Low-energy Compton scattering of polarized photons on polarized nucleons

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The general structure of the cross section of γN scattering with polarized photon and/or nucleon in initial and/or final state is systematically described and exposed through invariant amplitudes. A low-energy expansion of the cross section up to and including the order $\mathcal{O}(\omega^4)$ is given that involves ten structure parameters of the nucleon (dipole, quadrupole, dispersion, and spin polarizabilities). Their physical meaning is discussed in detail. Using fixed-t dispersion relations, predictions for these parameters are obtained and compared with results of chiral perturbation theory. It is emphasized that Compton scattering experiments at large angles can fix the most uncertain of these structure parameters. Predictions for the cross section and double-polarization asymmetries are given and the convergence of the expansion is investigated. The feasibility of the experimental determination of some of the structure parameters is discussed. [S0556-2813(98)02407-8]

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I. INTRODUCTION

Compton scattering on the proton at low and intermediate energies has thus far been studied mainly with unpolarized photons. Many recent data are available on the unpolarized differential cross section both in the region below pion threshold [1–3] and in the delta region [4–8]. They have led to a determination of the dipole electric and magnetic polarizabilities of the proton, have given useful constraints on pion photoproduction amplitudes near the delta peak, and have provided sensitive tests for different models of Compton scattering, such as those based on resonance saturation [9–12], chiral perturbation theory [13,14], and dispersion relations [15–19].

With the advent of new experimental tools such as highly polarized photon beams, polarized targets, and recoil polarimetry [20], it becomes possible to study the very rich spin structure of Compton scattering. In particular, many additional structure parameters of the nucleon, such as the spin [21–24] and quadrupole [25] polarizabilities, could be measured in such new-generation experiments and used for testing hadron models at low energies. The first attempt to determine the ''backward'' spin polarizability from unpolarized experiments has recently been reported [26]. Therefore, it is timely to give a detailed description of the appropriate polarization observables and their relationship to the low-energy parameters that might be measured in such experiments. That is the main purpose of the present report.

This paper is organized as follows. In Sec. II, we introduce the invariant Compton scattering amplitudes. In Sec. III we develop a general structure for the γN scattering cross section with polarized photons and/or polarized nucleons in the initial and/or final state. In Sec. IV we do a low-energy expansion of the invariant amplitudes and develop formulas for the low-energy expansion of the cross section and spin observables. In the process, we introduce and discuss the physical meaning of the parameters (polarizabilities) which are required to describe the cross section up to and including the order $\mathcal{O}(\omega^4)$. In Sec. V we give theoretical predictions for all these polarizabilities by using fixed-t dispersion relations and compare them with available predictions of chiral perturbation theory. We then investigate the range of validity of our low-energy expansion and show that it is generally valid below the pion threshold. Finally, we investigate quantitatively the dependence of different observables on the polarizabilities and recommend particularly sensitive experiments to perform. Some of the details related to the definitions and physical meaning of the polarizabilities are contained in the appendixes.

II. INVARIANT AMPLITUDES

The amplitude T_{fi} for Compton scattering on the nucleon,

$$\gamma(k)N(p)\to\gamma'(k')N'(p'), \tag{2.1}$$

is defined by

$$\langle f|S-1|i\rangle = i(2\pi)^4\delta^4(k+p-k'-p')T_{fi}. \tag{2.2}$$

Constrained by P and T invariance, it can be expressed in terms of six invariant amplitudes T_i as [27,28,15,17,18]

$$\begin{aligned} T_{fi} = \overline{u'}(p')e'^{*\mu}\Bigg\{ & -\frac{P'_\mu P'_\nu}{P'^2}(T_1+\gamma\cdot KT_2) \\ & -\frac{N_\mu N_\nu}{N^2}(T_3+\gamma\cdot KT_4)+i\frac{P'_\mu N_\nu - P'_\nu N_\mu}{P'^2K^2}\gamma_5 T_5 \\ & +i\frac{P'_\mu N_\nu + P'_\nu N_\mu}{P'^2K^2}\gamma_5\gamma\cdot KT_6\Bigg\}e^\nu u(p), \end{aligned} \tag{2.3}$$

where e and e' are the photon polarization vectors, u and u' are the bi-spinors of the nucleons ($\overline{u}u=2m$, m is the nucleon mass), and $\gamma_5=\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. The orthogonal four-vectors P', K, N, and Q are defined as

$$P'_\mu = P_\mu - K_\mu \frac{P\cdot K}{K^2}, \quad P=\frac{1}{2}(p+p'), \quad K=\frac{1}{2}(k'+k),$$

$$N_\mu = \epsilon_{\mu\alpha\beta\gamma} P'^{\alpha} Q^\beta K^\gamma, \quad Q=\frac{1}{2}(p-p')=\frac{1}{2}(k'-k), \tag{2.4}$$

where the antisymmetric tensor $\epsilon_{\mu\alpha\beta\gamma}$ is fixed by the condition $\epsilon_{0123}=1$.

Unfortunately, there is no accepted standard for the definitions of the invariant amplitudes in nucleon Compton scattering. The definitions we use here may differ from those used in other works. We follow the conventions of Refs. [17–19], which are related to the so-called Hearn-Leader amplitudes $A_i^{\rm HL}$ used in Refs. [15,28] by

$$T_1 = A_1^{\rm HL}, \quad T_3 = A_2^{\rm HL}, \quad T_5 = -A_3^{\rm HL},$$
$$T_2 = -A_4^{\rm HL}, \quad T_4 = -A_5^{\rm HL}, \quad T_6 = -A_6^{\rm HL}. \tag{2.5}$$

The amplitudes T_i are functions of the two variables $\nu=(s-u)/4m$ and t, where

$$s=(k+p)^2, \quad u=(k-p')^2, \quad t=(k-k')^2 \tag{2.6}$$

are the usual Mandelstam variables and $s+u+t=2m^2$. These functions have no kinematical singularities but they are subject to kinematical constraints arising from the vanishing of the denominators in the decomposition (2.3) in cases of forward or backward scattering. Therefore, it is useful to define the following linear combinations [18,19,29]:

$$A_1=\frac{1}{t}[T_1+T_3+\nu(T_2+T_4)], \quad A_2=\frac{1}{t}[2T_5+\nu(T_2+T_4)],$$
$$A_3=\frac{1}{\eta}\left[T_1-T_3-\frac{t}{4\nu}(T_2-T_4)\right],$$
$$A_4=\frac{1}{\eta}\left[2mT_6-\frac{t}{4\nu}(T_2-T_4)\right], \tag{2.7}$$
$$A_5=\frac{1}{4\nu}(T_2+T_4), \quad A_6=\frac{1}{4\nu}(T_2-T_4),$$

with

$$\eta=\frac{1}{m^2}(m^4-su)=4\nu^2+t-\frac{t^2}{4m^2}. \tag{2.8}$$

The amplitudes $A_i(\nu,t)$ are even functions of ν, they have no kinematical singularities or constraints, and they have dimension m^{-3}.

In the lab system (the nucleon N at rest) the kinematic invariants ν, t and η read

$$\nu=\frac{1}{2}(\omega+\omega'), \quad t=-2\omega\omega'(1-z), \quad \eta=2\omega\omega'(1+z), \tag{2.9}$$

where $\omega=k_0$, $\omega'=k'_0$ are the photon energies, $z=\cos\theta$ is the photon scattering angle, and

$$\omega'=\omega+\frac{t}{2m}=\omega\left[1+\frac{\omega}{m}(1-z)\right]^{-1}. \tag{2.10}$$

We will reserve the symbols ω, ω', and z for these lab-frame variables. Note that in the lab frame, $N^\mu=(0,\boldsymbol{N})$, where $\boldsymbol{N}=(m/2)\boldsymbol{k}'\times\boldsymbol{k}$ is orthogonal to the reaction plane.

In terms of the A_i, the Compton scattering amplitude T_{fi} in the lab frame assumes the following form:

$$\begin{aligned}
T_{fi}=\frac{1}{N(t)}\Bigg\{ & 2m\boldsymbol{e}'^*\cdot\boldsymbol{e}\ \omega\omega'\left[\left(1-\frac{t}{4m^2}\right)(-A_1-A_3)-\frac{\nu^2}{m^2}A_5-A_6\right] \\
& +2m\boldsymbol{s}'^*\cdot\boldsymbol{s}\ \omega\omega'\left[\left(1-\frac{t}{4m^2}\right)(A_1-A_3)+\frac{\nu^2}{m^2}A_5-A_6\right] \\
& -2i\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\boldsymbol{e}\ \nu\omega\omega'(A_5+A_6)+2i\boldsymbol{\sigma}\cdot\boldsymbol{s}'^*\times\boldsymbol{s}\ \nu\omega\omega'(A_5-A_6) \\
& +i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\ \boldsymbol{s}'^*\cdot\boldsymbol{e}\ \omega^2\omega'\left[A_2+\left(1-\frac{\omega'}{m}\right)A_4+\frac{\nu}{m}A_5+A_6\right] \\
& -i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}'\ \boldsymbol{e}'^*\cdot\boldsymbol{s}\ \omega\omega'^2\left[A_2+\left(1+\frac{\omega}{m}\right)A_4-\frac{\nu}{m}A_5+A_6\right] \\
& -i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\ \boldsymbol{e}'^*\cdot\boldsymbol{s}\ \omega^2\omega'\left[-A_2+\left(1-\frac{\omega'}{m}\right)A_4-\frac{\nu}{m}A_5+A_6\right] \\
& +i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}'\ \boldsymbol{s}'^*\cdot\boldsymbol{e}\ \omega\omega'^2\left[-A_2+\left(1+\frac{\omega}{m}\right)A_4+\frac{\nu}{m}A_5+A_6\right]\Bigg\},
\end{aligned} \tag{2.11}$$

where $N(t)=(1-t/4m^2)^{1/2}$ and the two magnetic vectors $\boldsymbol{s}$, $\boldsymbol{s}'$ are defined as

$$\boldsymbol{s}=\hat{\boldsymbol{k}}\times\boldsymbol{e}, \quad \boldsymbol{s}'=\hat{\boldsymbol{k}}'\times\boldsymbol{e}'. \tag{2.12}$$

III. CROSS SECTIONS AND ASYMMETRIES

A. General structure of the cross section

We will consider the general structure of the cross section and double-polarization observables in four related reactions:

$$\vec{\gamma}\vec{N}\to\gamma' N', \tag{3.1a}$$

$$\gamma\vec{N}\to\vec{\gamma}' N', \tag{3.1b}$$

$$\vec{\gamma}N\to\gamma'\vec{N}', \tag{3.1c}$$

$$\gamma N\to\vec{\gamma}'\vec{N}'. \tag{3.1d}$$

We start by introducing the polarization variables.

Photon polarization properties are conveniently described in terms of the Stokes parameters ξ_i $(i=1,2,3)$ [30] which define the photon polarization matrix density as follows [31]:

$$\langle e_\alpha e^*_\beta\rangle=\frac{1}{2}(1+\boldsymbol{\sigma}\cdot\boldsymbol{\xi})_{\alpha\beta}=\frac{1}{2}\begin{pmatrix}1+\xi_3 & \xi_1-i\xi_2\\ \xi_1+i\xi_2 & 1-\xi_3\end{pmatrix}_{\alpha\beta}. \tag{3.2}$$

Here the photon polarization vector e_μ is taken in the radiation gauge $\boldsymbol{e}\cdot\hat{\boldsymbol{k}}=0$ and $\alpha,\beta=1,2$ denote either of two orthogonal directions x_γ, y_γ which are in turn orthogonal to the photon momentum direction $z_\gamma=\hat{\boldsymbol{k}}$. Such a definition of ξ_i is manifestly frame dependent; nevertheless, the quantities

$$\xi_l=\sqrt{\xi_1^2+\xi_3^2} \tag{3.3}$$

and ξ_2 are Lorentz invariant. They give the degree of linear and circular polarization, respectively. Moreover, $\xi_2=\pm1$ corresponds to the right (left) helicity state, provided the xyz frame is right handed. The total degree of photon polarization is given by $\xi=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\leqslant1$. The values ξ_1 and ξ_3 separately are frame dependent, although they are still invariant with respect to boosts or rotations in the $x_\gamma z_\gamma$-plane. They define the angle φ that the electric field makes with the $x_\gamma z_\gamma$-plane:

$$\cos 2\varphi=\frac{\xi_3}{\xi_l}, \quad \sin 2\varphi=\frac{\xi_1}{\xi_l}. \tag{3.4}$$

To fix the azimuthal freedom in ξ_1 and ξ_3, we first choose a frame in which all the momenta $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{k}'$ and $\boldsymbol{p}'$ are coplanar. This choice is not too restrictive and includes both the lab and c.m. frames. In such a frame and for any polarized photon, either γ or γ', we take the y_γ axis in Eq. (3.2) to lie along the direction of $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}'$. Then the appropriate z_γ axis is given by $\hat{\boldsymbol{k}}$ or $\hat{\boldsymbol{k}}'$, and the x_γ axis is directed along $(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}')\times\hat{\boldsymbol{k}}$ or $(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}')\times\hat{\boldsymbol{k}}'$, respectively. The angle φ in Eq. (3.4) gives the angle between the electric field and the reaction plane. Thus defined, the Stokes parameters do not depend on further specification of the frame and are the same in the lab or c.m. frame. We will use the prime to distinguish the Stokes parameters for initial or final photon, ξ_i and ξ_i'.

Note that the above-defined Stokes parameters transform under parity as

$$\xi_1\overset{P}{\to}-\xi_1, \quad \xi_2\overset{P}{\to}-\xi_2, \quad \xi_3\overset{P}{\to}\xi_3, \tag{3.5}$$

under time inversion ($\hat{\boldsymbol{k}}\to-\hat{\boldsymbol{k}}$ $\boldsymbol{e}\to-\boldsymbol{e}^*$) as

$$\xi_1\overset{T}{\to}-\xi_1, \quad \xi_2\overset{T}{\to}\xi_2, \quad \xi_3\overset{T}{\to}\xi_3, \tag{3.6}$$

and under crossing ($\hat{\boldsymbol{k}}\to\hat{\boldsymbol{k}}'$ $\boldsymbol{e}\to\boldsymbol{e}'^*$) as

$$\xi_1\overset{\text{cross}}{\to}\xi_1', \quad \xi_2\overset{\text{cross}}{\to}-\xi_2', \quad \xi_3\overset{\text{cross}}{\to}\xi_3'. \tag{3.7}$$

The nucleon polarization density matrix is specified by a polarization four-vector S which is orthogonal to the nucleon four-momentum p [31]:

$$\langle u(p)\overline{u}(p)\rangle=\frac{1}{2}(\gamma\cdot p+m)(1+\gamma_5\gamma\cdot S). \tag{3.8}$$

Introducing also the polarization three-vector $\boldsymbol{\zeta}$ in the nucleon rest frame, one can relate $\boldsymbol{\zeta}$ and S through the boost transformation,

$$\boldsymbol{S}=\boldsymbol{\zeta}+\frac{S_0}{p_0+m}\boldsymbol{p}, \quad S_0=\frac{\boldsymbol{p}\cdot\boldsymbol{S}}{p_0}=\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{m}, \tag{3.9}$$

where $\sqrt{-S^2}=|\boldsymbol{\zeta}|\leqslant1$ gives the degree of nucleon polarization. We apply the notations S, $\boldsymbol{\zeta}$, and S', $\boldsymbol{\zeta}'$ for the initial and final nucleons, respectively.

Note that the vectors $\boldsymbol{\zeta}$, $\boldsymbol{\zeta}'$ are frame dependent and undergo Wigner rotation around the y axis when a boost in the reaction plane is applied. Nevertheless, $\boldsymbol{\zeta}$ is the same in the lab and c.m. frames, although that is not the case for $\boldsymbol{\zeta}'$.

In both the lab and c.m. frames, the differential cross section of the reactions (3.1) reads

$$\frac{d\sigma}{d\Omega}=\Phi^2|T_{fi}|^2 \quad \text{with} \quad \Phi=\begin{cases}\dfrac{1}{8\pi m}\dfrac{\omega'}{\omega}, & \text{lab frame,}\\[2ex] \dfrac{1}{8\pi\sqrt{s}}, & \text{c.m. frame.}\end{cases} \tag{3.10}$$

Here the square of the Lorentz-invariant matrix element T_{fi}, appropriately averaged and summed over polarizations, has the same generic form in all four cases (3.1),

$$|T_{fi}|^2=\sum_{i=0}^{3}[W_{0i}+K\cdot SW_{1i}+Q\cdot SW_{2i}+N\cdot SW_{3i}]\xi_i, \tag{3.11}$$

where we set $\xi_0=1$, and for the moment we disregard optional primes distinguishing polarization variables for initial and final particles. Since S and N are axial vectors and ξ_1, ξ_2 have odd P parity, some of the invariant functions W_{ij} must vanish:

$$W_{01}=W_{02}=W_{10}=W_{13}=W_{20}=W_{23}=W_{31}=W_{32}=0. \tag{3.12}$$

In terms of the remaining W_{ij}, the generic expression (3.11) gets the following specific form for the individual cases given in Eq. (3.1):

$$|T_{fi}(\vec{\gamma}\vec{N}\to\gamma' N')|^2 = W_{00}+W_{03}\xi_3+N\cdot S(W_{30}+W_{33}^{+}\xi_3) + K\cdot S(W_{11}^{+}\xi_1+W_{12}^{+}\xi_2) + Q\cdot S(W_{21}^{+}\xi_1+W_{22}^{+}\xi_2), \tag{3.13a}$$

$$|T_{fi}(\gamma\vec{N}\to\vec{\gamma}' N')|^2 = W_{00}+W_{03}\xi_3'+N\cdot S(W_{30}+W_{33}^{-}\xi_3') + K\cdot S(-W_{11}^{-}\xi_1'+W_{12}^{-}\xi_2') + Q\cdot S(W_{21}^{-}\xi_1'-W_{22}^{-}\xi_2'), \tag{3.13b}$$

$$|T_{fi}(\vec{\gamma}N\to\gamma'\vec{N}')|^2 = W_{00}+W_{03}\xi_3+N\cdot S'(W_{30}+W_{33}^{-}\xi_3) + K\cdot S'(W_{11}^{-}\xi_1+W_{12}^{-}\xi_2) + Q\cdot S'(W_{21}^{-}\xi_1+W_{22}^{-}\xi_2), \tag{3.13c}$$

$$|T_{fi}(\gamma N\to\vec{\gamma}'\vec{N}')|^2 = W_{00}+W_{03}\xi_3'+N\cdot S'(W_{30}+W_{33}^{+}\xi_3') + K\cdot S'(-W_{11}^{+}\xi_1'+W_{12}^{+}\xi_2') + Q\cdot S'(W_{21}^{+}\xi_1'-W_{22}^{+}\xi_2'). \tag{3.13d}$$

Note that the same functions W_{ij} determine the cross section in Eqs. (3.13a) and (3.13d) [as well as in Eqs. (3.13b) and (3.13c)] which are related through T invariance; the negative signs in Eqs. (3.13b), (3.13d) are easily found from Eq. (3.6). The relationship between the cases (3.13a) and (3.13b) is determined by the crossing symmetry of the amplitude T_{fi} and Eq. (3.7):

$$W_{00}(\nu,t)=W_{00}(-\nu,t),\quad W_{03}(\nu,t)=W_{03}(-\nu,t),$$
$$W_{30}(\nu,t)=-W_{30}(-\nu,t),\quad W_{33}^{-}(\nu,t)=-W_{33}^{+}(-\nu,t),$$
$$W_{11}^{-}(\nu,t)=W_{11}^{+}(-\nu,t),\quad W_{12}^{-}(\nu,t)=W_{12}^{+}(-\nu,t),$$
$$W_{21}^{-}(\nu,t)=W_{21}^{+}(-\nu,t),\quad W_{22}^{-}(\nu,t)=W_{22}^{+}(-\nu,t). \tag{3.14}$$

In terms of the invariant amplitudes T_i or A_i, the functions W_{ij} read (cf. Ref. [32]):

$$W_{00}=\frac{1}{2}(4m^2-t)(|T_1|^2+|T_3|^2)-\frac{1}{2}(s-m^2)(u-m^2)(|T_2|^2+|T_4|^2)+m(s-u)\ \mathrm{Re}\,(T_1T_2^*+T_3T_4^*) - t|T_5|^2+(m^4-su)|T_6|^2$$
$$=\frac{1}{4}(4m^2-t)(t^2|A_1|^2+\eta^2|A_3|^2)-\frac{1}{4}(t^3|A_2|^2-\eta^3|A_4|^2)-\nu^2t(t+8\nu^2)|A_5|^2+\frac{1}{2}\eta(t^2+2m^2\eta)|A_6|^2 + \mathrm{Re}\left\{2\nu^2t^2(A_1+A_2)A_5^*+\frac{1}{2}\eta^2(4m^2A_3+tA_4)A_6^*\right\}, \tag{3.15a}$$

$$W_{03}=\frac{1}{2}(4m^2-t)(|T_1|^2-|T_3|^2)-\frac{1}{2}(s-m^2)(u-m^2)(|T_2|^2-|T_4|^2)+m(s-u)\ \mathrm{Re}\,(T_1T_2^*-T_3T_4^*)$$
$$=\frac{\eta t}{2}\,\mathrm{Re}\,\{[(4m^2-t)A_1+4\nu^2A_5]A_3^*+4m^2A_1A_6^*\}, \tag{3.15b}$$

$$W_{30}=-4\,\mathrm{Im}\,(T_1T_2^*+T_3T_4^*) =-8\nu\,\mathrm{Im}\,(tA_1A_5^*+\eta A_3A_6^*), \tag{3.15c}$$

$$W_{33}^{\pm}=4\,\mathrm{Im}\,[-T_1T_2^*+T_3T_4^*\pm 2T_5T_6^*] = \mathrm{Im}\left\{-8\nu[[tA_1-(t+4\nu^2)A_5]A_6^*+\eta A_3A_5^*]\pm\frac{2}{m}(tA_2-4\nu^2A_5)(\eta A_4^*+tA_6^*)\right\}, \tag{3.15d}$$

$$W_{11}^{\pm}=\mathrm{Im}\,[(4m^2-t)(T_1+T_3)T_6^*+m(s-u)(T_2+T_4)T_6^*\mp t(T_2-T_4)T_5^*] = \mathrm{Im}\left\{\frac{t}{2m}[(4m^2-t)A_1+4\nu^2A_5](\eta A_4^*+tA_6^*)\pm 2\nu t(tA_2-4\nu^2A_5)A_6^*\right\}, \tag{3.15e}$$

$$W_{12}^{\pm}=\mathrm{Re}\,[(4m^2-t)(-T_1+T_3)T_6^*+m(s-u)(-T_2+T_4)T_6^*\pm t(T_2+T_4)T_5^*]$$
$$=\mathrm{Re}\left\{-\frac{\eta}{2m}[(4m^2-t)A_3+4m^2A_6](\eta A_4^*+tA_6^*)\pm 2\nu t(tA_2-4\nu^2A_5)A_5^*\right\}, \tag{3.15f}$$

$$W_{21}^{\pm}=\mathrm{Im}[4m(T_1-T_3)T_5^*+(s-u)(T_2-T_4)T_5^*\mp mt(T_2+T_4)T_6^*\pm(s-u)(T_1+T_3)T_6^*]$$
$$=2\mathrm{Im}\{-m(tA_2-4\nu^2A_5)[\eta A_3^*+(t+4\nu^2)A_6^*]\pm\nu[tA_1-(t+4\nu^2)A_5](\eta A_4^*+tA_6^*)\}, \tag{3.15g}$$

$$W_{22}^{\pm}=\mathrm{Re}[-4m(T_1+T_3)T_5^*-(s-u)(T_2+T_4)T_5^*\pm mt(T_2-T_4)T_6^*\mp(s-u)(T_1-T_3)T_6^*]$$
$$=2\mathrm{Re}\{-mt(tA_2-4\nu^2A_5)A_1^*\mp\nu\eta A_3(\eta A_4^*+tA_6^*)\}. \tag{3.15h}$$

Below the pion photoproduction threshold, the amplitudes T_i, A_i are real, and therefore only the six structures, W_{00}, W_{03}, $W_{12}^{\pm}$, and $W_{22}^{\pm}$, are different from zero.

