Randomly broken nuclei and disordered systems

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Similarities between models of fragmenting nuclei and disordered systems in condensed matter suggest corresponding methods. Several theoretical models of fragmentation investigated in this fashion show marked differences, indicating possible new methods for distinguishing models using yield data. Applying nuclear methods to disordered systems also yields interesting results. [S0556-2813(98)03301-9]

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I. INTRODUCTION

The breakup of nuclei into clusters of various sizes in high-energy heavy-ion collisions is currently being studied both experimentally [1–5] and theoretically [6–8]. The reasons for undertaking such studies are many, but one is to determine whether a liquid-gas-like phase transition occurs at densities and temperatures away from the normal conditions of the nucleus (achieved, albeit briefly, during the collision). By investigating the behavior of particular functions of the fragmentation pattern, the character of the transition can be determined, and the various critical point exponents can be measured. For example, the EOS Collaboration [5,9] studied a set of moments introduced by Campi [10] to determine the three critical exponents β, γ, τ and the critical multiplicity. Their results strongly suggest that nuclear fragmentation can be understood in terms of a liquid-gas or possibly a percolative phase transition.

Many non-nuclear systems also exhibit clustering. For example, in condensed matter physics the phase space of disordered systems [11,12] such as spin glasses [13,14] exhibits clustering of states of the system around energy minima. As in the nuclear case, the sizes and distributions of these clusters are studied in an effort to better understand the character of these systems and the nature of the disorder. As such one is not surprised to discover a great degree of overlap in studies of nuclear fragmentation and disordered systems. Differences in the approaches are considerable, however.

This paper investigates in detail the correspondences between fragmenting nuclear and disordered systems, distinguishing the similarities and the differences in the approaches. A similar investigation by Higgs [15] discussed such parallels between biological and disordered systems. Our purpose is to enlarge the avenues of investigation in nuclear fragmentation by examining areas where parallel methods of analysis are likely to be fruitful. Such approaches have been successful before, such as the application of percolation theory to nuclear fragmentation [16,10] and our preliminary results presented here and elsewhere [17] encourage such a detailed look. Our principal result, that all the models considered can be realized as a sequential breaking of the interval, reconciles the observation of these similarities to a distinct mechanism.

II. FRAGMENTATION OBSERVABLES

A nucleus of A nucleons (or in general a system of A objects) can be partitioned into n_k clusters of size k. The total number of clusters (in nuclear physics known as the multiplicity) then is given by $m=\Sigma_k n_k$ and the total number of nucleons $A=\Sigma_k k n_k$. The set of n_k's subject to $\Sigma_k k n_k = A$ determines the partitions of A, with $\mathbf{n}=(n_1,n_2,\ldots,n_A)$ describing a particular partition or fragmentation. Alternately, one can describe such a partition by $\lambda=(\lambda_1,\ldots,\lambda_m)$, where λ_k is the size of the kth cluster, $A=\Sigma_k \lambda_k$, and the clusters are ordered in some fashion, e.g., according to size $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$ or for dynamical fragmentation by order of appearance.

Either $\mathbf{n}$ or λ describes the clustering pattern, but we are more interested in functions of the pattern which elucidate the character of the fragmentation. The moments of a fragmentation pattern are often examined for this reason, as they distill the results of a complex fragmentation pattern into a single number which can be simply interpreted. We define the sth moment of a fragmentation pattern $\mathbf{n}$ by

$$Y_s(\mathbf{n}) = \sum_k k^s n_k . \tag{1}$$

The first two moments count the number of clusters and objects, i.e., $Y_0=m$, $Y_1=A$. The second moment $Y_2=Y$ indicates the average size of the cluster (Y/A) a randomly chosen object belongs to, and higher moments can be similarly understood. In spin glasses, Y/A^2 is known as the participation ratio in Anderson localization [18].

Sometimes we wish to reduce these moments Y_s, eliminating any contribution to them from the largest cluster, i.e., we define the reduced sth moment as

$$M_s(\mathbf{n}) = Y_s(\mathbf{n}) - k_{\max}^s , \tag{2}$$

where $k_{\max}$ is the size of the largest cluster. Such a moment seems natural if we consider the system as a liquid-gas system. Then the largest cluster can be considered the liquid phase, and such a reduction corresponds to considering the contributions from the gas vapor only. Alternately, percolation theory would identify such a cluster with the incipient infinite cluster, and the reduction would separate the system into percolating and nonpercolating regions. In spin glasses, such a reduction removes the contribution of the extended state, the deepest valley in the rugged energy landscape.

III. FRAGMENTATION MODELS

The discussion of observables considered only a single fragmentation event. Physical systems produce such events in great numbers, but not necessarily with a uniform distribution. For the observables discussed above, once the distribution of events is specified any question about the observable can in principle be answered. Clearly there is a physical mechanism which leads to such a distribution, but it is possible that several mechanisms may yield the same fragment distribution. Analyzing ensemble averages or distributions on fragment observables would then fail to distinguish between various mechanisms. Therefore, one can specify a model by its underlying fragment distribution without recourse to the mechanism which generates it. This section does this for a number of cases applicable to nuclear fragmentation and disordered systems, and shows that all these systems can be generated by a random breaking mechanism.

A. Nuclear systems

When two heavy ions collide at intermediate energies, the nuclei dissolve into a diverse mix of fragments. The fragment distribution is necessarily complex, but its general statistical features are determined by simple thermodynamics. For instance, the kinetic motion of each fragment limits the available phase space, and so the probability of seeing m fragments is proportional to x^m, where $x=V/\lambda_T^3$, the ratio of the available volume to the thermal volume. Internal excitations and other effects will also enter, but in the large T limit kinetic motion dominates, predicting a large numbers of fragments. Low-temperature considerations likewise constrains our fragment distribution. At low T a large fragment accompanied by some very small fragments is expected. The region between the two natural limits having several intermediate mass fragments is more difficult to describe and is the subject of current research.

