Unambiguous amplitude analysis of the $NN\rightarrow\Delta N$ transition from asymmetry measurements

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For particular Δ-production angles, an unambiguous determination of the $NN\rightarrow\Delta N$ transition amplitudes is performed, from $NN\rightarrow(N\pi)N$ experiments, in which the polarization states are measured in the entrance channel only. A three-step method is developed, which determines, first, the magnitudes of the amplitudes, second, independent relative phases, and third, some dependent relative phases for resolving the remaining discrete ambiguities. A rule of ambiguity elimination is applied, which is based on the closure of a chain of consecutive independent relative phases, by means of the *ad hoc* dependent one. A generalization of this rule is given for the case of a nondiagonal matrix connecting observables and bilinear combinations of amplitudes.
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I. INTRODUCTION

In a previous work [1], we proposed a two-step method for determining the $NN\rightarrow\Delta N$ transition amplitudes from $NN\rightarrow(N\pi)N$ experiments. The purpose was to select a complete set of 31 observables, which determines first the 16 magnitudes of the amplitudes and, in a subsequent step, 15 independent relative phases. The decay distribution was expressed in the density matrix formalism. Helicity and transversity frames were studied. The determination of the helicity amplitudes requires experiments involving polarized states of four nucleons at a time. In the transversity frame, the use of Bohr's rules simplifies the problem, avoiding the necessity of measuring the polarization of the outgoing decay nucleon.

At this stage of the analysis, however, many discrete ambiguities remain and it is not realistic to perform transformations between helicity and transversity amplitudes [2] or combinations of them. The remaining ambiguities may be eliminated in a third step. Note that in the method we are advocating, each stage provides information independently of the further stages [3,4]. Let us remark, also, that an $NN\rightarrow(N\pi)N$ experiment provides many $NN\rightarrow\Delta N$ observables which can be used in different stages of the analysis.

The main purpose of the present work is to apply this three-step method to the $NN\rightarrow\Delta N$ transition. In this framework, we ask the question of whether it is possible to perform an unambiguous determination of the $NN\rightarrow\Delta N$ amplitudes measuring the polarization states in the entrance channel only.

The problem is solved in terms of linear combinations of helicity amplitudes, chosen in such a way that the analysis avoids the detection of the polarization of the outgoing nucleons. Using ''the magnitude first'' method requires knowledge of the magnitudes of these amplitude combinations. With asymmetry experiments only, this resolution is obviously not possible in general, but we show that indeed this program is achievable at particular Δ-production angles, namely, at $\pi/2$, 0, and π. Precise determination of the amplitudes at these angles provides us with constraints on the general analysis.

An analysis in terms of helicity amplitude combinations, with many ambiguous phases, does not provide a unique set of helicity magnitudes. The uniqueness is recovered by resolving ambiguities.

The elimination of the discrete ambiguities is performed by the determination of some additional dependent relative phases. The required number of the latter depends on the set of independent phases chosen in the second stage of the analysis, and their choice obeys precise rules of ambiguity elimination.

In any formalism, observables are expressed in terms of bilinear combinations of amplitudes (''bicoms''). In general, the matrix connecting observables and bicoms is far from diagonal and thus a given observable depends on many bicoms and vice versa. If the matrix connecting observables and bicoms is diagonal, each observable is written as the real or imaginary part of a single bicom. In this case, the method of ambiguity elimination [4] consists in closing an open chain of consecutive independent relative phases by means of the *ad hoc* dependent relative phase, in such a way that the number of imaginary parts of bicoms along the closed chain is an odd number. For an even number, the ambiguity elimination is partial, the degeneracy being of order 2.

In the present analysis, the matrix connecting observables and bicoms being not diagonal, some observables only, called ''primary observables,'' are written as the real or imaginary part of a bicom. All others, called ''secondary observables,'' are written as the real or imaginary part of sum of bicoms. Whereas the ambiguity elimination obeys rather obvious rules in the case of primary observables, their generalization to the case of secondary observables is one of the key questions. The present work resolves this crucial point.

The paper is organized as follows. Section II is devoted to the relationships between observables and amplitudes, in the

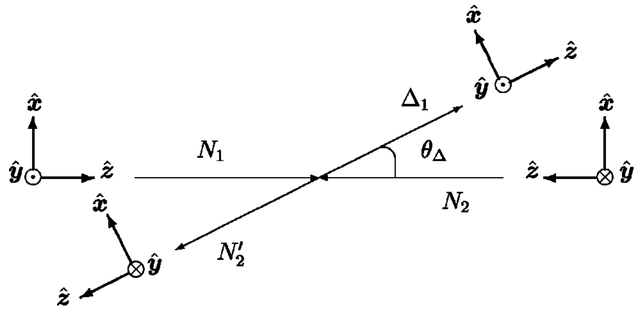

FIG. 1. Helicity frame.

helicity frame. The retained observables correspond to nonpolarized and polarized states of beam and target. In the general production angle case, it is not possible to apply the method. Nevertheless, it is possible for particular production angles. In Sec. III, the generalization of the rules of ambiguity elimination is performed for the case of secondary observables. In Sec. IV, an unambiguous helicity amplitude analysis is performed for $\theta_\Delta = \pi/2$, and in Sec. V, the case of forward and backward angles is treated.

II. OBSERVABLES AND AMPLITUDES

Relationships between observables and amplitudes are expressed in the helicity frame (see Fig. 1), for any Δ-production angle. The helicity amplitudes are denoted by $D(\lambda,l;\Lambda,L)$, where the indices λ, l, Λ, and L are the magnetic projections along the $\hat{\mathbf{z}}$ quantization axis of each particle, Δ, beam, recoil, and target nucleons, respectively. The three indices l, Λ, and L take the two values $+\frac{1}{2}$ and $-\frac{1}{2}$ and λ the four values $+\frac{3}{2}$, $+\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$. To denote the spin observables, use is made of the generalization of the standard nomenclature $\rho_{2\lambda,2\lambda'}$, for the spin-averaged density matrix elements. All the spin observables are denoted as $\mathrm{Re}(P^b_\alpha P^t_\beta P^r_\delta \rho_{2\lambda,2\lambda'})$ and $\mathrm{Im}(P^b_\alpha P^t_\beta P^r_\delta \rho_{2\lambda,2\lambda'})$, where P^b_α means that the beam nucleon is 100% polarized along its Cartesian α axis. Similarly, P^t_β and P^r_δ correspond to the target and recoil nucleon 100% polarized along its β and δ axes, respectively. In order to obtain a compact notation, the indices α, β, and δ are set equal to zero for unpolarized initial nucleons or undetected polarization of the recoil nucleon. Finally, the indices α, β, and δ take the four values 0, x, y, and z. More detailed information can be found in Ref. [1] and in quoted references therein.

The purpose of this work is to restrict the analysis to $NN \to (N\pi)N$ experiments with nonpolarized and polarized beam and target, the polarizations of the outgoing nucleons being undetected. The corresponding set of experiments involves the cross section σ and the asymmetries $A_{00\alpha\beta}$. From Eqs. (3.8) and (3.9) of Ref. [1], this set provides the following observables: $\mathrm{Re}(P^b_\alpha P^t_\beta P^r_0 \rho_{2\lambda,2\lambda'})$, where an even number of 0 and y appears globally among the two indices α and β, and $\mathrm{Im}(P^b_\alpha P^t_\beta P^r_0 \rho_{2\lambda,2\lambda'})$, where an odd number of 0 and y appears globally among the two indices α and β. The elimination of the nonselected observables leads [see Eqs. (3.17)–(3.20) of Ref. [1]] to the use of new amplitudes. These new amplitudes are defined as linear combinations of helicity amplitudes by

$$D_\pm(\lambda,l;\Lambda,L) = D(\lambda,l;\Lambda,L) \pm D(\lambda,-l;\Lambda,-L). \tag{2.1}$$

