^{-}(m+p'_T\cdot\gamma)v_{\lambda}(q)+\kappa^{+}(p'_T+q_T)\kappa^{-}\times(p'_T-p_T)\frac{1}{2q^{+}p'^{-}+p_T'^2+m^2+i\epsilon}\overline{u}_{\eta}(p)\sqrt{2}\alpha^{+}(m+p'_T\cdot\gamma)v_{\lambda}(q)\right\}. \tag{51}$$

Here the quantity

$$\kappa^{\pm}(p_T)=\int d^2x_Te^{-ip_Tx_T}e^{+i\Lambda^{\pm}(x_T)}. \tag{52}$$

IV. DISCUSSION

After we had finished the derivation of the amplitudes Eq. (43) and its equivalent Eq. (51) we became aware of similar results recently obtained by Segev and Wells [6]. These authors obtained an amplitude with the same structure as ours, but they did not obtain our closed form, Eqs. (40), (B6) or Eq. (B5) for the nonperturbative transverse momentum integral. They did argue that the transverse integral went appropriately to the ultrarelativistic perturbative limit in agreement with the result of Bottcher and Strayer [5]. In this limit their perturbative amplitude simply takes the form of Eq. (43) without the $i\eta$ in the exponent.

We now have the suggestion of an ansatz for a physical cutoff of the exact result in line with the discussion at the end of Sec. II. The perturbation theory result of Bottcher and Strayer not taken to the ultrarelativistic limit actually has denominators of the form $k_{\perp}^2+\omega^2/(\gamma^2-1)$ rather than $k_{\perp}^2$. Our ansatz then is to modify Eq. (43) to

$$M(p,q)=4\eta^2\int d^2k_{\perp}e^{ibk_{\perp}}\left[\left([p_{\perp}-k_{\perp}]^2+\frac{\omega^2}{\gamma^2}\right)\left([k_{\perp}+q_{\perp}]^2+\frac{\omega^2}{\gamma^2}\right)\right]^{i\eta-1}\left(\frac{\overline{u}(p,s_f)(1-\alpha_z)(-\not{k}_{\perp}+m)v(q,s_i)}{2p^{+}q^{-}+k_{\perp}^2+m^2}+\frac{\overline{u}(p,s_f)(1+\alpha_z)(-\not{p}_{\perp}+\not{q}_{\perp}+\not{k}_{\perp}+m)v(q,s_i)}{2p^{-}q^{+}+(p_{\perp}-q_{\perp}-k_{\perp})^2+m^2}\right). \tag{53}$$

We have effectively retained ϵ (set to ω/γ) in the factor taken to the $1-i\eta$ power of Eq. (B5) and ignored it elsewhere. Retaining η elsewhere in Eq. (B5) would cut off a little more sharply at small $k_\perp$ but is not necessary. It would not necessarily be more exact to retain all factors of η because the cutoff comes in response to the spatial region $\boldsymbol{\rho}=\gamma/\omega$ where both the singular and light cone potentials begin to lose their validity.

At this point one is left with a choice whether to perform the integral over the impact parameter either before or after the integral over the intermediate transverse momentum. If one is only interested in a the cross section for pair production in a given small momentum and/or energy bin then one must obtain the same answer independent of the order of integration. In that case performing the integral over the impact parameter first seems convenient. One has an impact parameter integral of the form

$$\int d^2\mathbf{b}\int |M(p,q)|^2 \sim \int d^2\mathbf{b}\int d^2k_\perp e^{ibk_\perp}f(k_\perp)\int d^2k'_\perp e^{-ibk'_\perp}f^*(k'_\perp). \tag{54}$$

The integral over $d^2\mathbf{b}$ gives $(2\pi)^2\delta^{(2)}(k_\perp-k'_\perp)$ and the $\pm i\eta$ exponents vanish, giving a result identical to what we would obtain in perturbation theory.

If we are interested in high multiplicity events or a total cross section for events in which at least one pair is produced then we must perform the integral over the intermediate transverse momentum first. For the highest multiplicity events we need to evaluate the square of the amplitude at small but nonintersecting values of the impact parameter: we need the mean number of pairs at a given impact parameter. From the mean number of pairs produced at each impact parameter, one can obtain the probability (less than one) of at least one pair being produced. Thus a total pair production cross section can be defined and computed.

Note that for a given impact parameter the square of the amplitude is not identical in the exact solution to what it is in the the perturbation theory solution. For the exact solution the $i\eta$ in the exponent gives a rotating phase in the the integral over $k_\perp$ that is absent in the perturbative case. For example if we look at a dominant contribution near one of the cutoffs the amplitude goes as

$$M\sim\int_0^k \frac{k_\perp dk_\perp}{(k_\perp^2+\omega^2/\gamma^2)^{1-i\eta}} = \frac{1}{2i\eta}\left(e^{i\eta\ln(k^2+\omega^2/\gamma^2)}-e^{i\eta\ln\omega^2/\gamma^2}\right), \tag{55}$$

which decreases with increasing η. Here the exact contribution is less than perturbation theory would be.