One might believe that the distribution of fragments in the intermediate region is not dominated by thermal factors, but rather by the available phase space associated with the number of possible patterns. Sobotka and Moretto [19] explored the simplest such model in which every pattern is given equal probability. Such a model shows an exponential fall off in the fragment yield, contrary to the experimentally observed power law. One can generalize this model in many ways, the most obvious of which is to consider a model where different cluster sizes would contribute different amounts to the fragment distribution phase space, i.e., $\Pr(\mathbf{n})=(1/Z_A)\Pi_k x_k^{n_k}$ so that $x_k=1$ reduces to Sobotka and Moretto's model. The partition function $Z_A=\Sigma_{\mathbf{n}}\Pi_k x_k^{n_k}$ in this case can be calculated using the recursion relation $Z_A=(1/A)\Sigma_{k>0}k\Sigma_{i>0}x_k^i Z_{A-ik}$ with $Z_0=1$.

Such models have many interesting properties, but need some modifications to avoid problems associated with the indistinguishability of clusters of the same size. This indistinguishability should reduce the size of the phase space available to the fragments, or equivalently, reduce the probability for a particular fragment distribution by factors of $1/n_k!$ after the argument of Gibbs. With this consideration, a simple and somewhat generic model of fragmentation would be to take the probability of a particular fragmentation pattern to be given by

$$\Pr(\mathbf{n})=\frac{1}{Z_A}\prod_{k>0}\frac{x_k^{n_k}}{n_k!}, \tag{3}$$

where the partition function or normalizing factor Z_A can be obtained using the recursion $Z_A=(1/A)\Sigma_k k x_k Z_{A-k}$ with $Z_0=1$. Such a model as applicable to nuclear fragmentation was explored in [20,21], and took $x_k=xy^{k-1}/\beta_k$ where $x=V/\lambda_T^3$ arises from the thermal motion, y describes effects due to binding and internal energies, and β_k allows different sized clusters to contribute different amounts to the phase space. Thermodynamic arguments discussed in [22] allow y to be expressed as $y=\exp\{a_V/k_BT+(k_BT/\varepsilon_0)[T_0/(T+T_0)]\}$, where a_V is the binding energy per nucleon [i.e., it is assumed that $E_B(k)\approx a_V(k-1)$ for a cluster of size k], ε_0 is the energy spacing between excited levels, and T_0 is a cutoff temperature for internal excitations. The parameters β_k can be taken as k^τ so as to reproduce the observed cluster distribution $\langle n_k\rangle\sim k^{-\tau}$, where τ is a critical exponent introduced by Fisher [23]. For nuclear fragmentation $\tau\approx 2.2$ [5].

Several models with $\beta_k=k^\tau$ have been explored by us and others. For example, $\beta_k=1$, called the linear chain model, was extensively studied by Gross *et al.* [24–26]. Another example, $\beta_k=k$, was the first model investigated by one of us (A.Z.M.) [27]. We have investigated $\beta_k=k^\tau$ with $\tau>2$, in particular $\tau=5/2$ more recently [20]. It has many features in common with Bose-Einstein condensation and percolation theory; namely, below a certain value of $x=x_c$, a cluster forms containing a finite fraction of the mass of the system even in the infinite A limit. This ''infinite'' cluster can be likened to the accumulation of particles in the ground state in a Bose gas below $T=T_c$, or to the formation of the incipient infinite cluster in percolating systems above $p=p_c$. Related weights including more factors (some dynamical in nature) have been studied by Gross *et al.* [6] and Bondorf *et al.* [7] in describing more detailed models of nuclear fragmentation.

B. Disordered systems

Disordered systems can also involve clustering, and in those cases the probability of seeing a particular clustering pattern is of interest. In the case of nuclear fragmentation, thermodynamic and phase space arguments were used to arrive at the pattern distribution, i.e., Eq. (3). Here we arrive at a similar description by analyzing a simple but general mechanism which generates disorder: the random sequential breaking of an interval. This model was studied extensively by Derrida and Flyvbjerg [28,29] and more recently by Frontera *et al.* [30]. Here we generalize their mechanism in such a way as to encompass Eq. (3) and many other models.

The sequential breaking of an interval is simple to describe. In the continuous case, the unit interval is sequentially broken into pieces of sizes $W_1, W_2, \ldots$, each chosen randomly in some fashion. Specifically, the sequence $W_1=z_1$, $W_2=(1-z_1)z_2$, $W_3=(1-z_1)(1-z_2)z_3$, etc., is generated where each z_k is chosen from some probability distribution $\rho_k(z)$. In other words, at step i one breaks off a randomly chosen fraction z_i of the remaining piece of size $1-W_1-\cdots W_{i-1}$. Typically the probability distribution on

z_i is independent of i, though that need not be the case. Derrida and Flyvbjerg studied the case where $\rho(z) = x(1-z)^{x-1}$ [29].

Let us consider an equivalent, but discrete model. In this model, an interval of length A is partitioned sequentially into integral sized pieces. Suppose a piece of size k is broken off from the interval of size A with probability

$$\Pr(\lambda_1 = k; A) = \binom{A-1}{k-1} \frac{(1)_{k-1}(x)_{A-k}}{(x+1)_{A-1}}, \tag{4}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ and $x>0$ is a free parameter. In the limit $A\to\infty$, $k/A\to W$, this distribution converges to the continuous distribution mentioned above.

Applying this process repeatedly until the interval is exhausted, the sequence $\lambda_1,\lambda_2,\ldots,\lambda_m$ is generated. The overall probability for this sequence appearing from this process is the product $\Pr(\lambda_1;A)\Pr(\lambda_2;A-\lambda_1)\Pr(\lambda_3;A-\lambda_1-\lambda_2)\cdots\Pr(\lambda_m;A-\lambda_1-\cdots-\lambda_{m-1})$ and is given by

$$\Pr(\lambda_1,\ldots,\lambda_m) = x^m \binom{A+x-1}{A}^{-1} \times \frac{1}{A(A-\lambda_1)\cdots(A-\lambda_1-\cdots-\lambda_{m-1})}. \tag{5}$$

By summing over all possible orders of such a sequence, the probability $\Pr(\mathbf{n}) = \Sigma_{\lambda\mapsto\mathbf{n}}\Pr(\lambda_1,\ldots,\lambda_m)$ that a particular unordered clustering pattern appears is determined, and is given by

$$\Pr(\mathbf{n}) = \binom{A+x-1}{A}^{-1} \prod_k \frac{1}{n_k!}\left(\frac{x}{k}\right)^{n_k}, \tag{6}$$

where we have applied Eq. (A1). This is in fact the distribution given by Eq. (3) with $x_k = x/k$. More generally, the sequential breaking process where the probability of breaking off a fragment of size k from an interval of size A is given by $\Pr(k;A) = (kx_k/A)Z_{A-k}/Z_A$ yields Eq. (3) as the fragmentation pattern. This gives a possible mechanism for obtaining such a fragment distribution, but it is not unique. Another mechanism based on Markov chains was explored by us before [21].

