Shell-model calculations for the three-nucleon system

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(Received 21 May 1997)

We use Faddeev's decomposition to solve the shell-model problem for three nucleons. The dependence on harmonic-oscillator excitations allowed in the model space, up to $32\hbar\Omega$ in the present calculations, and on the harmonic-oscillator frequency is studied. Effective interactions derived from Nijmegen II and Reid93 potentials are used in the calculations. The binding energies obtained are close to those calculated by other methods. The structure of the Faddeev equations is discussed and a simple formula for matrix elements of the permutation operators in a harmonic-oscillator basis is given. The Pauli principle is properly treated in the calculations. [S0556-2813(98)02102-5]

PACS number(s): 21.45.+v, 21.60.Cs, 27.10.+h

I. INTRODUCTION

Many different methods have been used to solve the three-nucleon problem in the past. The most viable approach appears to be the Faddeev method [1]. It has been successfully applied to solve the three-nucleon bound-state problem for various nucleon-nucleon potentials [2–5]. The most complex calculations of this kind include up to 34 channels, when all the $j \leq 4$ waves are taken into account. The precision achieved in these calculations is better than 1% [4,5].

On the other hand, when studying the properties of more complex nuclei one typically resorts to the shell model. In that approach, the harmonic-oscillator basis is used in a truncated model space. Instead of the free nucleon-nucleon potential, one uses effective interactions inside the model space. Examples of such calculations are the large-basis no-core shell-model calculations that have recently been performed [6–8]. In these calculations all nucleons are active, which simplifies the effective interaction as no hole states are present. The effective interaction is determined for a system of two nucleons in a harmonic-oscillator well interacting by the nucleon-nucleon potential and is subsequently used in the many-particle calculations.

In the present paper we combine the application of the shell-model approach to the three-nucleon system with the Faddeev method. We use Faddeev's decomposition for the basis states and remove the center-of-mass term. This leads to a significant simplification of the problem and allows us to extend considerably the model space. We can study the convergence properties of the results with the increasing model space. If convergence is achieved, then this approach leads to the exact solution of the three-nucleon problem and is, thus, complementary to the traditional calculations, based on the differential equation solutions.

In addition to the attempt of solving the three-nucleon problem exactly, the present method serves primarily as a test of the shell-model approach. In particular, it allows us to test effective interactions used in standard shell-model appli-

cations. The present approach has several advantages. First, any number of partial waves can be included. Second, the calculation is simplified by using a compact formula for the matrix elements of the permutation operators in the harmonic-oscillator basis. Also, because of the way we do the model-space truncation, we keep equivalence of the Faddeev and Schrödinger equations throughout the calculation.

In Sec. II we discuss the shell-model Hamiltonian with a bound center-of-mass, the Faddeev equation, and the methods used to derive the starting-energy-independent effective interaction. Results of the calculations for the three-nucleon system are presented in Sec. III. In particular, we discuss the harmonic-oscillator frequency and the model-space-size dependences. Conclusions are given in Sec. IV.

II. SHELL-MODEL HAMILTONIAN AND THE FADDEEV APPROACH

In most shell-model studies the one- plus two-body Hamiltonian for the A-nucleon system, i.e.,

$$H = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} + \sum_{i<j}^{A} V_N(\vec{r}_i - \vec{r}_j), \quad (1)$$

where m is the nucleon mass and $V_N(\vec{r}_i - \vec{r}_j)$ the nucleon-nucleon interaction, is modified by adding the center-of-mass harmonic-oscillator potential $\frac{1}{2}Am\Omega^2\vec{R}^2$, $\vec{R} = 1/A \sum_{i=1}^{A}\vec{r}_i$. This potential does not influence intrinsic properties of the many-body system. It provides, however, a mean field felt by each nucleon and allows us to work with a convenient harmonic-oscillator basis. The modified Hamiltonian, depending on the harmonic-oscillator frequency Ω, can be cast into the form

$$H^\Omega = \sum_{i=1}^{A} \left[\frac{\vec{p}_i^2}{2m} + \frac{1}{2}m\Omega^2\vec{r}_i^2 \right]$$
$$+ \sum_{i<j}^{A} \left[V_N(\vec{r}_i - \vec{r}_j) - \frac{m\Omega^2}{2A}(\vec{r}_i - \vec{r}_j)^2 \right]. \quad (2)$$

The one-body term of the Hamiltonian (2) can be rewritten as a sum of the center-of-mass term $H_{\text{c.m.}}^\Omega = \vec{P}_{\text{c.m.}}^2/2Am$

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$+\frac{1}{2}Am\Omega^2\vec{R}^2$, $\vec{P}_{\mathrm{c.m.}}=\Sigma_{i=1}^{A}\vec{p}_i$, and a term depending on relative coordinates only. In the present application we use a basis, which explicitly separates center-of-mass and relative-coordinate wave functions. Therefore, the center-of-mass term contribution is trivial and will be omitted from now on.