To sum up, we have a situation where one can say both that the exact cross section for pair production to any final state is identical to the perturbation theory cross section, and that measurements of high multiplicity pair events will show deviations from perturbation theory probably by being smaller.

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APPENDIX A: CONVENTIONS FOR LIGHT CONE VARIABLES AND THE USE OF PROJECTION OPERATORS

We will let the light cone coordinates for any vector be

$$V^\pm=\frac{1}{\sqrt{2}}(V^0\pm V^z). \tag{A1}$$

The metric convention we will use is $g^{00}=1$ and $g^{ij}=-\delta^{ij}$ for spatial components so that in light cone gauge the dot product is

$$A\cdot B=A^+B^-+A^-B^+-A_T\cdot B_T. \tag{A2}$$

We shall use ordinary gamma matrices for fermions so that

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}. \tag{A3}$$

We can define projection operators for fermions as

$$P^\pm=\frac{1}{\sqrt{2}}\gamma^0\gamma^\pm=\frac{1}{2}(1\pm\alpha_z). \tag{A4}$$

These projection operators satisfy $P^++P^-=1$, $P^{\pm 2}=P^\pm$, and $P^\pm P^\mp=0$. The plus and minus components of a fermion field are

$$\psi^\pm=P^\pm\psi, \tag{A5}$$

where $\psi^++\psi^-=\psi$.

In light cone coordinates, the Dirac equation for a free particle becomes

$$\psi^-=\frac{1}{\sqrt{2}p^+}(p_T\cdot\alpha_T+\beta m)\psi^+, \tag{A6}$$

where ψ^+ solve

$$(2p^+p^--p_T^2-m^2)\psi^+=0. \tag{A7}$$

In an external field which is independent of $x^\pm$, the transverse momentum is converted into a transverse covariant momentum. Also, the relationships can be reversed under the transformation $\psi^\pm\to\psi^\mp$.

APPENDIX B: EVALUATION OF THE TRANSVERSE INTEGRAL

We would like to evaluate the integral

$$I=\int d^2\rho\, e^{-i\mathbf{k}\cdot\rho}e^{-i\eta\ln(\rho\pm\mathbf{b}/2)^2/(b/2)^2}. \tag{B1}$$

A coordinate shift gives us

$$I=\left(\frac{b^2}{4}\right)^{i\eta}e^{\mp i[\mathbf{k}\cdot\mathbf{b}/2]}\int_0^{2\pi}d\phi\int_0^{\infty}\rho\, d\rho e^{-(ik_\perp\cos\phi+\epsilon|\cos\phi|)\rho}e^{-i\eta\ln\rho^2}, \tag{B2}$$

where we have added a convergence factor $\epsilon|\cos\phi|$. The integral over ρ can then be carried out ([8] 3.381.5) and we obtain

$$I=\left(\frac{b^2}{4}\right)^{i\eta}e^{\mp i[\mathbf{k}\cdot\mathbf{b}/2]}\int_0^{2\pi}d\phi\frac{\Gamma(2-2i\eta)}{[(k_\perp^2+\epsilon^2)\cos^2\phi]^{1-i\eta}}e^{(-2i-2\eta)\arctan(k_\perp\cos\phi/\epsilon|\cos\phi|)}. \tag{B3}$$

Making use of quadrant symmetry this may be rewritten

$$I=4e^{\mp i[\mathbf{k}\cdot\mathbf{b}/2]}\left(\frac{b^2}{4}\right)^{i\eta}\frac{(\epsilon^2-k_\perp^2)\cosh(2\eta\arctan[k_\perp/\epsilon])+i2\epsilon k_\perp\sinh(2\eta\arctan[k_\perp/\epsilon])}{(\epsilon^2+k_\perp^2)}\frac{\Gamma(2-2i\eta)}{(k_\perp^2+\epsilon^2)^{1-i\eta}}\int_0^{\pi/2}\frac{d\phi}{(\cos\phi)^{2-2i\eta}}. \tag{B4}$$

The integral over ϕ can then be carried out by Wallis' formula ([7] 6.1.49) giving

$$I=4e^{\mp i[\mathbf{k}\cdot\mathbf{b}/2]}\left(\frac{b^2}{4}\right)^{i\eta}\frac{(\epsilon^2-k_\perp^2)\cosh(2\eta\arctan[k_\perp/\epsilon])+i2\epsilon k_\perp\sinh(2\eta\arctan[k_\perp/\epsilon])}{(\epsilon^2+k_\perp^2)}\frac{\Gamma(2-2i\eta)}{(k_\perp^2+\epsilon^2)^{1-i\eta}}\frac{\sqrt{\pi}\Gamma(-0.5+i\eta)}{2\Gamma(i\eta)}. \tag{B5}$$

