Methods for analyzing anisotropic flow in relativistic nuclear collisions

A. M. Poskanzer[1] and S. A. Voloshin[2,*]
[1]*Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720*
[2]*Physikalisches Institut der Universität Heidelberg, Heidelberg, Germany*
(Received 20 May 1998)

The strategy and techniques for analyzing anisotropic flow (directed, elliptic, etc.) in relativistic nuclear collisions are presented. The emphasis is on the use of the Fourier expansion of azimuthal distributions. We present formulas relevant for this approach, and in particular, show how the event multiplicity enters into the event plane resolution. We also discuss the role of nonflow correlations and a method for introducing flow into a simulation. [S0556-2813(98)04109-0]

PACS number(s): 25.75.Ld

I. INTRODUCTION: SUBJECT AND TERMINOLOGY

Recently, the study of collective flow in nuclear collisions at high energies has attracted increased attention of both theoreticians and experimentalists [1]. There are several reasons for this: (i) the observation of anisotropic flow at the AGS [2–7] and at the SPS [8–11], (ii) progress in the theoretical understanding of the relation between the appearance and development of flow during the collision evolution, and processes such as thermalization, creation of the quark-gluon plasma, phase transitions, etc. [12–18], (iii) the study of mean field effects [14,19], (iv) the importance of flow for other measurements such as identical and nonidentical two-particle correlation analyses [20–23], (v) the development of new techniques suitable for flow studies at high energies [24–26,1,9]. Although all forms of flow are interrelated and represent only different parts of one global picture, usually people discuss different forms of collective flow, such as longitudinal expansion, radial transverse flow, and anisotropic transverse flow of which the most well established are directed flow [24,19] and elliptic flow [27].

The study of flow in ultrarelativistic nuclear collisions has just begun and it is very important to define the subject and establish the terminology. At high energies the longitudinal flow is well decoupled from transverse flow. The flow (polar) angles observed at low energies are relatively large and a rotation of the coordinate system was done in order to analyze the event shape in the plane perpendicular to the main axis of the flow ellipsoid. At high energies the flow angles become very small, $\theta_{\rm flow} \approx \langle p_x \rangle / p_{\rm beam} \ll 1$, so that not doing such a rotation ''induces'' an elliptic anisotropy of the order of the square of the flow angle and usually can be neglected. This means that at high energies one does not have to rotate to the flow axis to study the flow pattern [27], but one can use the plane transverse to the beam axis. Thus we discuss only anisotropic transverse flow from the particle azimuthal distributions at fixed rapidity or pseudorapidity. It appears very convenient to describe the azimuthal distributions by means of a Fourier expansion [28,25,24], and below we characterize different kinds of anisotropies as corresponding to different harmonics (in analogy and/or contrast to the description at low energies, where the three-dimensional event shape is characterized using multipole terminology). Anisotropic flow corresponding to the first two harmonics plays a very important role and we use special terms for them: directed and elliptic flow, respectively. The word ''directed'' comes from the fact that such flow has a direction, and the word ''elliptic'' is due to the fact that in polar coordinates the azimuthal distribution with nonzero second harmonic represents an ellipse.

II. CORRELATIONS WITH RESPECT TO AN EVENT PLANE

Strategy. In this section we summarize an approach to the study of anisotropic flow which is particularly suitable at high (AGS/SPS/RHIC) energies. It uses the Fourier expansion of azimuthal distributions, introduced in this way for the analysis in Ref. [25]. The essence of the method is to first estimate the reaction plane. The estimated reaction plane we call the event plane. The Fourier coefficients in the expansion of the azimuthal distribution of particles with respect to this event plane are evaluated. Because the finite number of detected particles produces limited resolution in the angle of the measured event plane, these coefficients must be corrected up to what they would be relative to the real reaction plane. This is done by dividing the observed coefficients by the event plane resolution, which is estimated from the correlation of the planes of independent subevents [29] (subgroups of the particles used for the event plane determination). The resolution obtained from the subevents can be converted to that for the full event by means of the multiplicity dependence of the resolution which will be described below. Also, if the detector does not have full azimuthal acceptance, the acceptance bias has to be removed.

Fourier expansion. The quantity under study in the most general case is the triple differential distribution. In this, the dependence on the particle emission azimuthal angle measured with respect to the reaction plane can be written in a form of Fourier series

$$E\frac{d^3N}{d^3p} = \frac{1}{2\pi}\frac{d^2N}{p_t dp_t dy}\left(1 + \sum_{n=1}^{\infty} 2v_n \cos[n(\phi - \Psi_r)]\right), \tag{1}$$

*On leave from Moscow Engineering Physics Institute, Moscow, 115409, Russia.

where Ψ_r denotes the (true) reaction plane angle, and the sine terms vanish due to the reflection symmetry with respect to the reaction plane. The main advantage of the Fourier method is that the Fourier coefficients, evaluated using observed event planes, can be corrected for the event plane resolution caused by the finite multiplicity of the events. This correction always raises the value of the coefficients. The great importance of this is that then the results for particles in a certain phase space region may be compared directly to theoretical predictions, or to simulations unfiltered for the detector acceptance, and for which the reaction plane has been taken to be the plane containing the theoretical impact parameter. Note the factor of 2 in front of each v_n coefficient. We propose to use it because in this case the meaning of the coefficients v_n becomes transparent [25], $v_n = \langle \cos[n(\phi-\Psi_r)] \rangle$, where $\langle \; \rangle$ indicates an average over all particles in all events. For the particle number distribution, the coefficient v_1 is $\langle p_x/p_t \rangle$ and v_2 is $\langle (p_x/p_t)^2 - (p_y/p_t)^2 \rangle$.

