Efimov effect in the distorted cluster state representation

Aleksandr Estrin, Marijan Koštrun, and Yukap Hahn
Physics Department, University of Connecticut, Storrs, Connecticut 06269
(Received 30 April 1997)

The Efimov effect for a three-body system was studied previously in an adiabatic representation, where the distance between two ''heavy'' particles was represented by **x**, and the third ''light'' particle was described by coordinate **y** relative to the center of mass of the heavy ones. When **x** is held fixed, an exact solution for the light particle can be obtained if the interaction is assumed to be of separable form. The resulting adiabatic potential between the heavy particles shows the critical $-1/x^2$ behavior that leads to an infinite number of bound states. However, subsequently the leading nonadiabatic effect was shown to generate an undesirable correction of $1/x$ type, the pseudo-Coulomb disease (PCD). To remedy the PCD, the Efimov effect is reexamined in the adiabatic state representation, but with the new Jacobi coordinates (**r**,**R**), where **r** describes the distance between one of the heavy particles and the light one and **R** is the position of the other heavy particle with respect to the center of mass of the subsystem of light-heavy particles. It is then shown that the adiabatic potential in this distorted cluster state representation behaves as $-1/R^2$ at large **R** if the pair described by **r** has a zero-energy bound state. Moreover, in this case the leading nonadiabatic correction term does not manifest the PCD, thus explicitly showing that the PCD is due to the ''wrong'' choice of coordinates (**x**,**y**). Alternatively, a particle boost factor is introduced to eliminate the PCD in the treatment with the original (**x**,**y**) coordinates. This is shown to be equivalent to the change in coordinates, described above. Such a factor is usually associated with high energy collisions, but for the first time shown here to play also an important role in near zero-energy situations. [S0556-2813(98)05401-6]

PACS number(s): 21.45.+v, 21.10.Dr, 25.70.−z, 33.15.Fm

I. INTRODUCTION

The Efimov effect [1–4] states that a three body system will have an arbitrarily large number of bound states if its two-body subsystem supports a zero-energy bound state. The large size of the two-body bound state with an infinite scattering length gives rise to a long-range effective potential of the form $-1/x^2$. Fonseca *et al.* [4] continued the discussion of the Efimov effect within the framework of the Born-Oppenheimer approximation (BOA). They constructed a simple three-body model in which one light particle is bound simultaneously to two heavy particles. The adiabatic potential obtained analytically in this solvable model exhibited directly the desired behavior $U_a \sim -1/x^2$, where x is the relative distance between the two heavy particles. Subsequently it was shown [5], however, that the first leading correction to the adiabatic picture contributed a term of the form $+1/x$, thus making the proof of Ref. [4] inoperative.

An attempt is made here to improve the result of previous works [5,4], mainly by the choice of proper adiabatic coordinates. In particular, we are concerned with the leading correction to the adiabatic potential which behaves at large x as $+1/x$. This pseudo-Coulomb disease (PCD) is disastrous, because the adiabatic potential $-1/x^2$ will be completely wiped out for x large enough. It was then suggested [5] that (a) the PCD was due to a bad choice of the coordinates (**x**,**y**) and that a properly chosen coordinate system may avoid the PCD difficulties altogether. Alternatively, (b) the higher order corrections to the adiabatic potential may be such as to cancel the PCD. (c) Finally, the PCD may also be cured by introducing a particle boost factor. The points (a) and (c) are the subject of study in this paper, while point (b) will be discussed elsewhere [6].

For clarity and to introduce notation, we will first summarize in Sec. II the proof of the Efimov effect in the ''conventional'' molecular coordinates, as was done in Refs. [4] and [5], and the leading correction [4] with the PCD. In Sec. III we obtain the Efimov effect using the new distorted cluster states [7] representation (DCS) in the BOA. Absence of the PCD in this representation is demonstrated. Section IV will show the connection between the results of Secs. II and III, by introducing a particle boost factor [8,6] (PBF). A drastic improvement in consistency of the results on the Efimov effect is obtained. The conclusion and discussion are presented in Sec. V.

II. THE EFIMOV EFFECT IN THE ADIABATIC REPRESENTATION

We consider a ''molecular model'' three-body system [5,4] of two heavy and one light particles, where a two-body subsystem supports a zero-energy bound state. It was shown in [1–3] that under these conditions an attractive potential of infinite range appears between heavy particles, giving rise to an infinite number of bound states in the three-body Hamiltonian. This is known as the Efimov effect. Our discussion of Efimov effect closely follows that of [5] and [4], adopting the notation of [5].

Following Fonseca *et al.* [4], we choose a system of spinless particles with masses of two heavy particles 1 and 2 to be equal to M and mass of the third light one to be m. The Jacobi coordinates **x** is for the pair (1 and 2) of heavy particles, and **y** for particle 3 with respect to the center of mass of (1 and 2), see Fig. 1(a). Their conjugate momenta are **P** and **p**, respectively. The three-body Hamiltonian of the system is then

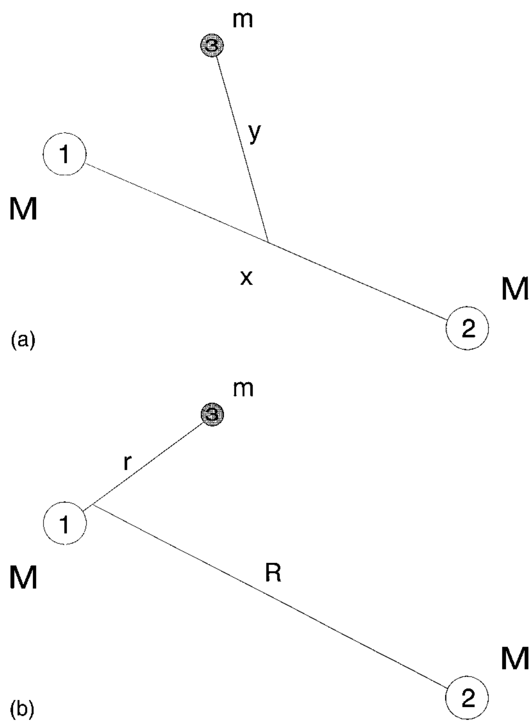

FIG. 1. (a) The Jacobi coordinates (**x**,**y**) used in Refs. [5] and [4]. (b) The alternative Jacobi coordinates (**r**,**R**) employed in Sec. III for the distorted cluster state calculation. This choice is not symmetric in particles 1 and 2.

$$H=\frac{1}{M}\,\mathbf{P}^2+\frac{1}{2\mu_3}\,\mathbf{p}^2+V_1\left(\mathbf{y}+\frac{\mathbf{x}}{2}\right)+V_2\left(\mathbf{y}-\frac{\mathbf{x}}{2}\right)+V_3(x)$$
$$=K_{\mathbf{x}}+h_3(\mathbf{y},\mathbf{x}),\qquad(1)$$

where $K_{\mathbf{x}}=\mathbf{P}^2/M$. If $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are the coordinates of the particles 1, 2, and 3 with respect to a fixed reference, then the Jacobi coordinates **x** and **y** are given by

$$\mathbf{x}=\mathbf{r}_1-\mathbf{r}_2,$$
$$\mathbf{y}=\mathbf{r}_3-\frac{\mathbf{r}_1+\mathbf{r}_2}{2}.\qquad(2)$$

The reduced mass μ_3 of the light particle is defined by

$$\frac{1}{\mu_3}=\frac{1}{2M}+\frac{1}{m}.\qquad(3)$$

We note that $h_3(\mathbf{y},\mathbf{x})$ is a two-body Hamiltonian in which the $4\equiv(1+2)$ subsystem is frozen; $4+3$ is then regarded as a two body. V_1 represents interaction between particles 2 and 3, V_2 is between the 1 and 3, and V_3 is between the 1 and 2. All interactions are assumed to be of short range. The particular form of interaction was chosen so that V is separable [5,4], i.e., $V_i=-\lambda|f_i\rangle\langle f_i|$, where $i=1,2,3$ is the pair index (1 for pair of particles 2 and 3, etc.). The coupling strength λ is positive and $|f\rangle$ is of the form

$$\langle\mathbf{y}|f\rangle=\frac{e^{-\gamma y}}{y},\quad y=|\mathbf{y}|,$$

$$\langle\mathbf{p}|f\rangle=\frac{1}{\pi(\mathbf{p}^2+\gamma^2)}.\qquad(4)$$

Operator of the interaction between the light and heavy particles assumes simple form

$$V\left(\mathbf{y}\mp\frac{1}{2}\,\mathbf{x}\right)=T^{\pm1}V(\mathbf{y})T^{\mp1}=-\lambda T^{\pm1}|f\rangle\langle f|T^{\mp1},\qquad(5)$$

where $T=e^{i\mathbf{p}\cdot\mathbf{x}/2}$ is the translation operator, and **p** is the momentum conjugate to **y**.