For a three-nucleon system, i.e., $A=3$, the transformation of the coordinates

$$\vec{r}=\sqrt{\frac{1}{2}}(\vec{r}_1-\vec{r}_2), \tag{3a}$$

$$\vec{y}=\sqrt{\frac{2}{3}}[\tfrac{1}{2}(\vec{r}_1+\vec{r}_2)-\vec{r}_3], \tag{3b}$$

and, similarly, of the momenta

$$\vec{p}=\sqrt{\frac{1}{2}}(\vec{p}_1-\vec{p}_2), \tag{4a}$$

$$\vec{q}=\sqrt{\frac{1}{6}}(\vec{p}_1+\vec{p}_2)-\sqrt{\frac{2}{3}}\vec{p}_3, \tag{4b}$$

can be introduced that brings the relative-coordinate part of the one-body harmonic-oscillator Hamiltonian into the form

$$H_0=\frac{\vec{p}^2}{2m}+\frac{1}{2}m\Omega^2\vec{r}^2+\frac{\vec{q}^2}{2m}+\frac{1}{2}m\Omega^2\vec{y}^2. \tag{5}$$

Eigenstates of this Hamiltonian,

$$|nlsjt,\mathcal{NLSJ}\tau,JT\rangle, \tag{6}$$

are then used as the basis for our calculation. Here n,l and $\mathcal{N},\mathcal{L}$ are the harmonic-oscillator quantum numbers corresponding to the harmonic oscillators associated with the coordinates and momenta $\vec{r},\vec{p}$ and $\vec{y},\vec{q}$, respectively. The quantum numbers s,t,j describe the spin, isospin, and angular momentum of the relative-coordinate partial channel of particles 1 and 2. $\mathcal{S}=\frac{1}{2}$ and $\tau=\frac{1}{2}$ are the spin and isospin of the third particle, while $\mathcal{J}$ is the angular momentum of the third particle relative to the center-of-mass of particles 1 and 2. The J and T are the total angular momentum and the total isospin, respectively. Note that for the ^{3}H nucleus $J=\frac{1}{2}$ and $T=\frac{1}{2}$.

The Faddeev equation for the bound system can be written in the form

$$\widetilde{H}|\phi\rangle=E|\phi\rangle, \tag{7}$$

with

$$\widetilde{H}=H_0+V(\vec{r})\mathcal{T}. \tag{8}$$

Here, $V(\vec{r})=V_{\mathrm{N}}(\sqrt{2}\vec{r})-\frac{1}{3}m\Omega^2\vec{r}^2$ is the potential and $\mathcal{T}$, which has the properties of a metric operator [3,9], is given by

$$\mathcal{T}=1+\mathcal{T}^{(-)}+\mathcal{T}^{(+)}, \tag{9}$$

with $\mathcal{T}^{(+)}$ and $\mathcal{T}^{(-)}$ the cyclic and the anticyclic permutation operators, respectively. We derived a simple formula for the matrix elements of $\mathcal{T}^{(-)}+\mathcal{T}^{(+)}$ in the basis (6), namely,

$$\langle n_1l_1s_1j_1t_1,\mathcal{N}_1\mathcal{L}_1\mathcal{S}\mathcal{J}_1\tau,JT|\mathcal{T}^{(-)}$$

$$+\mathcal{T}^{(+)}|n_2l_2s_2j_2t_2,\mathcal{N}_2\mathcal{L}_2\mathcal{S}\mathcal{J}_2\tau,JT\rangle$$

$$=-\delta_{N_1,N_2}\sum_{LS}\hat{L}^2\hat{S}^2\hat{j}_1\hat{j}_2\hat{\mathcal{J}}_1\hat{\mathcal{J}}_2\hat{s}_1\hat{s}_2\hat{t}_1\hat{t}_2$$