By modifying the random breaking mechanism we can obtain the other model discussed in the previous section (the equipartition model) as well. Suppose one allows multiple clusters of the same size to be broken off instead of just one. For example, take the probability that m clusters of size k are broken from an interval of length n as $\Pr(k,m;A) = (kx_k^m/A)(Z_{A-mk}/Z_A)$. Then the distribution of cluster sizes can be shown to be given by $\Pr(\mathbf{n}) = (1/Z_A)\Pi_k x_k^{n_k}$. This is in fact a generalization of a well-known method for generating equiprobable partitions introduced by Nijenhuis and Wilf [31].

Having obtained the models of the previous section as cases of this disordered system mechanism, we are encouraged to believe that other disordered systems can be likewise described in the context of randomly breaking the interval. Indeed, two other models considered by Derrida and Flyvbjerg fit nicely into this mechanism.

Consider the quenched random map model [28,32,33]. The A^A maps or functions of A points onto themselves are chosen uniformly at random, and their cluster structure is determined by the number of points in each basin of attraction, where a basin is defined as a limit cycle of the iterated action of the map. For example, the map $1\mapsto 2$, $2\mapsto 1$, $3\mapsto 2$, $4\mapsto 3$ has a single limit cycle $(1\mapsto 2\mapsto 1\cdots)$ which all the points eventually enter, so that the cluster structure is just one single cluster of size four. Such a simple model has interesting properties, and its cluster pattern is given by Eq. (3) with $x_k = \Sigma_{j=0}^{k-1} k^{j-1}/j! \approx e^k/2k$. This can be inferred from the fact that the probability that the map is indecomposable is known and would be given by x_k/Z_k where $Z_k = k^k/k!$ by simple counting. Interpreting Eq. (3) in terms of randomly breaking an interval has already been accomplished, so we see that the random map model is a special case of randomly breaking the interval. The asymptotic limit can be seen by considering the elements in the sum as terms in the Poisson distribution $e^{-x}x^j/j!$ with $x=k$. The factor e^k does not affect the cluster distributions, so that the model asymptotically converges to $x_k = 1/(2k)$, and hence is included in the original distribution considered by Derrida and Flyvbjerg [28] $\rho(z) = x(1-z)^{x-1}$ with $x = 1/2$.

Secondly, consider what happens when the sequential pieces are chosen from a changing distribution. For example, suppose the $i+1$st piece is chosen from the interval of size $A_i = A-\lambda_1-\cdots-\lambda_i$, $A_0 = A$ with probability

$$\Pr(k;A_i) = \binom{A_i-1}{k-1} \frac{(\gamma)_{k-1}[x+i(1-\gamma)]_{A_i-k}}{[x+\gamma+i(1-\gamma)]_{A_i-1}}. \tag{7}$$

In this case the sequence of pieces is generated with probability

$$\Pr(\lambda_1,\ldots,\lambda_m) = A!\,\frac{(x)_{(m-1;1-\gamma)}}{(x+\gamma)_{A-1}} \times \frac{1}{A(A-\lambda_1)(A-\lambda_1-\lambda_2)\cdots} \times \prod_k \frac{(\gamma)_{\lambda_k-1}}{(\lambda_k-1)!}, \tag{8}$$

where $x_{(n;\alpha)}$ is a generalized Pochhammer symbol

$$x_{(n;\alpha)} = \begin{cases} 1 & \text{for } n=0, \\ x(x+\alpha)\cdots[x+(n-1)\alpha] & \text{for } n>0. \end{cases}$$

Summing over all the possible orders of the λ's, gives the probability for the unordered fragmentation pattern

$$\Pr(\mathbf{n}) = A!\,\frac{(x)_{(m-1;1-\gamma)}}{(x+\gamma)_{A-1}} \prod_k \frac{1}{n_k!}\left(\frac{(\gamma)_{k-1}}{k!}\right)^{n_k}. \tag{9}$$

This is clear since the only term that depends on the order of the fragments is $1/[A(A-\lambda_1)\cdots]$; the other terms are invariant under permuting the order of the λ's, so that we can apply Eq. (A1) directly.

This distribution pattern has been studied in a statistics context by Pitman [34], but what we would like to emphasize here is that in the limit $x+\gamma=1$, this model reproduces as-

ymptotically all of the clustering features of a Sherrington-Kirkpatrick spin glass [35,36,29]. Unlike the other disordered systems, we cannot prove this result, but the evidence (discussed below) is compelling enough that we consider Eq. (9) with $x+\gamma=1$ as a model of a spin glass. Why the changing distributions produces the patterns seen in a spin glass is perhaps unexpected and should be explored further.

IV. TRADITIONAL ANALYSIS OF MODELS

Now that we have a description of the fragment distributions we are in a position to make definitive predictions. Since nuclear and disordered systems have spawned separate methods of analysis, we begin by briefly considering some features of these models from their traditional points of view.

A. Nuclear systems

Fragmenting nuclear systems are traditionally analyzed by studying expectation values of the various observables as functions of thermodynamic variables. Some of the important observables are the multiplicity m and its variance, the size of the largest cluster $k_{\max}$, and the reduced moment M_2. They are important because of their expected behavior in the region of a critical point phase transition. For example, from percolation theory we know that the reduced moments are functions of the critical exponents and $p-p_c$ in the region of the critical point [9], i.e.,

$$\begin{aligned}\langle M_0\rangle &= \langle m\rangle - 1 \sim |p-p_c|^{2-\alpha},\\ \langle M_1\rangle &= A - \langle k_{\max}\rangle \sim |p-p_c|^{\beta},\\ \langle M_2\rangle &= \langle Y_2\rangle - \langle k_{\max}^2\rangle \sim |p-p_c|^{-\gamma}.\end{aligned} \tag{10}$$

The EOS Collaboration [5] has argued that both nuclear fragmentation and percolation theory can be analyzed in a similar fashion, extending the observations and results of Campi [10,37]. Thus from observing the behavior of the moments the critical point and exponents can be deduced, and a number of papers in the literature have studied this in a variety of theoretical [37,38,26] and experimental [1,5] systems.