In the lowest order Born-Oppenheimer approximation (BOA), the wave function of this system is taken to be $\Psi(\mathbf{x},\mathbf{y})\simeq\chi_x(\mathbf{y})\psi(\mathbf{x})$. The "fast" part $\chi_x(\mathbf{y})$ describes the light particle motion with respect to the center of mass of two heavy particles, and $\psi(\mathbf{x})$ is the wave function of relative "slow" motion of the two heavy particles. The initial equation for the wave function Ψ

$$(H-E)\Psi(\mathbf{x},\mathbf{y})=0\qquad(6)$$

separates into two parts; the equation that describes the wave function $\psi(\mathbf{x})$, as

$$\langle\chi_x|(H-E)|\chi_x\rangle\psi(\mathbf{x})=\left(\frac{1}{M}\,\mathbf{P}^2-\frac{1}{2\mu_3}\,\omega^2+u+\eta_3-E\right)\psi(\mathbf{x})$$
$$=0,\qquad(7)$$

and the equation for $\chi_x(\mathbf{y})$ given by

$$\left[\frac{1}{2\mu_3}\,[\omega^2(x)+\mathbf{p}^2]+V_1\left(\frac{\mathbf{x}}{2}+\mathbf{y}\right)+V_2\left(\frac{\mathbf{x}}{2}-y\right)\right]\chi_x(\mathbf{y})=0.\qquad(8)$$

The energy $-\omega^2/(2\mu_3)$ that appears as the eigenvalue in Eq. (8) contains the **x** coordinate as parameter. In Eq. (7), $-\omega^2(x)/(2\mu_3)$ appears again, but due to its dependence on **x**, it becomes the effective interaction energy between the two heavy particles. The quantities u and η_3 that appear in Eq. (7) are defined by

$$u(x)=\frac{1}{M}\,\langle\chi_x|P^2|\chi_x\rangle,$$
$$\eta_3(x)=\langle\chi_x|V_3(\mathbf{x})|\chi_x\rangle=V_3.\qquad(9)$$

The quantity $u(x)$ is a nonadiabatic contribution to the adiabatic interaction potential $-\omega^2/2\mu_3$, while η_3 is neglected due to its short range, without affecting the final result.

We briefly review the previous results of Refs. [5,4], and discuss the conditions that lead to the Efimov effect, all within the adiabatic picture. The Efimov effect states that when a two-body subsystem supports a zero-energy bound state there can be an infinite number of bound states involving the third particle. But this has been interpreted differently by different authors. The approach of Fonseca *et al.* [4] considers a pseudo two-body subsystem, say 1 and 3, in which the variables (**x**,**y**) are retained but the reduced mass is changed. On the other hand, the approach of Giraud and Hahn, in [5], takes simply $h_3(\mathbf{y},\mathbf{x})$ as it appears in Eq. (1), which is also effectively a "two-body" Hamiltonian in which one body is $(1+2)$ and the other is particle 3. Thus, in order to literally conform to the original definition of the Efimov effect within adiabatic picture, Fonseca *et al.* in [4] introduced a two-body subsystem $(1+3)$ Hamiltonian, as

$$H^{(2)}=\frac{1}{2\mu_2}\,\mathbf{p}^2-\Lambda|f\rangle\langle f|, \tag{10}$$

where the reduced mass μ_2 is given by

$$\frac{1}{\mu_2}=\frac{1}{m}+\frac{1}{M}. \tag{11}$$

It is important to note that $H^{(2)}$ still contains the coordinate $\mathbf{x}$ in $|f\rangle$, and $\mathbf{p}$ is conjugate to $\mathbf{y}$. The only change from $h_3(\mathbf{y},\mathbf{x})$ in Eq. (1) is to replace μ_3 by μ_2 and keeping only V_2. Obviously this does not make the Hamiltonian truly two body. [See Sec. III for a correct two-body Hamiltonian describing $(m+M)$, with the variables $\mathbf{r}$ and $\mathbf{R}$.] Therefore $H^{(2)}$ is not physical. Nevertheless, $H^{(2)}$ is useful in clarifying the role played by the reduced mass, as shown below. Since $H^{(2)}$ is used here only for illustrating the importance of μ_3 in Eq. (1), the final conclusion of the Efimov effect is unaffected; within the picture with h_3, where $1+2\equiv 4$ is frozen, the two-body picture of $4+3$ is sufficient and introduction of $H^{(2)}$ is not needed although convenient for discussion.

The condition on the coupling strength that $H^{(2)}$ supports a zero-energy bound state is then

$$\frac{1}{2\mu_2\Lambda}=\frac{1}{\gamma^3}. \tag{12}$$

On the other hand, Eq. (8) describes the state of the light particle in the system of two heavy particles, which act as a frozen single particle. We have

$$(\omega^2+\mathbf{p}^2-2\mu_3\lambda[T|f\rangle\langle f|T^{-1}+T^{-1}|f\rangle\langle f|T])\chi_x(\mathbf{y})=0, \tag{13}$$

where $T=e^{i\mathbf{p}\cdot\mathbf{x}/2}$ is a translational factor and $\mathbf{p}$ is the momentum conjugate to $\mathbf{y}$. In momentum space its solution can be written implicitly as

$$\chi_x(\mathbf{p})=\frac{2\mu_3\lambda}{\mathbf{p}^2+\omega^2}\,[T|f\rangle\langle f|T^{-1}|\chi_x\rangle+T^{-1}|f\rangle\langle f|T|\chi_x\rangle]. \tag{14}$$

Using the symmetry of the ground state, $\langle f|T^{-1}|\chi_{\vec{x}}\rangle=\langle f|T|\chi_{\vec{x}}\rangle$, we obtain the condition for existence of a bound state with energy $-\omega^2(x)$

$$\frac{1}{2\mu_3\lambda}=\left\langle f\left|\frac{T^2+1}{\mathbf{p}^2+\omega^2}\right|f\right\rangle. \tag{15}$$

To further clarify the role of reduced mass μ_2 and μ_3, we impose that the coupling strength Λ of Eq. (12) in the two-body Hamiltonian and the coupling strength λ of Eq. (15) in the three-body Hamiltonian, are the same. This condition yields, after the integration of Eq. (15),

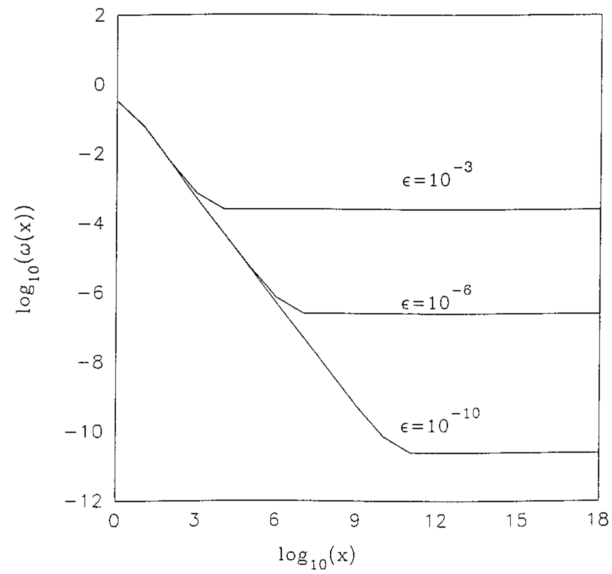

FIG. 2. Numerical solution for $\omega=\omega(x)$ in log-log scale, from Eq. (16). The values used for parameters are $\gamma=1$ and mass ratio $\epsilon=10^{-3}$, 10^{-6}, and 10^{-10}.

$$\frac{\mu_2}{\mu_3}\frac{1}{\gamma^3}=\frac{1}{\gamma(\gamma+\omega)^2}+\frac{2\gamma e^{-\omega x}+[(\omega^2-\gamma^2)x-2\gamma]e^{-\gamma x}}{\gamma x(\gamma^2-\omega^2)^2}. \tag{16}$$