$$\times(-1)^L\left\{\begin{array}{ccc} l_1 & s_1 & j_1 \\ \mathcal{L}_1 & \frac{1}{2} & \mathcal{J}_1 \\ L & S & J \end{array}\right\}\left\{\begin{array}{ccc} l_2 & s_2 & j_2 \\ \mathcal{L}_2 & \frac{1}{2} & \mathcal{J}_2 \\ L & S & J \end{array}\right\}$$

$$\times\left\{\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & s_1 \\ \frac{1}{2} & S & s_2 \end{array}\right\}\left\{\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & t_1 \\ \frac{1}{2} & T & t_2 \end{array}\right\}$$

$$\times[(-1)^{s_1+s_2+t_1+t_2-\mathcal{L}_1-l_1}$$

$$\times\langle\mathcal{N}_1\mathcal{L}_1n_1l_1L|n_2l_2\mathcal{N}_2\mathcal{L}_2L\rangle_{d=3}$$

$$+\langle n_1l_1\mathcal{N}_1\mathcal{L}_1L|\mathcal{N}_2\mathcal{L}_2n_2l_2L\rangle_{d=3}], \tag{10}$$

where $N_i=2n_i+l_i+2\mathcal{N}_i+\mathcal{L}_i$, $i\equiv1,2$, $\hat{j}=\sqrt{2j+1}$, and $\langle\mathcal{N}_1\mathcal{L}_1n_1l_1L|n_2l_2\mathcal{N}_2\mathcal{L}_2L\rangle_{d=3}$ is the general harmonic-oscillator bracket for two particles with mass ratio 3 as defined, e.g., in Ref. [10], where a compact formula is also given for calculating the brackets. The expression (10) can be derived by examining the action of $\mathcal{T}^{(+)}$ and $\mathcal{T}^{(-)}$ on the basis states (6). A similar derivation for a different basis is described, e.g., in Ref. [11].

Note that the eigensystem of the metric $\mathcal{T}$ (9) consists of two subspaces. The first subspace has eigenstates with the eigenvalue 3, which form totally antisymmetric physical states, while the second subspace has eigenstates with the eigenvalue 0, which form a not completely antisymmetric, unphysical subspace of states. It is possible to Hermitize the Hamiltonian (8) on the physical subspace, where it is quasi-Hermitian (see the discussion of quasi-Hermitian operators, e.g., in Ref. [9]). The Hermitized Hamiltonian takes the form

$$\bar{H}=H_0+\bar{\mathcal{T}}^{1/2}V(\vec{r})\bar{\mathcal{T}}^{1/2}, \tag{11}$$

where $\bar{\mathcal{T}}$ operates on the physical subspace only.

Apparently, the interaction $V(\vec{r})$ is diagonal in the quantum numbers $\mathcal{N},\mathcal{L},\mathcal{J}$ (and also in s,j,t due to the properties of the nucleon-nucleon potential). The metric $\mathcal{T}$ (9) is, on the other hand, diagonal in $N=2n+l+2\mathcal{N}+\mathcal{L}$. Note that any basis truncation other than one of the type $N\leq N_{\max}$ violates, in general, the Pauli principle and mixes physical and unphysical states. Here, $N_{\max}$ characterizes the maximum of total allowed harmonic-oscillator quanta in the model space and is an input parameter of the calculation. At the same time, the truncation into totally allowed oscillator quanta $N\leq N_{\max}$ preserves the equivalence of the Hamiltonians (8) and (11) on the physical subspace.

From solving two-nucleon systems in a harmonic-oscillator well, interacting by soft-core potentials, one learns that excitations up to about $300\hbar\Omega$ ($N_{\max}=300$) are required to get almost exact solutions. We anticipate, therefore, that at least the same number of excitations should be allowed to solve the three-nucleon system using the formalism discussed above. The Faddeev formulation has the obvious advantage compared with the traditional shell-model approach that the center-of-mass coordinate is explicitly removed. Even then, it is presently not feasible to solve the eigenvalue problem either for Eq. (8) or for Eq. (11) in such a large space. On the other hand, shell-model calculations are always performed by employing effective interactions tailored to a specific model space. In practice, these effective interactions can never be calculated exactly as, in general, for an A-nucleon system the A-body effective interaction is required. Consequently, large model spaces are desirable. In that case, the calculation should be less affected by any imprecision of the effective interaction. The same is true for the evaluation of any observable characterized by an operator. In the model space, renormalized effective operators are required. The larger the model space, the less renormalization is needed. We may take advantage of the present approach to perform shell-model calculations in significantly larger model spaces than are possible in conventional shell-model approach, particularly when using a Hermitized Hamiltonian (11). At the same time we can investigate convergence properties of effective interactions. If convergence is achieved, we should obtain the exact solution, since we recover the original full-space problem as $N_{\max}\to\infty$, provided that the condition $V_{\text{eff}}\to V$ is satisfied in this limit.