The corresponding "bicoms" are related to the selected observables by the three following expressions, for $\lambda=\lambda' = \frac{3}{2},\frac{1}{2}$ and for $\lambda=\frac{3}{2}$, $\lambda'=\pm\frac{1}{2}$:

$$\begin{pmatrix} \sum_\Lambda \mathrm{Re}\left[D_+\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)D^*_+\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Re}\left[D_+\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)D^*_+\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Re}\left[D_-\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)D^*_-\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Re}\left[D_-\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)D^*_-\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)\right] \end{pmatrix} = 2I_0 \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix} \begin{pmatrix} \mathrm{Re}(P^b_0 P^t_0 P^r_0 \rho_{2\lambda,2\lambda'}) \\ \mathrm{Re}(P^b_z P^t_z P^r_0 \rho_{2\lambda,2\lambda'}) \\ \mathrm{Re}(P^b_x P^t_x P^r_0 \rho_{2\lambda,2\lambda'}) \\ -\mathrm{Re}(P^b_y P^t_y P^r_0 \rho_{2\lambda,2\lambda'}) \end{pmatrix}, \tag{2.2}$$

$$\begin{pmatrix} \sum_\Lambda \mathrm{Re}\left[D_+\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)D^*_-\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)+D_-\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)D^*_+\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Re}\left[D_-\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)D^*_+\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)+D_+\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)D^*_-\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \end{pmatrix}$$

$$= 4I_0 \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{Re}(P^b_z P^t_x P^r_0 \rho_{2\lambda,2\lambda'}) \\ -\mathrm{Re}(P^b_x P^t_z P^r_0 \rho_{2\lambda,2\lambda'}) \end{pmatrix}, \tag{2.3}$$

and

$$\begin{pmatrix} \sum_\Lambda \mathrm{Im}\left[D_+\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)-D_-\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Im}\left[D_-\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)-D_+\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \end{pmatrix}$$
$$=4I_0\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{Re}(P_0^bP_y^tP_0^r\rho_{2\lambda,2\lambda'}) \\ -\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{2\lambda,2\lambda'}) \end{pmatrix}. \tag{2.4}$$

The summation Σ_Λ runs over $\pm\frac{1}{2}$. Let us remark that, for $\lambda=\lambda'$, Eq. (2.2) involves magnitude squares of the new amplitudes. However, the summation over Λ prevents us from obtaining each magnitude separately, and then, the ''magnitude first'' method is not applied to the θ_Δ general case.

For $\lambda=\frac{3}{2}$, $\lambda'=\pm\frac{1}{2}$, bicoms are also related to selected observables by the four following expressions:

$$\begin{pmatrix} \sum_\Lambda \mathrm{Re}\left[D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)-D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Re}\left[D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)-D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)\right] \end{pmatrix}$$
$$=4I_0\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{Im}(P_y^bP_x^tP_0^r\rho_{3,2\lambda'}) \\ \mathrm{Im}(P_x^bP_y^tP_0^r\rho_{3,2\lambda'}) \end{pmatrix}, \tag{2.5}$$

$$\begin{pmatrix} \sum_\Lambda \mathrm{Im}\left[D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)+D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Im}\left[D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)+D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)\right] \end{pmatrix}$$
$$=4I_0\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{Im}(P_z^bP_0^tP_0^r\rho_{3,2\lambda'}) \\ \mathrm{Im}(P_0^bP_z^tP_0^r\rho_{3,2\lambda'}) \end{pmatrix}, \tag{2.6}$$

$$\begin{pmatrix} \sum_\Lambda \mathrm{Re}\left[D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)-D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Re}\left[D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)-D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \end{pmatrix}$$
$$=-4I_0\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{Im}(P_z^bP_y^tP_0^r\rho_{3,2\lambda'}) \\ \mathrm{Im}(P_y^bP_z^tP_0^r\rho_{3,2\lambda'}) \end{pmatrix}, \tag{2.7}$$

and

$$\begin{pmatrix} \sum_\Lambda \mathrm{Im}\left[D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)+D_+\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_+^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \\ \sum_\Lambda \mathrm{Im}\left[D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,+\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,-\frac{1}{2}\right)+D_-\left(\frac{3}{2},\frac{1}{2};\Lambda,-\frac{1}{2}\right)D_-^*\left(\lambda',\frac{1}{2};\Lambda,+\frac{1}{2}\right)\right] \end{pmatrix}$$
$$=4I_0\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{Im}(P_0^bP_x^tP_0^r\rho_{3,2\lambda'}) \\ \mathrm{Im}(P_x^bP_0^tP_0^r\rho_{3,2\lambda'}) \end{pmatrix}. \tag{2.8}$$

From invariance under parity, helicity amplitudes satisfy

$$D(\lambda,l;\Lambda,L)=(-)^{\lambda+l+\Lambda+L+1}D(-\lambda,-l;-\Lambda,-L). \tag{2.9}$$

Then, in Eqs. (2.2)–(2.8), the new amplitudes, for $\lambda'=\pm\frac{1}{2}$, are related to one another by

$$D_{\pm}\left(-\frac{1}{2},\frac{1}{2};\Lambda,L\right)=\pm(-)^{\Lambda-L}D_{\pm}\left(+\frac{1}{2},\frac{1}{2};-\Lambda,L\right). \tag{2.10}$$

Accounting for this relationship, the 16 linearly independent amplitudes are chosen as $D_{\pm}(\lambda,\frac{1}{2};\Lambda,L)$, with $\lambda=\frac{3}{2},\frac{1}{2}$, $\Lambda=\pm\frac{1}{2}$, and $L=\pm\frac{1}{2}$.

Relationships between θ_Δ and $(\pi-\theta_\Delta)$ for helicity amplitudes, observables, and new amplitudes of Eq. (2.1) are written as

$$D(\lambda,l;\Lambda,L)(\theta_\Delta)=(-)^{\lambda-\Lambda}D(\lambda,L;\Lambda,l)(\pi-\theta_\Delta), \tag{2.11}$$

$$\begin{aligned}&\mathrm{Re}(P^b_\alpha P^t_\beta P^r_0\rho_{2\lambda,2\lambda'})(\theta_\Delta)\\ &\quad=(-)^{\lambda-\lambda'}\mathrm{Re}(P^b_\beta P^t_\alpha P^r_0\rho_{2\lambda,2\lambda'})(\pi-\theta_\Delta),\\ &\mathrm{Im}(P^b_\alpha P^t_\beta P^r_0\rho_{2\lambda,2\lambda'})(\theta_\Delta)\\ &\quad=(-)^{\lambda-\lambda'}\mathrm{Im}(P^b_\beta P^t_\alpha P^r_0\rho_{2\lambda,2\lambda'})(\pi-\theta_\Delta),\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&D_{\pm}\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)(\theta_\Delta)\\ &\quad=(-)^{\lambda-\Lambda}D_{\pm}\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)(\pi-\theta_\Delta),\\ &D_{\pm}\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)(\theta_\Delta)\\ &\quad=\pm(-)^{\lambda-\Lambda}D_{\pm}\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)(\pi-\theta_\Delta),\end{aligned} \tag{2.13}$$

respectively.

For the particular case $\theta_\Delta=\pi/2$, eight amplitudes vanish for $\Lambda=+\frac{1}{2}$ or $-\frac{1}{2}$:

$$\begin{aligned}&D_{\pm}\left(\lambda,\frac{1}{2};\Lambda,+\frac{1}{2}\right)(\pi/2)\\ &\quad=D_{+}\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)(\pi/2)=0\quad\text{for }(\lambda-\Lambda)\text{ odd},\\ &\quad D_{-}\left(\lambda,\frac{1}{2};\Lambda,-\frac{1}{2}\right)(\pi/2)=0\quad\text{for }(\lambda-\Lambda)\text{ even}.\end{aligned} \tag{2.14}$$

TABLE I. Abbreviated notation for helicity amplitudes $D(\lambda,l;\Lambda,L)$.

l	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
Λ	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
L	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$D\left(\frac{3}{2},l;\Lambda,L\right)$	G_1	G_2	F_1	F_2	F_3	F_4	G_3	G_4
$D\left(\frac{1}{2},l;\Lambda,L\right)$	F_5	F_6	G_5	G_6	G_7	G_8	F_7	F_8

The eight remaining amplitudes are $D_+(\lambda,\frac{1}{2};+\Lambda,+\frac{1}{2})$, $D_+(\lambda,\frac{1}{2};+\Lambda,-\frac{1}{2})$, $D_-(\lambda,\frac{1}{2};+\Lambda,+\frac{1}{2})$, and $D_-(\lambda,\frac{1}{2};-\Lambda,-\frac{1}{2})$, for $\lambda=\frac{3}{2},\frac{1}{2}$, and $(-)^{\lambda-\Lambda}=+1$. Consequently, the sum in Eqs. (2.2)–(2.8) over the two values of Λ vanishes, and Eq. (2.2) provides the magnitudes of these eight remaining amplitudes. Then, the magnitude first method may be applied at $\theta_\Delta=\pi/2$. An unambiguous analysis at this angle is presented in Sec. IV.