Equation (16) can be solved numerically for ω for different values of x, using the ratio μ_2/μ_3 as a parameter. We have, with $\epsilon=m/M$,

$$\frac{\mu_2}{\mu_3}=1-\frac{\epsilon}{2(1+\epsilon)}. \tag{17}$$

Different solutions for $\omega=\omega(x)$ are shown in Fig. 2, for various values of the mass ratio ϵ and for parameter $\gamma=1$. The main feature of the solution $\omega(x)$ is that, with respect to its behavior as function of the heavy particle separation $\mathbf{x}$, there are three regions, separated by two characteristic distances x_m and x_0, $x_m<x_0$. For $x<x_m$, the solution $\omega(x)$ is not important because the short-range effects of the potential $V_3(x)$ in Eq. (7), acting between the two heavy particles, cannot be neglected. In the region $x_m\ll x\ll x_0$, the solution of Eq. (16) is $\omega(x)\simeq c/x+O(x^{-2})$, where c is the solution of the transcendental equation $e^{-c}=c$. In the region $x_0\ll x$, the solution of Eq. (16) is $\omega(x)\simeq\gamma\epsilon/2$. While characteristic distance x_m depends mainly on the range of potential $V_3(x)$ and parameter γ, a simple analysis shows that the characteristic distance x_0 can be approximated by $x_0\simeq 2c/\gamma\epsilon$.

The main problem of the approach of Fonseca *et al.* [4] using $H^{(2)}$ now becomes apparent. Having the parameter $\epsilon>0$ at this stage of the analysis prevents any progress. If $\epsilon>0$ for values of x such that $x\gg x_0$ the attractive potential $-\omega^2/(2\mu_3)$ in Eq. (7) has an unphysical constant value of $-\gamma^2\epsilon^2/(8\mu_3)$ rather than having a physical x dependence. This leads then to the requirement that $\mu_2\to\mu_3$, or equivalently ϵ has to be zero, which makes this approach equivalent to the approach of the Giraud and Hahn [5] where $H^{(2)}$ was never introduced but stayed with h_3 and Eq. (1). That is, in

Ref. [5], a two-body subsystem is described by the effective two-body Hamiltonian of the form

$$H_{\rm eff}^{(2)}=\frac{1}{2\mu_3}\,\mathbf{p}^2-\Lambda' T|f\rangle\langle f|T^{-1}=h_3(y,x)-V_1-V_3. \tag{18}$$

Compared to the Hamiltonian $H^{(2)}$ of Eq. (10) used by Fonseca *et al.* [4], we see that the reduced mass of the light particle must be μ_3 and not μ_2. With the choice (18), the reduced mass μ_2 in Eqs. (10)–(17) is replaced by reduced mass μ_3, as it should, in accordance with Eq. (1).

We now impose the condition that the coupling strength Λ' in the Hamiltonian $H_{\rm eff}^{(2)}$ of Eq. (18) is the same as the coupling strength λ introduced in the three-body Hamiltonian, Eq. (8). The function $\omega=\omega(x)$ in Eq. (7) is then the solution to the transcendental equation

$$\frac{1}{\gamma^3}=\frac{1}{\gamma(\gamma+\omega)^2}+\frac{2\gamma e^{-\omega x}+[(\omega^2-\gamma^2)x-2\gamma]e^{-\gamma x}}{\gamma x(\gamma^2-\omega^2)^2}. \tag{19}$$

For the values of the large heavy particles separation distance x, i.e., $x\gg\gamma^{-1}$, the solution for $\omega=\omega(x)$ is of the form

$$\omega(x)=\frac{c}{x}+O(x^{-2}), \tag{20}$$

where c is again the solution of the equation $e^{-c}=c$. The attractive potential that appears in the Eq. (7), that describes the slow coordinate $\mathbf{x}$, is of the form $-\omega^2/(2\mu_3)=-c^2/(2\mu_3)x^{-2}$. This potential has all the desired properties; it is attractive and it goes to zero for large values of the heavy particle separation distance x. If nonadiabatic correction u in Eq. (7) is neglected, this results in an infinite number of the bound states, the Efimov effect. One important distinction is the fact that the "two-body" bound states here implies the fixed cluster $(1+2)$, to be counted as one particle.

Our next concern is the calculation of the nonadiabatic correction u in Eq. (7), as discussed in Ref. [5]. Using the solution for ω as a function of x, the heavy particle separation distance, Eq. (7) describes the two heavy particles in the adiabatic potential generated by the light particle, as

$$\left(\frac{1}{M}\,\mathbf{P}^2-\frac{1}{2\mu_3}\frac{c^2}{x^2}+u+\eta_3-E\right)\psi(\mathbf{x})=0, \tag{21}$$

where the nonadiabatic correction u is included.

As shown in Ref. [5], potential u can be evaluated within the adiabatic model. In the lowest order it was found that

$$u(x)=\frac{1}{M}\,\frac{\epsilon}{2+\epsilon}\,\frac{c\gamma}{(1+c)x}+O(x^{-2}), \tag{22}$$

which describes the pseudo-Coulomb disease (PCD). This term makes the proof of the existence of Efimov effect within the adiabatic model incomplete, because the correction $u(x)$ to the adiabatic potential is not small at large x, as compared to $-\omega^2(x)/2\mu_3$.

The main purpose of the present paper is to address this problem by first showing that a proper choice of coordinate system can cure the PCD (Sec. III). Alternatively, a corrective factor on the wave function χ_x in terms of a particle boost factor (PBF) can also remedy the disease asymptotically (Sec. IV).

III. DISTORTED CLUSTER STATES

As one of possible resolutions [5] of the PCD problem, we consider changing the coordinates. A new set of Jacobi coordinates for a three-body system, as shown in Fig. 1(b) is

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_3,$$

$$\mathbf{R}=\frac{m\mathbf{r}_3+M\mathbf{r}_1}{M+m}-\mathbf{r}_2. \tag{23}$$

These new coordinates [7] are different from the earlier set $(\mathbf{x},\mathbf{y})$ in the most critical way, as will become clear later. We especially note that this new set treats particles 1 and 2 differently. The adiabatic states generated for the pair $(3+1)$ are termed distorted cluster states (DCS). By symmetry, we can also easily consider the $\mathbf{r}'=\mathbf{r}_2-\mathbf{r}_3$ for the $(2+3)$ bound pair. In fact, the choice (23) treats the pair $(1+3)$ differently from the other pair $(2+3)$. This can be easily remedied by constructing a 2 by 2 matrix equation [7] of the Faddeev type. However, we retain this asymmetric form to illustrate the main point of this section. That is, the pair $(1+3)$ is treated here by a new coordinate, while the pair $(2+3)$ is treated in nearly identical way as in the original coordinates $(\mathbf{x},\mathbf{y})$. Therefore, the PCD disappears, as will be shown below, for pair $(1+3)$ but not for $(2+3)$.

The Hamiltonian in the lab system is

$$H_{\rm tot}=-\frac{1}{2M}\,(\nabla_{\mathbf{r}_1}^2+\nabla_{\mathbf{r}_2}^2)-\frac{1}{2m}\,\nabla_{\mathbf{r}_3}^2+V_1+V_2+V_3, \tag{24}$$

where we set $\hbar=1$. In the new coordinate system, the total kinetic energy in Eq. (24) is separable; neglecting the total center of mass motion, we have

$$H=-\frac{1}{2\mathcal{M}}\,\nabla_{\mathbf{R}}^2-\frac{1}{2\mu_2}\,\nabla_{\mathbf{r}}^2+V_1+V_2+V_3, \tag{25}$$

where $\mathcal{M}=M(M+m)/(2M+m)$ and $\mu_2=Mm/(M+m)$. In the following we again choose separable potentials for V_1 and V_2:

$$V_1=V_1(|\mathbf{r}_2-\mathbf{r}_3|)=V_1\left(\left|\mathbf{R}-\frac{M}{M+m}\,\mathbf{r}\right|\right)$$
$$=-\lambda T_A|f_A\rangle\langle f_A|T_A^{-1}, \tag{26}$$

$$V_2=V_2(|\mathbf{r}_1-\mathbf{r}_3|)=V_2(|\mathbf{r}|)=-\lambda|f\rangle\langle f|. \tag{27}$$

