Interquark potential model for multiquark systems

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A potential model for four interacting quarks is constructed in SU(2) from six basis states—the three partitions into quark pairs, where the gluon field is either in its ground state or first excited state. With four independent parameters to describe the interactions connecting these basis states, it is possible to fit 100 pieces of data—the ground and first excited states of configurations from six different four-quark geometries calculated on a $16^3 \times 32$ lattice. [S0556-2813(98)04006-0]

I. INTRODUCTION

Over the years there has been much progress in lattice QCD. However, this has been restricted to systems with only a few quarks—up to three in most cases. On the other hand, in particle and nuclear physics there is also considerable interest in *few-hadron* and *multihadron* systems beginning with the possibility of bound $K\bar{K}$ states. It seems unlikely that direct results for any but the simplest of these hadron systems could be obtained in the near future from lattice QCD simulations. Our approach is thus to construct a model that reproduces lattice results for the simplest multiquark systems in a way that can be relatively easily extended to more complex cases. The lattice data were obtained earlier by the Helsinki group [1] (and references therein) as energies of four static quarks in various geometries, such as quarks at the corners of squares, rectangles, and tetrahedra, and some other less symmetric geometries. These ''experimental'' data are then to be explained by a model. The choice of geometries is hopefully general enough for the model to reproduce also the energies for geometries intermediate to those actually considered.

It should be pointed out that the ''experimental'' data are not for full QCD. For numerical reasons, the SU(2) gauge group is used instead of SU(3), the use of the latter requiring an order of magnitude increase in computer resources. However, the indications are that, for the present type of discussion, SU(2) suffices. Another approximation is taking the quarks to have an infinite mass. This means that our constituent quarks are static and that there are no sea quarks (the so-called quenched approximation). At present, this limit is now being partially removed by applying the techniques of Ref. [2] to a system of two light and two heavy (static) quarks. The sea quarks will appear through using gauge configurations generated by the UKQCD Collaboration for full QCD. However, in this paper, we only attempt to understand systems with four static SU(2) quarks in the quenched approximation.

Any model that can be extended to multiquark systems must presumably depend explicitly only on the quark degrees of freedom, with the gluon degrees of freedom entering only implicitly. This is the same philosophy that has been successful for interacting multinucleon systems. In the latter, the meson fields generating the interactions are ''summarized'' as internucleon potentials, which take the form of two-nucleon potentials and, to a lesser extent, three- and four-nucleon potentials. However, an important difference between multiquark and multinucleon systems is the strongly self-interacting nature of the gluon fields mediating the interaction in the former case. It is, therefore, not at all clear that *any* effective model defined in terms of only quark degrees of freedom will be successful. It is the purpose of this article to see to what extent a model can be developed for the four-quark case. If this attempt fails, then there is no point in expecting it to work in even more complex quark problems; a successful model for four quarks is necessary but not sufficient before considering any extension to even more quarks.

It should be added that, in spite of the problems outlined above, many groups still consider that multiquark systems can be treated using simply the basic two-quark potential, e.g., [3,4]. In the opinion of the authors—and backed up by their earlier works—there seems to be no justification for such an approach.

In Sec. II three versions of the model, starting with the simplest, are introduced and in Sec. III results are given.

II. MODEL WITH TWO, THREE, OR SIX BASIS STATES

Since the four quarks in the lattice calculation are static, the corresponding model should not contain any kinetic energy. Also, because we only consider SU(2), there is no distinction between the group properties of quarks (q) and antiquarks ($\bar{q}$). Four such quarks can then be partitioned as pairs in three different ways:

$$A = (q_1 q_3)(q_2 q_4), \quad B = (q_1 q_4)(q_2 q_3),$$

$$\text{and} \quad C = (q_1 q_2)(q_3 q_4), \tag{2.1}$$

where each $(q_i q_j)$ is a color singlet. However, these three basis states are not orthogonal to each other. Also, remembering the fact that the quarks are indeed fermions gives, in the weak coupling limit, the condition in the Appendix of Ref. [5]:

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$$|A+B+C\rangle=0. \tag{2.2}$$

Since $\langle A|A\rangle=\langle B|B\rangle=\langle C|C\rangle=1$, we get—in this limit—the equalities $\langle A|B\rangle=\langle B|C\rangle=\langle A|C\rangle=-1/2$.