At $\theta_\Delta=0$ and π, 10 $D_{\pm}(\lambda,\frac{1}{2};\Lambda,L)$ amplitudes vanish for $\lambda-\Lambda\neq\pm(\frac{1}{2}-L)$. Among the six remaining amplitudes, two relationships give

$$\begin{aligned}&D_+\left(\frac{3}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right)=\exp(i\theta_\Delta)D_-\left(\frac{3}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right),\\ &D_+\left(\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2}\right)=\exp(i\theta_\Delta)D_-\left(\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2}\right).\end{aligned} \tag{2.15}$$

At forward and backward production angles, we choose the four linearly independent amplitudes $D_{\pm}(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$, $D_-(\frac{3}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2})$, and $D_-(\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2})$. Again, the sum in Eqs. (2.2)–(2.8) over the two values of Λ vanishes, making the magnitude first method applicable. From Eq. (2.13), amplitudes at $\theta_\Delta=\pi$ are obtained from those of the forward case. In Sec. V, the amplitude analysis is presented for $\theta_\Delta=0$.

In order to simplify the notation, in Secs. IV and V, where the explicit notation $D(\lambda,l;\Lambda,L)$ is not necessary, the helicity amplitudes will be denoted by [5] F_i and G_i for $i=1,\ldots,8$, the correspondence being given in Table I. Similarly, Table II gives abbreviated notation for combinations $D_{\pm}(\lambda,l;\Lambda,L)$ of Eq. (2.1).

III. AMBIGUITY ELIMINATION RULES

There have been contributions [6–13] in the literature on the formulation of rules which permit one to choose sets of observables for determining reaction amplitudes. However, selecting a set that eliminates all ambiguities is a problem which has been much less discussed. It has been studied in the particular case of a diagonal matrix, connecting observ-

TABLE II. Abbreviated notation for combinations $D_{\pm}(\lambda,l;\Lambda,L)$ of Eq. (2.1).

l	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
Λ	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
L	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
$D_{\pm}\left(\frac{3}{2},l;\Lambda,L\right)$	$G_1\pm G_2$	$F_1\pm F_2$	$F_3\pm F_4$	$G_3\pm G_4$
$D_{\pm}\left(\frac{1}{2},l;\Lambda,L\right)$	$F_5\pm F_6$	$G_5\pm G_6$	$G_7\pm G_8$	$F_7\pm F_8$

ables and ‘‘bicoms’’ [3,4]. It is a prime necessity to extend the investigation to the case of a nondiagonal matrix, which corresponds to observables expressed as sum of bicoms.

If the matrix connecting observables and bicoms is diagonal, each observable is a ‘‘primary observable’’ and is written as the real or imaginary part of a single bicom.

Briefly, in polar notation, a complex amplitude A_j is denoted $|A_j|\exp(i\phi_j)$. The magnitudes being determined, knowledge of an observable $\delta_R=\mathrm{Re}(A_1A_2^*)$ determines the relative phase as a twofold solution

$$(\phi_1-\phi_2)^{s_{1,2}}=s_{1,2}\arccos\frac{\delta_R}{|A_1||A_2|}, \tag{3.1}$$

with $s_{1,2}=\pm 1$, the sum of which is

$$S_{1,2}=(\phi_1-\phi_2)^{s_{1,2}}+(\phi_1-\phi_2)^{-s_{1,2}}=0. \tag{3.2}$$

Similarly, knowledge of $\delta_I=\mathrm{Im}(A_1A_2^*)$ determines the relative phase from

$$(\phi_1-\phi_2)^{s_{1,2}}=\frac{\pi}{2}+s_{1,2}\arccos\frac{\delta_I}{|A_1||A_2|}, \tag{3.3}$$

with $s_{1,2}=\pm 1$, the sum of which is

$$S_{1,2}=(\phi_1-\phi_2)^{s_{1,2}}+(\phi_1-\phi_2)^{-s_{1,2}}=\pi. \tag{3.4}$$

In each case, the sum of the two solutions has a particular value (0 or π). Such a sum will be called a ‘‘specific sum.’’ Contrarily, we shall refer to an ‘‘unspecific sum’’ if its value differs from 0 or π.

Let us consider an open chain of $(n-1)$ consecutive independent relative phases, $(\phi_1-\phi_2)$, $(\phi_2-\phi_3)$, ..., $(\phi_{n-1}-\phi_n)$, $n\geq 2$. Each phase, determined either from $\mathrm{Re}(A_iA_{i+1}^*)$ or from $\mathrm{Im}(A_iA_{i+1}^*)$, presents a twofold solution $(\phi_i-\phi_{i+1})^{s_{i,i+1}}$ with $s_{i,i+1}=\pm 1$, the sum

$$S_{i,i+1}=(\phi_i-\phi_{i+1})^{s_{i,i+1}}+(\phi_i-\phi_{i+1})^{-s_{i,i+1}} \tag{3.5}$$

being a specific sum.

The global phase of the open chain $(\phi_1-\phi_n)$ presents 2^{n-1} solutions, written as

$$(\phi_1-\phi_n)^{s_{1,2},s_{2,3},\dots,s_{n-1,n}}=(\phi_1-\phi_2)^{s_{1,2}}+(\phi_2-\phi_3)^{s_{2,3}}+\cdots+(\phi_{n-1}-\phi_n)^{s_{n-1,n}}. \tag{3.6}$$

We remark that the sum of the following two solutions

$$(\phi_1-\phi_n)^{s_{1,2},s_{2,3},\dots,s_{n-1,n}}+(\phi_1-\phi_n)^{-s_{1,2},-s_{2,3},\dots,-s_{n-1,n}}=S_{1,2}+S_{2,3}+\cdots+S_{n-1,n}, \tag{3.7}$$

is again specific; i.e., it is equal to 0 or π (mod 2π) according to the even or odd number of observables taken as the imaginary part of a bicom, along the open chain, respectively.

The ambiguity elimination is achieved by adding the dependent relative phase $(\phi_n-\phi_1)$ which closes the chain. This additional phase is determined by knowledge of an observable $\mathrm{Re}(A_nA_1^*)$ or $\mathrm{Im}(A_nA_1^*)$. It presents a twofold solution, the sum of which,

$$S_{n,1}=(\phi_n-\phi_1)^{s_{n,1}}+(\phi_n-\phi_1)^{-s_{n,1}}, \tag{3.8}$$

is specific. No ambiguity remains if the sum of the specific sums, along the closed chain, is different from 0 (mod 2π). Obviously, accidental or numerical degeneracy is discarded.

This gives the ‘‘rule of the odd number of imaginary parts’’ [4]: Let us consider a set of consecutive independent relative phases (open chain), each of them being known from a ‘‘primary observable.’’ The elimination of ambiguities is performed by determining an additional relative phase, which depends linearly on the others and closes the chain. All the ambiguities are eliminated if the number of imaginary parts of bicoms along the closed chain is an odd number. For an even number, the ambiguity elimination is partial, the degeneracy being of order 2.