Estimation of the reaction plane. The method uses the anisotropic flow itself to determine the event plane. It also means that the event plane can be determined independently for each harmonic of the anisotropic flow. The event flow vector Q_n and the event plane angle Ψ_n from the nth harmonic of the distribution are defined by the equations

$$Q_n \cos(n\Psi_n) = X_n = \sum_i w_i \cos(n\phi_i), \tag{2}$$

$$Q_n \sin(n\Psi_n) = Y_n = \sum_i w_i \sin(n\phi_i), \tag{3}$$

or

$$\Psi_n = \left(\tan^{-1} \frac{\sum_i w_i \sin(n\phi_i)}{\sum_i w_i \cos(n\phi_i)} \right) \Big/ n. \tag{4}$$

The sums go over the i particles used in the event plane determination and the w_i are weights. In general the weights are also optimized to make the reaction plane resolution the best that is possible. Sometimes it can be done by selecting the particles of one particular type, or weighting with transverse momentum of the particles, etc. Usually the weights for the odd and even harmonic planes are different. Optimal weights are discussed in footnote 2 of [17]. For symmetric collisions reflection symmetry says that particle distributions in the backward hemisphere of the center of mass should be the same as in the forward hemisphere if the azimuthal angles of all particles are shifted by π. This explains why for the odd harmonics the signs of the weights are reversed in the backward hemisphere while for the even harmonics the signs of the weights are not reversed. Note that the event plane angle Ψ_n determined from the nth harmonic is in the range $0 \leqslant \Psi_n < 2\pi/n$. For the case of $n=1$, Eqs. (2)–(4) are equivalent to obtaining Ψ_1 for number flow from [29]

$$\mathbf{Q} = \sum w \mathbf{p}_t / |p_t| \tag{5}$$

where the sum is over all the particles. The case of $n=2$ is equivalent to the event plane determined from the transverse sphericity matrix [24].

Acceptance correlations. Biases due to the finite acceptance of the detector which cause the particles to be azimuthally anisotropic in the laboratory system can be removed by making the distribution of event planes isotropic in the laboratory. We know of several different methods to remove the effects of anisotropy which have been used (sometimes in a combination with each other). Each of them has some advantages along with disadvantages.

The simplest one is to recenter [3,4,10,30] the distributions (X_n, Y_n) [Eqs. (2) and (3)] by subtracting the (X_n, Y_n) values averaged over all events. The main disadvantage of this method is that it does not remove higher harmonics from the resulting distribution of Ψ_n. If such harmonics are present then the method requires additional flattening of the event plane distribution by one of the other methods. The second, one of the most commonly used methods, is to use the distribution of the particles themselves as a measure of the acceptance [3,4,9]. One accumulates the laboratory azimuthal distribution of the particles for all events and uses the inverse of this as weights in the above calculation of the event planes. The limitation of this approach is that it does not take into account the multiplicity fluctuations around the mean value. The third method is to use mixed events [3,4,9]. Correlations with the raw event planes are stored in histograms and correlations with event planes from previous events are also stored. The real correlations are then divided by the mixed event correlations to remove the acceptance correlations. This third method suffers from the problem that, if one uses only one mixed event for each real event, the errors are $\sqrt{2}$ larger. If one uses many mixed events for each real event, the errors decrease as $n_{mix}^{1/4}$ instead of $\sqrt{n_{mix}}$ because the same events are being used n_{mix} times [31]. The fourth method fits the unweighted laboratory distribution of the event planes, summed over all events, to a Fourier expansion and devises an event-by-event shifting of the planes needed to make the final distribution isotropic [3,4].[1] In all these methods one has to check that the event plane distributions indeed become isotropic.

Particle distributions with respect to the event plane. The next step is to study the particle distributions with respect to the event planes. Note that for a given n the corresponding Fourier coefficient v_n can be evaluated using the reaction planes determined from any harmonic m, with $n \geqslant m$, if n is a multiple of m. If $n > m$, the sign of v_n is determined relative to v_m. That is, the first harmonic plane can be used, in principle, to evaluate all v_n. The second harmonic plane can be used to evaluate v_2, v_4, etc. For the event plane evaluated from the mth harmonic the Fourier expansion is

$$\frac{d(wN)}{d(\phi-\Psi_m)} = \frac{\langle wN \rangle}{2\pi} \left(1 + \sum_{k=1}^{\infty} 2 v_{km}^{obs} \cos[km(\phi-\Psi_m)] \right). \tag{7}$$

[1]The equation for the shift is (see Appendix in Ref. [3])

$$n\Delta\Psi_n = \sum_{i=1}^{i_{max}} \frac{2}{i} [-\langle \sin(in\Psi_n) \rangle \cos(in\Psi_n) + \langle \cos(in\Psi_n) \rangle \sin(in\Psi_n)]. \tag{6}$$

We have usually taken $i_{max} = 4/n$ for $n = 1,2$.