B. Asymmetries

The thirteen invariant quantities W_{ij} standing in Eq. (3.13) intervene directly in the definition of polarization observables which can be measured experimentally. Not all of them are independent, since with six complex invariant amplitudes there are only 11 independent observables. In the following we shall classify them according to the number of polarization degrees of freedom involved in the process. We shall define some n-index asymmetries Σ_i, Σ_α, and $\Sigma_{i\alpha}$, where $i=(1,2,3)$ or $(1',2',3')$ refers to the photon Stokes parameters ξ_i or ξ_i', and $\alpha=(x_N,y_N,z_N)$ or (x_N',y_N',z_N') refers to the right-handed axes along which the nucleon spin $\boldsymbol{\zeta}$ or $\boldsymbol{\zeta}'$ may be aligned. At this point we have to relate the products such as $K\cdot S$, $Q\cdot S$, and $N\cdot S$ in Eq. (3.13) to the spin vector $\boldsymbol{\zeta}$ (or $\boldsymbol{\zeta}'$), and eventually fix a frame. We choose the lab frame for practical reasons; for any other frame only the coefficients $C_{x,z,x',z'}^{K,Q}$ introduced below have to be recalculated. We will indicate all necessary changes to be done for the c.m. frame.

Directing the y_N and y_N' axes along $\boldsymbol{k}\times\boldsymbol{k}'$, we choose the z_N axis along the photon beam direction $\hat{\boldsymbol{k}}$. Note that since $\boldsymbol{\zeta}_{\mathrm{c.m.}}=\boldsymbol{\zeta}_{\mathrm{lab}}$, all the asymmetries obtained in this way in the lab frame for the reactions (3.1a),(3.1b) with the polarized nucleon N are the same in the c.m. frame. In the case of the polarized final nucleon, we choose the z_N' axis to lie along the nucleon recoil momentum $\boldsymbol{p}'$. Such an axis depends on whether we use the lab or c.m. frame so that we get different, though related, asymmetries.

We can further simplify the notation. The axes $(x_\gamma,y_\gamma,z_\gamma)$ and (x_N,y_N,z_N) are identical and we will denote them in the following by simply (x,y,z). For (x_N',y_N',z_N') we will use (x',y',z'). Note that, despite the fact that the y and y' axes are the same, we will keep the prime when using y' because it identifies which nucleon (N or N') we are referring to. All these axes are shown in Fig. 1.

The set of observables is defined as follows.

Polarization-independent observable ($n=0$), or simply the unpolarized cross section,

$$\frac{d\overline{\sigma}}{d\Omega}=\Phi^2W_{00}. \tag{3.16}$$

Single-polarization observables ($n=1$), of which there are two for each of the reactions (3.1):

(i) The beam asymmetry for photons which are linearly polarized either parallel or perpendicular to the scattering plane ($\xi_3=\pm1$) and unpolarized nucleons (target and recoil). The same quantity gives the degree of linear polarization (ξ_3') of the photon scattered from unpolarized nucleons:

$$\Sigma_3=\Sigma_{3'}=\left(\frac{\sigma^{\parallel}-\sigma^{\perp}}{\sigma^{\parallel}+\sigma^{\perp}}\right)_{\zeta=\zeta'=0}=\left(\frac{\sigma^{\parallel'}-\sigma^{\perp'}}{\sigma^{\parallel'}+\sigma^{\perp'}}\right)_{\zeta=\zeta'=0}=\frac{W_{03}}{W_{00}} \tag{3.17}$$

(the primed polarizations $\parallel'$ or $\perp'$ refer to the final photon state). This asymmetry is often designated as Σ.

(ii) The target asymmetry or recoil polarization for unpolarized photons, whereby either N or N' is polarized (ζ_y or $\zeta_y'=\pm1$) along the $\pm y=\pm y'$ directions:

$$\Sigma_y=\Sigma_{y'}=\left(\frac{\sigma_y-\sigma_{-y}}{\sigma_y+\sigma_{-y}}\right)_{\xi=\xi'=0}=\left(\frac{\sigma_{y'}-\sigma_{-y'}}{\sigma_{y'}+\sigma_{-y'}}\right)_{\xi=\xi'=0}$$
$$=C_y^N\frac{W_{30}}{W_{00}}. \tag{3.18}$$

Here the coefficient

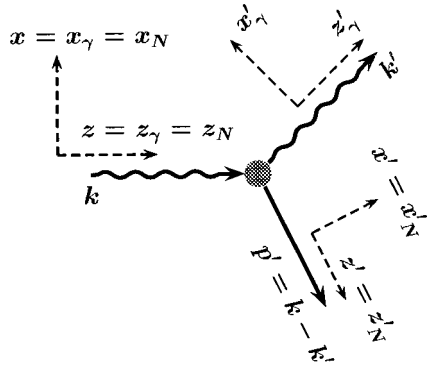

FIG. 1. Axes to project out polarizations in the lab frame. The various y axes (y_γ, y_N, y_γ', and y_N') all point out of the plane of the figure.

$$C_y^N = \frac{m}{2}\omega\omega' \sin\theta = \frac{m}{4}\sqrt{-\eta t} \tag{3.19}$$

(as given through both lab and invariant quantities) determines the scalar product $N\cdot S = C_y^N \zeta_y$. It is just the negative-y component of N^μ. The asymmetry $\Sigma_y = \Sigma_{y'}$ is often designated as $\mathcal{P}$, the recoil nucleon polarization.

Double-polarization observables ($n=2$), of which there are five for each of the reactions (3.1):

(i) The beam-target asymmetry for incoming linearly polarized photons, either parallel or perpendicular to the scattering plane ($\xi_3 = \pm 1$), and target nucleon polarized perpendicular to the scattering plane. The same quantity can be measured as the correlation of polarizations of the outgoing particles γ' and N':

$$\Sigma_{3y} = \Sigma_{3'y'} = \frac{(\sigma^\parallel - \sigma^\perp)_y - (\sigma^\parallel - \sigma^\perp)_{-y}}{(\sigma^\parallel + \sigma^\perp)_y + (\sigma^\parallel + \sigma^\perp)_{-y}} = \frac{(\sigma^{\parallel\prime} - \sigma^{\perp\prime})_{y'} - (\sigma^{\parallel\prime} - \sigma^{\perp\prime})_{-y'}}{(\sigma^{\parallel\prime} + \sigma^{\perp\prime})_{y'} + (\sigma^{\parallel\prime} + \sigma^{\perp\prime})_{-y'}} = C_y^N \frac{W_{33}^+}{W_{00}}. \tag{3.20}$$

Another independent quantity is the correlation of the beam polarization and the polarization of recoil nucleon, or the correlation of the target polarization with the final photon polarization:

$$\Sigma_{3y'} = \Sigma_{3'y} = \frac{(\sigma^\parallel - \sigma^\perp)_{y'} - (\sigma^\parallel - \sigma^\perp)_{-y'}}{(\sigma^\parallel + \sigma^\perp)_{y'} + (\sigma^\parallel + \sigma^\perp)_{-y'}} = \frac{(\sigma^{\parallel\prime} - \sigma^{\perp\prime})_{y} - (\sigma^{\parallel\prime} - \sigma^{\perp\prime})_{-y}}{(\sigma^{\parallel\prime} + \sigma^{\perp\prime})_{y} + (\sigma^{\parallel\prime} + \sigma^{\perp\prime})_{-y}} = C_y^N \frac{W_{33}^-}{W_{00}}. \tag{3.21}$$

(ii) The asymmetries with circular photon polarizations (ξ_2 or $\xi_2' = \pm 1$). The general expression for the asymmetry with circularly polarized photons follows from Eq. (3.13). For example, for the reaction (3.1a)

$$\frac{\sigma^R - \sigma^L}{\sigma^R + \sigma^L} = \frac{K\cdot S W_{12}^+ + Q\cdot S W_{22}^+}{W_{00} + N\cdot S W_{30}}. \tag{3.22}$$

Such a quantity survives when the nucleon spin lies in the reaction plane. Considering the cases with the target spin $\boldsymbol{\zeta}$ aligned in $\pm x$ or $\pm z$ directions, we can introduce the beam-target asymmetries

$$\Sigma_{2x} = \frac{\sigma_x^R - \sigma_x^L}{\sigma_x^R + \sigma_x^L} = \frac{C_x^K W_{12}^+ + C_x^Q W_{22}^+}{W_{00}},$$

$$\Sigma_{2z} = \frac{\sigma_z^R - \sigma_z^L}{\sigma_z^R + \sigma_z^L} = \frac{C_z^K W_{12}^+ + C_z^Q W_{22}^+}{W_{00}} \tag{3.23}$$

and the scattered photon-target asymmetries

$$\Sigma_{2'x} = \frac{\sigma_x^{R'} - \sigma_x^{L'}}{\sigma_x^{R'} + \sigma_x^{L'}} = \frac{C_x^K W_{12}^- - C_x^Q W_{22}^-}{W_{00}},$$

$$\Sigma_{2'z} = \frac{\sigma_z^{R'} - \sigma_z^{L'}}{\sigma_z^{R'} + \sigma_z^{L'}} = \frac{C_z^K W_{12}^- - C_z^Q W_{22}^-}{W_{00}}. \tag{3.24}$$

Here the coefficients $C_{x,z}^{K,Q}$ determine the scalar products

$$K\cdot S = C_x^K \zeta_x + C_z^K \zeta_z, \quad Q\cdot S = C_x^Q \zeta_x + C_z^Q \zeta_z. \tag{3.25}$$

In terms of lab or invariant quantities, they read

$$C_x^K = C_x^Q = -\frac{1}{2}\omega' \sin\theta = -\frac{m\sqrt{-\eta t}}{2(s-m^2)},$$

$$C_z^K = -\frac{1}{2}(\omega + \omega' \cos\theta) = -\frac{s-m^2}{2m} - \frac{t(s+m^2)}{4m(s-m^2)}, \tag{3.26}$$

$$C_z^Q = \frac{1}{2}(\omega - \omega' \cos\theta) = -\frac{t(s+m^2)}{4m(s-m^2)}.$$

Numerically, $C_{x,z}^{K,Q}$ are the same in the lab and c.m. frames.

Another four asymmetries, which are depend linearly on those in Eqs. (3.23) and (3.24), describe the correlations of the recoil polarization with the polarization of the photons γ or γ':

$$\Sigma_{2x'} = \frac{\sigma_{x'}^R - \sigma_{x'}^L}{\sigma_{x'}^R + \sigma_{x'}^L} = \frac{C_{x'}^K W_{12}^- + C_{x'}^Q W_{22}^-}{W_{00}},$$

$$\Sigma_{2z'} = \frac{\sigma_{z'}^R - \sigma_{z'}^L}{\sigma_{z'}^R + \sigma_{z'}^L} = \frac{C_{z'}^K W_{12}^- + C_{z'}^Q W_{22}^-}{W_{00}}, \tag{3.27}$$

$$\Sigma_{2'x'} = \frac{\sigma_{x'}^{R'} - \sigma_{x'}^{L'}}{\sigma_{x'}^{R'} + \sigma_{x'}^{L'}} = \frac{C_{x'}^K W_{12}^+ - C_{x'}^Q W_{22}^+}{W_{00}},$$

$$\Sigma_{2'z'} = \frac{\sigma_{z'}^{R'} - \sigma_{z'}^{L'}}{\sigma_{z'}^{R'} + \sigma_{z'}^{L'}} = \frac{C_{z'}^K W_{12}^+ - C_{z'}^Q W_{22}^+}{W_{00}}.$$

They are given by the coefficients $C_{x',z'}^{K,Q}$ appearing in the expansions

$$K\cdot S' = C_{x'}^K \zeta_{x'}' + C_{z'}^K \zeta_{z'}', \quad Q\cdot S' = C_{x'}^Q \zeta_{x'}' + C_{z'}^Q \zeta_{z'}', \tag{3.28}$$

which in the lab frame read

$$C_{x'}^K = -\frac{\sqrt{\eta}}{2N(t)}, \quad C_{x'}^Q = 0, \quad C_{z'}^K = \frac{\nu\sqrt{-t}}{2mN(t)},$$

$$C_{z'}^Q = \frac{\sqrt{-t}}{2}N(t) \quad \text{(lab)}, \tag{3.29}$$

where $N(t)=(1-t/4m^2)^{1/2}$. Since the z' axis in the c.m. frame is different from that in the lab frame, the corresponding $C^{K,Q}_{x',z'}$ are different too. Except for the signs, they coincide with $C^{K,Q}_{x,y}$ in Eq. (3.26):

$$C^K_{x'}=-C^Q_{x'}=C^K_x=C^Q_x, \quad C^K_{z'}=-C^K_z,$$

$$C^Q_{z'}=C^Q_z \quad \text{(c.m.)}. \tag{3.30}$$

(iii) The asymmetries with photons linearly polarized at $\varphi=\pm\pi/4$ with respect to the scattering plane ($\xi_1=\pm 1$). For example, in the case (3.1a),

$$\frac{\sigma^{\pi/4}-\sigma^{-\pi/4}}{\sigma^{\pi/4}+\sigma^{-\pi/4}}=\frac{K\cdot SW^+_{11}+Q\cdot SW^+_{21}}{W_{00}+N\cdot SW_{30}}. \tag{3.31}$$

With the same considerations discussed for Eq. (3.23), one can define the asymmetries

$$\Sigma_{1x}=\frac{\sigma^{\pi/4}_x-\sigma^{-\pi/4}_x}{\sigma^{\pi/4}_x+\sigma^{-\pi/4}_x}=\frac{C^K_xW^+_{11}+C^Q_xW^+_{21}}{W_{00}}$$

$$\Sigma_{1z}=\frac{\sigma^{\pi/4}_z-\sigma^{-\pi/4}_z}{\sigma^{\pi/4}_z+\sigma^{-\pi/4}_z}=\frac{C^K_zW^+_{11}+C^Q_zW^+_{21}}{W_{00}}, \tag{3.32}$$

$$\Sigma_{1'x}=\frac{\sigma^{\pi/4'}_x-\sigma^{-\pi/4'}_x}{\sigma^{\pi/4'}_x+\sigma^{-\pi/4'}_x}=\frac{-C^K_xW^-_{11}+C^Q_xW^-_{21}}{W_{00}},$$

$$\Sigma_{1'z}=\frac{\sigma^{\pi/4'}_z-\sigma^{-\pi/4'}_z}{\sigma^{\pi/4'}_z+\sigma^{-\pi/4'}_z}=\frac{-C^K_zW^-_{11}+C^Q_zW^-_{21}}{W_{00}},$$

and a similar linearly dependent set

$$\Sigma_{1x'}=\frac{\sigma^{\pi/4}_{x'}-\sigma^{-\pi/4}_{x'}}{\sigma^{\pi/4}_{x'}+\sigma^{-\pi/4}_{x'}}=\frac{C^K_{x'}W^-_{11}+C^Q_{x'}W^-_{21}}{W_{00}},$$

$$\Sigma_{1z'}=\frac{\sigma^{\pi/4}_{z'}-\sigma^{-\pi/4}_{z'}}{\sigma^{\pi/4}_{z'}+\sigma^{-\pi/4}_{z'}}=\frac{C^K_{z'}W^-_{11}+C^Q_{z'}W^-_{21}}{W_{00}}, \tag{3.33}$$

$$\Sigma_{1'x'}=\frac{\sigma^{\pi/4'}_{x'}-\sigma^{-\pi/4'}_{x'}}{\sigma^{\pi/4'}_{x'}+\sigma^{-\pi/4'}_{x'}}=\frac{-C^K_{x'}W^+_{11}+C^Q_{x'}W^+_{21}}{W_{00}},$$

$$\Sigma_{1'z'}=\frac{\sigma^{\pi/4'}_{z'}-\sigma^{-\pi/4'}_{z'}}{\sigma^{\pi/4'}_{z'}+\sigma^{-\pi/4'}_{z'}}=\frac{-C^K_{z'}W^+_{11}+C^Q_{z'}W^+_{21}}{W_{00}}.$$

One can get another representation (with $x\to -x$, etc.) for the above asymmetries by using the following relations among the cross sections that have opposite in-plane nucleon (target or recoil) polarizations:

$$\sigma^R_i=\sigma^L_{-i}, \quad \sigma^\varphi_i=\sigma^{-\varphi}_{-i}, \quad (i=\pm x,\pm z,\pm x',\pm z') \tag{3.34}$$

and similarly for the primed cross sections (with the polarized final photon). These relations are due to parity conservation.

Introducing the generic quantities

$$F_1=\frac{\sigma^{\pi/4}-\sigma^{-\pi/4}}{2\bar{\sigma}}=\Sigma_{1x}\zeta_x+\Sigma_{1z}\zeta_z,$$

$$F_2=\frac{\sigma^R-\sigma^L}{2\bar{\sigma}}=\Sigma_{2x}\zeta_x+\Sigma_{2z}\zeta_z, \tag{3.35}$$

$$F_3=\frac{\sigma^{\parallel}-\sigma^{\perp}}{2\bar{\sigma}}=\Sigma_3+\Sigma_{3y}\zeta_y,$$

one can write the differential cross section in the following compact form:

$$\frac{d\sigma}{d\Omega}=\frac{d\bar{\sigma}}{d\Omega}\{1+\Sigma_y\zeta_y+\boldsymbol{F}\cdot\boldsymbol{\xi}\}, \tag{3.36}$$

which, being supplied with appropriated primes, is valid for any combination of ingoing or outgoing polarized particles.

Thus, with unpolarized and linearly polarized initial or final photons, one can access two observables: $d\bar{\sigma}/d\Omega$ and Σ_3, respectively. When the nucleon (target or recoil) polarization is added, four more asymmetries appear, Σ_y and Σ_{1x}, Σ_{1z}, Σ_{3y} (and their primed companions), all of which vanish below the pion threshold. When circularly polarized photons are used with polarized nucleons, two more asymmetries appear, Σ_{2x} and Σ_{2z}, both of which survive below the pion threshold.

It is important to emphasize that the formulas such as (3.10) and (3.36), being used for polarized particle(s) in the final state, give the total yield of the particles and their polarization density matrix. To get a partial cross section of a particle produced with specific final polarization(s), one must use an average instead of a sum over polarizations. That is equivalent to inserting the statistical factor $1/g$ into these formulas, where $g=1,2,2,4$ for the cases (3.1a)–(3.1d), respectively.