V_3 is a short-range potential between heavy particles 1 and 2, and is neglected here. T_A is coordinate translation operator and A is a mass ratio, defined by

$$T_A=\exp\left[\frac{i\mathbf{qR}}{A}\right]$$

and

$$A=\frac{M}{M+m}, \tag{28}$$

and $\mathbf{q}$ is now the momentum conjugate to $\mathbf{r}$. Note that from the definition of A, $1/2\leqslant A<1$ for $m\leqslant M$. The potential form factor f is given explicitly by

$$\langle f_A|\mathbf{r}\rangle=\frac{\exp[-\gamma rA]}{rA},\langle f_A|\mathbf{q}\rangle=\frac{1}{A\pi(\mathbf{q}^2+\gamma^2A^2)}, \tag{29}$$

$$\langle f|\mathbf{r}\rangle=\frac{\exp[-\gamma r]}{r},\langle f|\mathbf{q}\rangle=\frac{1}{\pi(\mathbf{q}^2+\gamma^2)}. \tag{30}$$

In the adiabatic approximation the three-body wave function $\Psi(\mathbf{R},\mathbf{r})$ is again set in the BOA form $\Psi(\mathbf{R},\mathbf{r})\cong\chi_{\mathbf{R}}(\mathbf{r})\psi(\mathbf{R})$. Then, $(H-E)|\Psi\rangle=0$ may be cast [5] into a set of two equations; first we introduce the distorted cluster state function [7] defined by

$$\left[-\frac{1}{2\mu_2}\nabla_{\mathbf{r}}^2+\frac{\omega^2}{2\mu_2}-\lambda(T_A|f_A\rangle\langle f_A|T_A^{-1}+|f\rangle\langle f|)\right]\chi_{\mathbf{R}}(\mathbf{r})=0. \tag{31}$$

By projecting with the normalized $\langle\chi_{\mathbf{R}}|$, we also have for $|\psi\rangle$

$$\left[-\frac{1}{2\mathcal{M}}(\nabla_{\mathbf{R}}^2+\langle\chi_{\mathbf{R}}|\nabla_{\mathbf{R}}^2|\chi_{\mathbf{R}}\rangle)-\frac{\omega^2}{2\mu_2}+\langle\chi_{\mathbf{R}}|V_3|\chi_{\mathbf{R}}\rangle-E\right]\psi(\mathbf{R})=0. \tag{32}$$

Equation (31) in momentum space assumes the form

$$(\mathbf{q}^2+\omega^2)\chi_{\mathbf{R}}(\mathbf{q})=\frac{\lambda'}{\pi}\left[\frac{\alpha}{\mathbf{q}^2+\gamma^2}+\frac{\beta T_A}{A(\mathbf{q}^2+\gamma^2A^2)}\right], \tag{33}$$

where $\lambda'=2\mu_2\lambda$, $\langle f|\chi_{\mathbf{R}}\rangle=\alpha$ and $\langle f_A|T_A^{-1}|\chi_{\mathbf{R}}\rangle=\beta$. This is to be compared with Eq. (14). In the Fourier transformed space, Eq. (33) becomes

$$\chi_{\mathbf{R}}(\mathbf{q})=\frac{\lambda'}{\pi(\mathbf{q}^2+\omega^2)}\left[\frac{\alpha}{\mathbf{q}^2+\gamma^2}+\frac{\beta T_A}{A(\mathbf{q}^2+\gamma^2A^2)}\right], \tag{34}$$

so that explicitly

$$\alpha=\int d^3q\,\frac{\lambda'}{\pi^2(\mathbf{q}^2+\omega^2)(\mathbf{q}^2+\gamma^2)}\left[\frac{\alpha}{\mathbf{q}^2+\gamma^2}+\frac{\beta T_A}{A(\mathbf{q}^2+\gamma^2A^2)}\right], \tag{35}$$

$$\beta=\int d^3q\,\frac{\lambda'}{\pi^2A(\mathbf{q}^2+\omega^2)(\mathbf{q}^2+\gamma^2A^2)}\left[\frac{\alpha T_A^{-1}}{\mathbf{q}^2+\gamma^2}+\frac{\beta}{A(\mathbf{q}^2+\gamma^2A^2)}\right]. \tag{36}$$

The integrals in Eqs. (35) and (36) are evaluated in Appendix A, and the coupled equations (35) and (36) become

$$\alpha=\lambda'\left[\frac{1}{\gamma(\gamma+\omega)^2}\,\alpha+\frac{2e^{-R\omega/A}}{R(\gamma^2-\omega^2)(\gamma^2A^2-\omega^2)}\,\beta\right], \tag{37}$$

$$\beta=\lambda'\left[\frac{1}{\gamma A^3(\gamma A+\omega)^2}\,\beta+\frac{2e^{-R\omega/A}}{R(\gamma^2-\omega^2)(\gamma^2A^2-\omega^2)}\,\alpha\right]. \tag{38}$$

Solutions of the system of two equations (37) and (38) give conditions on λ' and on the adiabatic energy ω. To the lowest order and in the limits of $R\to\infty$ and $\omega\to0$, the coupling term becomes small and we have

$$\begin{pmatrix}\frac{1}{\lambda'}-\frac{1}{\gamma^3} & 0\\ 0 & \frac{1}{\lambda'}-\frac{1}{\gamma^3A^5}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\cong0. \tag{39}$$

That is, the critical values for λ' are $\lambda_1'=\gamma^3$ and $\lambda_2'=\gamma^3A^5$. Note that $A<1$, so that $\lambda_1'\geqslant\lambda_2'$. This is reasonable because the strength of the interaction can be smaller when one of the particles in the zero-energy bound state is much heavier. The same result is obtained for the two particles (heavy-light). Thus, at the critical values for λ', we have either

$$\frac{1}{\lambda_1'}=\int d^3q\,\frac{f^2(\mathbf{q})}{\mathbf{q}^2}=\frac{1}{\gamma^3}$$

or

$$\frac{1}{\lambda_2'}=\int d^3q\,\frac{f_A^2(\mathbf{q})}{\mathbf{q}^2}=\frac{1}{\gamma^3A^5}. \tag{40}$$

The next order in power of $1/R$ and ω gives

$$\begin{pmatrix}\frac{2\omega}{\gamma^4} & -\frac{2e^{-\omega R/A}}{R\gamma^4A^2}\\ -\frac{2e^{-\omega R/A}}{R\gamma^4A^2} & \frac{2\omega}{\gamma^4A^6}\end{pmatrix}=0, \tag{41}$$

or $\omega R/A=e^{-\omega R/A}$. As before, the solution of this equation is a constant $c\cong0.5671$. The ‘‘slow’’ BOA equation now becomes

$$\left[-\frac{1}{2\mathcal{M}}\nabla_{\mathbf{R}}^2+u'-\frac{c^2A^2}{2R^2\mu_2}+\eta'-E\right]\psi(\mathbf{R})=0, \tag{42}$$

where the adiabatic potential $c^2A^2/2\mu_2R^2$ will give the Efimov effect, provided that the correction $u'=-(1/2\mathcal{M})\langle\chi_{\mathbf{R}}|\nabla_{\mathbf{R}}^2|\chi_{\mathbf{R}}\rangle$ to the adiabatic potential does not show the PCD. The short-range potential $\eta'=\langle\chi_{\mathbf{R}}|V_3|\chi_{\mathbf{R}}\rangle\to0$ faster than $1/R^2$ as $R\to\infty$.

Before we proceed with the evaluation of this correction, we first normalize $\chi_{\mathbf{R}}$ as $\langle\chi_{\mathbf{R}}|\chi_{\mathbf{R}}\rangle=1$ for each fixed $\mathbf{R}$. (Here we assume that the binding energy of the state $\chi_{\mathbf{R}}$ is slightly negative. Otherwise, it is not normalizable in the limit of zero energy.) Then,

$$\langle\chi_{\mathbf{R}}|\chi_{\mathbf{R}}\rangle=\frac{\lambda'^2\beta^2}{\pi^2}\int d^3q\,\frac{1}{(\mathbf{q}^2+\omega^2)^2}\,\frac{1}{(\mathbf{q}^2+\gamma^2A^2)}+\frac{\alpha^2}{\beta^2}\frac{1}{(\mathbf{q}^2+\gamma^2)^2}+\frac{\alpha}{\beta}\frac{T_A+T_A^{-1}}{(\mathbf{q}^2+\gamma^2)(\mathbf{q}^2+\gamma^2A^2)}$$
$$=\lambda'^2\beta^2\frac{2}{\pi}\oint dq\,\frac{q^2}{(q^2+\omega^2)^2}\,\frac{1}{A^2(q^2+\gamma^2A^2)}+\frac{\alpha^2}{\beta^2}\frac{1}{(q^2+\gamma^2)^2}+\frac{\alpha}{\beta}\frac{2Ae^{(iqR/A)}}{iqR(q^2+\gamma^2)(q^2+\gamma^2A^2)},$$

where α/β is R dependent and $T_A=e^{i\mathbf{q}\cdot\mathbf{R}/A}$.