Let us remark that a relative phase exactly known, by means of any method, may be considered as a phase known from the real and imaginary parts of its bicom. Its presence in a closed chain eliminates all the ambiguities. Yet such an exactly known phase may be considered as a set of two equal solutions, the sum of which being unspecific. The presence of such a phase, in a closed chain, eliminates all the ambiguities.

In the situation we want to analyze (see Sec. II), the matrix connecting observables and bicoms being not diagonal, some observables only are written as the real or imaginary part of a bicom. All other ‘‘secondary observables’’ are written as the real or imaginary part of sum of two bicoms. We generalize the above prescription in the following way.

Knowledge of a secondary observable $\delta_R=\mathrm{Re}(A_1A_2^*+A_1'A_2'^*)$ may determine the relative phase $(\phi_1-\phi_2)$ if

$$\Phi=(\phi_2-\phi_2')-(\phi_1-\phi_1') \tag{3.9}$$

is known. Then, we define

$$\alpha=|A_1||A_2|+|A_1'||A_2'|\cos\Phi, \tag{3.10}$$

$$\beta=-|A_1'||A_2'|\sin\Phi, \tag{3.11}$$

and X (mod 2π) by

$$\cos X = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \quad \text{and} \quad \sin X = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}. \tag{3.12}$$

The relative phase $(\phi_1-\phi_2)$ is expressed as a twofold solution

$$(\phi_1-\phi_2)^{s_{1,2}} = X + s_{1,2}\arccos\frac{\delta_R}{\sqrt{\alpha^2+\beta^2}}, \tag{3.13}$$

with $s_{1,2}=\pm 1$, the sum of which, being equal to $2X$, is unspecific. Similarly, from $\delta_I=\mathrm{Im}(A_1A_2^*+A_1'A_2'^*)$, we obtain two solutions

$$(\phi_1-\phi_2)^{s_{1,2}} = \frac{\pi}{2} + X + s_{1,2}\arccos\frac{\delta_I}{\sqrt{\alpha^2+\beta^2}}, \tag{3.14}$$

the sum of which, being equal to $\pi+2X$, is unspecific.

The simplest method to obtain the quantity Φ of Eq. (3.9) is to determine $(\phi_1-\phi_1')$ and $(\phi_2-\phi_2')$ from the corresponding primary observables. Thus, let us consider a "triad," constituted by one secondary and two associated primary observables. More precisely, we know $\mathrm{Re}(A_1A_2^*+A_1'A_2'^*)$ or $\mathrm{Im}(A_1A_2^*+A_1'A_2'^*)$, $\mathrm{Re}(A_1A_1'^*)$ or $\mathrm{Im}(A_1A_1'^*)$, and $\mathrm{Re}(A_2A_2'^*)$ or $\mathrm{Im}(A_2A_2'^*)$. Knowledge of the two primary observables determines the solutions $(\phi_1-\phi_1')^{s_1}$ and $(\phi_2-\phi_2')^{s_2}$, with $s_1=\pm 1$, $s_2=\pm 1$, the sums

$$S_1=(\phi_1-\phi_1')^{s_1}+(\phi_1-\phi_1')^{-s_1}, \tag{3.15}$$

$$S_2=(\phi_2-\phi_2')^{s_2}+(\phi_2-\phi_2')^{-s_2} \tag{3.16}$$

being specific. Here, the simplified notation s_1, s_2, S_1, and S_2 stands for the quantities defined by Eqs. (3.1)–(3.4), $s_{1,1'}$, $s_{2,2'}$, $S_{1,1'}$, and $S_{2,2'}$, respectively. Then, the quantity Φ of Eq. (3.9) takes four possible values

$$\Phi_{s_1,s_2}=(\phi_2-\phi_2')^{s_2}-(\phi_1-\phi_1')^{s_1}. \tag{3.17}$$

From Eqs. (3.10)–(3.12), α, β, and X depend on s_1 and s_2, also. The relative phase $(\phi_1-\phi_2)$ is given, from such a triad, as an eightfold solution

$$(\phi_1-\phi_2)^{s_{1,2}}_{s_1,s_2} = X_{s_1,s_2} + s_{1,2}\arccos\frac{\delta_R}{\sqrt{\alpha^2_{s_1,s_2}+\beta^2_{s_1,s_2}}} \tag{3.18}$$

or

$$(\phi_1-\phi_2)^{s_{1,2}}_{s_1,s_2} = \frac{\pi}{2} + X_{s_1,s_2} + s_{1,2}\arccos\frac{\delta_I}{\sqrt{\alpha^2_{s_1,s_2}+\beta^2_{s_1,s_2}}}, \tag{3.19}$$

with $s_{1,2}=\pm 1$, the secondary observable being δ_R or δ_I, respectively. Note that the total ambiguity elimination among the eight solutions of $(\phi_1-\phi_2)$ also determines exactly $(\phi_1-\phi_1')$ and $(\phi_2-\phi_2')$.

The next key step answers the following question: among the eight solutions, do there exist paired solutions forming specific sums? One has

$$\Phi_{s_1,s_2}+\Phi_{-s_1,-s_2}=S_2-S_1. \tag{3.20}$$

If $S_1=S_2$, one gets, from Eqs. (3.10)–(3.12),

$$X_{s_1,s_2}+X_{-s_1,-s_2}=0 \tag{3.21}$$

and

$$\sqrt{\alpha^2_{s_1,s_2}+\beta^2_{s_1,s_2}}=\sqrt{\alpha^2_{-s_1,-s_2}+\beta^2_{-s_1,-s_2}}. \tag{3.22}$$

We find that the eight solutions $(\phi_1-\phi_2)^{s_{1,2}}_{s_1,s_2}$ constitute four sets of twofold solutions with a specific sum,

$$S_{1,2}=(\phi_1-\phi_2)^{s_{1,2}}_{s_1,s_2}+(\phi_1-\phi_2)^{-s_{1,2}}_{-s_1,-s_2}, \tag{3.23}$$

equal to 0 from Eq. (3.18) and equal to π from Eq. (3.19). Each set of twofold solutions is independent of the others. If $S_1\neq S_2$, no such specific sums exist. Then, for a triad with $S_1=S_2$, there exists a twofold solution with a specific sum for the secondary observable ($S_{1,2}=0$ or π), as for a primary one.

Consider again an open chain of $(n-1)$ consecutive independent relative phases, $(\phi_1-\phi_2)$, $(\phi_2-\phi_3)$, $\ldots$, $(\phi_{n-1}-\phi_n)$, for $n>2$. Each phase is determined either from a primary observable or from a triad. For two consecutive triads, one of the associated primary observable may be different or shared in common. To be explicit, the first triad is constituted by Re or $\mathrm{Im}(A_1A_2^*+A_1'A_2'^*)$, Re or $\mathrm{Im}(A_1A_1'^*)$, and Re or $\mathrm{Im}(A_2A_2'^*)$. Similarly, the second triad involves Re or $\mathrm{Im}(A_2A_3^*+A_2''A_3'^*)$, Re or $\mathrm{Im}(A_2A_2''^*)$, and Re or $\mathrm{Im}(A_3A_3'^*)$. The two possible cases A_2' and A_2'' being equal or different may be examined. If they are different, $8^2=64$ solutions exist for the two triads together; otherwise, the number of solutions reduces to $8^2/2=32$. Actually, A_2' is equal to A_2'' in all the cases encountered in the present work.

The generalization of the rule of ambiguity elimination for a mixed open chain is the following: Let us consider a set of $(n-1)$ consecutive independent relative phases (open chain), for $n>2$, each of them being known either from a primary observable or from a triad. A total or partial elimination of the ambiguities is performed by determining an additional relative phase, also known either from a primary observable or from a triad, which depends linearly on the others and closes the chain. A partial ambiguity elimination gives a degeneracy of order 2. The elimination of all the ambiguities is obtained by performing an investigation for each of the n relative phases along the closed chain. For each phase, the question is whether paired solutions forming specific sums exist among its solutions. If the answer is negative, at least for one phase, the elimination is total. If the answer is positive, for the n phases, total elimination is obtained for an odd number of imaginary parts along the closed chain.