IV. LOW-ENERGY EXPANSIONS

A. Invariant amplitudes and polarizabilities

A very general method for obtaining low-energy expansions of physical amplitudes for different reactions consists in introducing invariant amplitudes free from kinematical singularities and constraints and expanding them in a power series [33,34]. Following this method, we first decompose the invariant amplitudes A_i into Born and non-Born contributions,

$$A_i(\nu,t)=A^{\rm B}_i(\nu,t)+A^{\rm NB}_i(\nu,t) \quad (i=1,\ldots 6). \tag{4.1}$$

The Born contribution is associated with pole diagrams involving a single nucleon exchanged in s- or u-channels and γNN vertices taken in the on-shell regime,

$$\Gamma_\mu(p',p)=q\gamma_\mu-\frac{\kappa}{4m}[\gamma_\mu,\gamma\cdot(p'-p)]. \tag{4.2}$$

It is completely specified by the mass m, the electric charge eq, and the anomalous magnetic moment $e\kappa/2m$ of the nucleon:

$$A_i^{\mathrm{B}}(\nu,t)=\frac{me^2r_i}{(s-m^2)(u-m^2)}=-\frac{e^2r_i}{4m\omega\omega'}, \qquad (4.3)$$

where e is the elementary charge, $e^2\simeq 4\pi/137$, and $q=1,0$ for the proton and neutron, respectively. The pole residues r_i read

$$r_1=-2q^2+r_3\frac{t}{4m^2}, \quad r_2=2q\kappa+2q^2+r_3\frac{t}{4m^2},$$
$$r_3=r_5=\kappa^2+2q\kappa, \quad r_4=\kappa^2, \quad r_6=-2q^2-r_3. \qquad (4.4)$$

The Born contribution to T_{fi} possesses all the symmetries of the total amplitude T_{fi}, including gauge invariance, and takes all singularities of T_{fi} at low energies.

The non-Born parts A_i^{NB} of the invariant amplitudes, which contain all the structure-dependent information, are regular functions of ν^2 and t and can be expanded as a power series in ν^2 and t. Since

$$t=-2\omega\omega'(1-z), \quad \nu^2=\omega\omega'+\frac{t^2}{16m^2}, \qquad (4.5)$$

such an expansion can be recast as a power series in the cross-even parameter $\omega\omega'$:

$$A_i^{\mathrm{NB}}(\nu,t)=a_i+\omega\omega'(a_{i,\nu}-2(1-z)a_{i,t})+\cdots. \qquad (4.6)$$

Here the low-energy constants

$$a_i=A_i^{\mathrm{NB}}(0,0), \quad a_{i,\nu}=\left(\frac{\partial A_i^{\mathrm{NB}}}{\partial\nu^2}\right)_{\nu=t=0},$$
$$a_{i,t}=\left(\frac{\partial A_i^{\mathrm{NB}}}{\partial t}\right)_{\nu=t=0}, \ldots \qquad (4.7)$$

parametrize the structure of the nucleon as seen in its two-photon interactions.

The expansion (4.6) directly leads to a corresponding low-energy expansion of the total amplitude T_{fi} in the lab frame. We first consider the spin-independent part of Eq. (2.11). The spin-independent part of the Born term T_{fi}^{B} follows from Eq. (2.11):

$$\frac{N(t)}{8\pi m}T_{fi}^{\mathrm{B,nospin}}$$
$$=r_0\boldsymbol{e}'^*\cdot\boldsymbol{e}\left\{-q^2+\frac{\omega\omega'}{4m^2}[\kappa^2+2q\kappa-q^2(1-z)]\right\}$$
$$+r_0\boldsymbol{s}'^*\cdot\boldsymbol{s}\,\frac{\omega\omega'}{4m^2}[q^2-z(\kappa+q)^2]. \qquad (4.8)$$

Here $r_0=e^2/4\pi m$. The leading term in Eq. (4.8) reproduces the Thomson limit,

$$\frac{1}{8\pi m}T_{fi}^{\mathrm{Thomson}}=-r_0q^2\boldsymbol{e}'^*\cdot\boldsymbol{e}\ . \qquad (4.9)$$

The non-Born contribution to T_{fi} is determined by the structure constants introduced in the expansion (4.6). Its spin-independent part starts with a $\omega\omega'$ term,

$$\frac{1}{8\pi m}T_{fi}^{\mathrm{NB,nospin}}=\omega\omega'(\alpha_E\boldsymbol{e}'^*\cdot\boldsymbol{e}+\beta_M\boldsymbol{s}'^*\cdot\boldsymbol{s})+\mathcal{O}(\omega^2\omega'^2). \qquad (4.10)$$

Here the constants α_E and β_M,

$$4\pi\alpha_E=-a_3-a_6-a_1, \quad 4\pi\beta_M=-a_3-a_6+a_1, \qquad (4.11)$$

are identified as the dipole electric and magnetic polarizabilities of the nucleon, just in accordance with an $\mathcal{O}(\omega^2)$ effective dipole interaction

$$H_{\mathrm{eff}}^{(2),\mathrm{nospin}}=-\frac{1}{2}4\pi(\alpha_E\boldsymbol{E}^2+\beta_M\boldsymbol{H}^2) \qquad (4.12)$$

of the nucleon with external electric and magnetic fields, leading to the amplitude (4.10). Due to its interference with the Thomson amplitude (4.9), the contribution of the polarizabilities, Eq. (4.10), results in a $\mathcal{O}(\omega^2)$ effect in the differential cross section $d\bar{\sigma}/d\Omega$ in the case of the proton ($q=1$) and in a $\mathcal{O}(\omega^4)$ effect in the case of the neutron ($q=0$).

The $\mathcal{O}(\omega^4)$ terms in Eq. (4.10) are also easily read out from Eqs. (2.11) and (4.6). They are determined by the dipole polarizabilities and the following combinations of ν- and t-derivative constants $a_{i,\nu}$ and $a_{i,t}$ of the amplitudes A_1, A_3 and A_6:

$$4\pi\alpha_\nu=-a_{3,\nu}-a_{6,\nu}-a_{1,\nu}-\frac{a_5}{m^2},$$
$$4\pi\beta_\nu=-a_{3,\nu}-a_{6,\nu}+a_{1,\nu}+\frac{a_5}{m^2},$$
$$4\pi\alpha_t=-a_{3,t}-a_{6,t}-a_{1,t}+\frac{a_3}{4m^2}, \qquad (4.13)$$
$$4\pi\beta_t=-a_{3,t}-a_{6,t}+a_{1,t}+\frac{a_3}{4m^2}.$$

These polarizability-like quantities are constant coefficients of energy- and angle-dependent corrections to the dipole interaction (4.12) that enter to next order in photon energy. The recoil correction $\sim 1/m^2$ in Eqs. (4.13) is explained in Appendix C.

We now introduce linear combinations of the parameters (4.13) which have a more direct physical meaning. If we consider the partial-wave structure of the amplitude T_{fi} (see Appendix A), we can relate the t-derivative constants in Eq. (4.13) to the quadrupole polarizabilites of the nucleon [25]:

$$\alpha_{E2}=12\alpha_t, \quad \beta_{M2}=12\beta_t. \tag{4.14}$$

A nonrelativistic example is given in Appendix B. The quantities

$$\alpha_{E\nu}=\alpha_\nu-2\alpha_t+\beta_t, \quad \beta_{M\nu}=\beta_\nu-2\beta_t+\alpha_t, \tag{4.15}$$

which we call ''dispersion polarizabilities,'' describe the ω dependence of the dynamic dipole polarizabilities. In terms of the parameters α_{E2}, β_{M2}, $\alpha_{E\nu}$, and $\beta_{E\nu}$, the corresponding effective interaction of $\mathcal{O}(\omega^4)$ has the form

$$H_{\rm eff}^{(4),\rm nospin}=-\frac{1}{2}4\pi(\alpha_{E\nu}\dot{\boldsymbol{E}}^2+\beta_{M\nu}\dot{\boldsymbol{H}}^2) -\frac{1}{12}4\pi(\alpha_{E2}E_{ij}^2+\beta_{M2}H_{ij}^2), \tag{4.16}$$

where the dots mean a time derivative and

$$E_{ij}=\frac{1}{2}(\nabla_i E_j+\nabla_j E_i), \quad H_{ij}=\frac{1}{2}(\nabla_i H_j+\nabla_j H_i) \tag{4.17}$$

are the quadrupole strengths of the electric and magnetic fields.

We next consider the spin-dependent part of Eq. (2.11). The spin-dependent part of T_{fi} starts with $\mathcal{O}(\omega)$ terms that come from the Born contribution. To leading order one has [35]

$$\frac{1}{8\pi m}T_{fi}^{\rm B,,spin}=-ir_0\frac{\nu}{2m}(q^2\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\boldsymbol{e}+(\kappa+q)^2\boldsymbol{\sigma}\cdot\boldsymbol{s}'^*\times\boldsymbol{s}) +ir_0q\frac{\kappa+q}{2m}(\omega'\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}'\ \boldsymbol{s}'^*\cdot\boldsymbol{e}-\omega\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\ \boldsymbol{e}'^*\cdot\boldsymbol{s}) +\mathcal{O}(\omega^2). \tag{4.18}$$

The omitted higher-order terms can be read out from Eqs. (2.11) and (4.3).

As shown in Eq. (2.11), the non-Born part of $T_{fi}^{\rm spin}$ starts with $\mathcal{O}(\omega^3)$ terms. They are determined by the four constants a_2, a_4, a_5, and a_6. Due to their interference with the spin-dependent $\mathcal{O}(\omega)$ terms of Eq. (4.18), they give rise to a $\mathcal{O}(\omega^4)$ correction to the unpolarized cross section. In the case of a polarized proton and a circularly polarized photon, these terms appear at the order $\mathcal{O}(\omega^3)$.

The $\mathcal{O}(\omega^3)$ terms in $T_{fi}^{\rm NB,spin}$ were considered in several papers [21–23]. In the most recent (and best known) work [23] they are parametrized as

$$\frac{1}{8\pi m}T_{fi}^{\rm NB,spin}=i\omega^3\gamma_1\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\boldsymbol{e} +i\omega^3\gamma_2(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{k}}\ \boldsymbol{e}'^*\cdot\boldsymbol{e}-\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\boldsymbol{e}\ \hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}}) +i\omega^3\gamma_3(\boldsymbol{\sigma}\cdot\boldsymbol{s}\ \boldsymbol{e}'^*\cdot\hat{\boldsymbol{k}}-\boldsymbol{\sigma}\cdot\boldsymbol{s}'\ \boldsymbol{e}\cdot\hat{\boldsymbol{k}}') +i\omega^3\gamma_4(\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\hat{\boldsymbol{k}}\ \boldsymbol{e}\cdot\hat{\boldsymbol{k}}'-\boldsymbol{\sigma}\cdot\boldsymbol{e}\times\hat{\boldsymbol{k}}'\ \boldsymbol{e}'^*\cdot\hat{\boldsymbol{k}} -2\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\boldsymbol{e}\ \hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}})+\mathcal{O}(\omega^4), \tag{4.19}$$

where γ_i are structure parameters (often called the ''spin polarizabilities'') which are linear combinations of the constants a_i (see Appendix A). In the following we consider a linear combination of these parameters:

$$\gamma_{E1}=-\gamma_1-\gamma_3=\frac{1}{8\pi m}(a_6-a_4+2a_5+a_2),$$

$$\gamma_{M1}=\gamma_4=\frac{1}{8\pi m}(a_6-a_4-2a_5-a_2),$$

$$\gamma_{E2}=\gamma_2+\gamma_4=-\frac{1}{8\pi m}(a_4+a_6+a_2),$$

$$\gamma_{M2}=\gamma_3=-\frac{1}{8\pi m}(a_4+a_6-a_2), \tag{4.20}$$

which have a more transparent physical meaning, as explained in Appendixes A and B. The parameters γ_{E1} and γ_{M1} describe the spin dependence of the dipole electric and magnetic photon scattering, $E1\to E1$ and $M1\to M1$, whereas γ_{E2} and γ_{M2} describe the dipole-quadrupole amplitudes $M1\to E2$ and $E1\to M2$, respectively. The amplitude (4.19) implies an effective spin-dependent interaction of order $\mathcal{O}(\omega^3)$

$$H_{\rm eff}^{(3),\rm spin}=-\frac{1}{2}4\pi(\gamma_{E1}\boldsymbol{\sigma}\cdot\boldsymbol{E}\times\dot{\boldsymbol{E}}+\gamma_{M1}\boldsymbol{\sigma}\cdot\boldsymbol{H}\times\dot{\boldsymbol{H}} -2\gamma_{E2}E_{ij}\sigma_iH_j+2\gamma_{M2}H_{ij}\sigma_iE_j). \tag{4.21}$$

To summarize, we have obtained a low-energy expansion of the amplitudes for nucleon Compton scattering. To $\mathcal{O}(\omega^4)$, the cross section and polarization observables are determined by 10 polarizability parameters: (i) Two dipole polarizabilities: α_E and β_M, (ii) two dispersion corrections to the dipole polarizabilities: $\alpha_{E\nu}$ and $\beta_{M\nu}$, (iii) two quadrupole polarizabilities: α_{E2} and β_{M2}, and (iv) four spin polarizabilities: γ_{E1}, γ_{M1}, γ_{E2}, γ_{M2}. These polarizabilities have a simple physical interpretation in terms of the interaction of the nucleon with an external electromagnetic field. Equivalently, these ten parameters are linear combinations of the low-energy constants, Eq. (4.7), representing the zero-energy limit of the six invariant amplitudes A_i plus two combinations of both the ν and t derivatives of these amplitudes.

To illustrate the interplay among all these polarizabilities, we consider the two limiting cases of forward and backward scattering.

(i) The amplitude for forward scattering,

$$\frac{1}{8\pi m}[T_{fi}]_{\theta=0}=\frac{\omega^2}{2\pi}\left(-\boldsymbol{e}'^*\cdot\boldsymbol{e}\ (A_3+A_6) +\frac{\omega}{m}i\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\boldsymbol{e}\ A_4\right), \tag{4.22}$$

has the following low-energy decomposition:

$$\frac{1}{8\pi m}[T_{fi}]_{\theta=0}=\boldsymbol{e}'^{*}\cdot\boldsymbol{e}\,[-r_0q^2+\omega^2(\alpha_E+\beta_M)+\omega^4(\alpha_\nu+\beta_\nu)+\mathcal{O}(\omega^6)]+i\omega\boldsymbol{\sigma}\cdot\boldsymbol{e}'^{*}\times\boldsymbol{e}\left(-\frac{r_0\kappa^2}{2m}+\omega^2\gamma+\mathcal{O}(\omega^4)\right),\tag{4.23}$$

where

$$\gamma=-\gamma_{E1}-\gamma_{M1}-\gamma_{E2}-\gamma_{M2}=\frac{a_4}{2\pi m}\tag{4.24}$$

is the ''forward-angle spin polarizability,'' and

$$\alpha_\nu+\beta_\nu=\alpha_{E\nu}+\beta_{M\nu}+\tfrac{1}{12}\alpha_{E2}+\tfrac{1}{12}\beta_{M2}=-\frac{1}{2\pi}(a_{3,\nu}+a_{6,\nu}).\tag{4.25}$$

(ii) The amplitude for backward scattering,

$$\frac{1}{8\pi m}[T_{fi}]_{\theta=\pi}=-\frac{\omega\omega'}{2\pi N(t)}\left\{\boldsymbol{e}'^{*}\cdot\boldsymbol{e}\left(1-\frac{t}{4m^2}\right)\times\left(A_1-\frac{t}{4m^2}A_5\right)+i\frac{\nu}{m}\boldsymbol{\sigma}\cdot\boldsymbol{e}'^{*}\times\boldsymbol{e}\left[A_2+\left(1-\frac{t}{4m^2}\right)A_5\right]\right\}\tag{4.26}$$

has the following low-energy decomposition:

$$\frac{1}{8\pi m}[T_{fi}]_{\theta=\pi}=N(t)\boldsymbol{e}'^{*}\cdot\boldsymbol{e}\,\{-r_0q^2+\omega\omega'(\alpha_E-\beta_M)+\omega^2\omega'^2(\alpha_\nu-\beta_\nu-4\alpha_t+4\beta_t)+\mathcal{O}(\omega^3\omega'^3)\}+i\sqrt{\omega\omega'}\boldsymbol{\sigma}\cdot\boldsymbol{e}'^{*}\times\boldsymbol{e}\left\{\frac{r_0}{2m}(\kappa^2+4q\kappa+2q^2)+\omega\omega'\gamma_\pi+\mathcal{O}(\omega^2\omega'^2)\right\},\tag{4.27}$$

where

$$\gamma_\pi=-\gamma_{E1}+\gamma_{M1}+\gamma_{E2}-\gamma_{M2}=-\frac{a_2+a_5}{2\pi m}\tag{4.28}$$

is the ''backward-angle spin polarizability,'' and

$$\alpha_\nu-\beta_\nu-4\alpha_t+4\beta_t=\alpha_{E\nu}-\beta_{M\nu}-\tfrac{1}{12}\alpha_{E2}+\tfrac{1}{12}\beta_{M2}=\frac{1}{2\pi}\left(4a_{1,t}-a_{1,\nu}-\frac{a_5}{m^2}\right).\tag{4.29}$$

B. Cross sections

A decomposition analogous to Eq. (4.1) can be applied also to the invariant functions W_{ij}:

$$W_{ij}=W_{ij}^{\mathrm{B}}+W_{ij}^{\mathrm{NB}}.\tag{4.30}$$

The Born contribution is given by

$$\frac{1}{(8\pi m)^2}W_{00}^{\mathrm{B}}=\frac{r_0^2}{2}\left\{q^4(1+z^2)+\frac{\omega\omega'}{4m^2}[4q^3(q+2\kappa)(1-z)^2+2q^2(9-10z+z^2)\kappa^2+4q(3-2z-z^2)\kappa^3+(3-z^2)\kappa^4]\right\},\tag{4.31a}$$

$$\frac{1}{(8\pi m)^2}W_{03}^{\mathrm{B}}=-\frac{r_0^2}{2}(1-z^2)\left[q^4+\frac{\omega\omega'}{4m^2}(\kappa^2+2q\kappa)^2\right],\tag{4.31b}$$

$$\frac{1}{(8\pi m)^2}W_{12}^{\pm,\mathrm{B}}=\frac{r_0^2}{2m}\left\{(1+z)\left[q^2-\frac{\omega\omega'}{4m^2}(1-z)(\kappa^2+2q\kappa)\right][\kappa^2+q(1-z)(q+\kappa)]\pm\frac{\nu}{2m}(1-z)(\kappa^2+2q\kappa)[\kappa^2+2q\kappa+q(1-z)(q+\kappa)]\right\},\tag{4.31c}$$

$$\frac{1}{(8\pi m)^2}W_{22}^{\pm,\mathrm{B}}=\frac{r_0^2}{2m}\left\{(1-z)\left[q^2+\frac{\omega\omega'}{4m^2}(1-z)(\kappa^2+2q\kappa)\right][\kappa^2+2q\kappa+q(1-z)(q+\kappa)]\mp\frac{\nu}{2m}(1+z)(\kappa^2+2q\kappa)[\kappa^2+q(1-z)(q+\kappa)]\right\}.\tag{4.31d}$$

The non-Born contributions are more conveniently written by separating the terms of different order in $\omega\omega'$. In particular, for W_{00} and W_{03} we have

$$W_{0k}^{\mathrm{NB}}=U_{0k}^{(2)}\omega\omega'+U_{0k}^{(4)}\omega^2\omega'^2+\mathcal{O}(\omega^3\omega'^3)\quad(k=0,3), \tag{4.32}$$

where

$$\frac{1}{(8\pi m)^2}U_{00}^{(2)}=-r_0q^2[(1+z^2)\alpha_E+2z\beta_M], \tag{4.33a}$$

$$\frac{1}{(8\pi m)^2}U_{03}^{(2)}=r_0q^2(1-z^2)\alpha_E, \tag{4.33b}$$

$$\frac{1}{(8\pi m)^2}U_{00}^{(4)}=\frac{1}{2}(1+z^2)(\alpha_E^2+\beta_M^2)+2z\alpha_E\beta_M+\frac{r_0}{4m^2}(1-z)[(1+z)(\kappa^2+2q\kappa)(\alpha_E+z\beta_M)+2q^2(2z\alpha_E+(1+z^2)\beta_M)]$$
$$-r_0q^2\left[(1+z^2)\alpha_{E\nu}+2z\beta_{M\nu}+\frac{z^3}{6}\alpha_{E2}+\frac{3z^2-1}{12}\beta_{M2}\right]+\frac{r_0}{2m}P(z), \tag{4.33c}$$

$$\frac{1}{(8\pi m)^2}U_{03}^{(4)}=(1-z^2)\left\{-\frac{1}{2}(\alpha_E^2-\beta_M^2)-\frac{r_0}{4m^2}(\kappa^2+2q\kappa)(\alpha_E+z\beta_M)+r_0q^2\left[\alpha_{E\nu}+\frac{z}{6}\alpha_{E2}-\frac{1}{12}\beta_{M2}\right]+\frac{r_0}{2m}R\right\}. \tag{4.33d}$$