Writing the equation in terms of km instead of n insures, for instance, that when $m=1$ all terms are present, but when $m=2$ only the even terms are present. The quantity w is a weight which could be p_t of the particle if one studies transverse momentum flow, or just unity, if one studies flow of the number of particles. The coefficients v_n^{obs} are evaluated by $\langle\cos[n(\phi-\Psi_m)]\rangle$. The quantity $(\phi-\Psi_m)$ has a lowest order periodicity of $2\pi/m$. To graph the distribution one can shift the negative values to positive by adding 2π, and then fold the distribution with this periodicity using the module function. When a particle has been used in the calculation of an event plane, the auto-correlation effect in its distribution with respect to this plane is removed by recalculating that plane without this particle [29]. This is easily done if one saves the sums of sines and cosines from Eqs. (2)–(4), and subtracts the contribution of the particle and its weight from these sums. This method of removing autocorrelations assumes that contributions from conservation of momentum are small.

The event plane resolution. The coefficients in the Fourier expansion of the azimuthal distributions with respect to the *real* reaction plane are then evaluated by dividing by the event plane resolution [29,32,25,3,4,26],

$$v_n=v_n^{obs}/\langle\cos[km(\Psi_m-\Psi_r)]\rangle. \tag{8}$$

The mean cosine values are less than one and thus this correction always increases the flow coefficients.

The resolution depends both on the harmonic used to determine the event plane m and the order of the calculated coefficient n. It is generally true that better accuracy for the determination of v_n is achieved by using the event plane (Ψ_n) determined from the same harmonic $(m=n,\ k=1)$ because the resolution deteriorates as k increases (see below). For example, better accuracy for v_2 can be achieved using Ψ_2 even when the elliptic flow is somewhat weaker than the directed flow.

To calculate the resolution we start with the distribution of $m(\Psi_m-\Psi_r)$, which can be written as [25]

$$\frac{dP}{d[m(\Psi_m-\Psi_r)]}=\int\frac{v'_m dv'_m}{2\pi\sigma^2}\times\exp\left(-\frac{v_m^2+v_m'^2-2v_m v'_m\cos[m(\Psi_m-\Psi_r)]}{2\sigma^2}\right). \tag{9}$$

The parameter σ, which to second order in flow is common for all m, is inversely proportional to the square-root of N, the number of particles used to determine the event plane

$$\sigma^2=\frac{1}{2N}\frac{\langle w^2\rangle}{\langle w\rangle^2}. \tag{10}$$

The integral (9) can be evaluated analytically [24,25], and then the event plane resolution can be expressed as

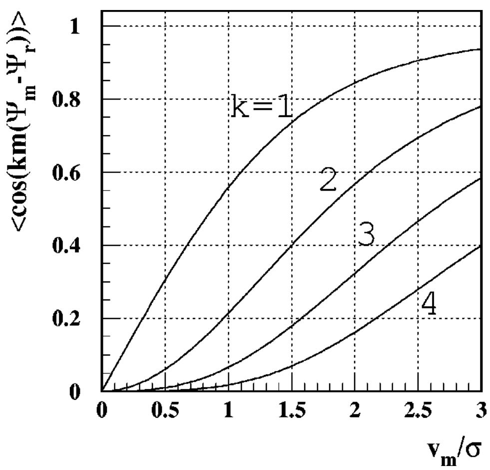

FIG. 1. The event plane resolution for the nth $(n=km)$ harmonic of the particle distribution with respect to the mth harmonic plane, as a function of $\chi_m=v_m/\sigma$.

$$\langle\cos[km(\Psi_m-\Psi_r)]\rangle=\frac{\sqrt{\pi}}{2\sqrt{2}}\chi_m\exp(-\chi_m^2/4)\times[I_{(k-1)/2}(\chi_m^2/4)+I_{(k+1)/2}(\chi_m^2/4)], \tag{11}$$

where $\chi_m\equiv v_m/\sigma$ $(=v_m\sqrt{2N}$ for number flow) and I_ν is the modified Bessel function of order ν. This resolution function is plotted in Fig. 1. Please note that χ used in this paper is $\sqrt{2}$ larger than that used by Ollitrault.[2]

Note that $\langle\cos[m(\Psi_m-\Psi_r)]\rangle$ is a correction for the reconstruction of the signal if the event plane is derived using the flow of the same harmonic $(k=1)$. In the case when the harmonic orders do not coincide, for example, when one uses the event plane derived from the first harmonic (directed flow) and studies the second harmonic (elliptic flow) of the particle distribution with respect to this plane [the $m=1,k=2$ term in Eq. (7)] the correction would be $\langle\cos[2(\Psi_1-\Psi_r)]\rangle$. For practical use all such functions can be calculated numerically as in the subsection on *Approximations* below.

The resolution correction used in earlier times at the Bevalac [28] is close to the present curve for values of $\chi_1=v_1\sqrt{2N}$ greater than about 1.5, which was generally true at the Bevalac. However, using the old procedure at the higher beam energies of the AGS or SPS would greatly overestimate the resolution and make the flow values too small.

The correlation between flow angles of independent sets of particles. If one constructs the event planes in two different windows, (a) and (b), or from two random subevents,[3] the corresponding correlation function also can be written analytically. But more important in this case is the simple relation for such correlations,

$$\langle\cos[n(\Psi_m^a-\Psi_m^b)]\rangle=\langle\cos[n(\Psi_m^a-\Psi_r)]\rangle\times\langle\cos[n(\Psi_m^b-\Psi_r)]\rangle. \tag{12}$$

[2]The parameter χ used in this paper is a factor of $\sqrt{2}$ larger than the one used in Refs. [1,26]; it is equivalent to the parameter $\tilde{\xi}$ in Ref. [25].