Some remarks must be made regarding the application of this rule. First, we recall that an exactly known phase, by means of any process, must be considered as paired solutions with unspecific sums. Second, while specifying "an odd number of imaginary parts along the closed chain," one must take into account, for a triad, only the imaginary part of

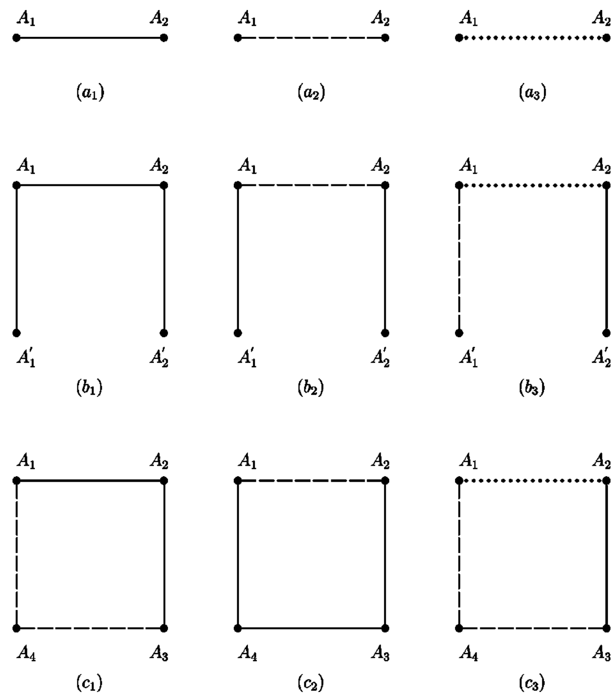

FIG. 2. Graphical code for relative phases.

the secondary observable, and not those of the associated primary ones. Third, we remark that a process of ambiguity elimination, performed by means of a triad, determines also the two associated relative phases, even if they are not included into the closed chain, but are dangling.

The generalized rule of ambiguity elimination is not applicable in the following academic case. Let us consider a closed chain, the n relative phases of which being known from triads, with the associated primary observables of the triads shared in common. If all the magnitudes are equal, another symmetry appears in the problem and the order of degeneracy increases. From Eqs. (3.9)–(3.12), one gets for the first triad

$$X_{s_1,s_2}=-\frac{\Phi_{s_1,s_2}}{2}, \tag{3.24}$$

with a Φ_{s_1,s_2} determination between $-\pi$ and π (mod 2π). Similarly, for the second triad, one has

$$X_{s_2,s_3}=-\frac{\Phi_{s_2,s_3}}{2}, \tag{3.25}$$

and so on. The sum along the closed chain,

$$\Phi_{s_1,s_2}+\Phi_{s_2,s_3}+\cdots+\Phi_{s_n,s_1}=0, \tag{3.26}$$

provides a similar relation for the X quantities:

$$X_{s_1,s_2}+X_{s_2,s_3}+\cdots+X_{s_n,s_1}=0, \quad (\mathrm{mod}\,2\pi). \tag{3.27}$$

An additional degeneracy appears; it is not developed here, because of the academic nature of the problem, which does not arise in a real amplitude analysis. Nevertheless, for $n=2$, the case is not academic and has been excluded in the terms of the rule.

To summarize, we propose a graphical code, which simplifies the use of this generalized rule of ambiguity elimination (see Fig. 2). Each amplitude is represented by a dot, and each relative phase by a line joining the two corresponding dots. Three symbols are used for the relative phase $(\phi_1-\phi_2)$. A solid or a dashed line indicates the presence of paired solutions with a specific sum equal to 0 [Fig. 2(a_1)] or π [Fig. 2(a_2)], respectively. A dotted line means the absence of paired solutions with a specific sum [Fig. 2(a_3)].

In the case in which the phase $(\phi_1-\phi_2)$ is known from a primary observable, Fig. 2(a_1) or 2(a_2) corresponds to knowledge of the real or imaginary part of $(A_1A_2^*)$, respectively. Figure 2(a_3) is equivalent to knowledge of the real and imaginary parts of $(A_1A_2^*)$.

If the phase $(\phi_1-\phi_2)$ is known from a triad, three examples are shown in Fig. 2(b). In these examples, the associated phases $(\phi_1-\phi_1')$ and $(\phi_2-\phi_2')$ are visualized. Figure 2(b_1) or 2(b_2) corresponds to knowledge of the real or imaginary part of $(A_1A_2^*+A_1'A_2'^*)$, respectively, the associated phases being known from $\mathrm{Re}(A_1A_1'^*)$, and $\mathrm{Re}(A_2A_2'^*)$. Figure 2(b_3) corresponds to knowledge of the real or imagi-

TABLE III. Combinations $D_\pm(\lambda,l;\Lambda,L)$ for $\theta_\Delta=\pi/2$.

l	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
Λ	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
L	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
$D_+\left(\frac{3}{2},l;\Lambda,L\right)$	$G_1+G_2=0$	$F_1+F_2=0$	F_3+F_4	$G_3+G_4=2G_3$
$D_-\left(\frac{3}{2},l;\Lambda,L\right)$	$G_1-G_2=0$	$F_1-F_2=2F_1$	F_3-F_4	$G_3-G_4=0$
$D_+\left(\frac{1}{2},l;\Lambda,L\right)$	F_5+F_6	$G_5+G_6=2G_5$	$G_7+G_8=0$	$F_7+F_8=0$
$D_-\left(\frac{1}{2},l;\Lambda,L\right)$	F_5-F_6	$G_5-G_6=0$	$G_7-G_8=0$	$F_7-F_8=2F_7$

nary part of $(A_1A_2^*+A_1'A_2'^*)$, the associated phases being known from $\mathrm{Im}(A_1A_1'^*)$, and $\mathrm{Re}(A_2A_2'^*)$.

In Fig. 2(c), the use of the generalized rule of ambiguity elimination is displayed for a square closed chain. Each phase is obtained from a primary observable or from a triad, the associated phases of the triad not being visualized. In Fig. 2(c_1), an example of a partial elimination is shown (degeneracy of order 2). In Figs. 2(c_2) and 2(c_3), examples of total ambiguity elimination are shown.

In terms of a closed chain, all the ambiguities are eliminated if the diagram satisfies one of the two following criteria: (i) there exists at least one dotted line along the closed chain [Fig. 2(c_3)] or (ii) there exists an odd number of dashed lines along the closed chain [Fig. 2(c_2)]. If the number of dashed lines is even, a partial ambiguity elimination is performed, with a degeneracy of order 2 [Fig. 2(c_1)].

IV. UNAMBIGUOUS ANALYSIS AT $\theta_\Delta=\pi/2$

In this section, use is made of the abbreviated notation defined in Table I for the helicity amplitudes $D(\lambda,l;\Lambda,L)$. For $\theta_\Delta=\pi/2$, amplitude combinations $D_\pm(\lambda,l;\Lambda,L)$ of Eq. (2.1) are displayed in Table III. Then, for $\lambda=\lambda'=\frac{3}{2},\frac{1}{2}$, Eq. (2.2) provides the eight magnitudes

$$\begin{pmatrix} |(F_3+F_4)|^2 \\ 4|G_3|^2 \\ |(F_3-F_4)|^2 \\ 4|F_1|^2 \end{pmatrix} = 2I_0 \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix} \begin{pmatrix} \mathrm{Re}(P_0^bP_0^tP_0^r\rho_{3,3}) \\ \mathrm{Re}(P_z^bP_z^tP_0^r\rho_{3,3}) \\ \mathrm{Re}(P_x^bP_x^tP_0^r\rho_{3,3}) \\ -\mathrm{Re}(P_y^bP_y^tP_0^r\rho_{3,3}) \end{pmatrix} \tag{4.1}$$

and

$$\begin{pmatrix} |(F_5+F_6)|^2 \\ 4|G_5|^2 \\ |(F_5-F_6)|^2 \\ 4|F_7|^2 \end{pmatrix} = 2I_0 \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix} \begin{pmatrix} \mathrm{Re}(P_0^bP_0^tP_0^r\rho_{1,1}) \\ \mathrm{Re}(P_z^bP_z^tP_0^r\rho_{1,1}) \\ \mathrm{Re}(P_x^bP_x^tP_0^r\rho_{1,1}) \\ -\mathrm{Re}(P_y^bP_y^tP_0^r\rho_{1,1}) \end{pmatrix}. \tag{4.2}$$