Usually, the effective interaction is approximated by a two-body effective interaction determined from a two-nucleon system. In the present calculations we replace matrix elements of the potential $V(\vec{r})$ by matrix elements of an effective two-body interaction, derived in a straightforward manner for each relative-coordinate partial channel. The relevant two-nucleon Hamiltonian is then

$$H_2\equiv H_{02}+V=\frac{\vec{p}^2}{2m}+\frac{1}{2}m\Omega^2\vec{r}^2+V_{\text{N}}(\sqrt{2}\vec{r})-\frac{m\Omega^2}{3}\vec{r}^2,$$

(12)

which can be solved as a differential equation or, alternatively, can be diagonalized in a sufficiently large harmonic oscillator basis. The latter possibility is, obviously, not applicable for hard-core nucleon-nucleon potentials.

To construct the effective interaction we employ the Lee-Suzuki [12] similarity transformation method, which gives the effective interaction in the form $PV_{\text{eff}}P=PVP+PVQ\omega P$, with ω the transformation operator satisfying $\omega=Q\omega P$, and P, $Q=1-P$, the projectors on the model and the complementary spaces, respectively. Our calculations start with exact solutions of the Hamiltonian (12) and, consequently, we construct the operator ω and, then, the effective interaction directly from these solutions. Let us denote the relative-coordinate two-nucleon harmonic-oscillator states, which form the model space, as $|\alpha_P\rangle$, and those which belong to the Q space, as $|\alpha_Q\rangle$. Then the Q-space components of the eigenvector $|k\rangle$ of the Hamiltonian (12) can be expressed as a combination of the P-space components with the help of the operator ω:

$$\langle\alpha_Q|k\rangle=\sum_{\alpha_P}\langle\alpha_Q|\omega|\alpha_P\rangle\langle\alpha_P|k\rangle.$$

(13)

If the dimension of the model space is d_P, we may choose a set $\mathcal{K}$ of d_P eigenvectors, for which the relation (13) will be satisfied. Under the condition that the $d_P\times d_P$ matrix $\langle\alpha_P|k\rangle$ for $|k\rangle\in\mathcal{K}$ is invertible, the operator ω can be determined from Eq. (13). In the present application we select the lowest states obtained in each channel. Their number is given by the number of basis states satisfying $2n+l\leq N_{\max}$. Once the operator ω is determined the effective Hamiltonian can be constructed as follows:

$$\langle\gamma_P|H_{2\text{eff}}|\alpha_P\rangle=\sum_{k\in\mathcal{K}}\left[\langle\gamma_P|k\rangle E_k\langle k|\alpha_P\rangle\right.$$
$$\left.+\sum_{\alpha_Q}\langle\gamma_P|k\rangle E_k\langle k|\alpha_Q\rangle\langle\alpha_Q|\omega|\alpha_P\rangle\right].$$

(14)

It should be noted that $P|k\rangle=\sum_{\alpha_P}|\alpha_P\rangle\langle\alpha_P|k\rangle$ for $|k\rangle\in\mathcal{K}$ is a right eigenvector of Eq. (14) with the eigenvalue E_k.

This Hamiltonian, when diagonalized in a model-space basis, reproduces exactly the set $\mathcal{K}$ of d_P eigenvalues E_k. Note that the effective Hamiltonian is, in general, quasi-Hermitian. It can be Hermitized by a similarity transformation determined from the metric operator $P(1+\omega^\dagger\omega)P$. The Hermitian Hamiltonian is then given by [13]

$$\bar{H}_{2\text{eff}}=[P(1+\omega^\dagger\omega)P]^{1/2}H_{2\text{eff}}[P(1+\omega^\dagger\omega)P]^{-1/2}.$$

(15)

Finally, the two-body effective interaction used in the present calculations is determined from the two-nucleon effective Hamiltonian (15) as $V_{\text{eff}}=\bar{H}_{2\text{eff}}-H_{02}$.

III. APPLICATION TO THE THREE-NUCLEON PROBLEM

In this section we discuss the results of application of the formalism outlined in Sec. II for the ^{3}H system. In the calculations we use the Nijmegen II nucleon-nucleon potential [14] corrected in the $^1P_1$ wave [15], and the Reid93 nucleon-nucleon potential [14]. We work in the isospin formalism; the charge invariant potential $V_{\text{N}}=\frac{2}{3}V_{nn}+\frac{1}{3}V_{np}$ is used for each $T=1$ wave [16].