A. Two basis states

Restricting the basis to only the two states A and B leads to the normalization matrix

$$\mathbf{N}=\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}. \tag{2.3}$$

In addition to this there is the potential matrix

$$\mathbf{V}=\begin{pmatrix} V_{AA} & V_{AB} \\ V_{BA} & V_{BB} \end{pmatrix}. \tag{2.4}$$

In the weak coupling limit—when all quarks are close to each other—we expect the potential matrix to simply become that form appropriate for one-gluon exchange, i.e.,

$$V_{ij}=-\frac{1}{3}\sum_{i<j} \vec{\tau}_i\cdot\vec{\tau}_j v_{ij}, \tag{2.5}$$

where $v_{ij}=-e/r_{ij}$. Then $V_{AA}=v_{13}+v_{24}$, etc., and

$$V_{AB}=V_{BA}=-\frac{1}{2}(v_{13}+v_{24}+v_{14}+v_{23}-v_{12}-v_{34}). \tag{2.6}$$

The object is then to compare the eigenvalues of

$$[\mathbf{V}-E(4)\mathbf{N}]=0 \tag{2.7}$$

with the lattice results—the success or failure of the model being to what extent the two agree. Here $E(4)$ is the total four-quark energy. However, in practice, it is more convenient to deal with the four-quark *binding* energy E defined as $E=E(4)-V_{AA}$, assuming that the A partition has the lowest energy. A perturbative calculation to $O(\alpha^2)$ has reproduced this two-state model [6].

If all three basis states are included, the model has a problem since the matrix in Eq. (2.7) is singular for the obvious reason that $|A+B+C\rangle=0$. In some of our earlier work this was interpreted to mean that it was unnecessary to include all three states and so the symmetry was broken by keeping the two states with the lowest energy, let us say A and B. A similar thing also occurred in the lattice simulations. There it was found that the energy of the lowest state was always the same in both a 2×2 and 3×3 description, providing A or B had the lowest energy. In addition the energy of the second state was, in most cases, more or less the same—the largest difference occurring with the tetrahedral geometry. Another modification to the basic model in Eqs. (2.5)–(2.7) is to replace the original one-gluon-exchange potential by the full two-quark potential

$$v_{ij}=-e/r_{ij}+b_s r_{ij}+c. \tag{2.8}$$

This is now a move away from the extreme weak coupling limit. In principle, everything is now known, since the parameters e, b_s, and c in v_{ij} can be determined from the same

lattice calculation as the four-quark energies. However, this model fails badly, since it still contains the van der Waals long range forces. The main feature of this failure is that the lowest eigenvalue (E_0) is too low and the other (E_1) too high. This suggests that the off-diagonal potential V_{AB} is too large. However, as we go away from the weak coupling limit, it is expected that $\langle A|B\rangle$ will decrease until eventually, in the extreme strong coupling limit, it would vanish. This, therefore, suggests that for some realistic intermediate coupling the off-diagonal normalization matrix element N_{AB} should also be reduced. This effect we simulate by introducing a factor f, which decreases as the distance between the quarks increases, i.e.,

$$\langle A|B\rangle=-f/2. \tag{2.9}$$

For internal consistency, this same factor must also be introduced into V_{AB}, i.e.,

$$V_{AB}\rightarrow fV_{AB}; \tag{2.10}$$

otherwise, the eigenvalues would depend on the self-energy term c in Eq. (2.8). In Ref. [1] this model had the good feature that, when fitting the data (E_0,E_1) for a given square, only a single f was necessary to fit both energies. Of course, f was dependent on the size of the square, but a reasonable parametrization was

$$f(Ia)=\exp(-b_s k_f S), \tag{2.11}$$

where b_s was the string energy of Eq. (2.8), S the area of the square, and $k_f\approx0.5$. The form of this parametrization was motivated by strong coupling ideas [7–9]. The original hope was that, with k_f determined from the squares, the model should automatically also fit other geometries with S being the appropriate area contained by the four quarks. When the four quarks lie in a plane, the definition of S is clear. However, in nonplanar cases the situation is more complicated and so here the area is simply taken to be the average of the sum of the four triangular areas defined by the positions of the four quarks, i.e., the faces of the tetrahedon. For example, in the notation of Eq. (2.1), the appropriate area $S(AB)$ for f is

$$S(AB)=0.5[S(431)+S(432)+S(123)+S(124)], \tag{2.12}$$

where $S(ijk)$ is the area of the triangle with corners at i,j, and k. For planar geometries this simply reduces to the expected area. But for nonplanar cases this is only an approximation to $S(AB)$—a more correct area being one that is not necessarily a combination of planar areas but of curved surfaces with minimum areas. These possibilities are discussed in Ref. [10]. It would be feasible to incorporate this refinement here, since only a few (≈50) such areas are needed. But for a general situation, in which the positions of the quarks are integrated over, it would become impractical, since the expression for the minimum area itself involves a double integration. In contrast, the area used here is an algebraic expression and is, therefore, more readily evaluated for any geometry. The above model will be referred to as version Ia.

This model has only one free parameter k_f in f. Another possibility with additional parameters f_0 and k_P is

$$f(Ib) = f_0 \exp(-b_s k_f S - \sqrt{b_s} k_P P) \quad \text{(version Ib)},$$
$$(2.13)$$

where P is the perimeter bounding S. However, as shown in Ref. [11], this reduces in the continuum limit to the same as version Ia—the differences at $\beta = 2.4$ being mainly due to lattice artifacts.