Taking into account Eq. (2.10), Eq. (2.2) gives also, for $\lambda=\frac{3}{2},\quad \lambda'=-\frac{1}{2}$, the four real parts of the ‘‘bicoms’’:

$$\begin{pmatrix} -\mathrm{Re}[(F_3+F_4)(F_5+F_6)^*] \\ 4\mathrm{Re}[G_3G_5^*] \\ \mathrm{Re}[(F_3-F_4)(F_5-F_6)^*] \\ 4\mathrm{Re}[F_1F_7^*] \end{pmatrix} = 2I_0 \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix} \begin{pmatrix} \mathrm{Re}(P_0^bP_0^tP_0^r\rho_{3,-1}) \\ \mathrm{Re}(P_z^bP_z^tP_0^r\rho_{3,-1}) \\ \mathrm{Re}(P_x^bP_x^tP_0^r\rho_{3,-1}) \\ -\mathrm{Re}(P_y^bP_y^tP_0^r\rho_{3,-1}) \end{pmatrix}. \tag{4.3}$$

From Eqs. (4.1)–(4.3), the determination of the eight magnitudes and four phases is obtained from the knowledge of the 12 observables $\mathrm{Re}(P_\alpha^bP_\alpha^tP_0^r\rho_{2\lambda,2\lambda'})$, with $\alpha=0,x,y,z$, and $\rho_{2\lambda,2\lambda'}=\rho_{3,3},\rho_{1,1},\rho_{3,-1}$. Such a determination is carried out using four $NN\to(N\pi)N$ experiments, namely, the cross section σ and the three asymmetries $\sigma A_{00\alpha\alpha}$, with $\alpha=x,y,z$. Let us remark that these experiments provide observables used in the first step of the analysis, as well as part of the second step.

For $\lambda=\lambda'=\frac{3}{2},\frac{1}{2}$, Eqs. (2.3) and (2.4) provide real and imaginary parts of the bicoms:

$$\mathrm{Re}[(F_3-F_4)G_3^*]=2I_0\mathrm{Re}(P_z^bP_x^tP_0^r\rho_{3,3}),$$

$$\mathrm{Im}[(F_3-F_4)G_3^*]=2I_0\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{3,3}) \tag{4.4}$$

and

$$\mathrm{Re}[(F_5-F_6)G_5^*]=2I_0\mathrm{Re}(P_z^bP_x^tP_0^r\rho_{1,1}),$$

$$\mathrm{Im}[(F_5-F_6)G_5^*]=2I_0\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{1,1}). \tag{4.5}$$

Also, for $\lambda=\frac{3}{2},\quad \lambda'=\pm\frac{1}{2}$, Eqs. (2.3) and (2.4) provide real and imaginary parts of sums of two bicoms:

$$\mathrm{Re}[F_1(F_5+F_6)^*+F_7(F_3+F_4)^*]=4I_0\mathrm{Re}(P_z^bP_x^tP_0^r\rho_{3,1}),$$

$$\mathrm{Im}[F_1(F_5+F_6)^*+F_7(F_3+F_4)^*]=4I_0\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{3,1}), \tag{4.6}$$

and

$$\mathrm{Re}[(F_3-F_4)G_5^*+(F_5-F_6)G_3^*]=4I_0\mathrm{Re}(P_z^bP_x^tP_0^r\rho_{3,-1}),$$

$$\mathrm{Im}[(F_3-F_4)G_5^*+(F_5-F_6)G_3^*]=4I_0\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{3,-1}). \tag{4.7}$$

Similarly, for $\lambda=\frac{3}{2},\quad \lambda'=\pm,\frac{1}{2}$, Eqs. (2.5)–(2.8) give real and imaginary parts of the following sums of two bicoms:

$$\mathrm{Re}[F_1G_5^*-F_7G_3^*]=2I_0\mathrm{Im}(P_x^bP_y^tP_0^r\rho_{3,1}),$$
$$\mathrm{Im}[F_1G_5^*-F_7G_3^*]=2I_0\mathrm{Im}(P_z^bP_0^tP_0^r\rho_{3,1}), \qquad (4.8)$$

$$\mathrm{Re}[(F_3+F_4)(F_5-F_6)^*+(F_5+F_6)(F_3-F_4)^*]$$
$$=8I_0\mathrm{Im}(P_x^bP_y^tP_0^r\rho_{3,-1}),$$
$$\mathrm{Im}[(F_3+F_4)(F_5-F_6)^*+(F_5+F_6)(F_3-F_4)^*]$$
$$=8I_0\mathrm{Im}(P_z^bP_0^tP_0^r\rho_{3,-1}), \qquad (4.9)$$

$$\mathrm{Re}[F_1(F_5-F_6)^*-F_7(F_3-F_4)^*]$$
$$=-4I_0\mathrm{Im}(P_y^bP_z^tP_0^r\rho_{3,1}),$$
$$\mathrm{Im}[F_1(F_5-F_6)^*-F_7(F_3-F_4)^*]$$
$$=-4I_0\mathrm{Im}(P_x^bP_0^tP_0^r\rho_{3,1}), \qquad (4.10)$$

and

$$\mathrm{Re}[G_5(F_3+F_4)^*+G_3(F_5+F_6)^*]$$
$$=-4I_0\mathrm{Im}(P_y^bP_z^tP_0^r\rho_{3,-1}),$$
$$\mathrm{Im}[G_5(F_3+F_4)^*+G_3(F_5+F_6)^*]$$
$$=-4I_0\mathrm{Im}(P_x^bP_0^tP_0^r\rho_{3,-1}). \qquad (4.11)$$

For each bicom or each sum of two bicoms of Eqs. (4.4)–(4.11), its imaginary part is obtained from an $A_{00\alpha 0}$ asymmetry and its real part from $A_{00\beta\gamma}$, with $\alpha\neq\beta\neq\gamma$. In the following example, which presents an amplitude analysis without ambiguity, asymmetries $A_{00\beta\gamma}$, with $\beta\neq\gamma$, are not taken into account. This is a choice among many others.

From the imaginary part of each sum of two bicoms, in Eqs. (4.6)–(4.11), a ''triad'' may be constituted by adding the two appropriated real parts of bicoms, given by Eq. (4.3). Such a triad presents four sets of twofold solutions with the ''specific sum'' equal to π.

The A_{00z0} asymmetry provides the observables $\mathrm{Im}(P_z^bP_0^tP_0^r\rho_{2\lambda,2\lambda'})$, for $\rho_{2\lambda,2\lambda'}=\rho_{3,1},\rho_{3,-1}$, which determine the phases between F_1 and G_5 [Eq. (4.8)], and between (F_3+F_4) and (F_5-F_6) [Eq. (4.9)], respectively. Similarly, the A_{00x0} asymmetry provides the observables $\mathrm{Im}(P_x^bP_0^tP_0^r\rho_{2\lambda,2\lambda'})$, for $\rho_{2\lambda,2\lambda'}=\rho_{3,1},\rho_{3,-1}$, which determine the phases between F_1 and (F_5-F_6) [Eq. (4.10)] and between G_5 and (F_3+F_4) [Eq. (4.11)], respectively.

Knowledge of these four phases achieves the second step of the analysis (determination of the three remaining independent phases) and constitutes a convenient starting point of the third step (elimination of the ambiguities). In terms of a graphical code, the addition of a fourth phase closes the square. These four triads are represented in Fig. 3. The number of dashed lines being equal to 4 along the square F_1, G_5, (F_3+F_4), (F_5-F_6), we are left with a degeneracy of order 2.