In the model given by Eq. (3), these quantities can be computed from their definitions and by utilizing the following results:

$$\langle n_k\rangle = x_k \frac{Z_{A-k}}{Z_A}, \tag{11}$$

$$\langle n_j(n_k-\delta_{jk})\rangle = x_j x_k \frac{Z_{A-j-k}}{Z_A}. \tag{12}$$

Computation of the reduced moments additionally requires that various expectation values of $k_{\max}$ be determined. This is most easily done by expressing the expectation values in terms of the probability distribution on $k_{\max}$. The probability that a random fragmentation of nucleons has a cluster of size $k_{\max}$ as its largest cluster is given by $\Pr(k_{\max})=\Delta Z_A(x_1,\ldots,x_{k_{\max}})/Z_A(x_1,\ldots,x_n)$, where $\Delta Z_A(x_1,\ldots,x_{k_{\max}})=Z_A(x_1,\ldots,x_{k_{\max}},0,\ldots)-Z_A(x_1,\ldots,x_{k_{\max}-1},0,\ldots)$. When $k_{\max}>A/2$ a simpler expression can be used. Since any cluster of size $k>A/2$ is automatically the largest, one can show that $\Delta Z_A=x_k Z_{A-k}$, so that $\Pr(k_{\max}=k)=x_k Z_{A-k}(\mathbf{x})/Z_A(\mathbf{x})$ when $k>A/2$.

For typical models such as $x_k=x/k^{\tau}$ with $\tau>2$, the variance of m and the reduced moment M_2 should peak at the point $x_c/A=1/\zeta(\tau-1)$. The finite size effects can be considerable, and the M_2 peak is significantly shifted from its infinite value in Fig. 1. At the same point $\langle k_{\max}\rangle$ tends to zero in the infinite A limit, and tends to be a better indicator of the transition. All these facts are consistent with the interpretation of $x=x_c$ as a critical point of a phase transition characterized by the appearance of an ''infinite'' or percolating cluster.

B. Disordered systems

Disordered systems are traditionally analyzed by considering the distribution of an observable at a particular point in parameter space. The important observables in disordered systems are $Y=Y_2$ and $k_{\max}$, and their distributions show significant non-Gaussian behavior, and this feature is in some sense a characteristic of the disorder.

For the model given by Eq. (9), these can be obtained by a recursive procedure. Define $x_k=(\gamma)_{k-1}/k!$. Then the probability for a pattern separates into a term such as Eq. (3) times a term which depends only on the multiplicity. The partition function $Z_A=(x+\gamma)_{A-1}/A!$ is then essentially a sum of terms contributed by each multiplicity class, namely, $Z_A=\Sigma_m x_{(m-1;1-\gamma)} Z_A^m$. Each of these terms can be computed for any x_k from the recursion $mZ_A^m=\Sigma_k x_k Z_{A-k}^{m-1}$. Then the probability distribution $\Pi(k_{\max})=\Delta Z_A/Z_A$ as in the previous section.

To determine the probability distribution $\Pi(Y)$, the partition function is broken into terms in which Y is fixed, i.e., $Z_A=\Sigma_Y Z_A^Y$, so that $\Pi(Y)=Z_A^Y/Z_A$. For computational purposes, it is convenient to continue to break these partition functions into components where the multiplicity is fixed, so that $Z_A^Y=\Sigma_m(x)_{(m-1,1-\gamma)} Z_A^{m,Y}$. Using x_k as defined above, we can compute $Z_A^{m,Y}$ via the recursion $mZ_A^{m,Y}=\Sigma_k x_k Z_{A-k}^{m-1,Y-k^2}$.

For many values of x and γ, these distributions are significantly non-Gaussian, for instance as seen in Fig. 2(a) which shows the distribution $\Pi(Y)$. First, one notes the large finite-size fluctuations seen in $\Pi(Y)$ when $Y/A^2>1/2$. These disappear in the infinite limit. The presence of cusps in $\Pi(Y)$ at $Y/A^2=1,1/2$ and $1/3$ can be seen in Fig. 2(a) and persists in the infinite limit. These cusps are not strictly an indicator of disorder, but the non-Gaussian nature of the distributions accentuates their presence.

Figure 2(b) which shows $\Pi(k_{\max})$ reveals that the distribution of sizes of the largest cluster varies widely for different models. All these cases are likely to have sizable largest clusters, however, there is a nontrivial distribution about $k_{\max}/A=1/2$ for these models, due in part to the large number of patterns with $k_{\max}=A/2$. The cusps at $k_{\max}/A=1/2,1/3$ etc., arise from singularities associated with breaking the system into fragments of equal size. For example, below $k_{\max}/A=1/2$, there is no contribution from binary fragmentation. The sudden cutoff of this contribution leads to a cusp.