Unfortunately, both of these models have the feature that, for regular tetrahedra, they are unable to reproduce a degenerate ground state with a *nonzero* energy, since the two eigenvalues are

$$E_0 = -\frac{f/2}{1+f/2}[V_{CC} - V_{AA}]$$

and

$$E_1 = \frac{f/2}{1-f/2}[V_{CC} - V_{AA}], \quad (2.14)$$

and for regular tetrahedra $V_{CC} = V_{AA}$.

B. Three basis states

In the previous subsection only two of the possible three basis states A, B, and C were used, the original reason for this being the condition in Eq. (2.2). However, once the factor f is introduced as in Eqs. (2.9) and (2.10) this condition no longer holds, so that all three states can be incorporated. This apparently leads to the need for two more factors f' and f'' defined by

$$\langle A|C \rangle = -f'/2 \quad \text{and} \quad \langle B|C \rangle = -f''/2. \quad (2.15)$$

However, with the parametrization of f as in Eq. (2.11) or (2.13) and the definition of S as in Eq. (2.12), it is seen that $f' = f'' = f$, since S is simply proportional to the area of the faces of the tetrahedron defined by the four quark positions and is *independent* of the state combination used. Therefore, a 3×3 model has the form where the N and V matrices are

$$\mathbf{N}(f) = \begin{pmatrix} 1 & -f/2 & -f/2 \\ -f/2 & 1 & -f/2 \\ -f/2 & -f/2 & 1 \end{pmatrix} \quad (2.16)$$

and

$$\mathbf{V}(f) = \begin{pmatrix} V_{AA} & fV_{AB} & fV_{AC} \\ fV_{BA} & V_{BB} & fV_{BC} \\ fV_{CA} & fV_{CB} & V_{CC} \end{pmatrix}. \quad (2.17)$$

Unfortunately, this model (II) also has some unpleasant features.

(i) Again for regular tetrahedra all three eigenvalues are zero.

(ii) For a linear geometry, since the "appropriate" area as defined by Eq. (2.12) vanishes, we get $f = 1$; i.e., we are back to the weak coupling limit and a singular matrix.

(iii) For squares the model gives $E_1 = -E_0$, whereas the predictions of model I in Eqs. (2.14) seem to be nearer the data.

The regular tetrahedron result is clear. There is only one energy scale in the model, since all the v_{ij} are the same. Therefore, there cannot be any excitations. The positive feature of this model is that all three states are treated equally—a point that seems to be necessary for regular tetrahedra.

C. Six basis states

The above models both have trouble in describing regular tetrahedra. This might be considered a minor point to worry about, since such a configuration is very special. However, our philosophy is that, if *any* configuration cannot be fitted, then the model fails, since then there is no reason to expect configurations not checked explicitly to be fitted. An interesting feature of the regular tetrahedron data is that the lowest state becomes *more* bound as the tetrahedron increases in size with the magnitude of E_0 increasing from $-0.0202(8)$ to $-0.028(3)$ as the d^3 cube containing the tetrahedron increases from $d = 2$ to $d = 4$. This is opposite to what happens with squares, where the magnitude of E_0 decreases from $-0.0572(4)$ to $-0.047(3)$ as d increases from 2 to 5. This indicates that there is coupling to some higher state(s) that becomes more effective as the size increases and suggests that these higher states contain a gluon excitation with respect to the A, B, and C configurations. Model III, introduced in Ref. [11], therefore, extends model II by adding three more states A^*, B^*, and C^*, where, in analogy with Eq. (2.1),

$$A^* = (q_1 q_3)_{E_u}(q_2 q_4)_{E_u}, \quad \text{etc.} \quad (2.18)$$

Here $(q_1 q_3)_{E_u}$ denotes a state where the gluon field is excited to the lowest state with the symmetry of the E_u representation of the lattice symmetry group D_{4h}. Because it is an odd parity excitation, A^*, B^*, and C^* must contain two such states in order to have the same parity as A, B, and C. The excitation energy of an E_u state over its ground state (A_{1g}) counterpart is $\approx \pi/R$ for two quarks a distance R apart. As R increases this excitation energy decreases making the effect of the A^*, B^*, and C^* states more important, leading to the effect mentioned above. Here we have assumed that these states arise from a combination of excited states with E_u. However, it is possible that they involve other excitations, e.g.,

$$A^* = (q_1 q_3)_{A'_{1g}}(q_2 q_4), \quad \text{etc.}, \quad (2.19)$$

where the A'_{1g} state is a gluonic excitation with the same quantum numbers as the ground state (A_{1g}). For this case the following formalism would be essentially the same. Another possibility, which is not considered here, is that the relevant excitations are flux configurations where all four quarks, instead of two, are involved in forming a color singlet. In the strong coupling approximation such states would reduce to two-body singlets due to Casimir scaling of the string tensions—the string tension for a higher representation would be more than double the value of the fundamental

string tension, thus preventing junctions of two strings in the fundamental and one in the higher representation. This would happen both in SU(2) and SU(3), the only exception being the unexcited C state in SU(3), which would involve an antitriplet string.