The A_{00y0} asymmetry provides the observables $\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{2\lambda,2\lambda'})$, for $\rho_{2\lambda,2\lambda'}=\rho_{3,3},\rho_{1,1},\rho_{3,1},\rho_{3,-1}$, which determine the phases between (F_3-F_4) and G_3 [Eq. (4.4)], between (F_5-F_6) and G_5 [Eq. (4.5)], between (F_5+F_6) and F_1 [Eq. (4.6)], and between (F_3-F_4) and G_5 [Eq. (4.7)], respectively. Starting from Fig. 3, and adding the line between G_5 and (F_5-F_6), yields Fig. 4. The number of dashed lines being 3 along the triangle F_1, G_5, (F_5-F_6) and 3 also along the triangle (F_3+F_4), G_5, (F_5-F_6), all the ambiguities are eliminated.

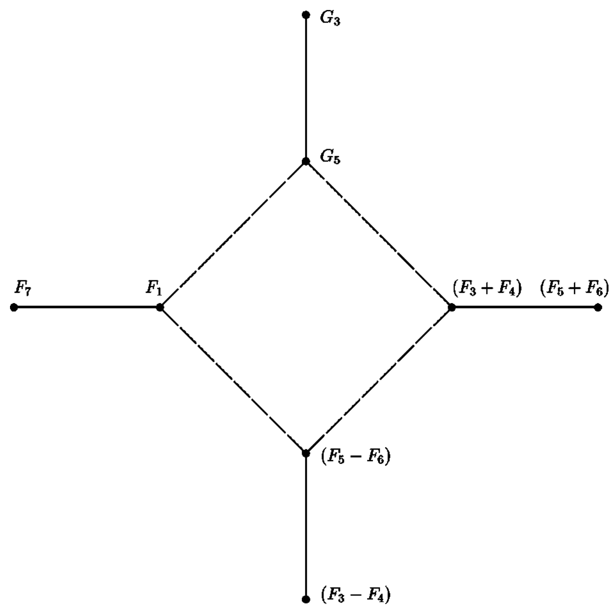

FIG. 3. Graphical code for phases at $\theta_\Delta=\pi/2$, with ambiguity of order 2.

Then, knowledge of seven experiments on the reaction $NN\to(N\pi)N$, namely, the cross section σ, the three polarized beam and target asymmetries σA_{00LL}, σA_{00SS}, and σA_{00NN}, and the three polarized beam asymmetries σA_{00L0}, σA_{00S0}, and σA_{00N0}, allows us to perform an unambiguous amplitude analysis in terms of helicity amplitude combinations. The transformation from these combinations to the helicity amplitudes is obvious. Here, in the four-subscript notation used for asymmetries, L, S, and N stand (see Fig. 1) for the direction (not oriented) of the $\hat{\mathbf{z}}$, $\hat{\mathbf{x}}$, and $\hat{\mathbf{y}}$ axes of the beam and the target.

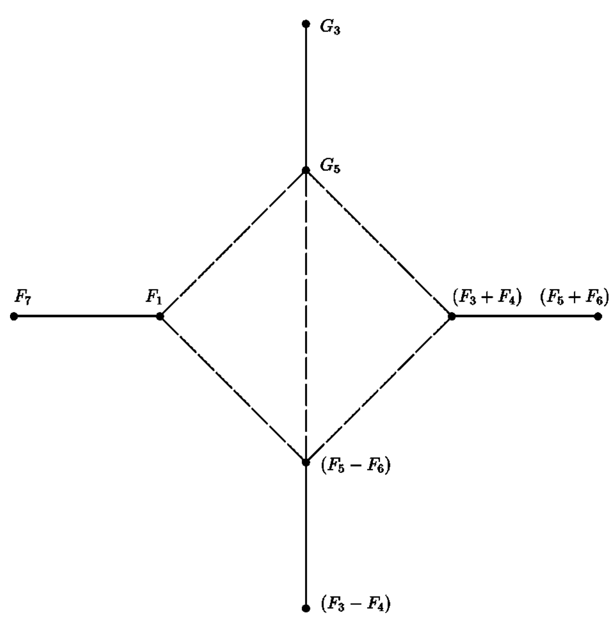

FIG. 4. Graphical code for unambiguous phases at $\theta_\Delta=\pi/2$.

TABLE IV. Combinations $D_{\pm}(\lambda,l;\Lambda,L)$ for $\theta_\Delta=0$.

l	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
Λ	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
L	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
$D_+\left(\frac{3}{2},l;\Lambda,L\right)$	$G_1+G_2=0$	$F_1+F_2=F_1$	$F_3+F_4=0$	$G_3+G_4=0$
$D_-\left(\frac{3}{2},l;\Lambda,L\right)$	$G_1-G_2=0$	$F_1-F_2=F_1$	$F_3-F_4=0$	$G_3-G_4=0$
$D_+\left(\frac{1}{2},l;\Lambda,L\right)$	F_5+F_6	$G_5+G_6=0$	$G_7+G_8=0$	$F_7+F_8=F_7$
$D_-\left(\frac{1}{2},l;\Lambda,L\right)$	F_5-F_6	$G_5-G_6=0$	$G_7-G_8=0$	$F_7-F_8=F_7$

V. CASE OF $\theta_\Delta=0$

In this section, use is made of the abbreviated notation, defined in Table I, for the helicity amplitudes $D(\lambda,l;\Lambda,L)$. For $\theta_\Delta=0$, amplitude combinations $D_{\pm}(\lambda,l;\Lambda,L)$ of Eq. (2.1) are displayed in Table IV.

Then, for $\lambda=\lambda'=\frac{3}{2},\frac{1}{2}$, Eq. (2.2) provides the four magnitudes

$$|F_1|^2=4I_0\mathrm{Re}(P_0^bP_0^tP_0^r\rho_{3,3}) \tag{5.1}$$

and

$$\begin{pmatrix} |(F_5+F_6)|^2 \\ |(F_5-F_6)|^2 \\ |F_7|^2 \end{pmatrix}=2I_0\begin{pmatrix} +1 & +1 & -2 \\ +1 & +1 & +2 \\ +1 & -1 & 0 \end{pmatrix}\times\begin{pmatrix} \mathrm{Re}(P_0^bP_0^tP_0^r\rho_{1,1}) \\ \mathrm{Re}(P_z^bP_z^tP_0^r\rho_{1,1}) \\ \mathrm{Re}(P_y^bP_y^tP_0^r\rho_{1,1}) \end{pmatrix}. \tag{5.2}$$

Taking into account Eq. (2.10), Eq. (2.2) gives also, for $\lambda=\frac{3}{2},\quad\lambda'=-\frac{1}{2}$, the real part of the ‘‘bicom’’:

$$\mathrm{Re}[F_1F_7^*]=-4I_0\mathrm{Re}(P_y^bP_y^tP_0^r\rho_{3,-1}). \tag{5.3}$$

Also, for $\lambda=\frac{3}{2},\quad\lambda'=\frac{1}{2}$, Eqs. (2.4) and (2.7) provide real and imaginary parts of the bicoms:

$$\mathrm{Re}[F_1(F_5+F_6)^*]=4I_0[\mathrm{Im}(P_z^bP_y^tP_0^r\rho_{3,1})+\mathrm{Im}(P_y^bP_z^tP_0^r\rho_{3,1})],$$

$$\mathrm{Im}[F_1(F_5+F_6)^*]=4I_0[\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{3,1})-\mathrm{Re}(P_0^bP_y^tP_0^r\rho_{3,1})] \tag{5.4}$$

and

$$\mathrm{Re}[F_1(F_5-F_6)^*]=4I_0[\mathrm{Im}(P_z^bP_y^tP_0^r\rho_{3,1})-\mathrm{Im}(P_y^bP_z^tP_0^r\rho_{3,1})],$$