The two-body effective interaction employed in the calculation is derived from Eqs. (13)–(15). Our model space is characterized by the condition $N\leq N_{\max}$, $N=2n+l+2\mathcal{N}+\mathcal{L}$. The condition for the relative-coordinate effective-interaction model space is then $2n+l\leq N_{\max}$. When diagonalizing the two-nucleon relative-coordinate Hamiltonian (12) in the full space we truncate the harmonic-oscillator basis by keeping only the states with $n\leq152$. The error caused by this truncation can be estimated, as the system can be solved as a differential equation. We found that the low-lying eigenvalues obtained in the two calculations do not differ by more than $\approx10^{-3}$ MeV and in most cases by much
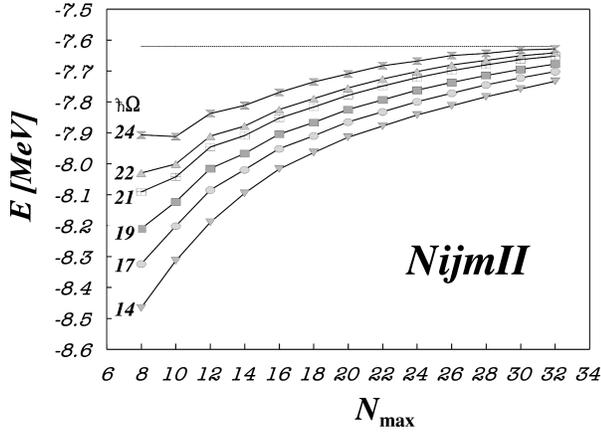
FIG. 1. Ground-state energy, in MeV, dependence on the maximal number of harmonic-oscillator excitations allowed in the model space for the Nijmegen II potential. Results for $\hbar\Omega$ = 14,17,19,21,22,24 MeV are presented. The dotted line represents the result −7.62 MeV of the 34-channel Faddeev calculation reported in Ref. [5].

less. The lowest eigenvalues are typically of the order of 10^1 MeV. Note that this error decreases with increasing Ω. We calculated the effective interactions up to $N_{max}=32$, as required in the present application.

Once the effective interaction is found we may directly diagonalize the non-Hermitian Hamiltonian (8) in the basis (6) truncated by N_{max} and with V replaced by V_{eff}.

On the other hand, a calculation with the Hermitized Hamiltonian (11) can be performed in three steps. First the effective interaction is calculated for each relative-coordinate partial channel. Second the metric $\mathcal{T}$ (9) is diagonalized for each N up to N_{max}. The physical eigenvectors corresponding to the eigenvalue 3 are selected and used, finally, as a new basis in which the Hamiltonian (11) is diagonalized. As the number of physical states is about a third of the number of all original basis states (6), it is more efficient to diagonalize the Hamiltonian (11) than the non-Hermitian Hamiltonian (8), in particular for higher values of N_{max}. In fact, for
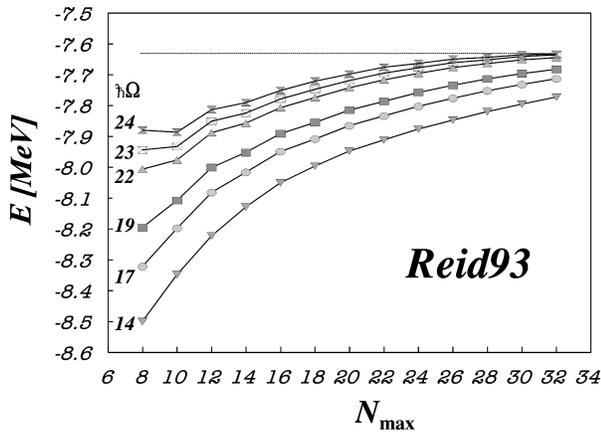
FIG. 2. Ground-state energy, in MeV, dependence on the maximal number of harmonic-oscillator excitations allowed in the model space for the Reid93 potential. Results for $\hbar\Omega$ = 14,17,19,22,23,24 MeV are presented. The dotted line represents the result −7.63 MeV of the 34-channel Faddeev calculation reported in Ref. [5].
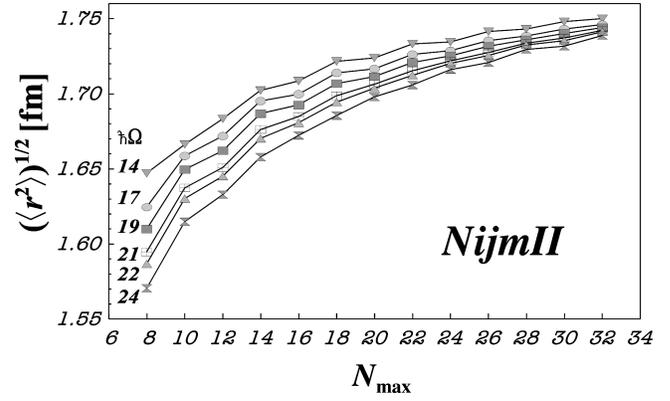
FIG. 3. Point-nucleon radius, in fm, dependence on the maximal number of harmonic-oscillator excitations allowed in the model space for the Nijmegen II potential. Results for $\hbar\Omega$ = 14,17,19,21,22,24 MeV are presented.