Here $P(z)$ and R are polynomials in z of the third and zero order, respectively, which are determined by the spin polarizabilities as follows:

$$\begin{aligned}P(z)=&[q^2(1+2z-3z^2)-2q\kappa(1-z)^2+2\kappa^2z]\gamma_{E1}\\&+[(q^2+2q\kappa)(3-2z-z^2)+\kappa^2(3-z^2)]\gamma_{M1}\\&+[-q^2(1-3z^2+2z^3)-2q\kappa(1+z-3z^2+z^3)\\&+\kappa^2(3z^2-1)]\gamma_{E2}\\&+[-q^2(1-z)^2+4q\kappa(z-z^2)+2\kappa^2z]\gamma_{M2},\end{aligned} \tag{4.34}$$

and

$$R=q^2(\gamma_{E1}-\gamma_{M2})-(\kappa+q)^2(\gamma_{M1}-\gamma_{E2}). \tag{4.35}$$

For the invariants W_{12} and W_{22} we can stop the expansion at $\mathcal{O}(\omega^3)$, since they appear in the squared amplitude multiplied by terms of $\mathcal{O}(\omega)$. Therefore we have

$$W_{k2}^{\pm,\mathrm{NB}}=U_{k2}^{(2)}\omega\omega'\pm U_{k2}^{(3)}\nu\omega\omega'+\mathcal{O}(\omega^2\omega'^2)\quad(k=1,2), \tag{4.36}$$

where

$$\frac{1}{(8\pi m)^2}U_{12}^{(2)}=-\frac{r_0}{2m}(1+z)[\kappa^2+q(1-z)(\kappa+q)](\alpha_E+\beta_M)+r_0q^2(1+z)[\gamma_{E1}+\gamma_{M1}+z(\gamma_{E2}+\gamma_{M2})], \tag{4.37a}$$

$$\frac{1}{(8\pi m)^2}U_{22}^{(2)}=-\frac{r_0}{2m}(1-z)[\kappa^2+2q\kappa+q(1-z)(\kappa+q)](\alpha_E-\beta_M)-r_0q^2(1-z)[\gamma_{E1}-\gamma_{M1}+z(\gamma_{E2}-\gamma_{M2})], \tag{4.37b}$$

$$\frac{1}{(8\pi m)^2}U_{12}^{(3)}=-\frac{r_0}{2m}(1-z)\{[2\kappa^2+4q\kappa+q(1-z)(\kappa+q)](\gamma_{E1}-\gamma_{M1})$$
$$+[(\kappa^2+2q\kappa)(1+z)+q(1-z)(\kappa+q)](\gamma_{E2}-\gamma_{M2})\}, \tag{4.37c}$$

$$\frac{1}{(8\pi m)^2}U_{22}^{(3)}=-\frac{r_0}{2m}(1+z)\left\{[\kappa^2+q(1-z)(\kappa+q)]\frac{\alpha_E+\beta_M}{m}+(\kappa+q)[2\kappa+q(1-z)](\gamma_{E1}+\gamma_{M1})\right.$$
$$\left.-[(\kappa+q)^2(1-z)-q\kappa(1+z)](\gamma_{E2}+\gamma_{M2})\right\}. \tag{4.37d}$$

In order to illustrate the interplay between polarizabilities and the cross sections (in the lab frame), we consider below the case of polarized photon beam and polarized target at $\theta = 0$, $\pi/2$, and π.

(i) Forward scattering:

$$\left[\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=0} = [r_0 q^2 - \omega^2(\alpha_E+\beta_M)]^2 + r_0^2\frac{\omega^2}{4m^2}\kappa^4 - 2r_0 q^2\omega^4(\alpha_{E\nu}+\beta_{M\nu}+\tfrac{1}{12}\alpha_{E2}+\tfrac{1}{12}\beta_{M2}) - r_0\frac{\omega^4}{m}\kappa^2\gamma + \mathcal{O}(\omega^6) \tag{4.38}$$

and

$$\left[\Sigma_{2z}\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=0} = -r_0^2 q^2\frac{\omega}{m}\kappa^2 + r_0\frac{\omega^3}{m}\kappa^2(\alpha_E+\beta_M) + 2r_0 q^2\omega^3\gamma + \mathcal{O}(\omega^5), \tag{4.39}$$

where γ is the forward-angle spin polarizability (4.24).

(ii) Backward scattering:

$$\left[\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=\pi} = \left(\frac{\omega'}{\omega}\right)^2\left\{q^2\left(1+\frac{\omega\omega'}{m^2}\right)\times[r_0^2 q^2 - 2r_0\omega\omega'(\alpha_E-\beta_M)] + r_0^2\frac{\omega\omega'}{4m^2}(\kappa^2+4\kappa q+2q^2)^2 + \omega^2\omega'^2(\alpha_E-\beta_M)^2 - 2r_0 q^2\omega^2\omega'^2\times(\alpha_{E\nu}-\beta_{M\nu}-\tfrac{1}{12}\alpha_{E2}+\tfrac{1}{12}\beta_{M2}) + r_0\frac{\omega^2\omega'^2}{m}(\kappa^2+4\kappa q+2q^2)\gamma_\pi\right\} + \mathcal{O}(\omega^6) \tag{4.40}$$

and

$$\left[\Sigma_{2z}\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=\pi} = \nu\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{r_0}{m}(\kappa^2+4\kappa q+2q^2)\times[r_0 q^2 - \omega\omega'(\alpha_E-\beta_M)] + 2r_0 q^2\omega\omega'\gamma_\pi\right\} + \mathcal{O}(\omega^5), \tag{4.41}$$

where γ_π is the backward-angle spin polarizability (4.28).

(iii) At 90°:

$$\left[\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=\pi/2} = \left(\frac{\omega'}{\omega}\right)^2\left\{\frac{r_0^2}{2}\left[q^4+\frac{\omega\omega'}{4m^2}(3(\kappa+q)^4-4\kappa q^3+q^4)\right] - r_0\omega\omega'\alpha_E\left[q^2-\frac{\omega\omega'}{4m^2}(\kappa^2+2\kappa q)\right] + r_0 q^2\frac{\omega^2\omega'^2}{2m^2}\beta_M + \frac{1}{2}\omega^2\omega'^2(\alpha_E^2+\beta_M^2) - r_0 q^2\omega^2\omega'^2\left(\alpha_{E\nu}-\frac{1}{12}\beta_{M2}\right) + \frac{r_0}{2m}\omega^2\omega'^2[-(2\kappa q-q^2)\gamma_{E1}+(\kappa+q)^2(3\gamma_{M1}-\gamma_{E2})-q^2\gamma_{M2}]\right\} + \mathcal{O}(\omega^6), \tag{4.42}$$

$$\left[\Sigma_3\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=\pi/2} = \left(\frac{\omega'}{\omega}\right)^2\left\{-\frac{r_0^2}{2}\left[q^4+\frac{\omega\omega'}{4m^2}(\kappa^2+2\kappa q)^2\right] + r_0\omega\omega'\alpha_E\left[q^2-\frac{\omega\omega'}{4m^2}(\kappa^2+2\kappa q)\right] - \frac{1}{2}\omega^2\omega'^2(\alpha_E^2-\beta_M^2) + r_0 q^2\omega^2\omega'^2\left(\alpha_{E\nu}-\frac{1}{12}\beta_{M2}\right) + \frac{r_0}{2m}\omega^2\omega'^2[q^2(\gamma_{E1}-\gamma_{M2})-(\kappa+q)^2(\gamma_{M1}-\gamma_{E2})]\right\} + \mathcal{O}(\omega^6), \tag{4.43}$$

$$\left[\Sigma_{2z}\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=\pi/2} = \left(\frac{\omega'}{\omega}\right)^2 r_0\omega\left\{\frac{r_0}{2m}\left[\kappa q^3-\frac{\omega'}{4m}(\kappa^2+2\kappa q)(\kappa+q)^2\right] + \frac{\omega\omega'}{2m}[(\kappa+q)^2\beta_M-\kappa q\alpha_E] - \frac{\nu}{4m^2}\omega\omega'(\kappa^2+\kappa q+q^2)(\alpha_E+\beta_M) - q^2\omega\omega'\gamma_{E1} + \frac{\nu}{2m}\omega\omega'[\kappa q(\gamma_{E1}-\gamma_{M2})-(2\kappa^2+4\kappa q+q^2)\gamma_{M1}+(\kappa+q)^2\gamma_{E2}]\right\} + \mathcal{O}(\omega^5), \tag{4.44}$$

$$\left[\Sigma_{2x}\frac{d\bar{\sigma}}{d\Omega}\right]_{\theta=\pi/2}=\left(\frac{\omega'}{\omega}\right)^2 r_0\omega'\left\{-\frac{r_0}{2m}\left[q^2(\kappa+q)^2+\frac{\omega}{4m}\kappa^2(\kappa q+2q^2)\right]\right.$$
$$+\frac{\omega\omega'}{2m}[(\kappa+q)^2\alpha_E-\kappa q\beta_M]+\frac{\nu}{4m^2}\omega\omega'(\kappa^2+\kappa q+q^2)(\alpha_E+\beta_M)-q^2\omega\omega'\gamma_{M1}$$
$$\left.-\frac{\nu}{2m}\omega\omega'[\kappa q(\gamma_{M1}-\gamma_{E2})-(2\kappa^2+4\kappa q+q^2)\gamma_{E1}+(\kappa+q)^2\gamma_{M2}]\right\}+\mathcal{O}(\omega^5). \quad (4.45)$$

For the low-energy expansion of the amplitudes and structure functions, two further remarks must be considered. First, as is evident from Eq. (2.3), the t-channel π^0 exchange contributes only to T_5 (and thus only to A_2). Owing to the small pion mass m_{π^0}, this diagram is a rapidly varying function of t and therefore it is profitable to keep its expression unexpanded. This can be achieved with the following replacement in the non-Born part of the amplitude A_2:

$$A_2^{\rm NB}(\nu,t)=A_2^{\pi^0}(t)-A_2^{\pi^0}(0)+\bar{A}_2^{\rm NB}(\nu,t), \quad (4.46)$$

where

$$A_2^{\pi^0}(t)=\frac{2}{t}T_5^{\pi^0}(t)=\frac{g_{\pi NN}F_{\pi^0\gamma\gamma}}{t-m_{\pi^0}^2}\tau_3, \quad (4.47)$$

with $\tau_3=1$ or -1 for the proton and neutron, respectively, and where

$$\Gamma_{\pi^0\to\gamma\gamma}=\frac{m_{\pi^0}^3}{64\pi}F_{\pi^0\gamma\gamma}^2 \quad (4.48)$$

is the π^0 two-photon decay width. The relative sign of the πNN coupling $g_{\pi NN}$ and the $\pi^0\gamma\gamma$ coupling $F_{\pi^0\gamma\gamma}$ is negative. In Eq. (4.46), $\bar{A}_2^{\rm NB}$ is a smoother function of ν,t than $A_2^{\rm NB}$ and is better approximated by a polynomial in $\omega\omega'$ (which is just the constant a_2 to the order we consider). In terms of the spin polarizabilities, this means that the following substitutions have to be done in all the expansions:

$$\gamma_{E1}\to\gamma_{E1}+X_{2a}f(t), \quad \gamma_{E2}\to\gamma_{E2}-X_{2a}f(t),$$
$$\gamma_{M1}\to\gamma_{M1}-X_{2a}f(t), \quad \gamma_{M2}\to\gamma_{M2}+X_{2a}f(t), \quad (4.49)$$

where

$$X_{2a}=\frac{A_2^{\pi^0}(0)}{8\pi m}, \quad f(t)=\frac{t}{m_{\pi^0}^2-t}. \quad (4.50)$$

Generally, after the separation of the π^0-exchange contribution, all the low-energy expansions discussed in this section become valid provided $\omega\omega'/m_\pi^2$ is a small parameter. The radius of convergence of the $\omega\omega'$ series is, up to a small correction of order $\mathcal{O}(m_\pi/m)$, equal to

$$|\omega|\leqslant m_\pi \quad (4.51)$$

and is determined by the pion photo-production threshold where the amplitudes have a singularity and acquire an imaginary part. Another close singularity is due to the t-channel exchange of two pions, which gives the same restriction (4.51).

The second remark is that the low-energy expansions of the structure functions W_{ij}, when used within the radius of convergence, give an accurate account of the dependence of the cross sections on the dipole polarizabilities α_E, β_M, and the spin polarizabilities γ_{E1}, γ_{M1}, γ_{E2}, γ_{M2}. In the expressions for $W_{12}^{\pm}$ and $W_{22}^{\pm}$, the spin polarizabilities enter to leading order $\mathcal{O}(\omega\omega')$ as an interference with the Thomson amplitude (4.9). The subleading terms of order $\mathcal{O}(\nu\omega\omega')$ appear due to interference with the amplitude (4.18). Numerically the latter terms are enhanced by the anomalous magnetic moment [for example, $\kappa^2+4q\kappa+2q^2=12.4$ in Eq. (4.27) for the proton] and are as important as the leading terms. On the other hand, in the expansions for W_{ij}, the leading order contribution of the quadrupole and dispersion polarizabilities α_{E2}, β_{M2}, $\alpha_{E\nu}$, $\beta_{M\nu}$ are already of order $\mathcal{O}(\omega^2\omega'^2)$, which is the highest order included in the expansion. Therefore the subleading terms which describe the interference of these polarizabilities with the amplitude (4.18) are not included, although they may be comparable in magnitude to the leading terms. For this reason, it is generally more accurate to use low-energy expansions for the invariant amplitudes $A_i(\nu,t)$, including all ten polarizabilities that appear to the order considered, and then to calculate the structure functions W_{ij} through Eqs. (3.15). This more accurate procedure will be used when we discuss the low-energy observables in Sec. V.

In principle, all the polarizabilities we have defined can be determined from experiment. For example, in the proton case once the dipole polarizabilities α_E and β_M have been determined [from low-energy experiments that are sensitive only to terms of order $\mathcal{O}(\omega^2)$], the angular behavior of the coefficients W_{12} and W_{22} (from the measurements of Σ_{2x} and Σ_{2z}) enables a determination of all the spin polarizabilities γ_{E1}, γ_{M1}, γ_{E2}, γ_{M2}. Then the angular dependence of the unpolarized cross section allows one to determine the remaining polarizabilities α_{E2}, β_{M2}, $\alpha_{E\nu}$, $\beta_{M\nu}$. In the next section, we investigate the feasibility of this approach.

V. DISCUSSION

In this section we summarize some theoretical predictions for the polarizabilities, examine the question of convergence of the low-energy expansion, and discuss the sensitivity of various observables to these polarizabilities.

A. Theoretical predictions for polarizabilities

We now consider various theoretical predictions for the polarizabilities in order to establish the accuracy that should be achieved in experiments to make them useful for resolving theoretical ambiguities. We restrict ourselves to estimates based on chiral perturbation theory and dispersion relations, since these have proved to be successful in applications to strong and electromagnetic interactions of hadrons at low energies.

In its standard form, chiral perturbation theory gives leading-order (LO) predictions for the polarizabilities entirely in terms of the pion mass m_π, the axial coupling constant of the nucleon, $g_A=1.2573\pm0.0028$, and the pion decay constant $f_\pi=92.4\pm0.3$ MeV. The nucleon mass m which describes recoil corrections does not enter into the LO predictions but appears in the next-to-leading order together with some additional parameters. The LO-contributions dominate in the chiral limit $m_\pi\to0$ and are expected to provide semiquantitative estimates in the real world. All results quoted here were obtained by Bernard *et al.*, as summarized in Ref. [13]. Using explicit formulas from that work, we have

$$\alpha_E^{(\mathrm{LO})}=10X_1,\quad \beta_M^{(\mathrm{LO})}=X_1,$$

$$\alpha_{E\nu}^{(\mathrm{LO})}=\frac{3}{4}X_3,\quad \beta_{M\nu}^{(\mathrm{LO})}=\frac{7}{6}X_3,$$

$$\alpha_{E2}^{(\mathrm{LO})}=7X_3,\quad \beta_{M2}^{(\mathrm{LO})}=-3X_3, \tag{5.1}$$

$$\gamma_{E1}^{(\mathrm{LO})}=-5X_2+X_{2a},\quad \gamma_{M1}^{(\mathrm{LO})}=-X_2-X_{2a},$$

$$\gamma_{E2}^{(\mathrm{LO})}=X_2-X_{2a},\quad \gamma_{M2}^{(\mathrm{LO})}=X_2+X_{2a}.$$

Here the quantities X_i are proportional to ith powers of the inverse pion mass:

$$X_1=\frac{E^2}{24m_\pi}=1.23\times10^{-4}\ \ \mathrm{fm}^3,$$

$$X_2=\frac{E^2}{12\pi m_\pi^2}=1.11\times10^{-4}\ \ \mathrm{fm}^4,$$

$$X_{2a}=\frac{E^2\tau_3}{\pi g_A m_{\pi^0}^2}=\pm11.3\times10^{-4}\ \ \mathrm{fm}^4,$$

$$X_3=\frac{E^2}{20m_\pi^3}=2.96\times10^{-4}\ \ \mathrm{fm}^5, \tag{5.2}$$

where $E=eg_A\sqrt{2}/(8\pi f_\pi)$ is the threshold photoproduction amplitude for charged pions in the chiral limit. Since the values X_1, X_2, X_3 are determined by pion loops with charged pions, we use the charged pion mass $m_\pi=139.57$ MeV for obtaining numbers for these X's. The terms with X_{2a} describe the π^0 exchange (4.47) with couplings fixed by the chiral anomaly (Wess-Zumino-Witten) and by the Goldberger-Treiman relation:

$$F_{\pi^0\gamma\gamma}=-\frac{e^2N_c}{12\pi^2f_\pi},\quad g_{\pi NN}=g_A\frac{m}{f_\pi}. \tag{5.3}$$

where $N_c=3$. Accordingly, we use the neutral pion mass $m_{\pi^0}=134.97$ MeV to evaluate X_{2a}.[1]

Calculation to the next order(s) involves new parameters (counterterms) that are usually estimated via an approximation that takes into account the nearest resonances and/or loops with strange particles. Such a procedure may lead to rather different results for polarizabilities, depending on further details (for example, compare Ref. [13] and Ref. [14]; see also below). Among the nucleon resonances, the $\Delta(1232)$ is special because it is separated from the nucleon by a relatively small energy gap $\Delta=m_{\Delta(1232)}-m\simeq2m_\pi$ and has a large coupling to the πN channel. Therefore this resonance is particularly important for low-energy phenomena such as the polarizabilities. In Refs. [14,36], the $\Delta(1232)$ was considered as a partner of the nucleon in the chiral expansion, which was reformulated in terms of a generic small energy scale $\epsilon=\mathcal{O}(m_\pi,\Delta)$. In the following we invoke only one component of such an approach, the $\Delta(1232)$-pole contribution. We ignore the contribution of pion loops with an intermediate $\Delta(1232)$. These terms are relatively small for the spin polarizabilities, although large for α_E and β_M. We take into account both $M1$ and $E2$ couplings of the Δ. Although the quadrupole contribution is of higher order in ϵ, numerically it is not negligible for the spin polarizability γ_{E2}. For the Δ-pole contribution we have

$$4\pi\beta_M^{(\Delta)}=\frac{2\mu_{N\Delta}^2}{\Delta},\quad 4\pi\beta_{M\nu}^{(\Delta)}=\frac{2\mu_{N\Delta}^2}{\Delta^3},\quad 4\pi\alpha_{E2}^{(\Delta)}=\frac{Q_{N\Delta}^2}{2\Delta},$$

$$4\pi\gamma_{M1}^{(\Delta)}=\frac{\mu_{N\Delta}^2}{\Delta^2},\quad 4\pi\gamma_{E2}^{(\Delta)}=\frac{-\mu_{N\Delta}Q_{N\Delta}}{2\Delta} \tag{5.4}$$

(and nothing for the other polarizabilities, as discussed in detail in Appendix D). Here $\mu_{N\Delta}$ and $Q_{N\Delta}$ are the magnetic and quadrupole transition moments, respectively. Depending on how the $M1$-coupling is extracted from the experimental radiative width of the Δ (with relativistic or nonrelativistic phase space), the Δ-pole contribution to β_M was evaluated in [14,36] as

$$\beta_M^{(\Delta)}=12.0\ \ [14]\quad \text{or}\quad 7.2\ \ [36] \tag{5.5}$$

[1]In principle, the difference between the masses of π^+ and π^0 runs beyond the accuracy of leading-order predictions. Its consistent treatment has to involve radiative corrections of $\mathcal{O}(e^2)$. However, we assume that the use of the experimental masses of the pions in the present context still makes sense because this is expected to reduce the size of the counterterm contributions that come from the full treatment.

TABLE I. Polarizabilities of the nucleon given by chiral perturbation theory to leading order in m_π (the columns labeled π^0 and ''loop'' [13,24]). Also the Δ-pole contribution is given with the larger strength of Eq. (5.5) (see discussion in text). Other predictions are based on dispersion relations [19,42]. Separately given are the π^0-exchange contribution ($A_2^{\rm as}$) and the contribution of excitations. The set HDT [42] uses pion photoproduction multipoles of Hanstein *et al.* [43]. The set SAID is the result of the present work based on the solution SP97K [41]. In the columns π^0 and $A_2^{\rm as}$, the proton or neutron case corresponds to $\tau_3=1$ or -1, respectively.