There are now several new matrix elements that need to be discussed for $\mathbf{N(f)}$ and $\mathbf{V(f)}$.

(a) With the addition of the A^*, B^*, and C^* states and the antisymmetry condition $A^* + B^* + C^* = 0$ analogous to Eq. (2.1), there are now two more functions $f^{a,c}$ defined as

$$\langle A^*|B^*\rangle = \langle A^*|C^*\rangle = \langle B^*|C^*\rangle = -f^c/2$$

and

$$\langle A^*|B\rangle = \langle A^*|C\rangle = \cdots = -f^a/2. \quad (2.20)$$

Here it is assumed that $f^{a,c}$ are both dependent on S as defined in Eq. (2.12). Since f^c involves only the excited states, it is reasonable to expect it has a form similar to f in Eq. (2.11), i.e.,

$$f^c = \exp(-b_s k_c S). \quad (2.21)$$

(b) By orthogonality $\langle A|A^*\rangle = \langle B|B^*\rangle = \langle C|C^*\rangle = 0$.

(c) In the weak coupling limit, from the $A^* + B^* + C^* = 0$ condition, we expect $\langle A|B^*\rangle = \langle B|C^*\rangle = \cdots = 0$ at small distances. To take this into account we parametrize f^a as

$$f^a = (f_1^a + b_s f_2^a S)\exp(-b_s k_a S). \quad (2.22)$$

A partial justification of this is given in Fig. 6 of Ref. [11]. In the following fit all three parameters f_1^a, f_2^a, and k_a are varied. However, it is found that f_1^a is always consistent with zero—as expected from the above condition $\langle A|B^*\rangle = \cdots = 0$. Therefore, in most of this work f_1^a is fixed at zero.

(d) For the potential matrix $\mathbf{V(f)}$ the diagonal matrix elements, after the lowest energy V_{DD} among the basis states is removed, are

$$\langle A^*|V - V_{DD}|A^*\rangle = v^*(13) + v^*(24) - V_{DD}, \quad \text{etc.,}$$

where $v^*(ij)$ is the potential of the E_u state—a quantity also measured on the lattice along with the four-quark energies—and

$$V_{DD} = \min[V_{AA} = v(13) + v(24),$$

$$V_{BB} = v(14) + v(23), \quad V_{CC} = v(12) + v(34)].$$

However, to allow more freedom in the following fits, we introduce a parameter b_0, where

$$\langle A^*|V - V_{DD}|A^*\rangle = V_{AA} - V_{DD} + b_0 V_{AA}^*,$$

$$\langle B^*|V - V_{DD}|B^*\rangle = V_{BB} - V_{DD} + b_0 V_{BB}^*,$$

$$\langle C^*|V - V_{DD}|C^*\rangle = V_{CC} - V_{DD} + b_0 V_{CC}^*, \quad (2.23)$$

where

$$V_{AA}^* = v^*(13) + v^*(24) - V_{AA},$$

$$V_{BB}^* = v^*(14) + v^*(23) - V_{BB},$$

$$V_{CC}^* = v^*(12) + v^*(34) - V_{CC}.$$

In the following fit b_0 is expected to be of the order unity and, indeed, fixing $b_0 = 1$ is found to be a good assumption.

The two-quark potentials $v(ij)$ are taken to be more elaborate than the three-term form of Eq. (2.8). They are fitted to the lattice data using [12]

$$v(r_{ij}) = 0.562 + 0.0696 r_{ij} - \frac{0.255}{r_{ij}} - \frac{0.045}{r_{ij}^2}. \quad (2.24)$$

Similarly, the excitation with E_u symmetry is fitted by

$$\Delta v = v^*(ij) - v(ij) = \frac{\pi}{r_{ij}} - \frac{4.24}{r_{ij}^2} + \frac{3.983}{r_{ij}^4}. \quad (2.25)$$

The extra terms containing r_{ij}^{-2} and r_{ij}^{-4} are purely for numerical reasons and ensure that the fitted values of $v(ij)$ and $v^*(ij)$ are, on average, well within 1% of the lattice values for all $r_{ij} \geq 2$.

(e) There are two types of off-diagonal elements:

$$\langle A^*|V|B^*\rangle \quad \text{and} \quad \langle A|V|B^*\rangle. \quad (2.26)$$