[3]The random subevents can be made to have equal multiplicity using the CERN library routine SORTZV.

Here, the assumption is made that there are no other correlations except the ones due to flow, or that such other correlations can be neglected. (For example, two-particle correlations due to resonance decays should scale with multiplicity as $1/N$ and usually can be neglected.) If this is not true, then special precautions have to be made to avoid or correct for such correlations. See the subsection below on *Nonflow correlations*.

Note that the correlation of two planes (the distribution in $\Psi^a - \Psi^b$) is *not* well represented by a Fourier expansion. For a strong correlation it should be Gaussian. For no flow the correlation between two random planes of the same order should be a triangular distribution. If one takes the absolute value of $\Psi_n^a - \Psi_n^b$ and folds this distribution about the angle π/n, the triangle becomes flat. This does not affect the $\langle\cos[n(\Psi_n^a - \Psi_n^b)]\rangle$ value needed for Eq. (12), but makes it easier to see a real flow effect by the non-flatness of the distribution in the graph.

For the correlation between angles determined from two subevents of different harmonics one can write similar relations. For example, the correlation between Ψ_1^a and Ψ_2^b is

$$\langle\cos[2(\Psi_1^a - \Psi_2^b)]\rangle = \langle\cos[2(\Psi_1^a - \Psi_r)]\rangle \times \langle\cos[2(\Psi_2^b - \Psi_r)]\rangle. \quad (13)$$

This expression is proportional to $v_1^2 v_2/\sigma^3$ and can be rather small in magnitude, but it can be useful for the determination of the relative orientation of the flow of different orders (see also the subsection on *Approximations* below). On the other hand, the relative orientation also can be determined using Eq. (7) with $k > 1$.

The determination of the event plane resolution. The above relations permit the evaluation of the event plane resolution directly from the data. For example, if one knows the correlation between two equal multiplicity subevents (where the resolution of each is expected to be the same) then from Eq. (12) the resolution of each of them is

$$\langle\cos[n(\Psi_m^a - \Psi_r)]\rangle = \sqrt{\langle\cos[n(\Psi_m^a - \Psi_m^b)]\rangle}, \quad (14)$$

where, as before, $n = km$, and k is not necessarily equal to 1. If the subevents are correlated, then the term inside the square-root is always positive.[4] Note that the event plane resolution determined in such a way is the event plane resolution of the subevents. If one wants to use the full event (all detected particles from the event) to determine the event plane, then the full event plane resolution can be calculated from the sub-event resolution using Eq. (11) or the approximations below, Eqs. (23) and (24), taking into account that the multiplicity of the full event is twice as large as the multiplicity of the sub-event. Because $\chi_m = v_m/\sigma$ is proportional to $\sqrt{N}$, in a case of low resolution where the curves in Fig. 1 are linear, this reduces to

$$\langle\cos[n(\Psi_m - \Psi_r)]\rangle = \sqrt{2}\langle\cos[n(\Psi_m^a - \Psi_r)]\rangle. \quad (15)$$

If the subevents are not "equal," or if you have only correlations between particles in different windows, and the resolution in each window can be different, then one needs at least three windows to determine the event plane resolution in each of them. In this case, for example, the resolution in the first window is determined as [3,4]

$$\langle\cos[n(\Psi_m^a - \Psi_r)]\rangle = \sqrt{\frac{\langle\cos[n(\Psi_m^a - \Psi_m^b)]\rangle\langle\cos[n(\Psi_m^a - \Psi_m^c)]\rangle}{\langle\cos[n(\Psi_m^b - \Psi_m^c)]\rangle}}. \quad (16)$$

Approximations. We want to discuss two limits. In a case of weak flow, $\chi_m \ll 1$ ($v_m \ll \sigma$, which also means that the event plane resolution is low) one can expand the exponent under the integral in Eq. (9) before the integration. It will follow that in this limit

$$\frac{dP}{d(m(\Psi_m - \Psi_r))} \propto 1 + \sqrt{\frac{\pi}{2}}\chi_m \cos[m(\Psi_m - \Psi_r)]. \quad (17)$$

Then the event plane resolution can be estimated analytically to be (the case $k = 1$)

$$\langle\cos[m(\Psi_m - \Psi_r)]\rangle \approx \sqrt{\frac{\pi}{8}}\chi_m. \quad (18)$$

In this case the resolution is linear in χ and, since σ is the same for all m, the resolution for different m scales with v_m.

Combining Eqs. (14), (10), and (18) one can estimate the flow signal directly from the correlation between two subevents

$$v_m \approx \sqrt{(4/\pi)\langle w^2\rangle/\langle w\rangle^2}\sqrt{\langle\cos[m(\Psi_m^a - \Psi_m^b)]\rangle/N_{sub}}, \quad (19)$$

where N_{sub} is the multiplicity of the subevents.