By contour integrations and in the limit of $R\to\infty$ and $\omega\to 0$, we obtain

$$\lambda'^2\beta^2=\frac{\omega\gamma^4}{\Gamma^2},\quad \Gamma^2=\left(\frac{\alpha^2}{\beta^2}+\frac{2c}{A^3}\frac{\alpha}{\beta}+\frac{1}{A^6}\right), \tag{43}$$

and the normalized $\chi_{\mathbf{R}}(\mathbf{q})$ then has the form $\chi_{\mathbf{R}}(\mathbf{q})=N(\mathbf{R})\xi(\mathbf{q},\mathbf{R})$, where

$$N(\mathbf{R})=\frac{\omega^{1/2}\gamma^2}{\pi\Gamma}, \tag{44}$$

$$\xi(\mathbf{q},\mathbf{R})=\frac{1}{(q^2+\omega^2)}\left[\frac{\alpha}{\beta}\frac{1}{q^2+\gamma^2}+\frac{T_A}{A(q^2+\gamma^2A^2)}\right]. \tag{45}$$

We note that

$$\langle\xi(\mathbf{q},\mathbf{R})|\xi(\mathbf{q},\mathbf{R})\rangle=N^{-2}$$

and

$$\nabla_{\mathbf{R}}(N^{-2})=2\langle\xi(\mathbf{q},\mathbf{R})|\nabla_{\mathbf{R}}\xi(\mathbf{q},\mathbf{R})\rangle. \tag{46}$$

Furthermore,

$$\langle\chi_{\mathbf{R}}|(-\nabla^2_{\mathbf{R}})|\chi_{\mathbf{R}}\rangle=N^2\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle-2\langle\xi|(\nabla_{\mathbf{R}}N)|\nabla_{\mathbf{R}}\xi\rangle+\langle\xi|(-\nabla^2_{\mathbf{R}}N)|\xi\rangle$$
$$=-\frac{1}{N^2}\left(\frac{dN}{dR}\right)^2+N^2\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle. \tag{47}$$

The quantities $\alpha/\beta, N(\mathbf{R}), \xi(\mathbf{q},\mathbf{R})$ depend on the critical values for λ'. So we consider the following two cases.

(i) $\lambda'=\lambda'_2=\gamma^3A^5$; the weak coupling. For this value of λ' we have in the limit of $R\to\infty$

$$\langle\chi_{\mathrm{R}}|-\nabla^2_{\mathbf{R}}|\chi_{\mathbf{R}}\rangle=-\frac{1}{4R^2}+N^2\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle, \tag{48}$$

with the explicit expressions for the quantities in Eqs. (44) and (45)

$$A=\frac{M}{M+m}<1,\quad \frac{\alpha}{\beta}=\frac{2A^2\omega}{(1-A^5)\gamma},$$
$$N(\mathbf{R})=\frac{\gamma^2A^3}{\pi}\left(\frac{c}{R}\right)^{(1/2)},\quad \Gamma=\frac{1}{A^3}.$$

The integral $N^2\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle$ for λ'_2 is evaluated in Appendix B, and the correction u' to the adiabatic potential assumes the form

$$u_2=\frac{1}{2\mathcal{M}}\,\gamma\,\frac{c}{R}. \tag{49}$$

Thus, this case is similar to the previous result summarized in Sec. II, and needs $M\gg m$ in order to suppress u_2 and to save the Efimov effect. That is, the new choice $(\mathbf{r},\mathbf{R})$ has failed to remedy the PCD. This is as expected, because the choice (23) for $\mathbf{r}$ treats heavy particle 1 and 2 differently. Apparently λ'_2 corresponds to bound state associated with particles $2+3$, not $1+3$.

(ii) $\lambda'=\lambda'_1=\gamma^3$; the strong coupling. In the limit of $R\to\infty$, we now have

$$\frac{\alpha}{\beta}=-\frac{(1-A^5)\gamma}{2A^2\omega},\quad N(\mathbf{R})=\frac{2\gamma^2A^2}{\pi(1-A^5)}\,\omega^{3/2},\quad \Gamma=\frac{\alpha}{\beta}$$

and thus

$$\langle\chi_{\mathbf{R}}|-\nabla^2_{\mathbf{R}}|\chi_{\mathbf{R}}\rangle=-\frac{9}{4R^2}+N^2\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle. \tag{50}$$

The integral $N^2\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle$ is evaluated in Appendix B, and the correction term is

$$u_1=\frac{1}{8\mathcal{M}R^2}. \tag{51}$$

Therefore, with λ'_1 and its solution, the PCD is eliminated. We emphasize that the case with λ'_1 is for the pair $1+3$, for which $\mathbf{r}$ is the proper coordinate. This is the main difference between the two different choices of the adiabatic coordinates. The results (49) and (51) further support our assertion that the PCD is caused by the ''bad'' choice of coordinates; the new asymmetric coordinates $(\mathbf{r},\mathbf{R})$ correct for the pair $1+3$, but not for $2+3$.

Finally, we then have

$$\left[-\frac{1}{2\mathcal{M}}\nabla^2_R+\frac{1}{8\mathcal{M}R^2}-\frac{c^2A^2}{2R^2\mu_2}+\eta'-E\right]\psi(\mathbf{R})=0 \tag{52}$$

or

$$\left[-\frac{1}{2\mathcal{M}}\nabla^2_{\mathbf{R}}-\frac{\omega^2}{2\mu_2}\,\delta+\eta'-E\right]\psi(\mathbf{R})=0.$$

The sign of the expression $\delta=(1-(1/4c^2)[m(m+2M)/M^2])=1-\epsilon(2+\epsilon)/4c^2$ depends on the mass ratio $\epsilon=m/M$; for $\epsilon\leqslant -1+\sqrt{1+4c^2}=0.51$ it is positive. So, for

$\epsilon \leq 0.51$, we have the correct sign and proper behavior for the effective potential. As noted earlier, potential u' represents the case in which the particle 2 is stationary, and the result (49) is thus similar to Eq. (22), with the PCD.

By a symmetric treatment of both particles 1 and 2, as shown in Ref. [7], we should be able to completely eliminate the PCD. Finally, we emphasize that the positive $1/R^2$ contribution of u_1 is still not very satisfactory, because it modifies the adiabatic potential. A more complete treatment of all nonadiabatic corrections is therefore needed to fully resolve the problem [6]. This is of course a more difficult task, because the simple BOA ansatz $\Psi(\mathbf{x},\mathbf{y}) \simeq \chi_x(\mathbf{y})\psi(\mathbf{x})$ must be extended [7] by including more terms for configuration mixing.

The result of this section in an unsymmetrized form should be especially useful in treating asymmetric system with different heavy particles 1 and 2, corresponding to a ''heteronuclear molecular'' system.

IV. PARTICLE BOOST FACTOR

We return to the original adiabatic coordinates $(\mathbf{x},\mathbf{y})$ of Sec. II, and analyze the situation with regard to the PCD employing the earlier suggestion (c) in Ref. [5]; that is, the introduction of a suitable particle boost factor (PBF) may remedy the PCD associated with the adiabatic representation. The discussion below is mathematically less rigorous than the treatment given in Sec. III, but brings out the physics more clearly. As it will be seen below, the two approaches are nevertheless similar, as they should be. The PBF is to be introduced to the BOA wave function because the original form is such that the light particle 3 may not be ''following'' particle 1 or 2 when the motion of the heavy particles are included by $\psi(\mathbf{x})$. We recall that the disease similar to this was found earlier in connection with high-energy ion-atom charge exchange collisions [8]. A similar difficulty will be shown to appear here also in the near-zero energy case.