Unlike V_{AB} in Eq. (2.6), there is now no guide from the one-gluon-exchange limit of Eq. (2.5). Therefore, a further assumption is necessary for the form of these matrix elements. Several different forms were tried, but the most successful seems to be the one where the matrix elements needed in $[\mathbf{V} - (E + V_{AA})\mathbf{N}] = 0$ are expressed as follows:

$$\langle A^*|V + \frac{f^c}{2} V_{DD}|B^*\rangle = -\frac{f^c}{2}[V_{ABCD} + c_0 V_{ABC}^*]$$

$$= \langle B^*|V + \frac{f^c}{2} V_{DD}|A^*\rangle,$$

$$\langle A^*|V + \frac{f^c}{2} V_{DD}|C^*\rangle = -\frac{f^c}{2}[V_{CABD} + c_0 V_{CAB}^*]$$

$$= \langle C^*|V + \frac{f^c}{2} V_{DD}|A^*\rangle,$$

$$\langle B^*|V + \frac{f^c}{2} V_{DD}|C^*\rangle = -\frac{f^c}{2}[V_{BCAD} + c_0 V_{BCA}^*]$$

$$= \langle C^*|V + \frac{f^c}{2} V_{DD}|B^*\rangle, \quad (2.27)$$

where

$$V_{ABCD} = V_{AA} + V_{BB} - V_{CC} - V_{DD},$$

$$V_{CABD} = V_{CC} + V_{AA} - V_{BB} - V_{DD},$$

$$V_{BCAD} = V_{BB} + V_{CC} - V_{AA} - V_{DD},$$

$$V^*_{ABC} = V^*_{AA} + V^*_{BB} - V^*_{CC},$$

$$V^*_{CAB} = V^*_{CC} + V^*_{AA} - V^*_{BB}, \quad V^*_{BCA} = V^*_{BB} + V^*_{CC} - V^*_{AA}.$$

These are seen to be simply the generalization of Eq. (2.6) to the interaction involving excited gluon states. Also in analogy with b_0 in Eqs. (2.23), c_0 is a free parameter—hopefully of order unity.

Likewise,

$$\langle A | V + \frac{f^a}{2} V_{DD} | B^* \rangle = -\frac{f^a}{2} \left[V_{ABCD} + a_0 \frac{V^*_{ABC}}{2} \right]$$

$$= \langle A^* | V + \frac{f^a}{2} V_{DD} | B \rangle, \qquad (2.28)$$

where, again in analogy with b_0 in Eqs. (2.23), a_0 is a free parameter—hopefully of order unity. However, it should be added that this hope is even less well founded than the one for c_0.

In the special case of regular tetrahedra $V_{DD} = V_{AA} = V_{BB} = V_{CC}$ and **V** reduces to the form

$$\mathbf{V} = \begin{bmatrix} V_{AA} & -fV_{AA}/2 & -fV_{AA}/2 & 0 & -f^a V_a/2 & -f^a V_a/2 \\ -fV_{AA}/2 & V_{AA} & -fV_{AA}/2 & -f^a V_a/2 & 0 & -f^a V_a/2 \\ -fV_{AA}/2 & -fV_{AA}/2 & V_{AA} & -f^a V_a/2 & -f^a V_a/2 & 0 \\ 0 & -f^a V_a/2 & -f^a V_a/2 & V_b & -f^c V_c/2 & -f^c V_c/2 \\ -f^a V_a/2 & 0 & -f^a V_a/2 & -f^c V_c/2 & V_b & -f^c V_c/2 \\ -f^a V_a/2 & -f^a V_a/2 & 0 & -f^c V_c/2 & -f^c V_c/2 & V_b \end{bmatrix}, \qquad (2.29)$$

where $V_a = V_{AA} + a_0 V^*_{AA}/2$, $V_b = V_{AA} + b_0 V^*_{AA}$, and $V_c = V_{AA} + c_0 V^*_{AA}$. As with all geometries,

$$\mathbf{N} = \begin{bmatrix} 1 & -f/2 & -f/2 & 0 & -f^a/2 & -f^a/2 \\ -f/2 & 1 & -f/2 & -f^a/2 & 0 & -f^a/2 \\ -f/2 & -f/2 & 1 & -f^a/2 & -f^a/2 & 0 \\ 0 & -f^a/2 & -f^a/2 & 1 & -f^c/2 & -f^c/2 \\ -f^a/2 & 0 & -f^a/2 & -f^c/2 & 1 & -f^c/2 \\ -f^a/2 & -f^a/2 & 0 & -f^c/2 & -f^c/2 & 1 \end{bmatrix}. \qquad (2.30)$$

The full 6×6 matrix $[\mathbf{V} - (E + V_{AA})\mathbf{N}]$ now factorizes into three 2×2 matrices, two of which are identical—giving the observed degeneracy. These have the form

$$[\mathbf{V} - (E + V_{AA})\mathbf{N}]$$

$$= \begin{bmatrix} -E(1 + f/2) & -f^a(E - V_a)/2 \\ -f^a(E - V_a)/2 & -E(1 + f^c/2) + V_b + f^c V_c/2 \end{bmatrix} = 0,$$

$$\qquad (2.31)$$

whereas the third 2×2 matrix is

$$[\mathbf{V} - (E + V_{AA})\mathbf{N}]$$

$$= \begin{bmatrix} -E(1 - f) & f^a(E - V_a) \\ f^a(E - V_a) & -E(1 - f^c) + V_b - f^c V_c \end{bmatrix} = 0. \qquad (2.32)$$

In this case the problem reduces to solving two quadratic equations for E. However, away from the regular tetrahedron the complete 6×6 matrix needs to be treated directly.