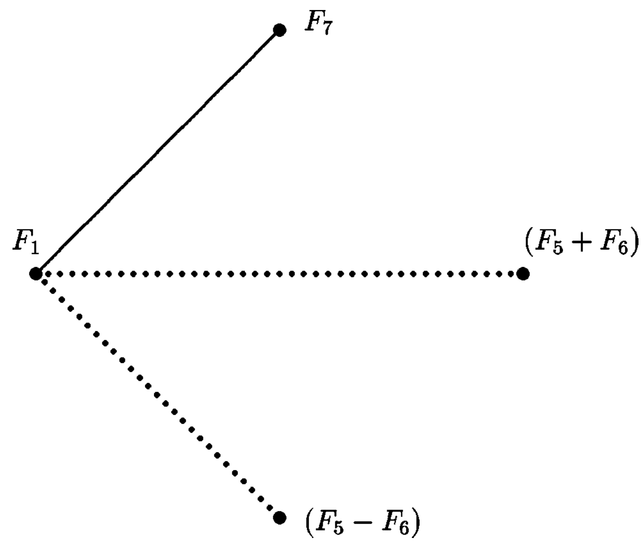

FIG. 5. Graphical code for phases at $\theta_\Delta=0$.

$$\mathrm{Im}[F_1(F_5-F_6)^*]=-4I_0[\mathrm{Re}(P_y^bP_0^tP_0^r\rho_{3,1})+\mathrm{Re}(P_0^bP_y^tP_0^r\rho_{3,1})]. \tag{5.5}$$

Knowledge of five experiments on the reaction $NN\rightarrow(N\pi)N$, namely, the cross section σ and the four asymmetries σA_{00LL}, σA_{00NN}, σA_{00N0}, and σA_{000N}, permits one to determine the four magnitudes of F_1, F_7, (F_5+F_6), and (F_5-F_6) and the three independent relative phases between F_1 and each of the three others. An ambiguity of order 2^3 remains.

From Eq. (5.4), knowledge of the real and imaginary parts of the bicom $F_1(F_5+F_6)^*$ determines exactly the corresponding relative phase, and similarly from Eq. (5.5). Then, additional knowledge of the two asymmetries σA_{00LN} and σA_{00NL} yields exactly the phases between F_1 and (F_5+F_6) and between F_1 and (F_5-F_6). The transformation from these combinations to the helicity amplitudes F_1, F_5, and F_6 may be performed without ambiguity. Nevertheless, the sign of the phase between F_1 and F_7 remains unknown (see Fig. 5).

VI. CONCLUSION

The present work is devoted to an unambiguous amplitude analysis of the $NN\rightarrow\Delta N$ transition, from $NN\rightarrow(N\pi)N$ asymmetry measurements. The study is performed in the helicity frame. Restricting the analysis to asymmetry measurements requires one to work with linear combinations of helicity amplitudes. Moreover, the magnitudes of these combinations may be obtained for particular Δ-production angles, only, namely, 0, $\pi/2$, and π. A three-step method is developed, which determines, first, the magnitudes, second, independent relative phases, and third, some dependent relative phases for resolving the remaining ambiguities.

The rule of ambiguity elimination is based on the closure of a chain of consecutive independent relative phases, by means of the *ad hoc* dependent one. If the matrix connecting observables and ‘‘bicoms’’ is diagonal, each observable is written as the real or imaginary part of a bicom. A relative phase, determined by such an observable, presents a twofold solution. All the ambiguities are eliminated if the number of imaginary parts of bicom along the closed chain is odd. For an even number, the ambiguity elimination is partial, the degeneracy being of order 2.

In the present analysis, the matrix connecting observables and bicoms is not diagonal. Many observables are written as the real or imaginary part of a sum of bicoms. A relative phase, determined from such an observable, presents many sets of twofold solutions. The generalization of the above ''rule of the odd number of imaginary parts'' is based on the study of the multiple solutions of a relative phase. More precisely, it relies on the existence, among all the solutions, of paired solutions forming a ''specific sum,'' i.e., a sum equal to 0 or π. Contrarily, we refer to an ''unspecific sum'' if the value of the sum differs from 0 or π. All the ambiguities are eliminated if, at least, one phase with an unspecific sum appears along the closed chain. If such a phase does not appear along the closed chain, total elimination is obtained for an odd number of phases with a specific sum equal to π. For an even number, the degeneracy is of order 2. A graphical code is proposed to simplify the use of this rule of ambiguity elimination.

At $\theta_\Delta=\pi/2$ production angle, an unambiguously determination of the $NN\to\Delta N$ transition amplitudes is performed. The eight magnitudes and four phases are determined by means of four experiments, namely, the cross section σ and the three polarized beam and target asymmetries σA_{00LL}, σA_{00SS}, and σA_{00NN}. Two asymmetries, chosen among the three polarized beam asymmetries, for example, σA_{00L0} and σA_{00S0}, determine four phases, the three last independent ones and one additional dependent phase. At this stage, the remaining ambiguity is of order 2. It is removed by adding the third polarized beam asymmetry σA_{00N0}.

At $\theta_\Delta=0$, the four magnitudes and a relative phase are determined by means of three experiments, the cross section σ and the two polarized beam and target asymmetries σA_{00LL} and σA_{00NN}. The two asymmetries σA_{000N} and σA_{00N0} determine two remaining independent phases. At this second step of the analysis, an ambiguity of order 2^3 remains. With two more experiments σA_{00LN} and σA_{00NL}, the helicity amplitudes are determined up to an ambiguity of sign of one relative phase.

We recall that knowledge of the full multidimensional form of the data [14–18] must be sufficient to yield the $NN\to\Delta N$ observables. For this, we must perform integration over the polar and azimuthal angles of the decay nucleon in the Δ rest frame, with respect to the direction of the Δ and to the production scattering plane [see Eqs. (3.8) and (3.9) of [1]]. Presently, for any incident energy, there is no such complete set of measurements.

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[1] J. P. Auger and C. Lazard, Phys. Rev. C **52**, 513 (1995).
[2] J. P. Auger and C. Lazard, Phys. Rev. D **45**, 821 (1992).
[3] K. Nam, M. J. Moravcsik, and G. R. Goldstein, Phys. Rev. Lett. **52**, 2305 (1984).
[4] M. J. Moravcsik, J. Math. Phys. **26**, 211 (1985).
[5] J. P. Auger and C. Lazard, Phys. Rev. D **47**, 894 (1993).
[6] M. J. Moravcsik and W. Yu, J. Math. Phys. **10**, 925 (1969).
[7] G. R. Goldstein, M. J. Moravcsik, and D. Bregman, Lett. Nuovo Cimento **11**, 137 (1974).
[8] C. Bourrely and J. Soffer, Phys. Rev. D **12**, 2932 (1975).
[9] P. W. Johnson, R. C. Miller, and G. H. Thomas, Phys. Rev. D **15**, 1895 (1977).
[10] G. R. Goldstein and M. J. Moravcsik, Ann. Phys. (N.Y.) **126**, 176 (1980); Phys. Lett. **102B**, 189 (1981).
[11] P. La France and P. Winternitz, Phys. Rev. D **27**, 112 (1983).
[12] M. J. Moravcsik, Phys. Rev. D **27**, 289 (1983); **29**, 2625 (1984).
[13] G. R. Goldstein, M. J. Moravcsik, L. Mathelitsch, and K. Schwarz, Ann. Phys. (N.Y.) **198**, 371 (1990).
[14] A. D. Hancock *et al.*, Phys. Rev. C **27**, 2742 (1983).
[15] T. S. Bhatia *et al.*, Phys. Rev. C **28**, 2071 (1983); G. Glass *et al.*, Phys. Lett. **129B**, 27 (1983).
[16] C. L. Hollas *et al.*, Phys. Rev. Lett. **55**, 29 (1985).
[17] A. B. Wicklund *et al.*, Phys. Rev. D **34**, 19 (1986); **35**, 2670 (1987).
[18] C. E. Waltham *et al.*, Nucl. Phys. **A433**, 649 (1985); R. L. Shypit *et al.*, Phys. Rev. Lett. **60**, 901 (1988); Phys. Rev. C **40**, 2203 (1989).