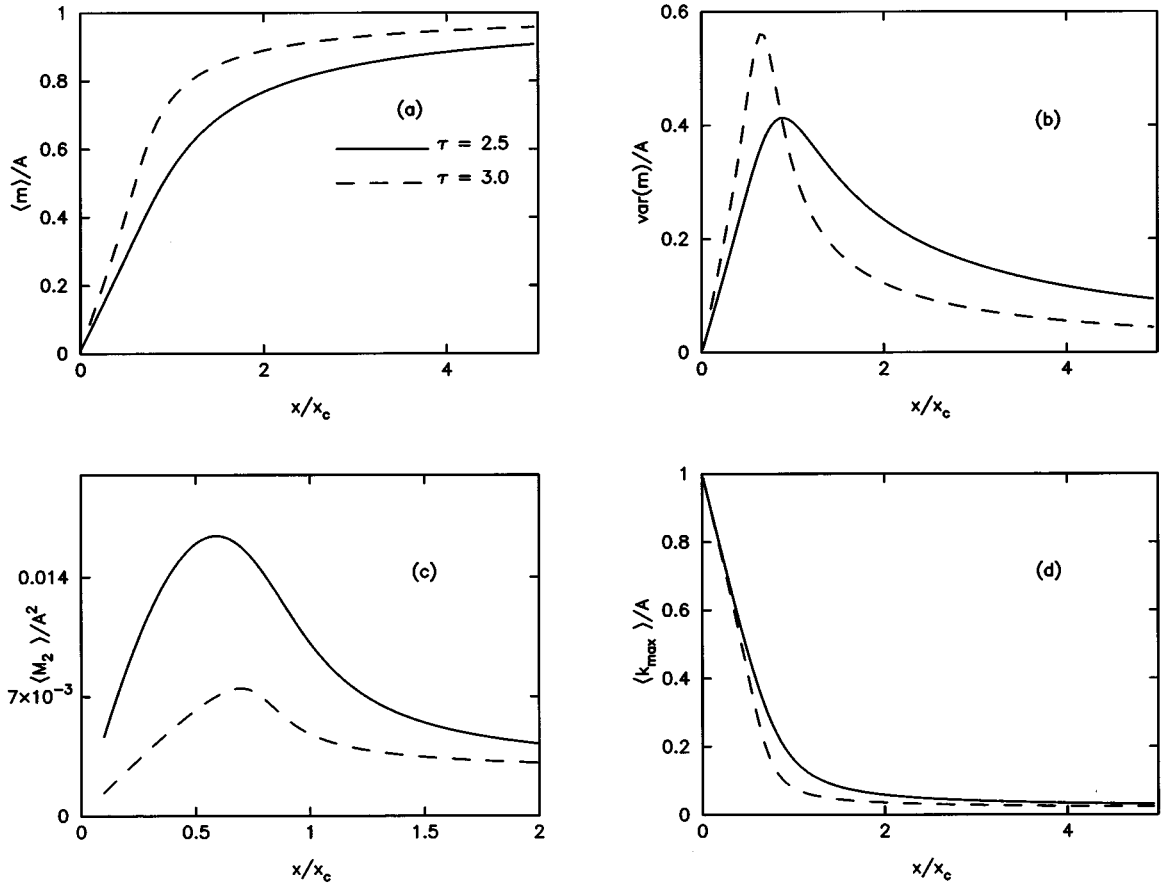

FIG. 1. The expected value (a) and variance (b) of the multiplicity and the expected values of the reduced second moment (c) and the largest cluster (d) for the model given by Eq. (3) with $x_k = x/k^{\tau}$, $\tau = 2.5$ (solid line), $\tau = 3.0$ (dashed line).

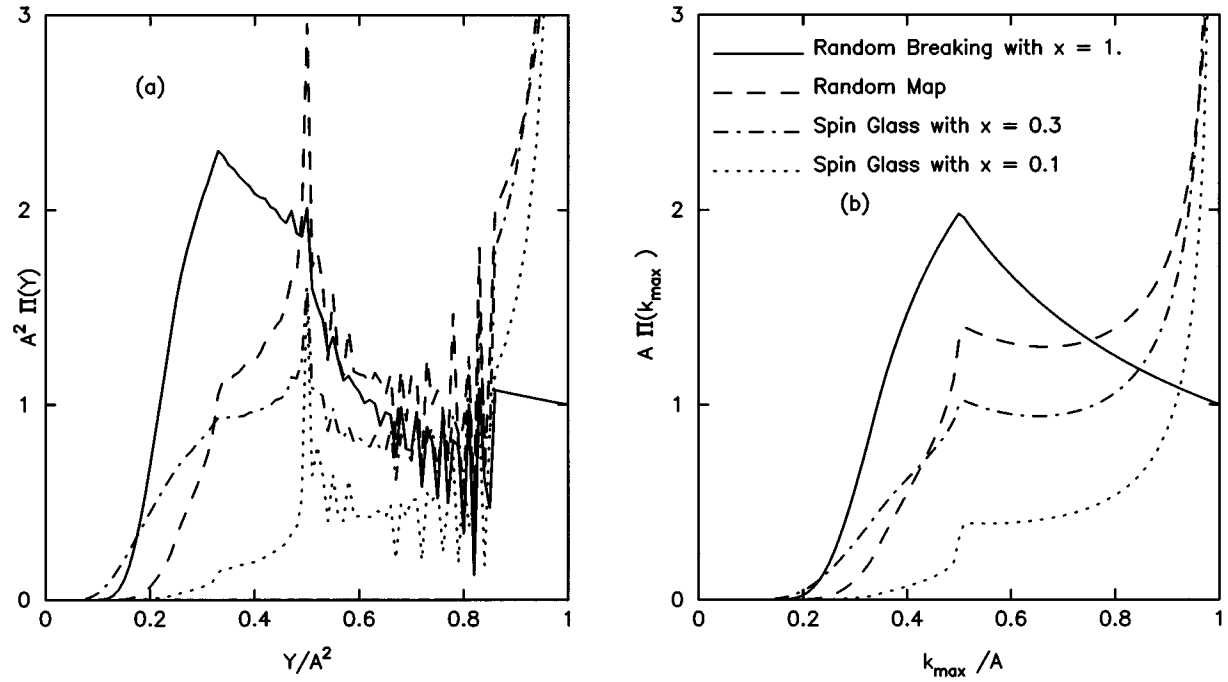

FIG. 2. The probability distribution of (a) the second moment $\Pi(Y)$ and (b) the size of the largest cluster $\Pi(k_{\max})$ for several disordered systems.