III. RESULTS

This section is in two parts. In Sec. III A the results for the six-basis-state model at $\beta = 2.4$ are given. There it is shown that only five of the possible eight parameters have any significant influence. Also it is pointed out that the restriction to the two-basis-state version of Sec. II A is distinctly inferior and that the further restriction to only two-body interactions [i.e., $k_f = 0$ in Eq. (2.11)] is for most geometries very poor. In Sec. III B, it is shown that the parameters of the model do not change significantly as β increases, i.e., as the continuum is approached. Therefore, the parameters extracted at $\beta = 2.4$ could be used directly in, for example, a resonating group calculation of a four-quark model of meson-meson scattering as in Ref. [4].

A. Six-basis-state model at $\beta = 2.4$

In Ref. [1] four quark energies have been extracted for a variety of geometries using a $16^3 \times 32$ lattice with $\beta = 2.4$. From these energies, only 100—distributed over six geometries—are selected for fitting. Configurations contain-

ing flux links of less than two lattice units were not included, since they have strong lattice artifacts. In detail, we use 15 tetrahedra (T), 6 squares (S), 12 rectangles (R), 4 quadrilaterals (Q), 9 nonplanar (NP), and 4 linear (L). Only the lowest two energies ($E_{0,1}$) from the lattice simulation are used. But, as mentioned earlier before Eq. (2.8), the values of these two energies are more or less the same from the 2×2 and 3×3 simulations. In the latter case a third energy (E_2) is in fact also available. However, it is not expected to be very reliable due to the higher excitations it contains. Its main purpose was to improve the estimate on $E_{0,1}$ by reducing the contamination from the higher states. The stability of ($E_{0,1}$) is understandable, since they are the lowest states of a given symmetry. For example, with squares their wave functions are simply $\psi(E_0) = (A + B)/\sqrt{2}$ and $\psi(E_1) = (A - B)/\sqrt{2}$.

The above 100 pieces of data are fitted with MINUIT—the Migrad option. In the model there are, in principle, eight parameters: k_f in Eq. (2.11), f_1^a, f_2^a, and k_a in Eq. (2.22), k_c in Eq. (2.21), b_0 in Eqs. (2.23), and a_0 and c_0 in Eqs. (2.27) and (2.28). However, in practice it is less. First, two can be fixed at the values $f_1^a = 0$ and $b_0 = 1$ as expected theoretically without any significant deterioration in the overall fit. Also two of the remaining parameters, f_2^a and a_0, are strongly correlated, since for some geometries they appear in the combination $a_0 f_2^a$. It is found that fixing a_0 at 4 and only varying f_1^a is a good compromise. Finally, all fits yield a value for k_c that is consistent with zero. This appears to be a common feature, suggesting that $f^c \approx 1$; i.e., the excited configurations interact among themselves simply through two-body potentials without the four-quark effect introduced when $k_c \neq 0$. In this way the data can be fitted with a $\chi^2/N_{\text{DF}} = 1.03$ using the parameter values $k_f = 1.51(8)$, $k_a = 0.55(3)$, $f_2^a = 0.51(2)$, and $c_0 = 3.9(3)$, in addition to those fixed at $k_c = 0$, $b_0 = 1$, and $a_0 = 4$. It should be added that Ref. [13] contains a preliminary fit to the above data, but with a model in which the off-diagonal matrix elements of Eqs. (2.27) and (2.28) have a slightly different form. The outcome, however, is qualitatively rather similar to the present results.

For comparison, model Ia with its one free parameter k_f in Eq. (2.11) gives from MINUIT a χ^2/N_{DF} of 3.16 for $k_f = 0.571(12)$. Most of this increase in the χ^2/N_{DF} comes from the tetrahedral geometry—especially from the regular tetrahedra. This was already anticipated at the end of Sec. II C.

Another comparison is the extreme limit of $f = 1$, where only two-quark potentials are used in model I. Here all the parameters are fixed, so that the $\chi^2/N_{\text{DF}} = 144$ is not the result of a minimization. Even so, this is clearly a very poor representation of the data. As expected, the larger the size of the geometry, the poorer the fit, since those are the cases that need most the smaller values of f. This comparison is the basis of our statement in the Introduction that multiquark models containing only the standard two-quark potential of Eq. (2.8) or even the more elaborate form in Eq. (2.24) seem to have no justification for the cases considered here.