	CHPT (leading order)		Δ pole	Dispersion relations				
				$A_2^{\rm as}$	excitations $+A_1^{\rm as}$			
					proton		neutron	
	π^0	loop			HDT	SAID	HDT	SAID
α_E		12.3				11.9[a,c]		13.3[b,c]
β_M		1.2	12.0			1.9[a,c]		1.8[b,c]
(10^{-4} fm^3)								
$\alpha_{E\nu}$		2.2				-3.8		-2.4
$\beta_{M\nu}$		3.5	5.3			9.1		9.2
α_{E2}		20.7	0.2			27.5		27.2
β_{M2}		-8.9				-22.4		-23.5
(10^{-4} fm^5)								
γ_{E1}	$11.3\tau_3$	-5.5		$11.2\tau_3$	-4.5	-3.4	-5.5	-5.6
γ_{M1}	$-11.3\tau_3$	-1.1	4.0	$-11.2\tau_3$	3.4	2.7	3.4	3.8
γ_{E2}	$-11.3\tau_3$	1.1	0.75	$-11.2\tau_3$	2.3	1.9	2.6	2.9
γ_{M2}	$11.3\tau_3$	1.1		$11.2\tau_3$	-0.6	0.3	-0.6	-0.7
γ		4.4	-4.8		-0.6	-1.5	-0.2	-0.4
γ_π	$-45.3\tau_3$	4.4	4.8	$-45.0\tau_3$	10.8	7.8[d]	12.1	13.0
(10^{-4} fm^4)								

[a]Experimental values for the proton are $\alpha_E=12.1\pm1$, $\beta_M=2.1\mp1$ [3].

[b]Experimental values for the neutron are $\alpha_E=12.6\pm2.5$ [44] and 0 ± 5 [45]. See also the criticism of the former result in Ref. [46].

[c]$\alpha_E-\beta_M=10.0$ and 11.5 is used as input for the proton and neutron, respectively. This guarantees that $A_1^{\rm as}(0)$ is the same for the proton and neutron.

[d]Experimental value reported for the non-π^0 part of γ_π for the proton is 17.3 ± 3.4 [26].

(units are 10^{-4} fm^3).[2] With the generally accepted phenomenological magnitude of $\mu_{N\Delta}$ (see, for example, [37]), $\beta_M^{(\Delta)}\simeq(12-13)$ [29,38]. In the Skyrme model [39] or large-N_c QCD [40], the transition magnetic moment is related to the isovector magnetic moment of the nucleon as $\mu_{N\Delta}=\sqrt{2}\mu_N^V=3.33(e/2m)$, so that $\beta_M^{(\Delta)}=12.0$. All these contributions are summarized in Table I, where the larger magnitude of Eq. (5.5) has been assumed. With the fixed ratio $Q_{N\Delta}/\mu_{N\Delta}=-0.25$ fm (see Appendix D), the use of the other value would reduce all the Δ-pole contributions by 40%.

Another way to calculate the polarizabilities is provided by dispersion relations for the amplitudes $A_i(\nu,t)$. In the following we give results of the fixed-t dispersion relations which have the form [19]

$$\mathrm{Re}\,A_i(\nu,t)=A_i^{\rm B}(\nu,t)+\frac{2}{\pi}P\int_{\nu_{\rm thr}}^{\nu_{\rm max}}\mathrm{Im}\,A_i(\nu',t)\frac{\nu'\,d\nu'}{\nu'^2-\nu^2}+A_i^{\rm as}(t). \tag{5.6}$$

Here the dispersion integral is taken from the pion photoproduction threshold to some large energy (actually, it is 1.5 GeV). The remaining so-called asymptotic contribution is t dependent but only weakly energy dependent. From unitarity, the imaginary part of the amplitudes A_i and the integral in Eq. (5.6) can be calculated by using experimental information on single- and double-pion photoproduction which is available at energies below $\nu_{\rm max}$. In the present context we use the latest version of the single-pion photoproduction amplitudes provided by the partial wave analysis of the VPI group [41] (the code SAID, solution SP97K). Also, double-pion photoproduction is taken into account in the framework of a model which includes both resonance mechanisms and nonresonant production of the $\pi\Delta$ and ρN states. Details of this procedure are given in Ref. [19].

The quantities $A_i^{\rm as}(t)$ take into account contributions of high energies into the dispersion relations. According to Regge phenomenology, only the amplitudes A_1 and A_2 can have a nonvanishing part at high energies and thus have a large asymptotic contribution. For the other amplitudes ($A_{3,4,5,6}$), the dispersion integrals provide a very good estimate of the corresponding non-Born parts. Thus, we can reliably find through Eq. (5.6) the constants $a_{3,4,5,6}$ and $a_{3,t}$, $a_{6,t}$, $a_{3,\nu}$, $a_{6,\nu}$. Moreover, the constant $a_{1,\nu}$ can be found as well, since it is not affected by the asymptotic contribution.

[2]This difference also illustrates how estimates of counterterms may depend on assumptions.

Uncertainties appear only for the constants a_1, a_2, and $a_{1,t}$, which get a sizable contribution from the asymptotic part of the amplitude.

Following the arguments of Ref. [19], we relate the asymptotic contributions $A_{1,2}^{\rm as}(t)$ to the t-channel exchange of the σ and π^0 mesons, respectively. The couplings of the π^0 are well known, and we include this contribution by virtue of Eq. (4.47), in which we use experimental values for couplings *and* introduce a monopole form factor with $\Lambda_\pi = 700$ MeV. The experimental magnitude of the product $g_{\pi NN}F_{\pi^0\gamma\gamma} = -0.33$ GeV^{-1} is $\simeq 3\%$ higher than that given by the anomaly equation (5.3). However, the form factor reduces the π^0 strength by $(m_\pi/\Lambda_\pi)^2 \simeq 4\%$ and makes the resulting contribution at $t=0$ very close to that given by the anomaly (see Table I). Nevertheless, beyond the π^0, the amplitude A_2 can have contributions from other heavier exchanges and thus have an additional piece that weakly depends on t.[3] Therefore the fixed-t dispersion relation for $A_2(\nu,t)$ should not be considered as a reliable source of information on a_2.

The amplitude $A_1(\nu,t)$ gets a large asymptotic contribution from σ exchange, which contributes to the constant a_1 and therefore to the difference of the dipole polarizabilities $\alpha_E - \beta_M$. Since this difference is experimentally known for the proton to be $\alpha_E - \beta_M = (10 \pm 2) \times 10^{-4}$ fm^3 [3], as determined by low-energy Compton scattering data, the constant a_1 is known too. However, the quantity $a_{1,t}$ is not fixed by these data and cannot be unambiguously predicted from the dispersion relations (5.6). Using data on γp scattering at higher energies (above 400 MeV), the t dependence of the asymptotic contribution $A_1^{\rm as}(t)$ at $-t \sim 0.5$ GeV^2 was estimated in [19]. It corresponds to a monopole form factor with the cutoff parameter $M_\sigma \simeq 600$ MeV (''mass'' of the σ). However, the t dependence of $A_1^{\rm as}(t)$ at smaller t might be somewhat steeper as suggested from independent estimates. Therefore, the fixed-t dispersion relation for $A_1(\nu,t)$ should not be considered as a reliable source of information on $a_{1,t}$.

With these precautions, we put into Table I the results of saturating the dispersion integrals by known photoproduction amplitudes and by the asymptotic contributions discussed above. We assume that $A_1^{\rm as}$ is the same for the proton and neutron. Depending on which photoproduction input is used, the integrals are slightly different. We give both our results, which use the SAID solution SP97K as an input and a model for double-pion photoproduction, and the results by Drechsel *et al.* [42], which use photoproduction amplitudes by Hanstein *et al.* (HDT) [43] (and ignore the double-pion channel). The main difference between these two evaluations comes from near-threshold energies where the SAID and HDT multipoles E_{0+} are rather different (see Ref. [42] for details). For some other (generally similar) dispersion estimates, see Refs. [22,47,48]. Available experimental data are also given in the footnotes to Table I, including measurements of α_E and β_M for the proton (see Ref. [3], and references therein), measurements of α_E for the neutron [44,45] (see also [46] for critical discussion), and a recent estimate of γ_π for the proton [26]. It is apparent from Table I that it is necessary to measure the spin polarizabilities to an accuracy $\sim 1 \times 10^{-4}$ fm^4 in order to obtain constraints valuable for the theory.

TABLE II. Combinations of polarizabilities of the nucleon which do not depend on the asymptotic contributions $A_{1,2}^{\rm as}(t)$. Notation is the same as in Table I.

	CHPT (LO)	Δ pole	DR (SAID)	
			proton	neutron
$\alpha_E + \beta_M$[a] (10^{-4} fm^3)	13.6	12.0	13.8	15.1
$\alpha_{E\nu} + \beta_{M\nu}$	5.7	5.3	5.3	6.8
$\alpha_{E2} + \beta_{M2}$	11.8	0.2	5.1	3.7
$\alpha_{E\nu} + \frac{1}{4}\alpha_{E2}$	7.4	0.05	3.1	4.4
$\beta_{M\nu} + \frac{1}{4}\beta_{M2}$ (10^{-4} fm^5)	1.2	5.3	3.5	3.4
$\gamma_{E1} + \gamma_{M1}$	-6.7	4.0	-0.7	-1.8
$\gamma_{E2} + \gamma_{M2}$	2.2	0.75	2.2	2.2
$\gamma_{E1} + \gamma_{E2}$	-4.4	0.75	-1.5	-2.6
$\gamma_{M1} + \gamma_{M2}$ (10^{-4} fm^4)	0	4.0	3.0	3.0

[a]Experimental value for the proton is $\alpha_E + \beta_M = 15.2 \pm 2.6$ [3]. A recent evaluation of this quantity in terms of the Baldin sum rule yields 13.69 ± 0.14 (stat) for the proton and 14.40 ± 0.66 (stat) for the neutron [48].

Similarly to Ref. [42], we can isolate quantities that can be reliably predicted by the dispersion relations (5.6) (up to small uncertainties coming from the photoproduction amplitudes), namely those that do not depend on a_1, a_2, and $a_{1,t}$ (see Table II). In the absence of precise data on double-polarized Compton scattering at low energies, these predictions can be used to diminish the number of unknown parameters during fits and to help in determination of unconstrained combinations. These latter combinations are $\alpha_E - \beta_M$ (which has already been measured), γ_π, and $\alpha_{E2} - \beta_{M2}$. Therefore, the most interesting and informative experiments on Compton scattering are those that are sensitive to the parameters γ_π and $\alpha_{E2} - \beta_{M2}$, which mainly contribute to the backward scattering amplitude. Below we discuss which observables are the best suitable to this aim. The above remarks apply principally to the proton. The case of the neutron can considered in a similar manner. However, since a free neutron target is not available, neutron polarizabilities are studied through elastic or inelastic *nuclear* Compton scattering, thereby introducing additional uncertainites due to Fermi motion, meson exchange currents, final-state interactions, etc. (see, e.g., Refs. [49,50]).

B. Convergence of the low-energy expansion

In order to investigate the convergence of the low-energy expansion, we use the same fixed-t dispersion relations de-

[3]A recent analysis [26] of unpolarized Compton scattering data suggests the existence of such an additional contribution, since the spin polarizability γ_π found there deviates considerably from that predicted by the dispersion integral (5.6) for the amplitude A_2 and by the π^0 asymptotic contribution (see Table I). Even after allowing for an uncertainty of $\sim \pm 2$ in the integral contribution given in Table I, as suggested by a sizable integrand at energies above 1 GeV where calculations of $\mathrm{Im}\, A_i$ are very model dependent, such a strong deviation is difficult to understand theoretically and needs to be confirmed by additional experimental measurements.

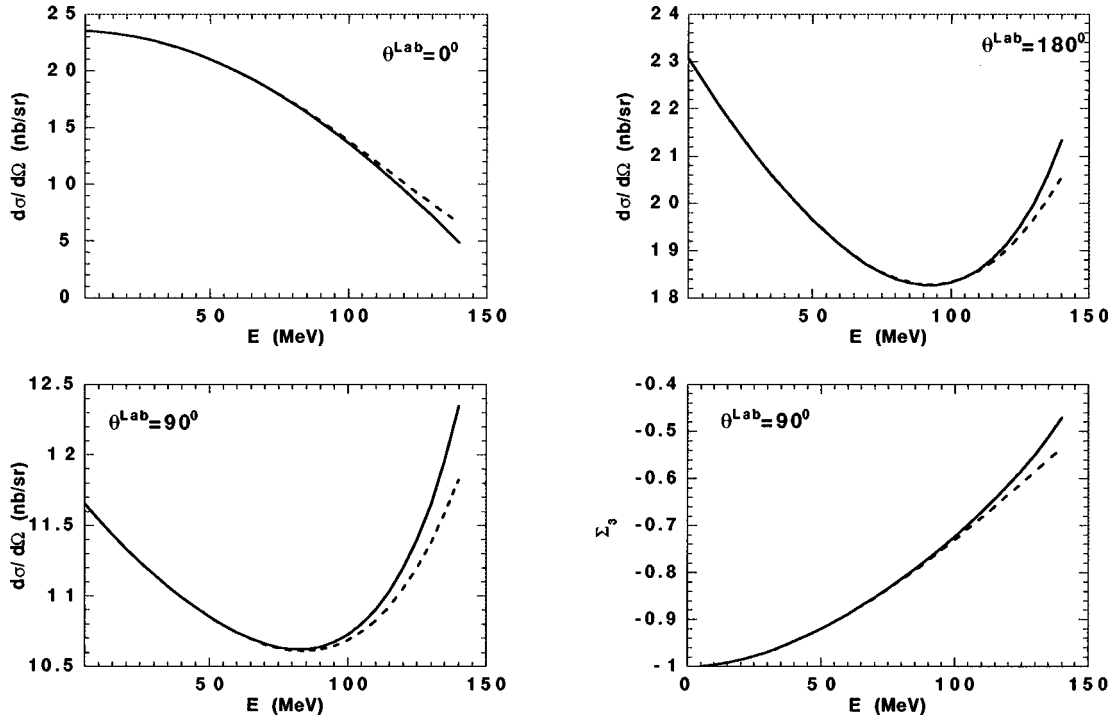

FIG. 2. Calculated cross sections and asymmetries for the proton. A comparison is given between a full dispersion calculation (full curves) and the low-energy expansion (dashed curves) in which the amplitudes are expanded to $\mathcal{O}(\omega^2\omega'^2)$. The upper panels are the unpolarized cross section at 0° (left) and 180° (right). The lower two panels are the unpolarized cross section (left) and Σ_3 (right) at 90°.

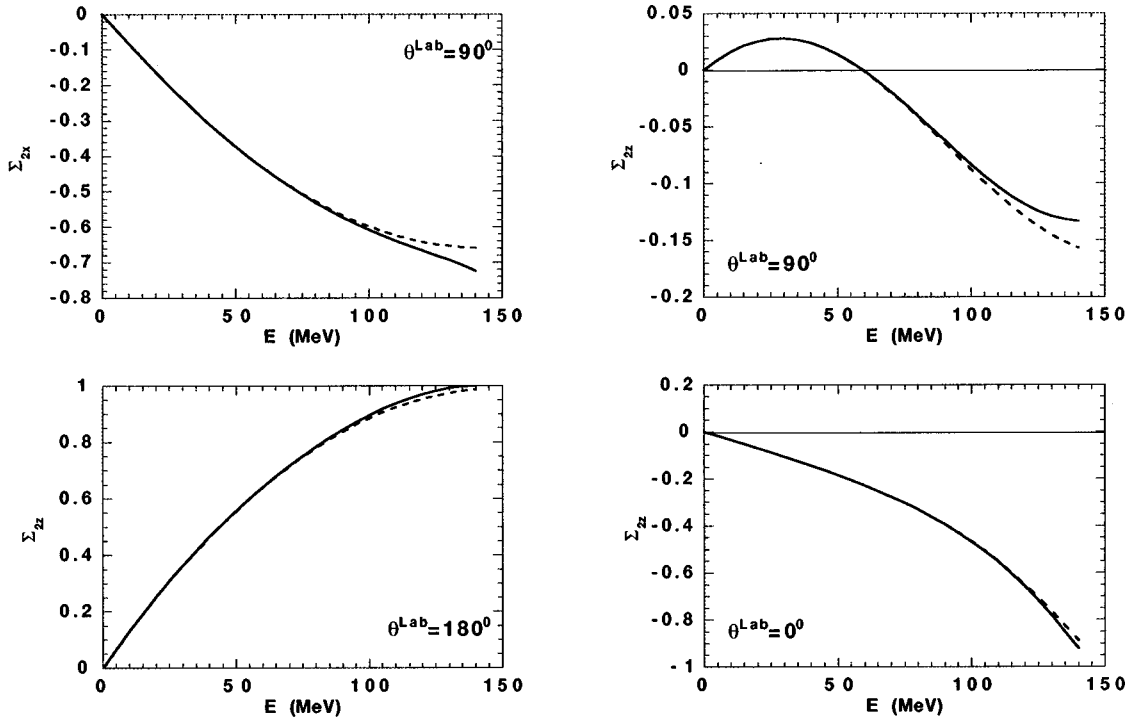

FIG. 3. Calculated asymmetries for the proton. A comparison between a full dispersion calculation (full curves) and the low-energy expansion (dashed curves) in which the amplitudes are expanded to $\mathcal{O}(\omega^2\omega'^2)$. The upper panels are Σ_{2x} (left) and Σ_{2z} (right) at 90°. The lower two panels are Σ_{2z} at 180° (left) and 0° (right).

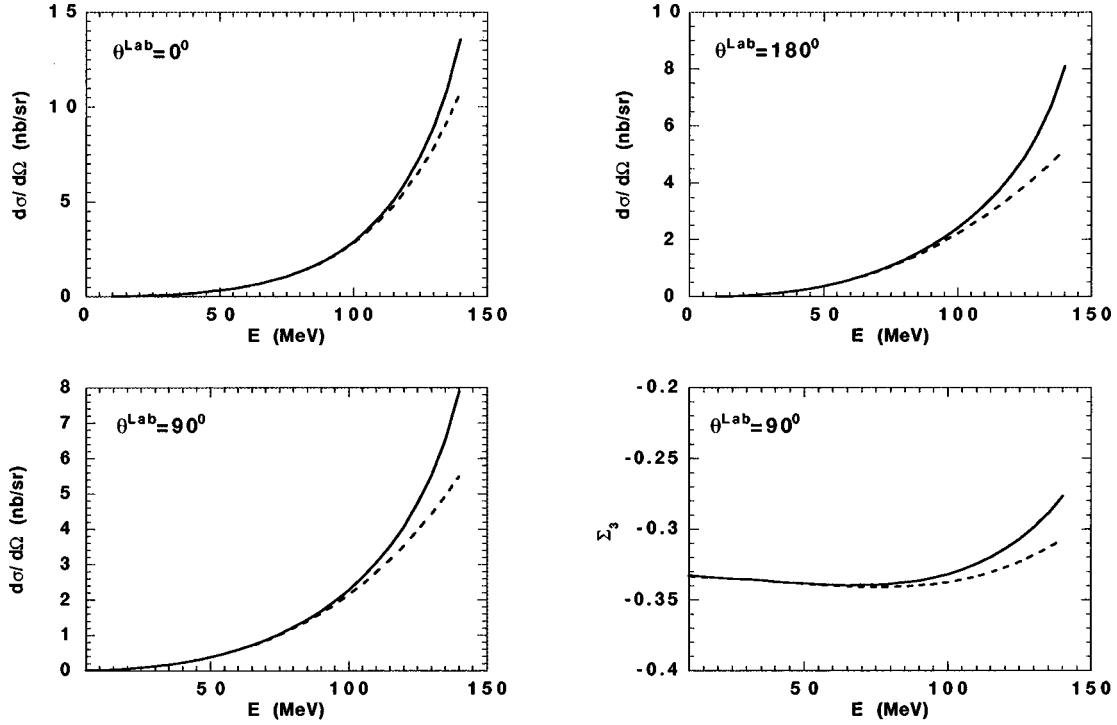

FIG. 4. Calculated cross sections and asymmetries for the neutron. A comparison is given between a full dispersion calculation (full curves) and the low-energy expansion (dashed curves) in which the amplitudes are expanded to $\mathcal{O}(\omega^2\omega'^2)$. The upper panels are the unpolarized cross section at 0° (left) and 180° (right). The lower two panels are the unpolarized cross section (left) and Σ_3 (right) at 90°.

scribed in the preceding section to make an exact prediction for the invariant amplitudes at any energy. From these the W_{ij} can be calculated exactly and predictions for the various observables can be made. These can be compared with predictions calculated by truncating the A_i to $\mathcal{O}(\omega^2\omega'^2)$, as described in Sec. IV. Such a comparison is shown in Figs. 2 and 3 for the proton and Figs. 4 and 5 for the neutron. For the proton, the expansion works very well for energies up to the pion threshold. For the neutron, it appears to work less well due to the fact that the Born contributions are considerably smaller than for the proton. The full calculation deviates rapidly from the expansion once the pion threshold is crossed.