The particle boost factor for the light particle W_ν may assume a simple form $W_\nu = e^{i\nu \mathbf{P}\cdot\mathbf{y}}$ at large x, where ν is an adjustable parameter and the conjugated variables are $(\mathbf{y},\mathbf{p})$ and $(\mathbf{x},\mathbf{P})$, for fast and slow, respectively. This form of W is arrived at by examining, for example, the asymptotic behavior of the wave function $\Psi(\mathbf{r},\mathbf{R})$ for the $(1+3)+2$; as $\mathbf{R}\to\infty$,

$$\Psi(\mathbf{r},\mathbf{R}) \to \chi_{x\to\infty}(r)[e^{i\mathbf{P}\cdot\mathbf{R}} + \cdots],$$

where χ describes the bound pair $(1+3)$. The plane wave part may be expressed, using Eqs. (1) and (23), as

$$e^{i\mathbf{P}\cdot\mathbf{R}} \simeq e^{i\mathbf{P}\cdot\mathbf{x}} e^{i\mathbf{P}\cdot\mathbf{y}\epsilon},$$

where $\mathbf{R} = [2(M+m)/2(M+m)]\mathbf{x} + [m/(M+m)]\mathbf{y}$. The second factor is identified to W and the new wave function in the Born-Oppenheimer approximation is given by $\hat{\Psi}^{\mathrm{BO}} \simeq \psi(\mathbf{x}) W \chi_x(\mathbf{y})$, where χ_x is a solution of Eq. (8).

The need for such a term also arises if we examine the Hamiltonian $\mathcal{H}$ obtained as $\mathcal{H} = T^{-1} H T$, where T is a translation operator given by $T = e^{i\mathbf{p}\cdot\mathbf{x}/2}$, i.e., for the transformation of coordinates such that $\mathbf{y}\to\mathbf{y}+\mathbf{x}/2(\equiv\mathbf{r})$. (It can be shown that we get exactly the same result if we consider transformation $\mathbf{y}\to\mathbf{y}-\mathbf{x}/2$.) We have

$$\mathcal{H} = T^{-1} H T = \frac{1}{M} P^2 + \frac{1}{M}\mathbf{p}\cdot\mathbf{P} + \frac{1}{2\mu_2}\mathbf{p}^2 - \lambda(T^{-2}|f\rangle\langle f|T^2 + |f\rangle\langle f|). \tag{53}$$

The Hamiltonian $\mathcal{H}$ differs from the initial three-body Hamiltonian H, Eq. (1), in the mass polarization term $(1/M)\mathbf{p}\cdot\mathbf{P}$ and in the reduced mass of the light particle which becomes μ_2 instead of μ_3.

The transformed Hamiltonian $\mathcal{H}$, Eq. (53), immediately suggests an improved ansatz for the wave function, i.e., we take $\tilde{\Psi}^{\mathrm{BO}} \simeq \psi(\mathbf{x}) T^{-1} \overline{\chi_x}(\mathbf{y})$, where $\overline{\chi_x}(\mathbf{y})$ is the solution of $[(1/2\mu_2)(\mathbf{p}^2+\omega^2) - \lambda'(T|f\rangle\langle f|T^{-1} + T^{-1}|f\rangle\langle f|T)]\overline{\chi_x}(\mathbf{y}) = 0$; it differs from $\chi_x(\mathbf{y})$ in reduced mass, here it is μ_2 while in Eq. (8) it is μ_3. Then,

$$\langle T^{-1}\overline{\chi_x}|\mathcal{H}|T^{-1}\overline{\chi_x}\rangle \psi(\mathbf{x}) = \langle T^{-1}\overline{\chi_x}|\frac{1}{M}\mathbf{P}^2 + \frac{1}{M}\mathbf{p}\cdot\mathbf{P} - \frac{\omega^2}{2\mu_2}|T^{-1}\overline{\chi_x}\rangle \psi(x). \tag{54}$$

We show below that the nonadiabatic contribution in Eq. (54) exhibits the desired behavior, i.e., it is of the order of $1/x^2$ or smaller, thus no PCD. If the operator T^{-1} in Eq. (54) is allowed to operate on $\mathcal{H}$, then we should get back to H with h_3, and the mass polarization term disappears. However, $\overline{\chi}$ is a solution of h_3 with reduced mass μ_3 replaced by μ_2; the difference is a small $\mathbf{p}^2$-dependent term which, when evaluated with respect to $|\chi|^2$, gives a $1/x$ contribution. To avoid this complication, we require that T^{-1} operates on $\overline{\chi}$ rather than on $\mathcal{H}$. The following calculation is carried out using explicitly this boosted function $T^{-1}\overline{\chi}$.

First we examine the adiabatic potential. From the transformed Hamiltonian $\mathcal{H}$, we choose as a fast Hamiltonian $\tilde{h}_3(\mathbf{y},\mathbf{x})$

$$\tilde{h}_3(\mathbf{y},\mathbf{x}) = \frac{1}{2\mu_2}\mathbf{p}^2 - \lambda'(T^{-2}|f\rangle\langle f|T^2 + |f\rangle\langle f|). \tag{55}$$

The reduced mass μ_2 in Eq. (55) consists of all prefactors to $\mathbf{p}^2$ in $\mathcal{H}$, as [7],

$$\frac{1}{\mu_2} = \frac{1}{\mu_3} + \frac{1}{2M} = \left(\frac{1}{m} + \frac{1}{2M}\right) + \frac{1}{2M}. \tag{56}$$

The conditions [5,4] for Efimov effect are now imposed to this fast Hamiltonian $\tilde{h}_3$, i.e., strength of the potential λ' has the critical value λ, so that the two body subsystem has a zero-energy bound state. Our new coordinate $\mathbf{r} = \mathbf{y} + \mathbf{x}/2$ allows us, as it was shown in Sec. III, to choose

$$2\mu_2\lambda' = \gamma^3. \tag{57}$$

Existence of the eigenstate $|T^{-1}\overline{\chi_x}\rangle$ with the energy $-\omega^2$ for the Hamiltonian $\tilde{h}_3$ leads to the condition for the ω dependence on $\mathbf{x}$, equivalent to Eq. (19):

$$\frac{1}{\gamma^3} = \frac{1}{\gamma(\gamma+\omega)^2} + \frac{2\gamma e^{-\omega x} + [(\omega^2-\gamma^2)x - 2\gamma]e^{-\gamma x}}{\gamma x(\gamma^2-\omega^2)^2}. \tag{58}$$

with the solution

$$\omega(x)=\frac{c}{x}+O(x^{-2}), \tag{59}$$

where c satisfies $e^{-c}=c$. We have thus obtained the desired behavior of the potential $-\omega^2(x)$, which is long range and attractive. Note that, unlike in Sec. II and Eq. (16), we do not have the spurious behavior of $\omega(\mathbf{x})$ at large x in the present case, because in Eqs. (54) and (55), only μ_2 arises. This is a direct consequence of the fact that $\overline{\chi}$ is used in Eq. (54), instead of χ.

The unnormalized eigenstate $|\widetilde{\varsigma}\rangle$ of the $\widetilde{h}_3$ with the energy $-\omega^2$ is given as

$$T^{-1}\overline{\chi}\sim|\widetilde{\varsigma}\rangle=\frac{1+T^{-2}}{p^2+\omega^2}\,|f\rangle. \tag{60}$$