B. Continuum limit

The above results are for $\beta = 2.4$, which corresponds to a lattice spacing of $a \approx 0.12$ fm. But, in practice, if the model is

to be used, for example, in some type of resonating group approach [4], then it is the continuum limit that is needed. Such an extrapolation would, in principle, require repeating the above lattice calculations with all six geometries for a series of increasing values of β. This we do not plan to do here. Instead we exploit the conclusions of Ref. [11]. There it was shown that the four-quark *binding energies* already scale at $\beta = 2.4$, even though the total two- and four-quark energies do not scale separately. Therefore, here we simply assume that, if some binding energy $E(\beta)$ would be scaled to the $\beta = 2.4$ lattice by the transformation $\rho E[\beta, r(\beta)/\rho] \rightarrow E(2.4)$, where $\rho = a(2.4)/a(\beta)$, then we would find that this scaled $E(2.4)$ equals the binding energy $E(r)$ we have calculated directly on the $\beta = 2.4$ lattice. Here the $r(\beta)$ and r are distances in the lattice units of the two different lattices. However, to completely specify the model at some other value of β also needs a knowledge of $v(ij, \beta)$ and $\Delta v(ij, \beta)$ corresponding to the $\beta = 2.4$ expressions in Eqs. (2.24) and (2.25). In Ref. [11] it was shown that $v(ij, \beta)$ did not scale. But this is not a problem, since $v(ij, \beta)$ is documented in the literature for other values of β—see Refs. [14–16]. These can then be scaled in the same way as the $E(\beta)$ to give $v(ij, 2.4)$. However, as found in Ref. [11] we do not expect these $v(ij, 2.4)$ to equal the $v(ij)$ in Eq. (2.24). It remains to specify $\Delta v(ij, \beta)$. Here we simply assume that it scales, so that the values at $\beta = 2.4$ can be used directly in precisely the same way as the binding energies $E(\beta)$. The rational behind this last assumption is twofold.

(i) In the model it is the *differences* Δv and $v(ij, \beta) - v(kl, \beta)$ that enter as in Eqs. (2.23), (2.27), and (2.28). Therefore, some of the lack of scaling in $v*(ij, \beta)$ and $v(ij, \beta)$ may well cancel—as in the four-quark binding energies.

(ii) Any lack of scaling that remains will be reflected in a lack of scaling of the a_0, b_0, and c_0 multiplying the Δv and $V*$'s.

The actual values of β used are 2.5 [14], 2.5(denoted by $2.5'$), 2.635 and 2.74 [15], and 2.85 [16]. In the last case, the authors present two potentials [$V(R, 5)$ and $V(\infty)$] that are appropriate for interquark distances R greater than $6a(2.85) \approx 0.17$ fm. Their own preference is $V(R, 5)$—the one we use.

The results are shown in Table I. The main observation is that, of the free parameters, only k_f appears to be a significant function of β—but even there all values are within the error bars. Therefore, the final version of the model has only four independent parameters.

A similar continuum extrapolation was carried out in Ref. [11]. However, there are several major differences compared with the above analysis.

(a) Much fewer data for each value of β was included in the fits, since only squares and tilted rectangles were analyzed. Furthermore, only the E_0's were fitted. This means that, for each β, there were only 17 pieces of data compared with the present case of 100 pieces covering 6 geometries with both E_0 and E_1.

(b) All the two- and four-quark energies were simulated at five values of β and interpolated to the same physical sizes. As scaling was found for the binding energies, they are here obtained directly from the $\beta = 2.4$ values.

TABLE I. The values of the parameters defining the interaction fixing $f_1^a = k_c = 0$, $b_0 = 1.0$, and $a_0 = 4.0$ for a series of β's.

Parameter	$\beta = 2.4$	$\beta = 2.5$	$\beta = 2.5'$	$\beta = 2.635$	$\beta = 2.74$	$\beta = 2.85$
k_f	1.51(8)	1.48(8)	1.50(8)	1.49(8)	1.45(8)	1.43(8)
k_a	0.55(3)	0.56(3)	0.56(3)	0.56(3)	0.57(3)	0.57(3)
f_2^a	0.51(2)	0.51(2)	0.51(2)	0.51(2)	0.51(2)	0.51(2)
c_0	3.9(3)	3.9(3)	3.9(3)	3.9(3)	3.9(3)	3.9(3)
χ^2/N_{DF}	1.03	1.03	1.11	1.09	1.15	1.07

(c) In order to study scaling and the effect of lattice artifacts, the errors on the data were taken to be purely statistical and so were much smaller than the present ones.

(d) The model analysis was carried out with the 2×2 model. Since this results in a quadratic equation for the energies, the step $E_0 \to f$ could be done analytically. After this the parameters k_f, k_P, and f_0 in Eq. (2.13) were extracted by a fitting procedure.

C. Comparison with other work

A fit with a potential model to some of the above four-quark binding energies was attempted earlier in Ref. [17]. However, there are several important differences compared with the present fit.

(1) The model contained only 2×2 or 3×3 bases that were essentially constructed from A, B, and C.

(2) The tetrahedral data were not included.

(3) Most of the data was for $\beta = 2.4$. Only squares included that from $\beta = 2.5$.

(4) The overlap factor is that in Eq. (2.13) but with $f_0 = 1$.