C. Sensitivity of observables to polarizabilities

We now address the question of the measurement of the polarizabilities we have defined, with the principal focus being on the spin polarizabilities. To the order of our expansion, the double polarization observables are not sensitive to the quadrupole and dispersion polarizabilities, whereas the unpolarized cross section and Σ_3 are sensitive to all of the polarizabilities. Therefore it seems reasonable to use double polarization measurements to constrain the γ_i, then unpolarized measurements to measure the remaining polarizabilities. At extreme forward and backward angles, the observable Σ_{2z} is sensitive to the γ_i in the combinations that give γ [Eq. (4.24)] and γ_π [Eq. (4.28)], respectively. Two other linear combinations can be obtained from measurements of Σ_{2x} and Σ_{2z} at 90°. Indeed, the formulas given at the end of Sec. IV suggest that these measurements are primarily sensitive to γ_{M1} and γ_{E1}, respectively, at least to the lowest order. In Figs. 6 and 7 we show these observables as a function of energy, as calculated using our low-energy expansion with polarizabilities fixed by the SAID dispersion relation values given in Table I. In order to evaluate the sensitivity of the observable to the polarizability, we have adjusted various quantities by 4×10^{-4} fm^4 relative to the value in Table I. This amount is comparable to the pion loop contribution to each of the polarizabilities but is somewhat larger than the typical discrepancy among the competing theories.

We conclude that in the energy regime below pion threshold where the low-energy expansion is valid, it will be very difficult to measure the spin polarizabilities to an accuracy that can discriminate among the theories, at least at the extreme forward and backward angles. In particular, the backward spin asymmetry Σ_{2z} is almost completely insensitive to theoretically motivated changes to γ_π, whereas the forward spin asymmetry is only moderately sensitive to changes in γ. Somewhat more sensitive are the asymmetries at 90°, which might provide some useful constraints on γ_{E1} and γ_{M1}. Most promising is Σ_{2x} for the neutron, which is remarkably sensitive to changes in γ_{M1}.

At higher energy, the spin asymmetries are more sensitive to the spin polarizabilities, but of course the low-energy expansion is no longer valid. Dispersion theory provides a convenient formalism for interpreting Compton scattering data beyond the low-energy approximation, but only for those polarizabilties not already constrained by the same disper-

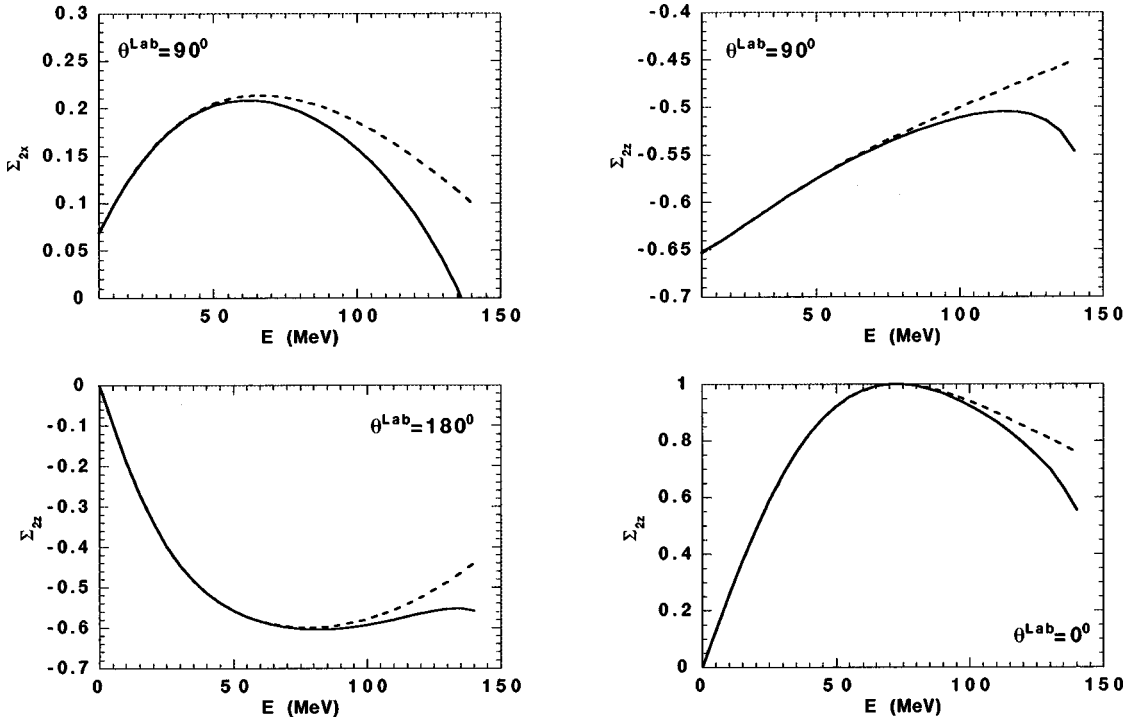

FIG. 5. Calculated asymmetries for the neutron. A comparison between a full dispersion calculation (full curves) and the low-energy expansion (dashed curves) in which the amplitudes are expanded to $\mathcal{O}(\omega^2\omega'^2)$. The upper panels are Σ_{2x} (left) and Σ_{2z} (right) at 90°. The lower two panels are Σ_{2z} at 180° (left) and 0° (right).

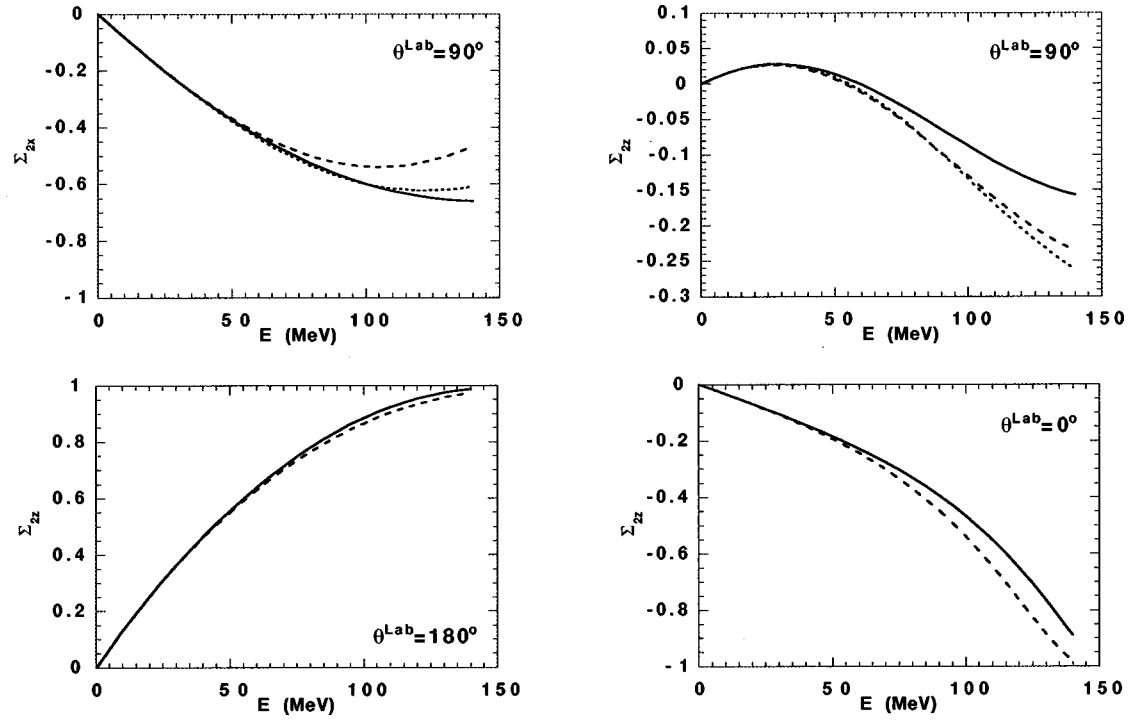

FIG. 6. Plots for the proton indicating the sensitivity of the double polarization observables to the spin polarizabilities as a function of energy. The solid curves use the SAID dispersion values for all of the spin polarizabilities (Table I). The upper panels show Σ_{2x} (left) and Σ_{2z} (right) at 90°. The long and short dashed curves increase by 4×10^{-4} fm^4 the values of γ_{E1} and γ_{M1}, respectively. The lower panels show Σ_{2z} at 180° (left) and 0° (right). The dashed curves decrease by 4×10^{-4} fm^4 the values of γ (right) or γ_π (left).

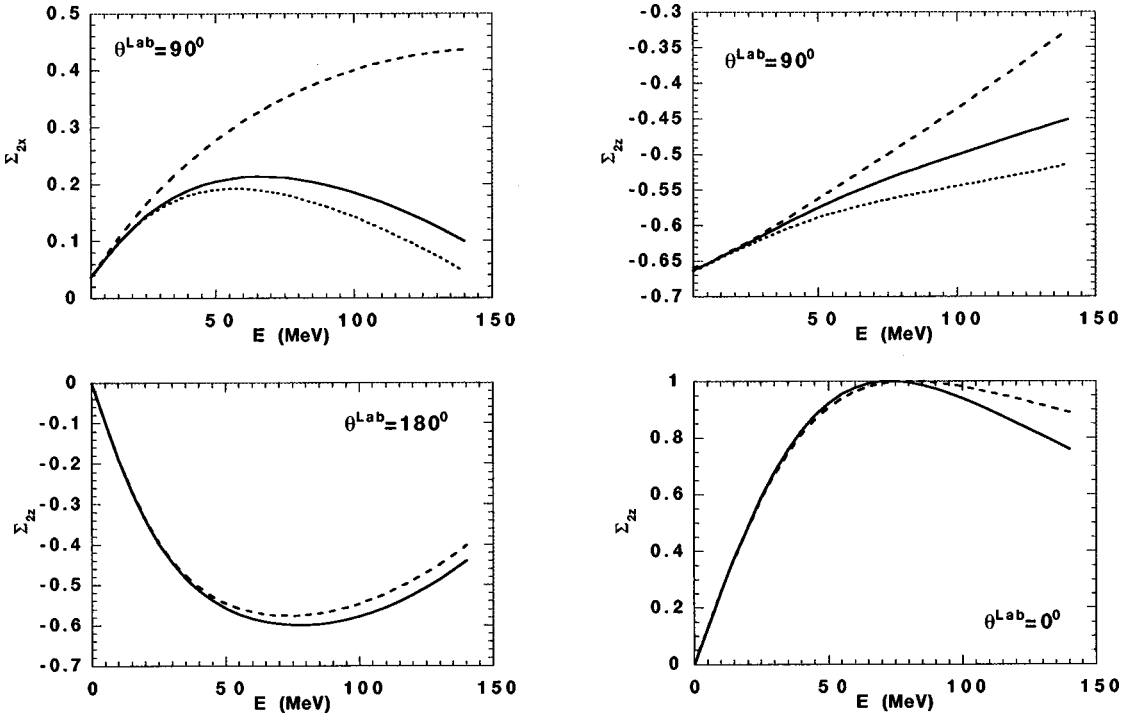

FIG. 7. Plots for the neutron indicating the sensitivity of the double polarization observables to the spin polarizabilities as a function of energy. The solid curves use the SAID dispersion values for all of the spin polarizabilities (Table I). The upper panels show Σ_{2x} (left) and Σ_{2z} (right) at 90°. The long and short dashed curves increase by 4×10^{-4} fm^4 the values of γ_{E1} and γ_{M1}, respectively. The lower panels show Σ_{2z} at 180° (left) and 0° (right). The dashed curves decrease by 4×10^{-4} fm^4 the values of γ (right) or γ_π (left).

sion relations. As discussed above, γ_π and $\alpha_{E2}-\beta_{M2}$ are not well constrained by the dispersion relations due to potentially unknown asymptotic contributions to a_2 and $a_{1,t}$, respectively. We therefore investigate whether Compton scattering in the so-called dip region between the $\Delta(1232)$ and higher resonances might usefully constrain these two parameters. In Figs. 8 and 9 we present calculations of the unpolarized cross section and the spin observables Σ_3, Σ_{2x}, and Σ_{2z} in the energy range 200–500 MeV. In these calculations, the parameter $a_{1,t}$ was adjusted by changing the σ mass in the asymptotic contribution $A_1^{\rm as}(t)=A_1^{\rm as}(0)/(1-t/M_\sigma^2)$, where $A_1^{\rm as}(0)$ was already fixed by experimental data on $\alpha_E-\beta_M$. The parameter a_2 was adjusted by adding to the asymptotic contribution from the π^0 exchange a contribution of heavier exchanges, i.e., by using in the dispersion relations the ansatz $A_2^{\rm as}(t)=(A_2^{\pi^0}(t)+C)F(t)$, where C was an adjusted constant and $F(t)$ was a monopole form factor with the cutoff parameter $\Lambda_\pi\approx700$ MeV. In the unpolarized cross section for the proton, a change of γ_π from -37 to -41 is indistinguishable from a change in the σ mass from 500 to 700 MeV (which changes $\alpha_{E2}-\beta_{M2}$ from 49 to 38). However, these possibilities are easily distinguished with Σ_{2z}, so that a combination of unpolarized and polarized measurements in this energy range offers the possibility of placing strong constraints on both γ_π and $\alpha_{E2}-\beta_{M2}$. Of course, any practical determination of the polarizabilities from Compton scattering data, especially at energies above the Δ peak, has to take into account uncertainties in the photopion multipoles used to evaluate the dispersion integrals. At the moment, these uncertainties are not negligible.[4]

VI. SUMMARY

The general structure of the Compton scattering amplitude from the nucleon with polarized photons and/or polarized nucleons in the initial and/or final state has been developed. A low-energy expansion of the amplitude to $\mathcal{O}(\omega^4)$ has been given in terms of ten polarizabilities: two dipole polarizabilities α_E and β_M, two dispersion corrections to the dipole polarizabilities $\alpha_{E\nu}$ and $\beta_{M\nu}$, two quadrupole polarizabilities α_{E2} and β_{M2}, and four spin polarizabilities γ_{E1}, γ_{M1}, γ_{E2}, and γ_{M2}. The physical significance of the parameters has been discussed, and the relationship between these and the cross section and spin observables below the pion threshold has been established. We have also presented theoretical predictions of these parameters based both on fixed-t dispersion relations and chiral perturbation theory. We have established that the range of validity of our expansion extends to the pion threshold. We have shown that low-energy experiments will have to be very precise to resolve the theoretical ambiguities in the polarizabilities. However, we have suggested that measurements at higher energy might help fix the most theoretically uncertain of them, particulary the

[4]For example, see Refs. [8,26] for arguments against the latest solution SP97K of Ref. [41] regarding the strength of the M_{1+} multipole and Refs. [42,43] for possible problems with the E_{0+} multipole.

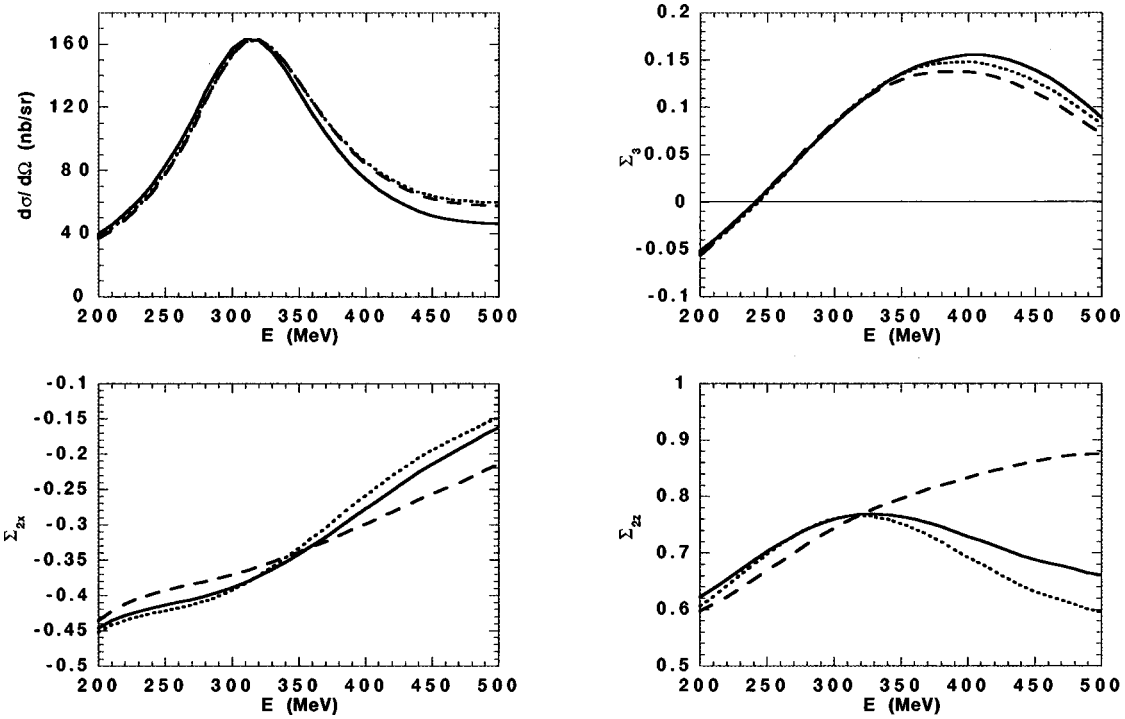

FIG. 8. Plots for the proton indicating the sensitivity of Σ_{2x} (bottom left), Σ_{2z} (bottom right), Σ_3 (top right), and the unpolarized cross section (top left) to changes in the the backward spin polarizability γ_π and the parameter $a_{1,t}$ at $\theta^{\mathrm{Lab}}=135°$ as a function of energy. The solid curve uses $a_{1,t}$ corresponding to $M_\sigma=500$ MeV and $\gamma_\pi=-37\times10^{-4}$ fm^4; the long dashes correspond to 500 MeV and -41; and the short dashes correspond to 700 MeV and -37.

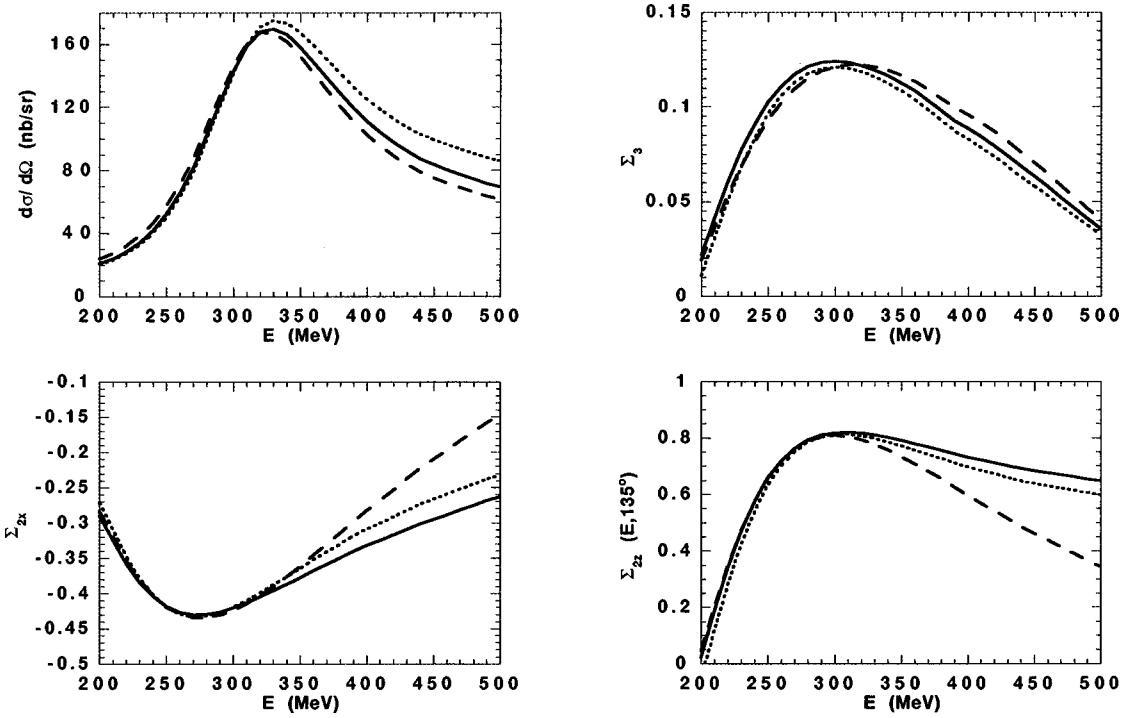

FIG. 9. Plots for the neutron indicating the sensitivity of Σ_{2x} (bottom left), Σ_{2z} (bottom right), Σ_3 (top right), and the unpolarized cross section (top left) to changes in the the backward spin polarizability γ_π and the parameter $a_{1,t}$ at $\theta^{\mathrm{lab}}=135°$ as a function of energy. The solid curve uses $a_{1,t}$ corresponding to $M_\sigma=500$ MeV and $\gamma_\pi=58\times10^{-4}$ fm^4; the long dashes correspond to 500 MeV and 62; and the short dashes correspond to 700 MeV and 58.

backward spin polarizabilitiy γ_π and the difference of quadrupole polarizabilities $\alpha_{E2}-\beta_{M2}$.