Keeping the second order terms in the expansion of Eq. (9) it can be shown that

$$\langle\cos[2m(\Psi_m - \Psi_r)]\rangle \approx \frac{\chi_m^2}{4} \approx \frac{2}{\pi}\{\langle\cos[m(\Psi_m - \Psi_r)]\rangle\}^2. \quad (20)$$

The latter relation for $k = 2$ is needed as a resolution correction for v_2 in the case where the event plane was determined by the directed flow ($m = 1$). The approximate relations (18)–(20) are rather accurate for $\chi_m < 0.5$.

Using the above approximations the correlation between flow angles of subevents of different harmonics ($m = 1$ and $m = 2$) can be written as

$$\langle\cos[2(\Psi_1^a - \Psi_2^b)]\rangle \approx \pm\frac{2}{\pi}\langle\cos(\Psi_1^a - \Psi_1^b)\rangle \times \sqrt{\langle\cos[2(\Psi_2^a - \Psi_2^b)]\rangle}. \quad (21)$$

As stated above, the sign of the left-hand side shows the relative orientation of the flow of the different harmonics, while the right-hand side after the $\pm$ sign is always positive. Note that all quantities in this equation are evaluated from

[4]For small amounts of flow, fluctuations and/or nonflow correlations can cause this term to be negative when the total p_t is required to be zero.

the data independently, so the comparison of the magnitudes is an additional consistency check.

In the second limiting case of strong flow $\chi_m \gg 1$, one can expand the cosine in the exponent of Eq. (9) and get approximately

$$\frac{dP}{d(m(\Psi_m-\Psi_r))} \approx \frac{\chi_m}{\sqrt{2\pi}} \exp\left[-\frac{\chi_m^2(m(\Psi_m-\Psi_r)^2)}{2}\right], \tag{22}$$

which also can be used to calculate the reaction plane resolution. The case of strong flow is important mostly for relatively low energy collisions and we will not discuss it in detail. Note that in this case (and *only* in this case) the distribution of $m(\Psi_m-\Psi_r)$ can be described by a Gaussian.

Here we present interpolation formulas of Eq. (11) for the most needed cases of $k=1,2$

$$\langle \cos[m(\Psi_m-\Psi_r)]\rangle = 0.626657\chi_m - 0.09694\chi_m^3 + 0.02754\chi_m^4 - 0.002283\chi_m^5, \tag{23}$$

$$\langle \cos[2m(\Psi_m-\Psi_r)]\rangle = 0.25\chi_m^2 - 0.011414\chi_m^3 - 0.034726\chi_m^4 + 0.006815\chi_m^5. \tag{24}$$

The interpolation formulas are valid for $\chi_m < 3$.

In order to use the above equations to go from the subevent resolution to the full-event resolution, one has to set the equation equal to the subevent resolution and take the root to obtain χ_m. One multiplies this by $\sqrt{2}$ because χ_m is proportional to $\sqrt{N}$, and then evaluates the full-event resolution from the same equation. Our routines for finding the roots of these equations use an iterative method, and the routines for propagating the errors involve a calculation for the change in the result with a small finite change in the input.

Another approximate method is to evaluate $\chi_m = v_m/\sigma$ of the full event from the fraction of events where the correlation of the planes of the subevents is greater than $\pi/2$ [26],

$$\frac{N_{\text{events}}(m|\Psi_m^a-\Psi_m^b| > \pi/2)}{N_{\text{total}}} = \frac{e^{-\chi_m^2/4}}{2}. \tag{25}$$

This fraction, and therefore the equation, is only accurate when χ_m is small.

III. EVENT-BY-EVENT ANALYSIS OF AZIMUTHAL DISTRIBUTIONS

Here we study the anisotropies in the azimuthal distributions of particles within a relatively large rapidity (pseudorapidity) window without determining an event plane. Finite multiplicities used in each event for the evaluation of the event flow vector $\mathbf{Q}_n$, defined in Eqs. (2) and (3), causes both finite event plane resolution and also fluctuations in the vector magnitude Q_n. In the case of zero flow $\langle Q_n^2\rangle = \langle N\rangle$. (In this section for simplicity we assume $w_i=1$.) Anisotropic flow, which shifts the vector $\mathbf{Q}_n$ in each event in the (random) flow direction by a value $v_n N$, results effectively in broadening the distribution in Q_n. Keeping the first nonvanishing contribution from flow one has

$$\langle Q_n^2\rangle = \langle N\rangle + \overline{v_n^2}\langle N^2\rangle. \tag{26}$$

This equation can be used to estimate the flow signal $\overline{v_n}$, which is the average of v_n over the rapidity region used for the Q_n calculation.

Another method [25] involves fitting the distribution in $r_n = Q_n/N$ to the theoretical formula, which has $\overline{v_n}$ (more exactly its absolute value) as a parameter. In this method the distribution of r_n has to be fitted by the function

$$\frac{dP}{r_n\, dr_n} = \frac{1}{\sigma^2}\exp\left(-\frac{\overline{v_n^2}+r_n^2}{2\sigma^2}\right) I_0\left(\frac{r_n\overline{v_n}}{\sigma^2}\right), \tag{27}$$

where I_0 is the modified Bessel function, $\overline{v_n}$ is the parameter of interest, and σ is the parameter related to finite multiplicity fluctuations. The distribution (27) was derived using the central limit theorem, requiring the particle multiplicity N to be large. Note that in principle Eq. (26) does not have this limitation, although effectively the ''signal to noise ratio'' is proportional to N and one would need N large to apply the equation to data.