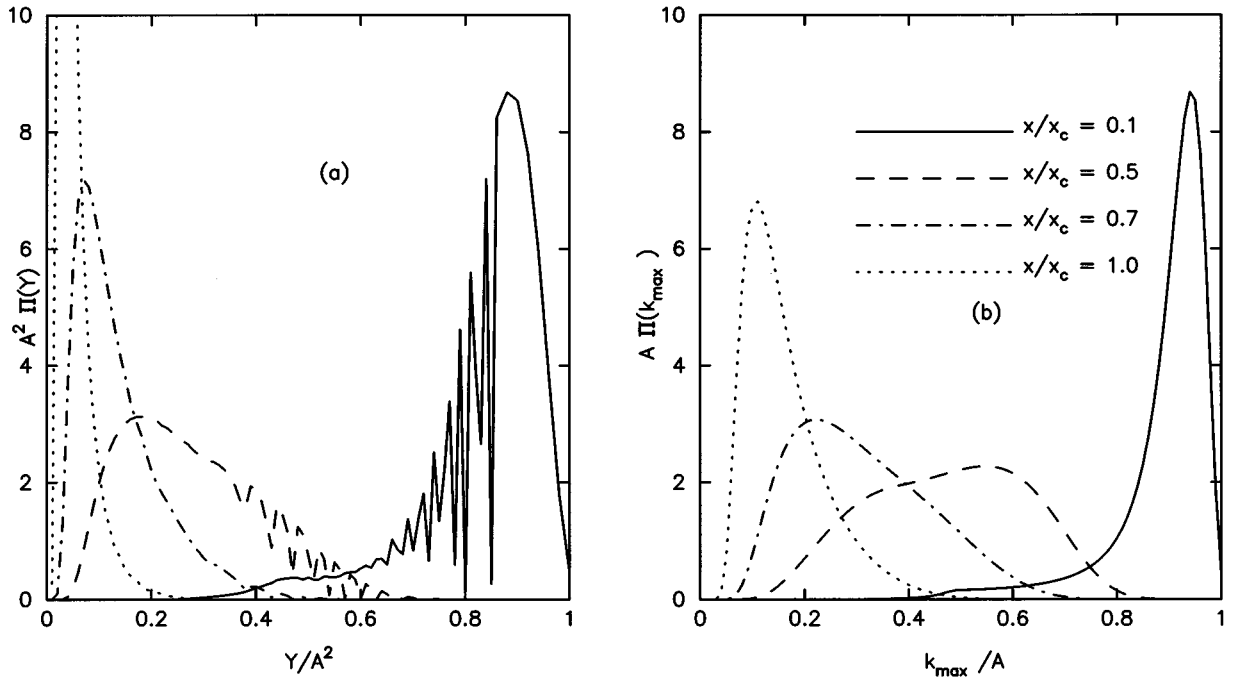

FIG. 3. The probability distribution of (a) the second moment $\Pi(Y)$ and (b) the size of the largest cluster $\Pi(k_{\max})$ for the model given by Eq. (3) with $x_k = x/k^{5/2}$ for various x.

V. CORRESPONDING ANALYSIS OF MODELS

In this section we reverse our methods of analysis, treating nuclear models as if they were disordered systems, and vice versa. Such an approach is warranted since both descriptions can be described without reference to the fragmenting mechanisms, and therefore the methods should be insensitive to the source of the data and sensitive instead to the underlying property we are trying to isolate in the data (i.e., criticality and disorder). Whether the methods are indeed selective should be revealed by such a (potentially) blind test.

A. Nuclear systems

The corresponding analysis of fragmenting nuclei considers the distribution of the second moment Y and the size of the largest cluster $k_{\max}$. Computing $\Pi(Y)$ and $\Pi(k_{\max})$ is not complicated for models with fragmentation pattern given by Eq. (3). We have already discussed how to obtain $\Pi(k_{\max})$ from the partition function in Sec. IV A. The distribution $\Pi(Y)$ is obtained by breaking the contributions to the partition function Z_A into classes with fixed Y, $Z_A = \Sigma_Y Z_A^Y$, and applying $\Pi(Y) = Z_A^Y / Z_A$. As in Sec. IV B, these partition functions can be computed by recursion, i.e., $AZ_A^Y = \Sigma_k k x_k Z_{A-k}^{Y-k^2}$.

If we analyze the fragmentation distribution given by Eq. (3) with a typical $x_k = x/k^\tau$ with $\tau > 2$ by plotting the distribution on the largest cluster size and the second moment, we find over most of the range that the distributions are essentially Gaussian centered about their expectation values. Figure 3 shows this. When x is small and the pattern is usually dominated by a single large fragment, the atypical events with smaller large fragments are distributed in a manner similar to a disordered system with a low value of x, only with a much diminished probability. This arises almost entirely from binary fragmentation events.

B. Disordered systems

The corresponding analysis of disordered systems examines the expectation values of the reduced moments as a function of a changing parameter. We consider two cases given by Eq. (9): the randomly broken interval with $\rho(z) = x(1-z)^{x-1}$ with $x > 0$ $(\gamma = 1)$, and the spin-glass model with $0 < x < 1$ $(x + \gamma = 1)$. To facilitate computing the reduced moments for the models described by Eq. (9) we have the following results for ensemble averages of n_k:

$$\langle n_k \rangle = \binom{A}{k} \frac{(x)_{A-k} (\gamma)_{k-1}}{(x+\gamma)_{A-1}}, \tag{13}$$

$$\langle n_j (n_k - \delta_{jk}) \rangle = x \frac{A!}{j!k!(A-j-k)!} (\gamma)_{j-1} (\gamma)_{k-1} \times \frac{(x-\gamma+1)_{A-j-k}}{(x+\gamma)_{A-1}}. \tag{14}$$

With these we can compute $\langle Y_s \rangle$ and its variance. Before we do so, it is interesting to note some properties of $\langle n_k \rangle$ itself.

Since $\Sigma_k k n_k = A$, $k\langle n_k \rangle / A$ is properly a probability distribution. In this case, the distribution has appeared before as a distribution for a replacement urn model introduced by Pólya [39]. This correspondence was noted by us before in the special case of $\gamma = 1$ [40]. A central feature in that case was the appearance of a power law in the fragment distribution $\langle n_k \rangle = 1/k$ for $x = 1$. Such a power law applies in this more general case, since in the large A limit, we can approximate $\langle n_k \rangle$ by

$$\langle n_k \rangle \approx \frac{\Gamma(x+\gamma)}{\Gamma(x)\Gamma(\gamma)} \frac{1}{k} \left(1-\frac{k}{A}\right)^{x-1} \left(\frac{k}{A}\right)^{\gamma-1}. \quad (15)$$

Setting $x=1$, $\langle n_k \rangle \approx \gamma A^{1-\gamma} k^{\gamma-2}$ which shows a power law falloff with exponent $\tau=2-\gamma$.