The leading nonadiabatic correction comes from two contributions, a term proportional to $\mathbf{P}^2$ and a term proportional to $\mathbf{P}\cdot\mathbf{p}$. We name them $\widetilde{u}(x)$ and $\widetilde{w}(x)$, respectively. We have

$$\widetilde{u}(x)=\frac{1}{M}\langle\widetilde{\varsigma}|P^2|\widetilde{\varsigma}\rangle=\frac{1}{M}\left(\frac{\langle\nabla_x\widetilde{\varsigma}|\nabla_x\widetilde{\varsigma}\rangle}{\langle\widetilde{\varsigma}|\widetilde{\varsigma}\rangle}-\frac{|\langle\widetilde{\varsigma}|\nabla_x\widetilde{\varsigma}\rangle|^2}{\langle\widetilde{\varsigma}|\widetilde{\varsigma}\rangle^2}\right)$$
$$=\frac{1}{M\langle\widetilde{\varsigma}|\widetilde{\varsigma}\rangle}\frac{1}{(\gamma+\omega)^3} \tag{61}$$

and

$$\widetilde{w}(x)=\frac{1}{M}\langle T^{-1}\overline{\chi}|\mathbf{P}\cdot\mathbf{p}|T^{-1}\overline{\chi}\rangle=\frac{1}{M}\frac{\langle\widetilde{\varsigma}|-i\mathbf{p}|\nabla_x\widetilde{\varsigma}\rangle}{\langle\widetilde{\varsigma}|\widetilde{\varsigma}\rangle}$$
$$=-\frac{1}{M(\widetilde{\varsigma}|\widetilde{\varsigma})}\left(\frac{1}{(\gamma+\omega)^3}+\frac{2\gamma^2+2\omega^2+(\omega^2-\gamma^2)\omega x}{e^{\omega x}x(\gamma^2-\omega^2)^3}\right). \tag{62}$$

The terms that appear in Eqs. (61) and (62) have the following forms in the limit when $x\to\infty$:

$$\frac{1}{(\gamma+\omega)^3}=\frac{1}{\gamma^3}-\frac{3c}{\gamma^4x}+O(x^{-2}),$$

$$\langle\widetilde{\varsigma}|\widetilde{\varsigma}\rangle=\frac{2(1+c)}{c\gamma^4}\,x+O(x^0),$$

$$\frac{2\gamma^2+2\omega^2+(\omega^2-\gamma^2)\omega x}{e^{\omega x}x(\gamma^2-\omega^2)^3}=\frac{(2-c)c}{\gamma^4x}+O(x^{-2}). \tag{63}$$

The total potential nonadiabatic potential $\widetilde{U}(x)$ is then the sum of Eqs. (61) and (62),

$$\widetilde{U}(x)=\widetilde{u}(x)+\widetilde{w}(x)=-\frac{1}{M\langle\widetilde{\varsigma}|\widetilde{\varsigma}\rangle}\frac{2\gamma^2+2\omega^2+(\omega^2-\gamma^2)\omega x}{e^{wx}x(\gamma^2-\omega^2)^3}$$
$$\simeq-\frac{1}{2M}\frac{2-c}{1+c}\frac{1}{x^2}+O(x^{-3}) \tag{64}$$

which no longer exhibits the $1/x$ behavior, i.e., the PCD is absent.

When $\widetilde{U}$ is combined with $-\omega^2(x)$, a total resulting potential U is

$$U(x)=\widetilde{U}(x)-\frac{\omega^2(x)}{2\mu_2}=-\left(\frac{c^2}{2\mu_2}+\frac{1}{2M}\frac{2-c}{1+c}\right)\frac{1}{x^2}+O(x^{-3}), \tag{65}$$

where the coefficient of the $-1/x^2$ part is positive because $c=0.5671<2$. Hence both, the infinite range and the attractiveness for all distances $\mathbf{x}$ are present, resulting in the Efimov effect, although the strength of the potential is somewhat changed. A direct comparison of Eq. (65) with that in Eq. (52) is difficult because two different variables $\mathbf{x}$ and $\mathbf{R}$ are involved.

We emphasize that the result of this section using the modified BO ansatz is similar to the treatment given in Sec. III, mainly because the transformation T changes the variables $\mathbf{y}$ to $\mathbf{r}=\mathbf{y}+\mathbf{x}/2$. Therefore, the DCS approach adopts the ($\mathbf{r}$,$\mathbf{R}$) set, while in Sec. IV we have effectively ($\mathbf{r}$,$\mathbf{x}$) rather than the set ($\mathbf{y}$,$\mathbf{x}$) used in Sec. II and in previous work. The mixed set ($\mathbf{r}$,$\mathbf{x}$) obviously introduces the mass polarization term.

V. CONCLUSION

We have reexamined the Efimov effect within the framework of the adiabatic state representation, where the necessary effective potential with the $-1/x^2$ behavior is obtained explicitly in the lowest order of the approximation. However, it is important to make sure that the nonadiabatic correction term does not affect this general behavior of the adiabatic potential. In Ref. [5], the leading correction term was shown to exhibit the pseudo-Coulomb disease (PCD), although it may be cancelled by the higher order corrections [6].

We have shown in Sec. III that a proper choice of the Jacobi coordinates improved the behavior of the leading nonadiabatic correction potential u and thus eliminated the PCD. In Sec. IV it was further shown that by giving a boost to the ''conventional'' adiabatic system, i.e., by allowing the heavy particles to move from their fixed position via the particle boost factor (PBF), the model has been improved again and the PCD disappeared. Of course these two approaches (DCS and PBF) are related; the PBF is effectively similar to the choice of new coordinates from Sec. III.

It is surprising that such a PBF is needed in the present case of nearly zero-energy collision. In fact, the energy scale involved here for the motion of the heavy particles relative to the near-zero energy of the pair may be such that the relative smallness is the essential factor; the energy of the heavy particle motion is still ''high'' relative to the small near-zero binding energy for the pair.

Note added in proof. The nonlocal model studied here and in Refs. [4] and [5] requires careful analysis, because the long range aspect of the problem is crucial. A numerical study of this important question has been carried out recently [9]. The discussion on the reduced mass in the fast Hamiltonian in that paper is consistent with the discussion given in Sec. II of the present paper, in that, in order to obtain a consistent solution, the same reduced mass (either μ_2 or μ_3) should be used in the fast Hamiltonian and in the definition of the coupling constant for the two-body zero-energy bound state.

ACKNOWLEDGMENTS

One of the authors (Y.H.) would like to thank Dr. B. Giraud of Saclay for many fruitful discussions and collaboration, and also Dr. Y. Abe for the hospitality at the Yukawa Theory Institute, Kyoto, where the problem was discussed. Correspondence with Dr. LeTourneux of the McGill University is also gratefully acknowledged.

APPENDIX A: α AND β INTEGRALS

The adiabatic wave function $\chi_R(\mathbf{q})$ defined by Eq. (33) in the momentum space is explicitly evaluated. It involves two constants α and β. First, α is defined as

$$\alpha=\langle f|\chi_{\mathbf{R}}\rangle=\int d^3q \frac{\lambda'}{\pi^2(q^2+\omega^2)(q^2+\gamma^2)}\left[\frac{\alpha}{q^2+\gamma^2}+\frac{\beta T_A}{A(q^2+\gamma^2A^2)}\right]. \tag{A1}$$

The angular integration gives

$$\frac{2\lambda'}{\pi}\int_0^\infty dq \frac{q^2}{(q^2+\omega^2)(q^2+\gamma^2)}\left[\frac{2\alpha}{q^2+\gamma^2}+\frac{\beta(e^{(iqR/A)}-e^{-(iqR/A)})}{iqR(q^2+\gamma^2A^2)}\right], \tag{A2}$$

which is transformed into a contour integral

$$\frac{2\lambda'}{\pi}\oint dq \frac{q}{(q^2+\omega^2)(q^2+\gamma^2)}\left[\frac{\alpha q}{q^2+\gamma^2}+\frac{\beta e^{(iqR/A)}}{iR(q^2+\gamma^2A^2)}\right]. \tag{A3}$$

The contour integration in the limit of $R\to\infty$ gives

$$\alpha=\lambda'\left(\frac{\alpha}{\gamma(\gamma+\omega)^2}+\frac{2\beta}{R}\frac{e^{(-\omega R/A)}}{(\gamma^2A^2-\omega^2)(\gamma^2-\omega^2)}\right). \tag{A4}$$

Next, the constant β is defined by

$$\beta=\langle f_A|T_A^{-1}|\chi_{\mathbf{R}}\rangle=\int d^3q \frac{\lambda'}{\pi^2A(q^2+\omega^2)(q^2+\gamma^2A^2)}\left[\frac{\alpha T_A^{-1}}{q^2+\gamma^2}+\frac{\beta}{A(q^2+\gamma^2A^2)}\right]. \tag{A5}$$

By comparison of Eq. (A5) with Eq. (A1) we see that the substitutions $\lambda'\to\lambda'/A$, $\alpha\leftrightarrows\beta/A$, $\gamma\leftrightarrows\gamma A$, make the integral for β exactly as that for α. We thus have immediately

$$\beta=\frac{\lambda'}{A}\left(\frac{\beta}{\gamma A^2(A\gamma+\omega)^2}+\frac{2\alpha A}{R}\frac{e^{(-\omega R/A)}}{(\gamma^2A^2-\omega^2)(\gamma^2-\omega^2)}\right). \tag{A6}$$

APPENDIX B: EVALUATION OF THE INTEGRAL $\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle$

The leading nonadiabatic correction given by Eqs. (32), (42) contains an integral $\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle$ which is evaluated here in the two cases, the ''weak'' and ''strong'' couplings.