Note also that both methods do not require the determination of the event plane and can be performed within one (pseudo)rapidity window, but they do require this window to be rather wide in order to have a relatively large number of particles. The distribution (27) can be used to select events with larger or smaller values of flow.

IV. NONFLOW CORRELATION CONTRIBUTION

The methods described so far in this paper are correct when correlations induced by flow dominate all others. By all others we mean correlations due to momentum conservation [29], long- and short-range two- and many-particle correlations (due to quantum statistics, resonances, mini and real jet production, etc.). Below, we discuss the contribution of such nonflow correlations to the correlation of the event planes determined from two independent subevents. Very often the contribution of nonflow correlations scales as $1/N$, where N is the multiplicity of particles used to determine the event plane. But one should remember that the contribution due to momentum conservation increases with the fraction of particles detected, and that the relative contribution of Bose-Einstein correlations can be independent of multiplicity (the later can be important if the subevents are formed by random division of all particles from the same phase space region into two subgroups, where the particles contributing to the different subevents can be very close to each other in phase space). The study of the effects of nonflow correlations (from the point of view of a flow analysis) in real data is rather complicated, so we start with the analysis of Monte-Carlo events.

Nonflow correlations in Monte Carlo generated events. When the true reaction plane is known (as it is in any generated event) the contribution of nonflow correlations can be studied by analyzing correlations along the axis perpendicular to the reaction plane (y axis). For example, let us consider the correlation between $\mathbf{Q}^a$ and $\mathbf{Q}^b$, the vectors [see Eq.

(5)] defined by two independent subevents. One can think of these vectors as the total transverse momentum of all particles of the sub-event [which is the case if the transverse momentum is used as a weight in Eq. (5)]. Then if there are no other correlations except flow

$$\langle \mathbf{Q}^a \mathbf{Q}^b \rangle = \langle \mathbf{Q}^a \rangle \langle \mathbf{Q}^b \rangle = \langle Q_x^a \rangle \langle Q_x^b \rangle, \tag{28}$$

the very relation our first method is based on. It was assumed here that the two Q vectors are totally uncorrelated except that both of them are correlated with the reaction plane.[5] If this is not true and there exist other correlations, then their contribution in first order would be the same to the correlations of x components and y components. Then

$$\begin{aligned}\langle Q_x^a Q_x^b \rangle &= \langle Q_x^a \rangle \langle Q_x^b \rangle + \langle Q_x^a Q_x^b \rangle_{\text{nonflow}} \\ &\approx \langle Q_x^a \rangle \langle Q_x^b \rangle + \langle Q_y^a Q_y^b \rangle_{\text{nonflow}} \\ &= \langle Q_x^a \rangle \langle Q_x^b \rangle + \langle Q_y^a Q_y^b \rangle \end{aligned} \tag{29}$$

$$\langle Q_x^a \rangle \langle Q_x^b \rangle \approx \langle Q_x^a Q_x^b \rangle - \langle Q_y^a Q_y^b \rangle. \tag{30}$$

Analyzing the correlation $\langle Q_y^a Q_y^b \rangle$ (and in particular $\langle \sin[n(\Psi_n^a - \Psi_r)] \sin[n(\Psi_n^b - \Psi_r)] \rangle$) one can estimate the corrections to the formulas (8) and (12)–(15).

Nonflow correlations in real data. The direct application of the above described method to real data is not possible. What can be done is the analysis of similar correlations using, instead of Ψ_r, the event plane derived from the second harmonic, where as the analysis of different models shows, the contribution of nonflow effects is significantly less. Then with the (second harmonic) event plane resolution known one can carry out the above analysis.

There exists in the literature other methods for estimating and accounting for nonflow correlations, see, for example Refs. [30,33]. Here we briefly describe the method [33], which was applied to the data of the WA93 Collaboration [34]. It was proposed [33,34] to characterize the nonflow correlation contribution by the value of the parameter

$$c = \frac{\langle \mathbf{Q}^a \mathbf{Q}^b \rangle - \langle Q_x^a \rangle \langle Q_x^b \rangle}{\sqrt{\langle (\mathbf{Q}^a)^2 \rangle} \sqrt{\langle (\mathbf{Q}^b)^2 \rangle}} \approx \frac{\langle \mathbf{Q}^a \mathbf{Q}^b \rangle - \langle Q_x^a \rangle \langle Q_x^b \rangle}{N}, \tag{31}$$

where N is the subevent multiplicity and, as in Sec. III, for simplicity we assume $w_i = 1$. The parameter c can strongly depend on the particular choice of subevents, but if the nonflow contribution is dominated by two-particle correlations, it is largely independent of multiplicity [33]. The nonflow correlation changes the distribution of event flow vectors, and in particular Eq. (26) now reads

$$\langle Q_n^2 \rangle \approx \langle N \rangle + \overline{v_n^2} \langle N^2 \rangle + 2c \langle N \rangle. \tag{32}$$

If the parameter c is relatively large, it can bias the results derived from an application of Eq. (26) to the data. However, the nonflow correlations contribute to the *shape* of the distribution in Q_n [Eq. (27)] in a different way than flow. They produce mostly a change in σ, the "Gaussian width" of the distribution [Eq. (10)], which is modified to

$$\sigma^2 = \frac{1}{2N}(1 + 2c), \tag{33}$$

and the parameter c can be directly extracted from the data. The analysis [34] of S+S and S+Au data gave (for their particular subevent selection) $c \approx 0.034 \pm 0.025$ which is not negligible compared with $v_2 \approx 0.04$–0.05 found in this analysis at multiplicities of about 30–50. Another possible consequence of nonflow contributions could be the change in the shape of the distribution of the difference of flow angles of subevents [33].