$N_{max}>22$ we used the Hermitian Hamiltonian only. Note that the Hamiltonians (8) and (11) have identical spectra of the physical states, provided that no other truncation than $N\leqslant N_{max}$ is allowed. The unphysical eigenstates of $\bar{H}$ (8) have energies corresponding to the unperturbed harmonic oscillator, starting at $3\hbar\Omega$.

In Figs. 1–4 we present the results for the ground-state energies and point-nucleon radii, calculated from $\langle r^2\rangle$ = $(1/A)\Sigma_{i=1}^A\langle(\vec{r}_i-\vec{R})^2\rangle$, obtained with the Nijmegen II and the Reid93 nucleon-nucleon potentials, respectively. Our calculation starts at $N_{max}=8$, which corresponds to a model space easily accessible with the traditional shell-model calculations. In Ref. [8] we performed an $8\hbar\Omega$ calculation for ^{3}H using a slightly different effective interaction than we employ here but derived in an analogous way. Note that it is straightforward to transform the relative-coordinate effective interaction used in the present calculations to the two-particle basis used for the shell-model input by the standard transformation [17]. We used the transformed interaction in the $8\hbar\Omega$ space to test our results. The shell-model diagonalization was performed by using the many-fermion-dynamics shell-model code [18] and we obtain the same answers from both the present calculation and the shell-model calculation.
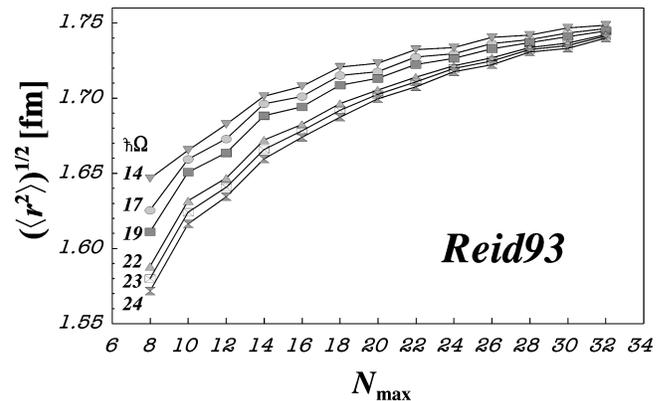
FIG. 4. Point-nucleon radius, in fm, dependence on the maximal number of harmonic-oscillator excitations allowed in the model space for the Reid93 potential. Results for $\hbar\Omega$ = 14,17,19,22,23,24 MeV are presented.
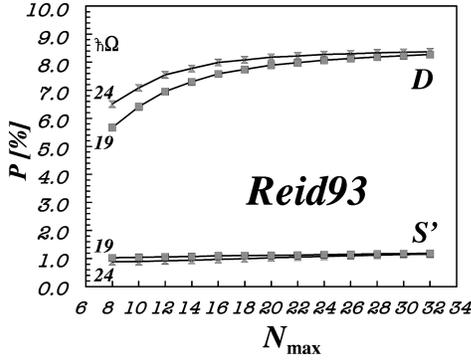
FIG. 5. D-state and S'-state probability, in %, dependence on the maximal number of harmonic-oscillator excitations allowed in the model space for the Reid93 potential. Results for $\hbar\Omega = 19,24$ MeV are presented.

FIG. 6. Dependence of the energy difference, in MeV, between the binding energies of ^{3}H and ^{3}He on the maximal number of harmonic-oscillator excitations allowed in the model space for the Reid93 potential. Results for $\hbar\Omega = 19,24$ MeV are presented.