(i) $\lambda'=\gamma^3A^5$ ''*weak*.'' We have from Eqs. (35), (36),

$$\frac{\alpha}{\beta}=\frac{2A^2\omega}{(1-A^5)\gamma},\quad N(\mathbf{R})=\frac{\gamma^2A^3}{\pi}\left(\frac{c}{R}\right)^{(1/2)},$$

$$\xi(\mathbf{q},\mathbf{R})=\frac{1}{(q^2+\omega^2)}\left[\frac{\alpha}{\beta}\frac{1}{q^2+\gamma^2}+\frac{T_A}{A(q^2+\gamma^2A^2)}\right], \tag{B1}$$

$$\nabla_{\mathbf{R}}\xi(\mathbf{q},\mathbf{R})=\frac{\alpha}{\beta}\frac{\hat{\mathbf{R}}}{(q^2+\gamma^2)(q^2+\omega^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)+\frac{T_A}{A(q^2+\gamma^2A^2)(q^2+\omega^2)}\left(\frac{2\omega^2\hat{\mathbf{R}}}{R(q^2+\omega^2)}+\frac{i\mathbf{q}}{A}\right). \tag{B2}$$

Thus the integral becomes

$$\begin{aligned}\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle=&\int d^3q\left\{\frac{\alpha}{\beta}\frac{1}{(q^2+\gamma^2)(q^2+\omega^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)\right\}^2\\&+\int d^3q\frac{\alpha}{\beta}\frac{2\omega^2(T_A+T_A^{-1})}{RA(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)\\&+\int d^3q\frac{\alpha}{\beta}\frac{i\mathbf{q}\cdot\hat{\mathbf{R}}(T_A-T_A^{-1})}{A^2(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)\\&+\int d^3q\frac{1}{A^2R(q^2+w^2)^2(q^2+\gamma^2A^2)^2}\left(\frac{4\omega^4}{R^2(q^2+\omega^2)^2}+\frac{q^2}{A^2}\right).\end{aligned} \tag{B3}$$

After performing an angular integration, Eq. (B3) can be expressed in a form involving contour integrals

$$\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle=2\pi\oint dq\left[\frac{\alpha}{\beta}\frac{q}{(q^2+\gamma^2)(q^2+\omega^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)\right]^2$$
$$+4\pi\oint dq\,\frac{\alpha}{\beta}\frac{q^2e^{iqR/A)}}{AR(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)$$
$$+4\pi\oint dq\,\frac{\alpha}{\beta}\frac{qe^{iqR/A}}{iR^2(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)$$
$$+8\pi\oint dq\,\frac{\alpha}{\beta}\frac{\omega^2qe^{iqR/A)}}{iR^2(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}-\frac{1}{R}\right)$$
$$+2\pi\oint dq\,\frac{q^2}{A^2R(q^2+\omega^2)^2(q^2+\gamma^2A^2)^2}\left(\frac{4\omega^4}{R^2(q^2+\omega^2)^2}+\frac{q^2}{A^2}\right). \tag{B4}$$

In the limit of $R\to\infty$, the main contribution is given by the last integral in Eq. (B4) and all other terms go to zero faster. Thus,

$$\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle\cong\frac{1}{N^2}\,\gamma\,\frac{c}{R}. \tag{B5}$$

(ii) $\lambda'=\gamma^3$ "*strong*." In this case, we have

$$\frac{\alpha}{\beta}=-\frac{(1-A^5)\gamma}{2A^2\omega};\quad N(\mathbf{R})=\frac{2\gamma^2A^2}{\pi(1-A^5)}\,\omega^{3/2}, \tag{B6}$$

$$\xi(\mathbf{q},\mathbf{R})=\frac{1}{(q^2+\omega^2)}\left[\frac{\alpha}{\beta}\frac{1}{q^2+\gamma^2}+\frac{T_A}{A(q^2+\gamma^2+A^2)}\right],$$

$$\nabla_{\mathbf{R}}\xi(\mathbf{q},\mathbf{R})=\frac{\alpha}{\beta}\frac{\hat{\mathbf{R}}}{(q^2+\gamma^2)(q^2+\omega^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)+\frac{T_A}{A(q^2+\gamma^2A^2)(q^2+\omega^2)}\left(\frac{2\omega^2\hat{\mathbf{x}}}{x(q^2+\omega^2)}+\frac{i\mathbf{q}}{A}\right). \tag{B7}$$

Therefore, the integral becomes

$$\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle=\int d^3q\left\{\frac{\alpha}{\beta}\frac{1}{(q^2+\gamma^2)(q^2+\omega^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)\right\}^2$$
$$+\int d^3q\,\frac{\alpha}{\beta}\frac{2\omega^2(T_A+T_A^{-1})}{RA(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)$$
$$+\int d^3q\,\frac{\alpha}{\beta}\frac{i\mathbf{q}\cdot\hat{\mathbf{R}}(T_A-T_A^{-1})}{A^2(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)$$
$$+\int d^3q\,\frac{1}{A^2R(q^2+\omega^2)^2(q^2+\gamma^2A^2)^2}\left(\frac{4\omega^4}{R^2(q^2+\omega^2)^2}+\frac{q^2}{A^2}\right). \tag{B8}$$

After performing an angular integration, this integral can be expressed as a contour integral,

$$\langle\nabla_{\mathbf{R}}\xi|\nabla_{\mathbf{R}}\xi\rangle=2\pi\oint dq\left[\frac{\alpha}{\beta}\frac{q}{(q^2+\gamma^2)(q^2+\omega^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)\right]^2$$
$$+4\pi\oint dq\,\frac{\alpha}{\beta}\frac{q^2e^{iqR/A}}{AR(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)$$
$$+4\pi\oint dq\,\frac{\alpha}{\beta}\frac{qe^{iqR/A}}{iR^2(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)$$
$$+8\pi\oint dq\,\frac{\alpha}{\beta}\frac{\omega^2qe^{iqR/A}}{iR^2(q^2+\gamma^2)(q^2+\omega^2)^2(q^2+\gamma^2A^2)}\left(\frac{2\omega^2}{R(q^2+\omega^2)}+\frac{1}{R}\right)$$
$$+2\pi\oint dq\,\frac{q^2}{A^2R(q^2+\omega^2)^2(q^2+\gamma^2A^2)^2}\left(\frac{4\omega^4}{R^2(q^2+\omega^2)^2}+\frac{q^2}{A^2}\right). \tag{B9}$$

In the limit of $R \to \infty$, the main contribution is given by the first integral in Eq. (B9), and all others go to zero. Thus

$$\langle \nabla_{\mathbf{R}} \xi | \nabla_{\mathbf{R}} \xi \rangle \cong \frac{1}{N^2} \frac{2}{R^2}. \tag{B10}$$

[1] V. Efimov, Yad. Fiz. **12**, 1080 (1970) [Sov. J. Nucl. Phys. **12**, 589 (1971)]; V. Efimov, Phys. Lett. **33B**, 563 (1970); V. Efimov, Nucl. Phys. **A210**, 579 (1973).

[2] R. D. Amado and F. C. Greenwood, Phys. Rev. D **7**, 2517 (1973).

[3] A. C. Phillips, Rep. Prog. Phys. **40**, 905 (1977).

[4] A. C. Fonseca, E. F. Redish, and P. E. Shanley, Nucl. Phys. **A320**, 273 (1979).

[5] B. G. Giraud and Y. Hahn, Nucl. Phys. **A588**, 653 (1995).

[6] Y. Hahn and B. G. Giraud, Z. Phys. A (to be published).

[7] Y. Hahn, Phys. Rev. **154**, 981 (1967).

[8] D. R. Bates and R. McCarroll, Adv. Phys. **11**, 39 (1962).

[9] Y. Abe, B. G. Giraud, Y. Hahn, H. Inamori, and J. LeTourneaux, Phys. Rev. C **56**, 2557 (1997).