If we now allow $A\to\infty$, $k/A\to W$, the discrete model becomes a continuum one. In this case, $f(W)=\lim_{A\to\infty} A\langle n_{WA}\rangle$ is the continuum distribution of fragment sizes, with $\int_0^1 W f(W) dW = 1$ and $\lim_{A\to\infty}\langle Y_s\rangle/A^s = \int_0^1 W^s f(W) dW$. Using Eq. (15), we arrive at

$$f(W) = \frac{\Gamma(x+\gamma)}{\Gamma(x)\Gamma(\gamma)} (1-W)^{x-1} W^{\gamma-2}, \quad (16)$$

i.e., $Wf(W)$ is a β distribution. Derrida and Flyvbjerg [11,29,33], extending results on spin glasses due to Mézard *et al.* [41] and other disordered systems, arrived at these distributions before. All the cases they considered are in fact special cases of the above expression. Specifically, $\gamma=1$, $x=1/2$ reproduces the quenched random map, $\gamma=1$, $x>0$ reproduces the randomly broken interval and $\gamma=1-x$ reproduces the Sherrington-Kirkpatrick spin-glass model [41]. We have already seen why the first two models reduce to this model. The spin-glass result, on the other hand, defies a simple explanation. Distributions on the largest and second largest cluster also agree with the known spin-glass results, and so it appears that this model does indeed capture the clustering properties of the spin glass.

Returning now to the question of moments, we see that in this case we can arrive at closed form solutions. Ensemble averages for the multiplicity and its variance can be derived by applying Eqs. (13) and (14) and some combinatorial identities [Eqs. (B1) and (B2)] to arrive at

$$\langle m \rangle = \frac{x+\gamma-1}{\gamma-1} \left(1-\frac{(x)_A}{(x+\gamma-1)_A}\right), \quad (17)$$

$$\mathrm{var}(m) = \frac{1}{\gamma-1} \frac{(x)_A}{(x+\gamma)_{A-1}} \left(1-\frac{1}{\gamma-1}\frac{(x)_A}{(x+\gamma)_{A-1}} + \frac{1}{\gamma-1}\frac{(x-\gamma+1)_A}{(x+1)_{A-1}}\right). \quad (18)$$

Similarly, the mean and variance of Y can be determined:

$$\langle Y \rangle = \frac{\gamma}{x+\gamma}\left(1+\frac{1}{A}\frac{x}{\gamma}\right), \quad (19)$$

$$\mathrm{var}(Y) = \frac{2x\gamma(A-1)(x+\gamma+A)(x+\gamma+A-1)}{A^3(x+\gamma)^2(x+\gamma+1)(x+\gamma+2)}. \quad (20)$$

From this the relative fluctuations in Y, $f(Y)=\langle Y^2\rangle/\langle Y\rangle^2-1$ can be computed. They are important when determining whether a system is self-averaging [13]. Consider the case $\gamma=1$, where $f(Y)\approx 2x/[(x+2)(x+3)]$ in the large A limit. For $x\approx 1$, the fluctuations are large and the system is said to lack self-averaging properties. For $x\propto A$ the fluctuations are small and $f(Y)\sim 1/A$ as expected. This is true in general in this model, and for the spin-glass case $x+\gamma=1$ leads to the known result $\langle Y^2\rangle = 1/3\langle Y\rangle(1+2\langle Y\rangle)$ [41] in the infinite A limit.

The self-averaging property is not difficult to understand. At one extreme, the nucleus can be broken into individual nucleons so that $Y/A^2=1/A$; thus, Y varies inversely as the size of the system. At the other extreme, when the A nucleons are in one cluster $Y/A^2=1$. This behavior is analogous to the two extreme limits for the participation ratio in Anderson localization. For a localized state $Y/A^2\sim 1/A$ and for an extended state $Y/A^2\sim 1$. A localized state corresponds to the localization of the electron wave functions around many individual sites, while an extended state is the opposite case. A percolation description of localized versus extended states can be found in Ziman [12]. Specifically, for site percolation an extended state includes an incipient infinite cluster.

Applying these results, we plot the various expectation values in Fig. 4. From these graphs one can comment on the possibility of criticality in this model. It is well known that the $x>0$, $\gamma=1$ case is not critical in the traditional sense. The zeros of the partition function occur on the negative real x axis, well isolated from the relevant parameter domain. For the $x+\gamma=1$, $0<x<1$ model, the reduced moments appear to peak near $x=1/2$, which suggests that $x=1/2$ may be a critical point for the model. However, var(m) peaks closer to $x=0.9$. At a true critical point one would expect this fluctuation to be maximal at the critical point. Since for any model the various expectation values must peak somewhere, the existence of a maximum is insufficient evidence for criticality. Further evidence is inferred from the fact that the partition function $(x+\gamma)_{A-1}/A!$ has zeros on the negative real $x+\gamma$ axis in a complexified parameter space. These zeroes are isolated from $x+\gamma=1$ real space, and suggests that a traditional critical point in which the zeros of the partition function encroach the thermodynamically relevant parameter domain is completely absent in this model.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have shown that the random breaking of an interval process when suitably generalized encompasses a wide range of physical models both in nuclear fragmentation and in disordered systems. The fact that all these models are implicit to a particular process strongly recommends that the analysis of their properties should be the same. Traditionally, however, this has not been the case. The models have arisen separately, motivated by particular features which theorists wished to capture. In nuclear fragmentation, the feature to reproduce is a specific criticality marked by the appearance of an ''infinite'' cluster. Interest has therefore focussed on reduced moments whose behavior changes with the appearance of such a cluster. In disordered systems the interest instead has been on whether such systems lack self-averaging properties, i.e., have significant non-Gaussian fluctuations. In this case, the focus is on probability distributions of the measures which might express such non-Gaussian fluctuations, e.g., the largest cluster and the second moment.