V. SIMPLE WAY TO INTRODUCE FLOW IN A MONTE-CARLO EVENT GENERATOR

Sometimes in order to investigate different detector effects or the reliability of the method, one needs to introduce anisotropic flow into a Monte Carlo event generator. It can be done by changing the azimuthal angle of each particle (and consequently changing the density in the azimuthal angle space)

$$\phi \to \phi' = \phi + \Delta\phi, \tag{34}$$

where

$$\Delta\phi = \sum_n \frac{-2}{n} \tilde{v}_n \sin[n(\phi - \psi_0)], \tag{35}$$

$\tilde{v}_n$ are the n parameters of the transformation, and ψ_0 is the direction of the added flow. The $\tilde{v}_n$ can be functions of rapidity and transverse momentum and, in particular, the $\tilde{v}_n$ for n odd should reverse sign in the backward hemisphere. One can check that such a change in the azimuthal angle results in the required change in the distribution. To first order in $\tilde{v}_n$, $v_n \equiv \langle \cos[n(\phi' - \Psi_r)] \rangle = \tilde{v}_n$. Small higher order corrections ($\sim \tilde{v}_n^2$), if needed, can also be easily calculated, for example, numerically.

VI. DISCUSSION

Higher harmonics ($n \geqslant 3$). Note that the flow analysis methods presented in this paper are valid for all harmonic orders of anisotropy. The higher harmonics look at the event with higher symmetry on a finer scale. It should be emphasized that the study of anisotropic flow corresponding to harmonic orders of $n \geqslant 3$ has interesting aspects. For instance, one would expect large differences between the theoretical predictions of hydro- and cascade-type models in the higher harmonics of the particle azimuthal distributions. Also there are physics processes which could lead to nonzero higher harmonics. It is widely accepted that one of the main reasons for pion "antiflow," directed flow in the direction opposite to that of the protons, is caused by pion shadowing by comoving nucleons. Such shadowing could affect pion higher harmonic azimuthal distributions, and affect them differently at different rapidities. Another effect is the transition from out-of-plane elliptic flow (squeeze-out) which is very impor-

[5] Strictly speaking there is one more assumption here, namely that the strength of the flow does not fluctuate event by event.

tant at low energies to in-plane elliptic flow which is the main effect at high energies. At the transition beam energy both effects may be important and the fourth harmonic may peak at this energy. Even at high beam energies, the out-of-plane squeeze-out effect could dominate at short times and the in-plane expansion at long times, leading to an overall fourth order harmonic coefficient.

Transverse radial and anisotropic flow. We would like also to emphasize that the above methods permit the reconstruction of the triple differential distribution with respect to the reaction plane, and in particular are convenient for the analysis of the p_t dependence of the anisotropy. The importance of such a study was stressed in Ref. [35], where the effect of the interplay of (transverse) radial and directed flow has been studied. In that analysis transverse directed flow is considered a result of the movement of an effective source in the transverse plane. It is assumed that in the source rest frame the first moment of the azimuthal distribution (v_1) is zero, and the final anisotropy appears only as a consequence of the source movement in the transverse direction.

In this case the transverse momentum dependence of v_1 has a rather specific shape. The radial expansion results in decreasing v_1 at low p_t. For some sets of parameters (temperature, radial and directed velocities) it becomes negative. Physically it corresponds to the case, when particle production with such a value of p_t is more probable from the part of the effective source which moves in the opposite direction from the flow direction, because in this case the directed flow and the radial flow tend to compensate each other.

The same considerations can be applied to elliptic flow as well. If one assumes that elliptic flow is a consequence of more rapid expansion of the effective source in some plane, then the p_t dependence of v_2 would exhibit exactly the same features as has been observed for v_1 in the case of directed flow.

Pair-wise azimuthal correlations. Two-particle large-angle azimuthal correlations are often proposed as a tool to study anisotropic flow [36]. Not rejecting this possibility we note that the expected signal in such correlations can be very small in magnitude. It can be easily shown that the pair-wise distribution in the azimuthal angle difference ($\Delta\phi=\phi_1-\phi_2$) is

$$\frac{dN^{pairs}}{d\Delta\phi}\propto\left(1+\sum_{n=1}^{\infty}2v_n^2\cos(n\Delta\phi)\right). \tag{36}$$

Note that the ''signal'' is v_n^2, and thus is small. It does not imply though, that the ''signal to noise ratio'' is small, and the method, in principle, can be successfully applied to data to obtain the flow signal. However, the reconstruction of the triple differential distribution with respect to the reaction plane (the goal of the flow analysis) becomes more involved.

This method does not require the determination of the event plane. Usually such a distribution is constructed using all possible pair combinations in each event. Note that in this case each particle enters into many pairs (on the order of the mean multiplicity of an event) and, consequently, the pairs are not statistically independent. Thus special precautions have to be taken for the evaluation of the error of the results [31].