The present calculation has, obviously, much smaller dimension.

As the results depend on $N_{\max}$ and Ω introduced in Eq. (2), we must test the convergence with regard to both of these parameters. With increasing $N_{\max}$ the calculations grow tedious. We performed the calculations up to $N_{\max} = 32$ for a wide range of the harmonic-oscillator frequencies Ω with values typical for standard shell-model calculations varying from $\hbar\Omega = 14$ MeV to $\hbar\Omega = 24$ MeV. In general, we observe a slow convergence with increasing $N_{\max}$. An unusual feature is the convergence from below. This is caused by the use of effective interactions instead of the free nucleon-nucleon interaction. The effective interactions we employ are too strong. We have not reached the convergence with respect to Ω in the whole range studied. However, for the values $\hbar\Omega = 22-24$ MeV our results almost reach convergence with $N_{\max} = 32$, in particular for the Reid93 potential. We note that the traditional 34-channel Faddeev calculation, as reported in Ref. [5], gives the binding energies 7.62 and 7.63 MeV for the Nijmegen II and the Reid93 nucleon-nucleon potentials, respectively. We present these values in Figs. 1 and 2 as dotted lines for comparison. From the figures it is apparent that we are obtaining virtually the same values in the calculations which start to converge. When comparing the results for the two different potentials used we can see larger sensitivity to Ω of the Reid93 calculation. On the other hand, the calculation using the Nijmegen II potential is slower in reaching the stability with respect to $N_{\max}$. While the ground-state energy calculation begins to stabilize for the largest values of Ω employed, the radius calculation has not reached complete stability for any of the Ω values within the model spaces we used.

As a further test on the stability and convergence of the method we analyzed the ground-state wave functions and calculated the probability of S,P,D,S',S'' states. In Fig. 5 we show the D-state and S'-state probabilities as a function of $N_{\max}$ for the Reid93 calculations with $\hbar\Omega = 19$ and 24 MeV. We observe a good stability with respect to $N_{\max}$ and little dependence on Ω for larger model spaces. The D-state probability approaches 8.4%, S'-state probability percentage reaches 1.2%. Not shown in the figure are the calculated P-state and S''-state percentage probabilities, for which we get 0.06 and $\approx 10^{-5}$ %, respectively. The present numbers are in agreement with those obtained using other nucleon-

nucleon potentials [4]. The D-state percentage is approximately 1.5 times the corresponding D-state percentage of deuteron (5.7%).
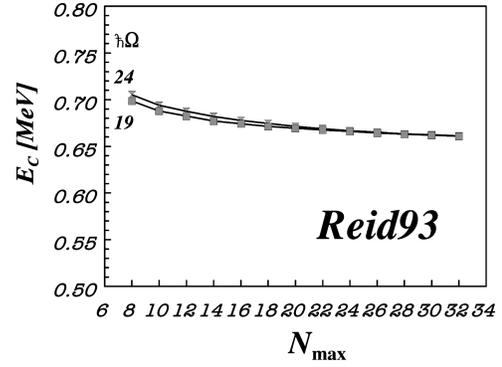
In addition to the calculations discussed so far, we also computed properties of ^{3}He, with the focus on obtaining the binding-energy difference between ^{3}H and ^{3}He. In those calculations the Coulomb potential was added to the proton-proton potential and the averaged potential $V_N = \frac{2}{3}V_{pp} + \frac{1}{3}V_{np}$ was eventually used for each $T=1$ wave [16]. The binding-energy differences obtained using the Reid93 potential and $\hbar\Omega = 19$ and 24 MeV are presented in Fig. 6. For larger model spaces we get an almost Ω-independent difference. The binding-energy splitting shows convergence with increasing $N_{\max}$. It decreases with $N_{\max}$ in correlation with increasing point-nucleon radius and approaches 0.66 MeV. This result is again in agreement with those obtained using other nucleon-nucleon potentials [4]. Note that the experimental value of the binding-energy difference is 0.764 MeV. To test the quality of the approximation used for the potential averaging and limitation to $T=\frac{1}{2}$, we performed an $N_{\max} = 8$ calculation with complete isospin breaking using proton-neutron formalism. The present method can be used to perform calculations with isospin breaking. For this particular calculation, however, we employed the many-fermion-dynamics shell-model code. The effective interaction was calculated separately for the proton-proton and proton-neutron systems, respectively, and transformed to the two-particle basis as discussed earlier in this section. In this way we found that the binding energy obtained with and without isospin breaking differs by 11 keV and the nucleon radius differs by less than 0.001 fm in a calculation with $\hbar\Omega = 19$ MeV. This confirms, that the limitation to $T=\frac{1}{2}$ together with the potential averaging provides an excellent approximation.