Given then this separate emphasis, it is understood why

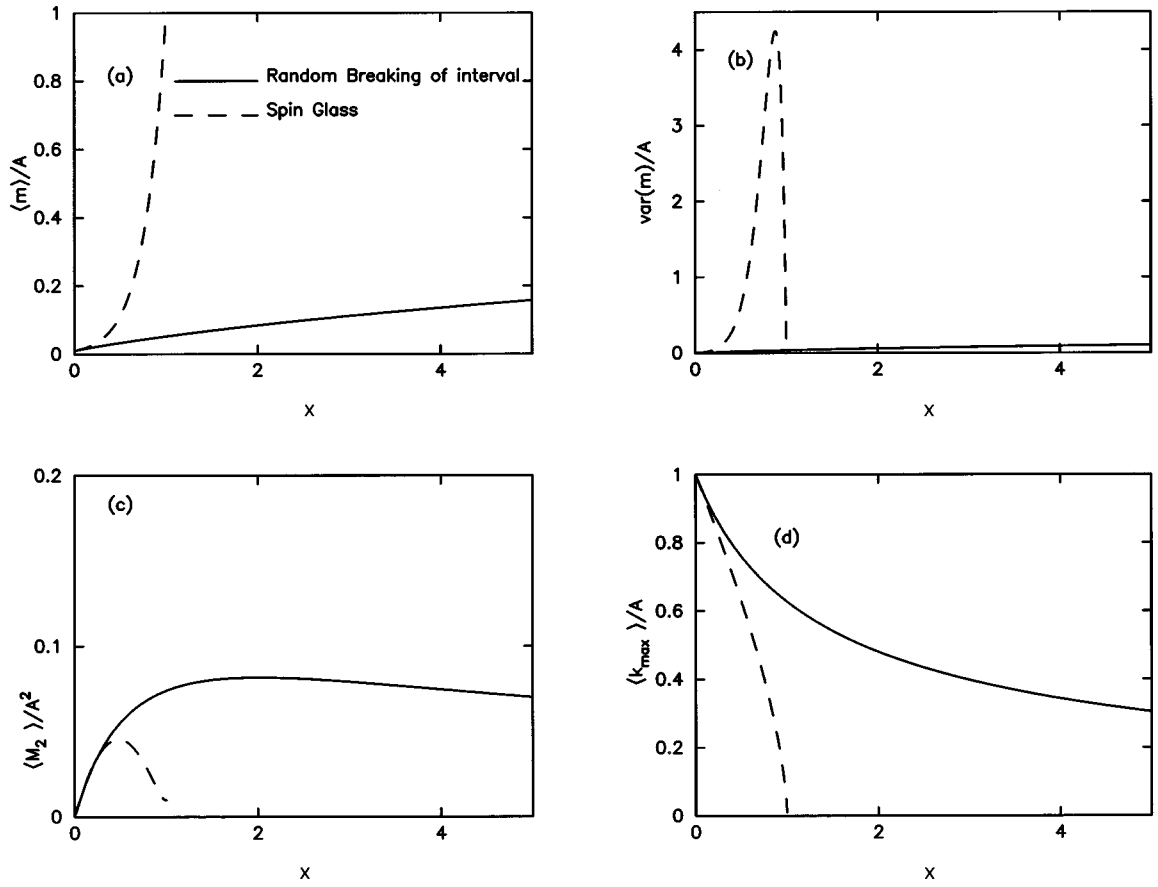

FIG. 4. The expected value (a) and variance (b) of the multiplicity and the expected values of the reduced second moment (c) and the largest cluster (d) for the random broken interval (solid line) and spin-glass (dashed line) models. Note that the spin glass is only defined when $0<x<1$.

the same variables in the models are studied in different ways. The corresponding analysis taken here reinforces the distinct approaches, as it shows that the general behavior of the two classes of models differ considerably. Nevertheless, the convergence of their description within the random breaking of an interval mechanism makes such a distinction somewhat puzzling. This mechanism is unlikely to be physical in the nuclear case, but this is not an explanation for the different behavior in the two cases. Indeed, the explanation may have a great deal to do with criticality or its absence. Models with exponent $\tau<2$ tend to look disordered, $\tau>2$ tend to have distinct critical points and Gaussian distributions. Since $\tau=2+1/\delta$, the two regions are characterized by positive and negative δ. It would be interesting to confirm this observation as a general feature.

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APPENDIX A: A COMBINATORIAL IDENTITY

Recall that a permutation on n elements can be described by the cycles containing the elements which permute among themselves under the iterated action of the permutation. For example, the permutation $p=\binom{12345678}{31254687}$ is usually written in cycle notation as (132)(45)(6)(78). If each cycle of length k is associated with a cluster of the size k, and these clusters are ordered by the largest element in each cluster, then each permutation can be mapped to a sequential partitioning. The above example would correspond to the sequential partition (2,1,2,3).

Suppose one chooses a permutation $p\in S_n$ uniformly at random, i.e., with probability $1/n!$. What is the probability that the first cluster in the sequence has size $\lambda_1=k$? There are $\binom{n-1}{k-1}$ ways of choosing the $k-1$ elements which are not element n but are in n's cycle map. There are $(k-1)!$ different cycles which permute the k objects and $(n-k)!$ ways of permuting the remaining elements which are not in this cycle. So the probability is $(k-1)!(n-k)!\binom{n-1}{k-1}/n!=1/n$. What about the second cluster? The $(n-k)!$ ways of permuting the elements not in the first cluster is a permutation $p'\in S_{n-\lambda_1}$ chosen uniformly at random. In this permutation, λ_2 is the first element, and by the above argument has size $\lambda_2=k$ with probability $1/(n-\lambda_1)$. Iterating yields the probability of the whole sequence as being $\Pr(\lambda_1,\ldots,\lambda_m)=1/[n(n-\lambda_1)(n-\lambda_1-\lambda_2)\cdots]$. Summing this over all possible orders one obtains $1/n!$ times the number of permutations with this particular cycle class structure. It is well known that this is given by Cauchy's formula [42] $n!/\Pi_k n_k! k^{n_k}$, so that

$$\sum_{\lambda \mapsto \mathbf{n}} \frac{1}{n(n-\lambda_1)(n-\lambda_1-\lambda_2)\cdots} = \prod_k \frac{1}{n_k!}\left(\frac{1}{k}\right)^{n_k}. \tag{A1}$$

APPENDIX B: SUM RULES

For systems satisfying Eqs. (13) and (14) one has the following sum rules:

$$S_p = \sum_{k=p}^{A} \frac{[k]_p}{[A]_p}\langle n_k \rangle = \frac{(\gamma)_{p-1}}{(x+\gamma)_{p-1}}, \tag{B1}$$

$$S_{pq} = \sum_{jk} \frac{[j]_p[k]_q}{[A]_{p+q}}\langle n_j(n_k-\delta_{jk})\rangle = x\frac{(\gamma)_{p-1}(\gamma)_{q-1}}{(x+\gamma)_{p+q-1}}, \tag{B2}$$

where $p,q>0$, $[k]_0=1$, $[k]_p=k(k-1)\cdots(k-p+1)$. Notice that the right-hand sides are independent of A. These rules are helpful in determining expectation values of m,Y, etc., and can be derived using a common combinatorial identity known as Norlünd's formula [43].

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