Letting ϵ approach zero from the positive direction and exploiting Γ function relations finally leads to

$$I=-e^{\mp i[\mathbf{k}\cdot\mathbf{b}/2]}\left(\frac{b^2}{16}\right)^{i\eta}\frac{4\pi}{k_\perp^{2-2i\eta}}\frac{\Gamma(1-i\eta)}{\Gamma(i\eta)}. \tag{B6}$$

APPENDIX C: SOLVING THE DIRAC EQUATION FOR THE LIÉNARD-WIECHERT POTENTIAL

In this appendix, we discuss solving the Dirac equation for Liénard-Wiechert potentials appropriate for the central region. We will show how to construct a solution for a light cone potential of the form

$$A_T^i=-\theta(x^-)\nabla^i\Lambda(x_T). \tag{C1}$$

We will then show how to convert this solution into that for the singular gauge and how this problem in singular gauge translates into a boundary condition on the Dirac wave function at $x^-=0$. In the text, we will use this to construct solutions across the boundaries at $x^\pm=0$.

We assume that for $x^-<0$, we are given a plane wave solution of the form

$$\psi(x)=e^{iqx}u_\lambda(x), \tag{C2}$$

where λ is a polarization label on the electron spinor. At $x^->0$, the solution must be a linear combination of plane wave solutions to the Dirac equation modulo a gauge rotation

$$\psi'(x)=e^{i\Lambda(x_T)}\int\frac{d^2p_T}{(2\pi)^2}F_{\lambda\lambda'}(p_T,q_T)\times u'_\lambda(p)e^{(ip_Tx_T-iq^-x^+-ip^+x^-)}, \tag{C3}$$

where

$$p^+=\frac{p_T^2+m^2}{2q^-}. \tag{C4}$$

The integral above is two dimensional since q^- labels the solution on both sides of the boundary as the potential is x^+ independent, and because the value of plus component of light cone momenta is determined by the mass shell condition.

What are the boundary conditions at $x^-=0$? Recall that the form of the Dirac equation is

$$(P_T\cdot\alpha_T+\beta m-\sqrt{2}p^+P^--\sqrt{2}p^-P^+)\psi=0, \tag{C5}$$

where $P^\pm$ are the projection operators described in Appendix A. Here P_T is the covariant momentum operator. We see there for the ψ^- can be chosen to be continuous across the boundary. On the other hand, the plus component of the wave function must be discontinuous since by the Dirac equation

$$\psi^+=\frac{1}{\sqrt{2p^-}}(P_T\cdot\alpha_T+\beta m)\psi^- \tag{C6}$$

and the covariant momentum operator contains the vector potential which is discontinuous.

This tells us what the proper boundary condition is in singular gauge. Recall that to transform between the two gauges

$$\psi_{\rm singular}(x) = e^{i\theta(x^-)\lambda(x_T)}\psi_{\rm lightcone}(x). \tag{C7}$$

We therefore have that the boundary condition in singular gauge is the the component

$$\psi^-_{\rm singular}(x)|_{x^-=0^+} = e^{i\lambda(x_T)}\psi^-_{\rm singular}(x)|_{x^-=0^-}, \tag{C8}$$

with the plus components determined by the Dirac equation. All components of the wave function in singular gauge are therefore discontinuous.

It is now straightforward to determine the solution for $x^- > 0$. Consider

$$\psi(x) = \frac{1}{\sqrt{2}} e^{i\Lambda(x_T)} \int \frac{d^2 p_T}{(2\pi)^2} d^2 z_T e^{-i\Lambda(z_T)} e^{ip_T x_T - iq^- x^+ - ip^+ x^-} \times e^{iz_T(q_T - p_T)} \left\{ 1 + \frac{\alpha_T \cdot p_T + \beta m}{\sqrt{2} q^-} \right\} \alpha^- u_\lambda(q), \tag{C9}$$

where

$$\alpha^\pm = \frac{1}{\sqrt{2}}(1 \pm \alpha_z) = \sqrt{2} P^\pm. \tag{C10}$$

Again $p^+ = (p_T^2 + m^2)/2q^-$. It is easy to see that this function solves the Dirac equation. At $x^- = 0$, the term involving p^+ disappears so that the integrals over p_T and z_T can be done with the result

$$\psi(x) = \frac{1}{\sqrt{2}} \left(\alpha^- + \alpha^+ \frac{\alpha_T \cdot Q_T + \beta m}{\sqrt{2} q^-} \right) e^{iqx} u_\lambda(q). \tag{C11}$$

Using the definition of the projection operators $P^\pm$, and the relationship between $\psi\pm$, this is just $e^{iqx}u_\lambda(q)$. To derive this, we must use the definition of the vector potential in terms of Λ. This solution therefore solves the boundary conditions.

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