ACKNOWLEDGMENTS

The discussions with G. Cooper, S. Esumi, G. Rai, H.-G. Ritter, and T. Wienold as well as with many other members of the E877, NA49, and E895 Collaborations are appreciatively acknowledged. One of the authors (S.V.) thanks the Nuclear Science Division at LBNL for financial support during his visit to LBNL, where this work was started. This work was supported by the U.S. Department of Energy under Contract No. DE-ACO3-76SFOOO98.

[1] For a recent review, see J.-Y. Ollitrault, in Proceedings of the 13th International Conference on Ultra-Relativistic Nucleus-Nucleus Collisions, Quark Matter '97, Tsukuba, Japan, 1997; Nucl. Phys. A (to be published).
[2] E877 Collaboration, J. Barrette *et al.*, Phys. Rev. Lett. **70**, 2996 (1993).
[3] E877 Collaboration, J. Barrette *et al.*, Phys. Rev. C **56**, 3254 (1997).
[4] E877 Collaboration, J. Barrette *et al.*, Phys. Rev. C **55**, 1420 (1997).
[5] E802 Collaboration, L. Ahle *et al.*, Phys. Rev. C **57**, 1416 (1998).
[6] E877 Collaboration, S.A. Voloshin, in Ref. [1].
[7] E895 Collaboration, H. Liu and G. Rai, in Ref. [1].
[8] NA49 Collaboration, T. Wienold *et al.*, Nucl. Phys. **A610**, 76c (1996).
[9] NA49 Collaboration, A.M. Poskanzer, in Ref. [1]; NA49 Collaboration, H. Appelhauser *et al.*, Phys. Rev. Lett. **80**, 4136 (1998).
[10] NA45 Collaboration, F. Ceretto, in Ref. [1].
[11] WA98 Collaboration, S. Nishimura, in Ref. [1].
[12] J. Stachel, Nucl. Phys. **A610**, 509c (1996).
[13] L. Bravina, L.P. Csernai, P. Lévai, and D. Strottman, Phys. Rev. C **50**, 2161 (1994); L. Bravina, Phys. Lett. B **344**, 49 (1995).
[14] B.A. Li and C.M. Ko, Phys. Rev. C **52**, 2037 (1995); **53**, R22 (1996); B.A. Li, C.M. Ko, and G.Q. Li, *ibid.* **54**, 844 (1996).
[15] C.M. Hung and E.V. Shuryak, Phys. Rev. Lett. **75**, 4003 (1995).
[16] D.H. Rischke, Nucl. Phys. **A610**, 88c (1996).
[17] P. Danielewicz, Phys. Rev. C **51**, 716 (1995).
[18] P. Danielewicz, R.A. Lacey, P.-B. Gossiaux, C. Pinkenburg, P. Chung, J.M. Alexander, and R.L. McGrath, nucl-th/9803047.
[19] H. Sorge, Phys. Rev. Lett. **78**, 2309 (1996); Phys. Lett. B **402**, 251 (1997); in *Advances in Nuclear Dynamics 3*, edited by Bauer and Mignerey (Plenum, New York, 1997), p. 1.
[20] T. Csörgő and B. Lorstad, Phys. Rev. C **54**, 1390 (1996).
[21] S.A. Voloshin and W.E. Cleland, Phys. Rev. C **53**, 896 (1996); **54**, 3212 (1996).
[22] U.A. Wiedemann, Phys. Rev. C **57**, 266 (1997).
[23] S. Voloshin, R. Lednicky, S. Panitkin, and Nu Xu, Phys. Rev. Lett. **79**, 4766 (1997).

[24] J.-Y. Ollitrault, Phys. Rev. D **46**, 229 (1992); **48**, 1132 (1993).
[25] S. Voloshin and Y. Zhang, Z. Phys. C **70**, 665 (1996).
[26] J.-Y. Ollitrault, Phys. Rev. C (to be published), preprint nucl-ex/9711003.
[27] W. Reisdorf and H.G. Ritter, Annu. Rev. Nucl. Part. Sci. **47**, 663 (1997).
[28] Plastic Ball Collaboration, H.H. Gutbrod, K.H. Kampert, B. Kolb, A.M. Poskanzer, H.G. Ritter, R. Schicker, H.R. Schmidt, Phys. Rev. C **42**, 640 (1990).
[29] P. Danielewicz and G. Odyniec, Phys. Lett. **157B**, 146 (1985).
[30] Streamer Chamber Collaboration, P. Danielewicz *et al.*, Phys. Rev. C **38**, 120 (1988).
[31] S.A. Voloshin, hep-ph/9407282.
[32] Diogene Collaboration, M. Demoulins *et al.*, Phys. Lett. B **241**, 476 (1990).
[33] J.-Y. Ollitrault, Nucl. Phys. **A590**, 561c (1995).
[34] W.H. van Heeringen, Ph.D. thesis, Utrecht University, 1996; WA93 Collaboration, M.M. Aggarwal *et al.*, Phys. Lett. B **403**, 390 (1997).
[35] S.A. Voloshin, Phys. Rev. C **55**, R1630 (1997).
[36] S. Wang, Y.Z. Jiang, Y.M. Liu, D. Keane, D. Beavis, S.Y. Chu, S.Y. Fung, M. Vient, C. Hartnack, and H. Stöcker, Phys. Rev. C **44**, 1091 (1991).