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APPENDIX A: MULTIPOLE CONTENT OF POLARIZABILITIES

1. Center-of-mass amplitudes

In the c.m. frame, the amplitude T_{fi} of nucleon Compton scattering can be represented by six functions R_i of the energy ω and the c.m. angle θ^* as [51,53]

$$T_{fi}=8\pi W\sum_{i=1}^{6}\rho_i R_i(\omega,\theta^*), \tag{A1}$$

where $W=\sqrt{s}$ and the spin basis ρ_i reads

$$\rho_1=\boldsymbol{e}'^*\cdot\boldsymbol{e}\,,\quad \rho_2=\boldsymbol{s}'^*\cdot\boldsymbol{s}\,,\quad \rho_3=i\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\boldsymbol{e}\,,$$

$$\rho_4=i\boldsymbol{\sigma}\cdot\boldsymbol{s}'^*\times\boldsymbol{s}\,,$$

$$\rho_5=i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\,\boldsymbol{s}'^*\cdot\boldsymbol{e}\,-\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}'\,\boldsymbol{e}'^*\cdot\boldsymbol{s}\,),$$

$$\rho_6=i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}'\,\boldsymbol{s}'^*\cdot\boldsymbol{e}\,-\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}\,\boldsymbol{e}'^*\cdot\boldsymbol{s}\,). \tag{A2}$$

In the particular cases of forward or backward scattering,

$$\frac{1}{8\pi W}[T_{fi}]_{\theta=0}=\rho_1(R_1+R_2)+\rho_3(R_3+R_4+2R_5+2R_6),$$

$$\frac{1}{8\pi W}[T_{fi}]_{\theta=\pi}=\rho_1(R_1-R_2)+\rho_3(R_3-R_4-2R_5+2R_6). \tag{A3}$$

Some authors [28,23,13,36] utilize a different spin basis, and the following identities provide links with the notation of those works:

$$\boldsymbol{e}'^*\cdot\hat{\boldsymbol{k}}\;\boldsymbol{e}\cdot\hat{\boldsymbol{k}}'\;=x\rho_1-\rho_2,\quad i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{k}}\;\boldsymbol{e}'^*\cdot\boldsymbol{e}\;=x\rho_3+\rho_4-\rho_5,$$

$$i(\boldsymbol{\sigma}\cdot\boldsymbol{e}'^*\times\hat{\boldsymbol{k}}\;\boldsymbol{e}\cdot\hat{\boldsymbol{k}}'\;-\boldsymbol{\sigma}\cdot\boldsymbol{e}\times\hat{\boldsymbol{k}}'\;\boldsymbol{e}'^*\cdot\hat{\boldsymbol{k}}\;)=2x\rho_3-\rho_5, \tag{A4}$$

$$i(\boldsymbol{\sigma}\cdot\boldsymbol{s}\;\boldsymbol{e}'^*\cdot\hat{\boldsymbol{k}}\;-\boldsymbol{\sigma}\cdot\boldsymbol{s}'\;\boldsymbol{e}\cdot\hat{\boldsymbol{k}}'\;)=2\rho_3-\rho_6,$$

where $x=\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}}$ and where we have used $\boldsymbol{e}\cdot\hat{\boldsymbol{k}}=\boldsymbol{e}'^*\cdot\hat{\boldsymbol{k}}'=0$.[5] In particular, the amplitudes A_i from [36] (we denote them here by A_i^{H}) read

$$A_1^{\mathrm{H}}=c(R_1+xR_2),\quad A_2^{\mathrm{H}}=-cR_2,$$

$$A_3^{\mathrm{H}}=c(R_3+xR_4+2xR_5+2R_6),$$

$$A_4^{\mathrm{H}}=cR_4,\quad A_5^{\mathrm{H}}=-c(R_4+R_5),\quad A_6^{\mathrm{H}}=-cR_6, \tag{A5}$$

where $c=4\pi W/m$.

The c.m. amplitudes R_i are related to the invariant amplitudes (2.7) by

$$\begin{aligned}
R_1&=C\left\{c_1\left(-A_1-\frac{W^2}{m^2}A_3\right)-\frac{\nu}{m}c_2A_5-\frac{W}{m}c_3A_6\right\},\\
R_2&=C\left\{c_1\left(A_1-\frac{W^2}{m^2}A_3\right)+\frac{\nu}{m}c_2A_5-\frac{W}{m}c_3A_6\right\},\\
R_3&=C\left\{(W-m)^2\left((x-1)A_1+(1+x)\frac{W^2}{m^2}A_3\right)-\frac{\nu}{m}c_3A_5-\frac{W}{m}c_2A_6\right\},\\
R_4&=C\left\{(W-m)^2\left((1-x)A_1+(1+x)\frac{W^2}{m^2}A_3\right)+\frac{\nu}{m}c_3A_5-\frac{W}{m}c_2A_6\right\},\\
R_5&=C\left\{(W-m)^2\left(-A_1-\frac{W^2}{m^2}A_3\right)+(W^2-m^2)\left(A_2+\frac{W^3}{m^3}A_4\right)+2(W-m)\left(-\nu A_5+\frac{W^2}{m}A_6\right)\right\},\\
R_6&=C\left\{(W-m)^2\left(A_1-\frac{W^2}{m^2}A_3\right)+(W^2-m^2)\left(-A_2+\frac{W^3}{m^3}A_4\right)+2(W-m)\left(\nu A_5+\frac{W^2}{m}A_6\right)\right\}.
\end{aligned} \tag{A6}$$

[5] A few other relations can be obtained from Eq. (A4) by doing a dual transformation, $\boldsymbol{e}\to\boldsymbol{s}=\hat{\boldsymbol{k}}\times\boldsymbol{e}$, $\boldsymbol{s}\to-\boldsymbol{e}=\hat{\boldsymbol{k}}\times\boldsymbol{s}$ (and the same for primed vectors) which is just a $\pi/2$ rotation of the polarizations. Under such a transformation, $\rho_1\leftrightarrow\rho_2$, $\rho_3\leftrightarrow\rho_4$, and $\rho_5\leftrightarrow\rho_6$.

Here $x=\cos\theta^*$ and

$$C=\frac{(s-m^2)^2}{64\pi s^2},\quad c_1=4mW+(W-m)^2(1-x),$$
$$c_2=4W(W-m)-(W-m)^2(1-x),$$
$$c_3=4W^2-(W-m)^2(1-x). \tag{A7}$$

Note also that the invariants ν, t, η are

$$\nu=\frac{s-m^2+t/2}{2m},\quad t=\frac{(s-m^2)^2}{2s}(x-1),$$
$$\eta=\frac{(s-m^2)^2}{2m^2}(x+1). \tag{A8}$$

In the important case of low energies (or very heavy nucleon), one has

$$R_1=-c(A_3+A_6+A_1),\quad R_2=-c(A_3+A_6-A_1),$$
$$R_3=-c'(A_6+A_5),\quad R_4=-c'(A_6-A_5), \tag{A9}$$
$$2R_5=c'(A_4+A_6+A_2),\quad 2R_6=c'(A_4+A_6-A_2)$$

up to higher orders in ω/m. Here $c=\omega^2/4\pi$ and $c'=\omega^3/4\pi m$.

A multipole expansion of the amplitudes R_i has the form [52–54]

$$R_1=\sum_{l\geqslant 1}\{[(l+1)f_{EE}^{l+}+lf_{EE}^{l-}](lP_l'+P_{l-1}'')-[(l+1)f_{MM}^{l+}+lf_{MM}^{l-}]P_l''\},$$
$$R_2=\sum_{l\geqslant 1}\{[(l+1)f_{MM}^{l+}+lf_{MM}^{l-}](lP_l'+P_{l-1}'')-[(l+1)f_{EE}^{l+}+lf_{EE}^{l-}]P_l''\},$$
$$R_3=\sum_{l\geqslant 1}\{[f_{EE}^{l+}-f_{EE}^{l-}](P_{l-1}''-l^2P_l')-[f_{MM}^{l+}-f_{MM}^{l-}]P_l''+2f_{EM}^{l+}P_{l+1}''-2f_{ME}^{l+}P_l''\},$$
$$R_4=\sum_{l\geqslant 1}\{[f_{MM}^{l+}-f_{MM}^{l-}](P_{l-1}''-l^2P_l')-[f_{EE}^{l+}-f_{EE}^{l-}]P_l''+2f_{ME}^{l+}P_{l+1}''-2f_{EM}^{l+}P_l''\}, \tag{A10}$$
$$R_5=\sum_{l\geqslant 1}\{[f_{EE}^{l+}-f_{EE}^{l-}](lP_l''+P_{l-1}''')-[f_{MM}^{l+}-f_{MM}^{l-}]P_l'''+f_{EM}^{l+}[(3l+1)P_l''+2P_{l-1}''']-2f_{ME}^{l+}[(l+1)P_{l+1}''+2P_l''']\},$$
$$R_6=\sum_{l\geqslant 1}\{[f_{MM}^{l+}-f_{MM}^{l-}](lP_l''+P_{l-1}''')-[f_{EE}^{l+}-f_{EE}^{l-}]P_l'''+f_{ME}^{l+}[(3l+1)P_l''+2P_{l-1}''']-2f_{EM}^{l+}[(l+1)P_{l+1}''+2P_l''']\}.$$

Here $P_l=P_l(x)$ are Legendre polynomials of $x=\cos\theta^*$. The multipole amplitudes $f_{TT'}^{l\pm}$, with $T,T'=E,M$ correspond to transitions $Tl\to T'l'$ and the superscript indicates the angular momentum l of the initial photon and the total angular momentum $j=l\pm\frac{1}{2}$. Due to T invariance,

$$f_{EM}^{l+}=f_{ME}^{(l+1)-},\quad f_{ME}^{l+}=f_{EM}^{(l+1)-}. \tag{A11}$$

Keeping only dipole-dipole and dipole-quadrupole transitions in these formulas, we obtain

$$R_1=2f_{EE}^{1+}+f_{EE}^{1-},\quad R_2=2f_{MM}^{1+}+f_{MM}^{1-},$$
$$R_3=f_{EE}^{1-}-f_{EE}^{1+}+6f_{EM}^{1+},\quad R_4=f_{MM}^{1-}-f_{MM}^{1+}+6f_{ME}^{1+}, \tag{A12}$$
$$R_5=-6f_{ME}^{1+},\quad R_6=-6f_{EM}^{1+}$$

(plus higher multipoles, which introduce an angular dependence to the amplitudes R_i).

2. Polarizabilities to order $\mathcal{O}(\omega^3)$

Low-energy expansions of the amplitudes R_i are obtained from Eqs. (A6) and (4.6) [22]. Leading nonvanishing terms of R_i are given by the Born term, in accordance with the low-energy theorem by Gell-Mann–Goldberger–Low:

$$R_1^{\mathrm{B}}=r_0q^2\left(-1+(1+x)\frac{\omega}{m}\right)+\mathcal{O}(\omega^2),$$
$$R_2^{\mathrm{B}}=-r_0q^2\frac{\omega}{m}+\mathcal{O}(\omega^2),$$
$$R_3^{\mathrm{B}}=-r_0q^2\frac{\omega}{2m}+\mathcal{O}(\omega^2),$$
$$R_4^{\mathrm{B}}=-r_0(\kappa+q)^2\frac{\omega}{2m}+\mathcal{O}(\omega^2), \tag{A13}$$

$$R_5^{\rm B}=\mathcal{O}(\omega^2),\quad R_6^{\rm B}=r_0 q(\kappa+q)\frac{\omega}{2m}+\mathcal{O}(\omega^2),$$

where $r_0=e^2/4\pi m$ and eq is the electric charge of the nucleon. Structure-dependent (i.e., non-Born) contributions to R_i start with the terms

$$\begin{aligned}&R_1^{\rm NB}=\omega^2\alpha_E+\mathcal{O}(\omega^3),\quad R_2^{\rm NB}=\omega^2\beta_M+\mathcal{O}(\omega^3),\\&R_3^{\rm NB}=\omega^3(\gamma_1+2\gamma_3)+\mathcal{O}(\omega^4),\quad R_4^{\rm NB}=\omega^3\gamma_2+\mathcal{O}(\omega^4),\\&R_5^{\rm NB}=-\omega^3(\gamma_2+\gamma_4)+\mathcal{O}(\omega^4),\quad R_6^{\rm NB}=-\omega^3\gamma_3+\mathcal{O}(\omega^4),\end{aligned}\tag{A14}$$

where coefficients are directly read out from Eq. (A9):

$$\begin{aligned}&\alpha_E=-\frac{1}{4\pi}(a_3+a_6+a_1),\quad \beta_M=-\frac{1}{4\pi}(a_3+a_6-a_1),\\&\gamma_1=\frac{1}{4\pi m}(a_4-a_5-a_2),\quad \gamma_2=\frac{1}{4\pi m}(a_5-a_6),\\&\gamma_3=\frac{1}{8\pi m}(a_2-a_4-a_6),\quad \gamma_4=\frac{1}{8\pi m}(a_6-a_2-a_4-2a_5).\end{aligned}\tag{A15}$$

Here the constants γ_i are those that appeared in Eq. (4.19).

The physical meaning of these polarizabilities can be understood by using the multipole expansion (A12). Comparing with Eq. (4.10), we identify the polarizabilities as leading terms in the structure-dependent multipoles:

$$\begin{aligned}&\omega^2\alpha_E\simeq(2f_{EE}^{1+}+f_{EE}^{1-})^{\rm NB},\\&\omega^2\beta_M\simeq(2f_{MM}^{1+}+f_{MM}^{1-})^{\rm NB},\\&\omega^3\gamma_1\simeq(f_{EE}^{1-}-f_{EE}^{1+}-6f_{EM}^{1+})^{\rm NB},\\&\omega^3\gamma_2\simeq(f_{MM}^{1-}-f_{MM}^{1+}+6f_{ME}^{1+})^{\rm NB},\\&\omega^3\gamma_3\simeq(6f_{EM}^{1+})^{\rm NB},\\&\omega^3\gamma_4\simeq(f_{MM}^{1+}-f_{MM}^{1-})^{\rm NB}.\end{aligned}\tag{A16}$$

It is seen that some of these γ_i describe mixed effects of spin-dependent dipole scattering and dipole-quadrupole transitions. A more transparent physical meaning can be ascribed to the quantities

$$\begin{aligned}\gamma_{E1}&=-\gamma_1-\gamma_3=\frac{1}{8\pi m}(a_6-a_4+2a_5+a_2)\\&\simeq\omega^{-3}(f_{EE}^{1+}-f_{EE}^{1-})^{\rm NB},\end{aligned}$$

$$\gamma_{M1}=\gamma_4=\frac{1}{8\pi m}(a_6-a_4-2a_5-a_2)\simeq\omega^{-3}(f_{MM}^{1+}-f_{MM}^{1-})^{\rm NB},\tag{A17}$$

which describe a spin dependence of the dipole transitions $E1\to E1$ and $M1\to M1$, and to the quantities

$$\begin{aligned}&\gamma_{E2}=\gamma_2+\gamma_4=-\frac{1}{8\pi m}(a_4+a_6+a_2)\simeq 6\omega^{-3}(f_{ME}^{1+})^{\rm NB},\\&\gamma_{M2}=\gamma_3=-\frac{1}{8\pi m}(a_4+a_6-a_2)\simeq 6\omega^{-3}(f_{EM}^{1+})^{\rm NB},\end{aligned}\tag{A18}$$

which describe transitions to quadrupole states, $M1\to E2$ and $E1\to M2$. In terms of these quantitities, the structure-dependent parts of the amplitudes R_3 to R_6 read

$$\begin{aligned}&R_3^{\rm NB}=\omega^3(-\gamma_{E1}+\gamma_{M2})+\mathcal{O}(\omega^4),\\&R_4^{\rm NB}=\omega^3(-\gamma_{M1}+\gamma_{E2})+\mathcal{O}(\omega^4),\\&R_5^{\rm NB}=-\omega^3\gamma_{E2}+\mathcal{O}(\omega^4),\\&R_6^{\rm NB}=-\omega^3\gamma_{M2}+\mathcal{O}(\omega^4).\end{aligned}\tag{A19}$$

The quantities γ_{E1}, γ_{M1}, γ_{E2}, and γ_{M2} are related to the spin polarizabilities δ, κ, δ_1 and κ_1 of Levchuk and Moroz [22] (they call them gyrations) by

$$\begin{aligned}&\delta=-\gamma_{E1}+\gamma_{M2},\quad \kappa=-\gamma_{M1}+\gamma_{E2},\\&\delta_1=-\gamma_{E2},\quad \kappa_1=-\gamma_{M2},\end{aligned}\tag{A20}$$

to the spin polarizabilities γ_i of Ragusa [23] by

$$\begin{aligned}&\gamma_1=-\gamma_{E1}-\gamma_{M2},\quad \gamma_2=-\gamma_{M1}+\gamma_{E2},\\&\gamma_3=\gamma_{M2},\quad \gamma_4=\gamma_{M1},\end{aligned}\tag{A21}$$

to the spin polarizabilities α_1, β_1, α_2, β_2 of Babusci *et al.* [24] by

$$\begin{aligned}&\alpha_1=4\gamma_{M2},\quad \beta_1=-4\gamma_{E2},\quad \alpha_2=-2\gamma_{E1}-2\gamma_{M2},\\&\beta_2=2\gamma_{M1}+2\gamma_{E2},\end{aligned}\tag{A22}$$

and to the forward- and backward-angle spin polarizabilities by

$$\begin{aligned}&\gamma=-\gamma_{E1}-\gamma_{M1}-\gamma_{E2}-\gamma_{M2},\\&\gamma_\pi=-\gamma_{E1}+\gamma_{M1}+\gamma_{E2}-\gamma_{M2}.\end{aligned}\tag{A23}$$

The parameter δ in [24] is $\delta=-\gamma_\pi$.

3. Quadrupole and dispersion polarizabilities

Effects described by the constants α_ν, α_t, β_ν, and β_t correspond to the following contributions of order $\mathcal{O}(\omega^4)$ to the spin-independent amplitudes R_1 and R_2:

$$\delta R_1^{(4)}=\omega^4[\alpha_\nu+(2x-2)\alpha_t],\quad \delta R_2^{(4)}=\omega^4[\beta_\nu+(2x-2)\beta_t]\tag{A24}$$

[these are not all terms of order $\mathcal{O}(\omega^4)$ in $R_{1,2}$, as is discussed in Appendix C]. The constants in Eq. (A24) are re-

lated to $\mathcal{O}(\omega^4)$ terms in dipole-dipole and quadrupole-quadrupole transitions. Introducing weighted sums over projections of the total angular momentum j,

$$f_{El}=(l+1)f_{EE}^{l+}+lf_{EE}^{l-},\quad f_{Ml}=(l+1)f_{MM}^{l+}+lf_{MM}^{l-}, \tag{A25}$$

we have

$$R_1=f_{E1}+2xf_{E2}-f_{M2},\quad R_2=f_{M1}+2xf_{M2}-f_{E2}. \tag{A26}$$

Comparing with Eq. (A24), we conclude that the constants α_t and β_t are proportional to the electric and magnetic quadrupole polarizabilities of the nucleon [25],

$$\alpha_{E2}=12\alpha_t\simeq 12\omega^{-4}(f_{E2})^{\mathrm{NB}},$$
$$\beta_{E2}=12\beta_t\simeq 12\omega^{-4}(f_{M2})^{\mathrm{NB}}. \tag{A27}$$

The normalization coefficient here is explained in Appendix B. It is chosen to conform to the definitions used in atomic physics where, for example, dynamic electric polarizabilities of the hydrogen read [25,55]

$$4\pi\alpha_{El}(\omega)=e^2\sum_{n\neq 0}\left(\frac{1}{E_n-E_0-\omega}+\frac{1}{E_n-E_0+\omega}\right)|[r^lP_l(\cos\theta)]_{n0}|^2. \tag{A28}$$

The factor 4π arises because we use units in which $e^2\simeq 4\pi/137$.

The combinations

$$\alpha_{E\nu}=\alpha_\nu-2\alpha_t+\beta_t,$$
$$\beta_{M\nu}=\beta_\nu-2\beta_t+\alpha_t \tag{A29}$$

are identified as $\mathcal{O}(\omega^4)$ terms in the dipole amplitudes f_{E1} and f_{M1}; that is, dispersion effects in the dynamic dipole polarizabilities

$$\alpha_{E1}(\omega)=\alpha_E+\omega^2\alpha_{E\nu}+\cdots,$$
$$\beta_{M1}(\omega)=\beta_M+\omega^2\beta_{M\nu}+\cdots. \tag{A30}$$

For the hydrogen atom,

$$4\pi\alpha_E=2\sum_{n\neq 0}\frac{|(D_z)_{n0}|^2}{E_n-E_0},\quad 4\pi\alpha_{E\nu}=2\sum_{n\neq 0}\frac{|(D_z)_{n0}|^2}{(E_n-E_0)^3},$$
$$\boldsymbol{D}=e\boldsymbol{r}. \tag{A31}$$

APPENDIX B: NORMALIZATION OF THE POLARIZABILITIES

In this appendix we explain the normalizations of the polarizabilities and effective interactions, which are defined in Sec. IV, by using a simple nonrelativistic model. We discuss the quadrupole polarizability α_{E2} and the spin polarizabilities γ_{E2} and γ_{M1}.