For the four geometries S, R, Q, and L the area S is the same as that used here in Eq. (2.12) and for geometry NP a curved surface is used. However, the perimeters P are determined by the underlying lattice structure of the geometry and not the usually expected perimeter. For example, for quadrilaterals, P is the perimeter of the *rectangle* enclosing the quadrilateral. The reason for this choice is an attempt to take into account lattice artifacts, which are present for the smallest configurations. These artifacts are more important in Ref. [17], since configurations containing flux lines of only one lattice link are included. In the present work the cutoff is at two lattice links. For squares the outcome in [17] was $k_f = 0.296(11)$ and $k_P = 0.080(2)$, compared with $k_f = 0.571(12)$ using model Ia above in Sec. II A. The difference between the values of these area constants is because the perimeter term, which is introduced to make artifact corrections for *small* configurations, is still important for large configurations, where rotational symmetry is restored and their presence not needed. For example, with a square of side $d = 5$, the two terms in the exponent of Eq. (2.13) are comparable, even though the lattice data for this square are $-0.042(5)$ and $-0.047(5)$ for the corresponding tilted square. In the lattice simulation the basis states have a similar form compared with the ones used in [17] but with the major difference that they are fuzzed to such an extent that these underlying basis states get transformed. Therefore, in [17], for larger configurations it would be necessary to include many other terms with different perimeters, so that on

the average the usual perimeter for a particular geometry would probably be more appropriate. This latter choice was tested in Ref. [11], where it was found to reduce in the continuum limit to the form of model Ia. Thus the perimeter term seems to measure lattice artifacts also when the physical perimeter is used.

In Ref. [10], the model in Ref. [17] is extended to include the tetrahedral data by using an 8×8 basis, where now all the states are constructed in terms of the underlying lattice. This, therefore, suffers from the above problem that the results are always dependent on the lattice even for large configurations, where rotational invariance is restored.

IV. CONCLUSIONS

In this article an interquark potential model is constructed than can explain, on average, the energies of a series of four-quark systems, in which the four static quarks are in any one of six geometries. The input energies used as data for this construction [1] were calculated on a $16^3 \times 32$ lattice for four static SU(2) quarks with $\beta = 2.4$. Two or three basis states were used in the lattice simulation, enabling reliable estimates to be made of the lowest two energy eigenvalues ($E_{0,1}$). In all, 100 pieces of data were considered suitable for confronting the model. Other pieces were rejected, if they involved configurations suspected of having strong lattice artifacts (e.g., with states constructed from less than two lattice links) or had vanishingly small binding energies (e.g., elongated rectangles with $r \gg d$).

The full model utilized six basis states A, B, C, A^*, B^*, and C^* and in its most general form had eight parameters. However, in practice, only four of these (k_f, k_a, f_2^a, and c_0) need be considered as free when fitting the data.

The parameters that are, perhaps, of most interest are those connected with the ranges of the various interactions, namely, k_f, k_a, and k_c. Here we define ''range'' as $r_{f,a,c} = \sqrt{1/b_s k_{f,a,c}}$. In model Ia, where $k_{a,c}$ are effectively infinite, we get $k_f(\text{Ia}) = 0.57(1)$, i.e., $r_f(\text{Ia}) = 5.0$ in lattice units. However, in model III as the excited states A^*, B^*, and C^* are introduced, the interaction between the basic states A, B, and C decreases by raising k_f to 1.51, giving $r_f(\text{III}) = 3.1$. But at the same time this loss of binding by the direct interaction between A, B, and C is compensated by their coupling to the A^*, B^*, and C^* states. This coupling is found to have about the *same* range $r_a = 5.1$ as $r_f(\text{Ia})$ above, whereas the direct interaction between the A^*, B^*, and C^* states seems, in all fits, to be satisfied with simply a two-quark description without any four-quark correction (i.e., $k_c = 0$). The observation that $r_f(\text{Ia}) \approx r_a$ suggests that the en-

ergy density has a range dictated by the longest range available—namely, r_a. Therefore, when the A^*, B^*, and C^* states are not explicitly present, as in model Ia, the only available range $r_f(\text{Ia})$ has to simulate the role of r_a. In the binding energies the contributions from the A^*, B^*, and C^* states rapidly dominate over those from the A, B, and C states. For example, with squares of side R, the A, B, and C states contribute only 85%, 40%, and 10% to the binding energy for $R=2$, 4, and 6, respectively. Of course, at the largest distances (≈ 0.7 fm) the quenched approximation is expected to break down and the role of quark-pair creation to become important. We therefore come to the following scenario for the four-quark interaction. At the shortest distances, up to about 0.2 fm, perturbation theory is reasonable with the binding being given mainly by the A, B, and C states interacting simply through the two-quark potentials with little effect from four-quark potentials. However, for intermediate ranges, from about 0.2 to 0.5 fm, the four-quark potentials act in such a way as to reduce the effect of the A, B, and C states so that the binding is dominated by the A^*, B^*, and C^* states, which now interact among themselves again simply through the two-quark potentials *with little effect from four-quark potentials*. At larger ranges quark-pair creation can no longer be neglected.

The next step in this work would be to carry out a resonating group calculation in the spirit of Ref. [4]. This would then test the universality of the above model by extending it from the six geometries discussed here to general four-quark geometries.

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