We stressed before that no other basis truncation than $N \leqslant N_{\max}$ was used. That means that we keep all the relative-coordinate channels in the basis. In most calculations, however, we set the nucleon-nucleon potential V_N to zero for $j > 6$. We also performed calculations with V_N set to zero for $j > 4$. The largest contribution of the $j = 5,6$ waves to the binding energy we observed was about 5 keV. This contribution increases with $N_{\max}$ and Ω. Moreover, we performed several calculations with V_N nonzero up to $j = 9$, and found that the ground-state energy is affected by less than 0.3 keV

compared to the $j \leqslant 6$ calculations. Note that such a trunca-tion of V_N does not imply that $V = 0$, see Eq. (8). Also, this type of potential truncation is not the same as used in the traditional Faddeev calculations [2–5]. The difference is, that there is no truncation in the treatment of $\mathcal{T}$ (9) in the present calculations.

It should be noted that the calculated binding energy ob-tained from calculations employing nucleon-nucleon poten-tials fitted to the two-nucleon scattering underbind ^{3}H by about 0.8 MeV, as its experimental binding energy is 8.48 MeV. Suggested solutions to this problem include the use of three-body forces, nonlocal potentials, and relativistic cor-rections [19].

IV. CONCLUSIONS

In the present paper we have discussed the three-nucleon bound system solution by combining the shell-model ap-proach with the Faddeev method. The use of Faddeev's de-composition reduces the basis and allows to perform shell-model calculations in significantly larger model spaces than in the traditional shell-model approach. We were able to cal-culate with the model spaces which included up to $32\hbar\Omega$ ($N_{\max} = 32$) harmonic-oscillator excitations.

We employed effective interactions, which take into ac-count the two-body correlations, in the calculations. These effective interactions were derived in the two-particle rela-tive coordinate channels from the Nijmegen II and Reid93 nucleon-nucleon potentials and subsequently used in the three-body calculation.

As our results depend on the model-space size parameter $N_{\max}$ and on the harmonic-oscillator frequency Ω, we tested the convergence in both these parameters. Even for the larg-est model spaces we have not reached complete convergence with respect to Ω in the whole range of the used values. However, for $\hbar\Omega = 22 - 24$ MeV our results start to converge to the binding energies obtained in the standard Faddeev calculations. As we include more partial waves in the nucleon-nucleon potential, typically up to $j = 6$, our results seem to confirm the statements in Ref. [4] that the 34-channel standard Faddeev calculation converged within 0.01

MeV. We have seen, in fact, that the higher partial waves of the nucleon-nucleon interaction are not significant. However, we believe a proper treatment of the metric operator (9) is important.

We observed that the convergence was rather slow with some dependence on the type of the nucleon-nucleon poten-tial. In smaller model spaces we cannot reproduce correctly the ground-state energy and the radius at the same time for any choice of Ω. The wave function probability distribution shows good stability as well as the binding-energy difference between ^{3}H and ^{3}He. Our results show, that an $8\hbar\Omega$ calcu-lation accessible by the standard shell-model approach de-scribes the ground states properties within 10% of exact val-ues. Modifications of effective interactions are possible to improve the description in small spaces. Examples of such modifications can be found in Refs. [7,8].

An unusual feature of the present approach is the conver-gence from below. This is caused by the use of effective interactions instead of the free nucleon-nucleon interactions. The effective interactions we employed were too strong. Ob-viously, it is possible to test convergence properties of alter-native effective interactions, as discussed, e.g., in Ref. [20]. It should be noted, however, that the effective interaction should meet the criterion $V_{\text{eff}} \rightarrow V$ for $N_{\max} \rightarrow \infty$ in order to converge to the exact solutions.

The formalism discussed here may be generalized for more complex systems as well. In particular, we are working on a generalization of the formalism for the $A = 4$ system, which relies on some results presented in this work. Also, it may be used to solve the three-nucleon system bound in a harmonic-oscillator well. Then, from those solutions three-body effective interactions can be constructed. Such interac-tions, after transformation to an appropriate three-particle ba-sis can serve as an input to standard shell-model calculations for light nuclei.

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