1. Quadrupole polarizability α_{E2}

Let us consider a charged particle in the bound s-wave state $|0\rangle$ affected by an external electric potential $A_0(\boldsymbol{r})$. Expanding the interaction $eA_0(\boldsymbol{r})$ in powers of $\boldsymbol{r}$, we get the quadrupole interaction with the external field,

$$V=\frac{1}{2}er_ir_j\nabla_i\nabla_jA_0(0)=-\frac{1}{6}Q_{ij}E_{ij}. \tag{B1}$$

Here $Q_{ij}=e(3r_ir_j-r^2\delta_{ij})$ is the quadrupole moment of the system and

$$E_{ij}=-\nabla_i\nabla_jA_0=\frac{1}{2}(\nabla_iE_j+\nabla_jE_i),\quad E_i^{\,i}=0, \tag{B2}$$

is the quadrupole strength of the field. The energy shift of the particle caused by the quadrupole interaction (B1) is given by second-order perturbation theory,

$$\Delta E=-\frac{1}{36}\sum_{n\neq 0}\frac{(Q_{ij}E_{ij})_{0n}(Q_{pq}E_{pq})_{n0}}{E_n-E_0}=-\frac{1}{36}XE_{ij}E_{ij}. \tag{B3}$$

Here n numerates excited states $|n\rangle$ and their energies E_n. The quantity X is defined as a coefficient in the expression

$$\sum_{n\neq 0}\frac{1}{E_n-E_0}[(Q_{ij})_{0n}(Q_{pq})_{n0}+\mathrm{H.c.}]=X\left(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp}-\frac{2}{3}\delta_{ij}\delta_{pq}\right). \tag{B4}$$

The right-hand side (RHS) of Eq. (B4) is the most general tensor T_{ijpq} which has vanishing traces $T_{i.pq}^{\,i}=T_{ijp.}^{\,\,\,p}=0$ and is symmetric under $i\leftrightarrow j$, $p\leftrightarrow q$, or $ij\leftrightarrow pq$.

Taking $i,j,p,q=z$, we relate X to the quadrupole polarizability α_{E2}:

$$X=12\pi\alpha_{E2},\quad 4\pi\alpha_{E2}\equiv\frac{1}{2}\sum_{n\neq 0}\frac{|(Q_{zz})_{n0}|^2}{E_n-E_0}. \tag{B5}$$

Finally, the energy shift (B3) takes the form of an effective quadrupole potential

$$H_{\mathrm{eff}}^{E2\ \mathrm{nospin}}=-\frac{1}{12}4\pi\alpha_{E2}E_{ij}E_{ij}. \tag{B6}$$

2. Quadrupole spin polarizability γ_{E2}

Now let us consider a particle moving around a heavy nucleus. We assume that both the particle and the nucleus have spin and that the total spin of the system in the ground state $|0\rangle$ is 1/2. In the presence of both an electric quadrupole and magnetic dipole interaction,

$$V=-\frac{1}{6}Q_{ij}E_{ij}-M_iH_i, \tag{B7}$$

where M_i is the magnetic moment operator, the corresponding energy shift of the system, $\Sigma_{n\neq 0}V_{0n}V_{n0}/(E_0-E_n)$, has a mixed $E2$-$M1$ term,

$$\Delta E = -\frac{1}{6}\sum_{n\neq 0}\frac{1}{E_n - E_0}[(Q_{ij}E_{ij})_{0n}(M_k H_k)_{n0} + \text{H.c.}]$$

$$= -\frac{1}{6} Y E_{ij}(\sigma_i H_j + \sigma_j H_i). \tag{B8}$$

Here Y is defined as

$$\sum_{n\neq 0}\frac{1}{E_n - E_0}[(Q_{ij})_{0n}(M_k)_{n0} + \text{H.c.}]$$

$$= Y\left(\sigma_i\delta_{jk} + \sigma_j\delta_{ik} - \frac{2}{3}\sigma_k\delta_{ij}\right). \tag{B9}$$

The RHS of Eq. (B9) is the most general tensor T_{ijk} which has vanishing trace $T^{i}_{i.k}=0$ and is symmetric under $i\leftrightarrow j$.

Taking $i,j,k=z$, we relate Y to the quadrupole spin polarizability γ_{E2}:

$$Y = -12\pi\gamma_{E2},$$

$$4\pi\gamma_{E2}\sigma_z \equiv -\frac{1}{4}\sum_{n\neq 0}\frac{1}{E_n - E_0}[(Q_{zz})_{0n}(M_z)_{n0} + \text{H.c.}], \tag{B10}$$

where the last equation explicitly shows the normalization and physical meaning of γ_{E2}. Such a polarizability can exist if there are tensor forces inside the system. Finally, the energy shift (B3) takes the form of an effective potential

$$H_{\text{eff}}^{E2,\text{spin}} = \frac{1}{2}4\pi\gamma_{E2}E_{ij}(\sigma_i H_j + \sigma_j H_i) = 4\pi\gamma_{E2}E_{ij}\sigma_i H_j. \tag{B11}$$

3. Dipole spin polarizability γ_{M1}

Now, let us assume that the above spin-1/2 system scatters a photon through a magnetic dipole interaction $-\boldsymbol{M}\cdot\boldsymbol{H}(t)$. The corresponding Compton scattering amplitude through excited intermediate states reads

$$\frac{1}{2m}T_{fi}^{M1} = \omega^2 s_j'^{*} s_i \sum_{n\neq 0}\left(\frac{(M_j)_{0n}(M_i)_{n0}}{E_n - E_0 - \omega} + \frac{(M_i)_{0n}(M_j)_{n0}}{E_n - E_0 + \omega}\right). \tag{B12}$$

Its spin dependent part at low energies is

$$\frac{1}{8\pi m}T_{fi}^{M1,\text{spin}} = -i\gamma_{M1}\omega^3\boldsymbol{\sigma}\cdot\boldsymbol{s}'^{*}\times\boldsymbol{s}, \tag{B13}$$

where the parameter γ_{M1} is defined as a coefficient in the equation

$$\sum_{n\neq 0}\frac{1}{(E_n - E_0)^2}[(M_j)_{0n}(M_i)_{n0} - \text{H.c.}] = 4\pi\gamma_{M1} i\epsilon_{ijk}\sigma_k. \tag{B14}$$

The scattering amplitude (B13) can be associated with an effective spin-dependent interaction

$$H_{\text{eff}}^{M1,\text{spin}} = -\frac{1}{2}4\pi\gamma_{M1}\boldsymbol{\sigma}\cdot\boldsymbol{H}\times\dot{\boldsymbol{H}}. \tag{B15}$$

4. Compton scattering amplitude

Now we give a summary of interactions and scattering amplitudes based on the above normalizations. Note that the corresponding electric and magnetic effective interactions are related through the duality transformation, $E_i\rightarrow H_i$, $H_i\rightarrow -E_i$.

With the effective quadrupole interaction

$$H_{\text{eff}}^{(E2,M2),\text{no spin}} = -\frac{1}{12}4\pi(\alpha_{E2}E_{ij}E_{ij} + \beta_{M2}H_{ij}H_{ij}), \tag{B16}$$

where $H_{ij}=\frac{1}{2}(\nabla_i H_j + \nabla_j H_i)$ and $H_i{}^{i}=0$, the Compton scattering amplitude reads

$$\frac{1}{8\pi m}T_{fi}^{(E2,M2),\text{no spin}} = \frac{1}{12}\omega^4\alpha_{E2}(2z\rho_1 - \rho_2) + \frac{1}{12}\omega^4\beta_{E2}(2z\rho_2 - \rho_1), \tag{B17}$$

where ρ_i are given in Eq. (A2). The spin-dependent dipole-quadrupole interaction

$$H_{\text{eff}}^{(E2,M2),\text{spin}} = 4\pi(\gamma_{E2}E_{ij}\sigma_i H_j - \gamma_{M2}H_{ij}\sigma_i E_j) \tag{B18}$$

results in the Compton scattering amplitude

$$\frac{1}{8\pi m}T_{fi}^{(E2,M2),\text{spin}} = \omega^3\gamma_{E2}(\rho_4 - \rho_5) + \omega^3\gamma_{M2}(\rho_3 - \rho_6). \tag{B19}$$

The spin-dependent dipole interaction

$$H_{\text{eff}}^{(E1,M1),\text{spin}} = -\frac{1}{2}4\pi\epsilon_{ijk}\sigma_k(\gamma_{E1}E_i\dot{E}_j + \gamma_{M1}H_i\dot{H}_j) \tag{B20}$$

gives the Compton scattering amplitude

$$\frac{1}{8\pi m}T_{fi}^{(E1,M1),\text{spin}} = -\omega^3(\gamma_{E1}\rho_3 + \gamma_{M1}\rho_4). \tag{B21}$$

APPENDIX C: QUADRUPOLE POLARIZABILITIES AND RELATIVISTIC CORRECTIONS TO THE DIPOLE INTERACTION

The polarizabilities of the nucleon can only be given an exact meaning through definition. The simplest definition of the multipole polarizabilities α_{El} and β_{Ml} is that they are the appropriately normalized coefficients of the ω^{2l} terms in the partial-wave amplitudes of Compton scattering, $(f_{El})^{\text{NB}}$ and $(f_{Ml})^{\text{NB}}$ [25,56]. However, we do not follow this approach for the quadrupole polarizabilities because it leads to some unwanted features when relativistic effects are taken into account. Considering $\mathcal{O}(\omega^4)$ terms in the amplitudes, we want to exclude contributions that arise merely as relativistic recoil corrections to the dipole polarizabilities.

We would like to associate with the polarizabilities α_{El} and β_{Ml} those nucleon-structure effects in the amplitude T_{fi} that are even functions of the photon energy or momentum

and do not depend on the nucleon spin. However, both the energy and the spin depend on the reference frame. If the frame is changed, the energy undergoes a Lorentz transformation and the Pauli spinors of the nucleon undergo a Wigner rotation. If the amplitude T_{fi} associated with the polarizabilities α_{El}, β_{Ml} is chosen to be spin independent in the c.m. frame, it would be spin dependent in other frames, including the lab and Breit frames. Moreover, since the electric and magnetic fields are not invariant under Lorentz transformations, the splitting of structure effects into electric and magnetic parts, $\sim\alpha_{El}$ and $\sim\beta_{Ml}$, may also depend on the frame. Giving a relativistically sound definition, we have to be cautious when choosing a frame and using correspondence with notions of classical physics.

The c.m. frame is not good in this respect. The c.m. amplitudes R_i do not possess all the symmetries that the amplitude T_{fi} itself has. The crossing transformation,

$$e,k\leftrightarrow e'^*,-k', \tag{C1}$$

brings the total momentum of the γN system, $k+p$, out of rest, so that the amplitudes R_i are neither odd nor even functions of the energy. Therefore, an effective covariant interaction (i.e., an effective Lagrangian), which describes the polarizabilities and possesses the symmetries of the total amplitude T_{fi}, would result in c.m. amplitudes R_i that contain terms of mixed order in ω, both even and odd. It would be difficult to rely on individual terms in R_i when identifying the polarizabilities, except for terms of lowest order.

The amplitudes in the lab frame are also not good, because of the lack of symmetry between the initial and final nucleon. In particular, the PT transformation,

$$e,k\leftrightarrow e',k', \quad \boldsymbol{\sigma},\boldsymbol{p}\leftrightarrow-\boldsymbol{\sigma},\boldsymbol{p}', \tag{C2}$$

applied to the γN system, brings the initial nucleon out of rest. That is why Eq. (2.11) contains both even and odd powers of the photon energies.

The best choice is provided by the Breit frame, in which the nucleon before and after photon scattering has the momentum

$$\boldsymbol{p}_{\rm B}=\frac{1}{2}(\boldsymbol{k}'-\boldsymbol{k})_{\rm B}=\boldsymbol{Q}_{\rm B}, \quad \boldsymbol{p}'_{\rm B}=-\boldsymbol{Q}_{\rm B}, \tag{C3}$$

respectively. In such a frame, both T invariance and crossing symmetry are fulfilled in the simplest way and, importantly, the nucleon is at rest on average, $(\boldsymbol{p}+\boldsymbol{p}')_{\rm B}=0$. That is why, in the course of an analysis of elastic eN scattering, the Breit frame rather than the c.m. frame is used to relate the amplitude of the reaction $eN\to eN$ with physically meaningful structure functions of the nucleon, the electromagnetic form factors G_E and G_M. For some deeper motivation in favor of the Breit frame and its relation with the language of wave packets, see Ref. [57].

Therefore in constructing our definitions, we choose to postulate that the polarizability interaction and the related Compton scattering amplitude are spin independent in the Breit frame. It will be spin dependent in the c.m. frame. Since the nucleon spin in the lab or Breit frame is the same,[6] the amplitude in the lab frame will be spin independent also. Nevertheless, we do not wish to directly relate individual spin-independent terms in Eq. (2.11) of different orders in $\omega\omega'$ with appropriate polarizabilities. That is partly because the amplitude (2.11) is not symmetric with respect to the initial and final nucleon. The factors $\omega\omega'\boldsymbol{e}'^*\cdot\boldsymbol{e}$ and $\omega\omega'\boldsymbol{s}'^*\cdot\boldsymbol{s}$ in Eq. (2.11) represent the electric and magnetic fields taken in the rest frame of the initial nucleon, whereas a sound definition should use the fields in the frame in which the nucleon is at rest, at least on average. Some of the $\mathcal{O}(\omega^2\omega'^2)$ terms in Eq. (2.11) are actually the result of a Lorentz transformation of $\mathcal{O}(\omega^2)$ terms in the Breit frame.

For the above reasons, we choose as a definition of the Compton scattering amplitude related with the dipole polarizabilities the expression

$$T_{fi,{\rm B}}^{(\alpha_E,\beta_M)}\equiv 4\pi\omega_{\rm B}^2\overline{u'}u(\boldsymbol{e}'^*_{\rm B}\cdot\boldsymbol{e}_{\rm B}\alpha_E+\boldsymbol{s}'^*_{\rm B}\cdot\boldsymbol{s}_{\rm B}\beta_M), \tag{C4}$$

where both the energy $\omega_{\rm B}$ and all polarizations are taken in the Breit frame. Neither $\mathcal{O}(\omega^4)$ terms nor recoil corrections $\sim t/m^2$ are explicitly included here. The factor $\overline{u'}u=\sqrt{4m^2-t}=2mN(t)$ is spin independent in the Breit frame and serves only for a covariant normalization.[7] Note also that $\omega_{\rm B}=\nu/N(t)$. Since the spin-independent part of the Compton scattering amplitude in the Breit frame reads

$$\begin{aligned}T_{fi,{\rm B}}^{\rm no\ spin}=\overline{u'}(p')e'^*{}^{\mu}&\left\{-\frac{P'_\mu P'_\nu}{P'^2}\left(T_1+\frac{m^2\nu T_2}{m^2-t/4}\right)\right.\\&\left.-\frac{N_\mu N_\nu}{N^2}\left(T_3+\frac{m^2\nu T_4}{m^2-t/4}\right)\right\}e^\nu u\\&=\frac{2m\nu^2}{N(t)}\left\{\boldsymbol{e}'^*\cdot\boldsymbol{e}\left[-A_1-\left(1-\frac{t}{4m^2}\right)A_3\right.\right.\\&\left.-\frac{\nu^2A_5}{m^2-t/4}-A_6\right]\\&\left.+\boldsymbol{s}'^*\cdot\boldsymbol{s}\left[A_1-\left(1-\frac{t}{4m^2}\right)A_3+\frac{\nu^2A_5}{m^2-t/4}-A_6\right]\right\},\end{aligned} \tag{C5}$$

the invariant amplitudes A_i corresponding to Eq. (C4) are

[6]That is because the Wigner angles for the nucleon-spin rotation between the lab and Breit frames vanish for both the initial and final nucleon. The Wigner angle, $\boldsymbol{\theta}_W\propto\boldsymbol{V}\times\boldsymbol{v}$, depends on the velocity $\boldsymbol{v}$ of the nucleon itself and the relative velocity $\boldsymbol{V}$ of the frames. In the case of the transformation between the lab and Breit frames, $\boldsymbol{\theta}_W=0$ for the initial nucleon N, because it is at rest, $\boldsymbol{v}=\boldsymbol{p}=0$. Also, $\boldsymbol{\theta}_W=0$ for the final nucleon N', because $\boldsymbol{V}\propto\boldsymbol{p}'+\boldsymbol{p}=\boldsymbol{p}'$ is parallel to the nucleon velocity $\boldsymbol{v}\propto\boldsymbol{p}'$. The similar Wigner angles for photons are generally not zero, so that the photon polarizations are different in the lab and Breit frames.

[7]Cf. the definition of the electromagnetic form factors of the nucleon [57].

$$A_1^{(\alpha_E,\beta_M)}=-2\pi(\alpha_E-\beta_M),\quad A_3^{(\alpha_E,\beta_M)}=-\frac{2\pi(\alpha_E+\beta_M)}{1-t/4m^2}, \tag{C6}$$

and other $A_i^{(\alpha_E,\beta_M)}$ are zero. Using Eq. (2.11), we find the corresponding scattering amplitude in the lab frame:

$$T_{fi}^{(\alpha_E,\beta_M)}=\frac{8\pi m\omega\omega'}{N(t)}\left\{\boldsymbol{e}'^*\cdot\boldsymbol{e}\ \alpha_E+\boldsymbol{s}'^*\cdot\boldsymbol{s}\ \beta_M - \frac{t}{4m^2}(\boldsymbol{e}'^*\cdot\boldsymbol{e}\ -\boldsymbol{s}'^*\cdot\boldsymbol{s}\ )(\alpha_E-\beta_M)\right\}, \tag{C7}$$

where a recoil correction $\sim t/m^2$ appears as a result of the no-recoil ansatz in the Breit frame, Eq. (C4).

With the above definition of the contribution of the dipole polarizabilities, we write the remaining terms of the non-Born amplitude $T_{fi}^{\mathrm{NB,no\ spin}}$ as

$$T_{fi}^{\mathrm{NB,no\ spin}}-T_{fi}^{(\alpha_E,\beta_M)}=8\pi m\omega\omega'\{\boldsymbol{e}'^*\cdot\boldsymbol{e}\ (\nu^2\alpha_\nu+t\alpha_t) +\boldsymbol{s}'^*\cdot\boldsymbol{s}\ (\nu^2\beta_\nu+t\beta_t)\}+\mathcal{O}(\omega^6). \tag{C8}$$

They are given by the parameters α_ν, α_t, β_ν, and β_t in Eq. (4.13), which determine quadrupole and dispersion polarizabilities, as discussed in Appendix A.

APPENDIX D: POLE CONTRIBUTION OF THE Δ(1232) TO POLARIZABILITIES

To calculate the contribution of the Δ-isobar excitation into the polarizabilities, we write an effective $\gamma N\Delta$ interaction in the form similar to Eq. (B7):

$$H_{\mathrm{eff}}=-\boldsymbol{M}\cdot\boldsymbol{H}-\frac{1}{6}Q_{ij}\nabla_i E_j\,. \tag{D1}$$

Here $\boldsymbol{M}$ and Q_{ij} are the magnetic dipole and electric quadrupole transition operators and are characterized by the matrix elements

$$\langle\Delta,\pm\tfrac{1}{2}|M_z|N,\pm\tfrac{1}{2}\rangle=\mu_{N\Delta},\quad \langle\Delta,\pm\tfrac{1}{2}|Q_{zz}|N,\pm\tfrac{1}{2}\rangle=\pm Q_{N\Delta}\,. \tag{D2}$$

Since the interaction (D1) involves $M1$ and $E2$ transitions into the $j=3/2$ state, it contributes to the multipoles f_{MM}^{1+}, f_{EE}^{2-}, f_{ME}^{1+} and therefore to the polarizabilities β_M, γ_{M1}, α_{E2}, γ_{E2}.

Using these matrix elements and Eq. (B4) from Appendix B, we find $4\pi\alpha_{E2}=Q_{N\Delta}^2/(2\Delta)$. From Eq. (B9) we get $4\pi\gamma_{E2}=-\mu_{N\Delta}Q_{N\Delta}/(2\Delta)$. Using the same matrix elements, the Wigner-Eckart theorem, and Eq. (B14), we find $4\pi\gamma_{M1}=\mu_{N\Delta}^2/\Delta^2$. Finally, the equations $4\pi\beta_M=2\mu_{N\Delta}^2/\Delta$ and $4\pi\beta_{M\nu}=2\mu_{N\Delta}^2/\Delta^3$ are magnetic analogs of Eq. (A31).

In terms of the ratio $R=E_{1+}/M_{1+}$ of the resonance multipoles of pion photoproduction taken at the resonance energy ($E_{\gamma,\mathrm{lab}}=340$ MeV),

$$\frac{Q_{N\Delta}}{\mu_{N\Delta}}=\frac{12}{k}R\simeq-0.25\ \ \mathrm{fm}, \tag{D3}$$

where k is the photon energy of the decay $\Delta\to\gamma N$ in the rest frame of the Δ and $R\simeq-2.75\%$ (we take an average of -2.5% [58] and -3.0% [8]).

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