# Analytical investigation for multiplicity difference correlators under quark-gluon plasma phase transition

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It is suggested that the study of multiplicity difference correlators between two well-separated bins in high-energy heavy-ion collisions can be used as a means to detect evidence of a quark-hadron phase transition. Analytical expressions for the scaled factorial moments of multiplicity difference distribution are obtained for small bin size with mean multiplicity $\bar{s} \lesssim 0.3$ within Ginzburg-Landau description. It is shown that the scaling behaviors between the moments are still valid, though the behaviors of the moments with respect to the bin size are completely different from the so-called intermittency patterns. A universal exponent $\gamma$ is given to describe the dynamical fluctuations in the phase transition. [S0556-2813(98)01404-6]

One of the primary motivations of the study of high-energy heavy-ion collisions is to investigate the properties of a quark-gluon system at extremely high temperature and high density. Such a system may be in the state of quark-gluon-plasma (QGP), and with expanding and cooling the system will undergo a quark-hadron phase transition and turn out to be hadrons detected in experiments. One of the theoretical aims is to find a signal about the phase transition. As is well known for a long time, fluctuations are large for statistical systems near their critical points. Thus the study of fluctuations in the process might reveal some features for the phase transition [1,2]. Monte Carlo simulations [3] on intermittency [4] without phase transition for $pp$ collisions [5] show quantitatively different results on multiplicity fluctuations from theoretical predictions with the onset of phase transition [2]. These different results stimulated a lot of theoretical works on multiplicity fluctuations with phase transition of second-order [2,6] and first-order [7,8] within the Ginzburg-Landau model which is suitable for the study of phase transition for macroscopic systems. Most of these works give remarkable scaling behaviors between $F_q$ and $F_2$, and there seems to exist a universal exponent $\nu$ [2,6,8]. It is suggested that the exponent $\nu$ can be used as a useful diagnostic tool to detect the formation of QGP. In [7] $\ln F_q$ are expanded as power series of $\delta^{1/3}$, and it is shown that the coefficient of the $\delta^{1/3}$ term can be used as a criterion for the onset and the order of the phase transition. All those works show the violation of intermittency patterns in the phase transition.

It is known for a long time that the investigation of multiplicity fluctuations is very different in heavy-ion collisions, though the power-law dependence of $\ln F_q$ on $\delta$, $F_q \propto \delta^{-\varphi_q}$, has been found ubiquitous in hadronic and leptonic processes [9]. The main differences between heavy-ion physics and hadronic and leptonic ones on multiplicity fluctuations were noticed earlier in Ref. [10]. In Ref. [11] an alternative way was proposed to study the fluctuations by means of factorial moments of the multiplicity difference (FMMD) between two well-separated bins. This alternative is a hybrid of the usual factorial correlators [4] and wavelets [12] because $W_{jk}$ in Haar wavelet analysis is just the difference of multiplici-

ties in two nearest bins. Present discussions, of course, will not be limited in the nearest bins. Let the two bins, each of size $\delta^2$ and separated by $\Delta$, have multiplicities $n_1$ and $n_2$, and define their multiplicity difference $m = |n_1 - n_2|$. Scaled FMMD are defined as

$$\mathcal{F}_q = f_q / f_1^q, \quad f_q = \sum_m m(m-1) \cdots (m-q+1) Q_m, \quad (1)$$

with $Q_m$ the distribution of multiplicity difference which may be dependent on $\Delta, \delta$ and details of the process. Moments defined above are similar to but not the same as the Bialas-Peschanski correlators [4] $F_{q_1 q_2}$, for $\mathcal{F}_q$ may depend on both $\Delta$ and $\delta$ while $F_{q_1 q_2}$ depends only on $\Delta$.

In Ref. [11], $\mathcal{F}_q$ are numerically studied within the Ginzburg-Landau model. The scaling behaviors between $F_q$ and $F_2$, $\mathcal{F}_q \propto \mathcal{F}_2^{\beta_q}$, are shown with $\beta_q = (q-1)^\gamma$ and a universal exponent $\gamma = 1.099$.

In this paper, $\mathcal{F}_q$ are studied analytically for very small bin size $\delta$. Then the dynamical fluctuation components $\mathcal{F}_q^{(\text{dyn})}$ of FMMD are defined. It is shown that both $\ln \mathcal{F}_q$ and $\ln \mathcal{F}_q^{(\text{dyn})}$ increase linearly with the bin size $\delta$ when $\delta$ is very small, completely different from the usual intermittency behaviors of $\ln F_q$ which increase with the decrease of bin size. But the scaling laws between $\mathcal{F}_q$ and $\mathcal{F}_2$, and between $\mathcal{F}_q^{(\text{dyn})}$ and $\mathcal{F}_2^{(\text{dyn})}$ are still valid, although the corresponding $\beta_q$ and $\beta_q^{(\text{dyn})}$ are different. A universal exponent $\gamma$ for $\beta_q^{(\text{dyn})}$ is given which has no dependence on any parameter in the model and is different from that in [11].

As a starting point, let us first discuss the trivial and simplest case. Suppose that the two bins considered are well separated so that there are no correlations between them. Let the mean multiplicities in each bin be $s_1, s_2$, respectively. If there is no dynamical reason, the multiplicity distribution for each bin is a Poisson one

$$P_{n_i}(s_i) = \frac{s_i^{n_i}}{n_i!} \exp(-s_i) \quad (i=1,2). \quad (2)$$

From this distribution, one can deduce the multiplicity difference distribution as

$$P_m(s_1,s_2)=\cosh\left(\frac{m}{2}\ln\frac{s_1}{s_2}\right)I_m(2\sqrt{s_1s_2})e^{-s_1-s_2}(2-\delta_{m0}), \tag{3}$$

where $I_m(z)$ is the modified Bessel function of order $m$,

$$I_m(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k+m}}{k!(k+m)!}.$$

FMMD for pure statistical fluctuations are

$$f_q^{(\mathrm{stat})}=\sum_{m\geq q}m(m-1)\cdots(m-q+1)P_m(s_1,s_2). \tag{4}$$

For simplicity, let us discuss the case with $s_1=s_2$. This condition can always be satisfied if one chooses the two bins properly. Since only $m\geq q$ contribute to $f_q$, the summation over $m$ in the last equation can be extended to $m=0$. This summation converges very slowly because $I_m(2s)$ decreases with $m$ approximately as $s^m/m!$ for large $m$ and small $s$, but the product $m(m-1)\cdots(m-q+1)$ increases with $m$ quickly. So contributions from all $m\geq q$ must be taken into account. This will cause some difficulties in numerical calculations if one starts directly from the definition of the moments. In this paper, we will alternatively sum over $m$ analytically, and then do numerical calculations from the final expression. In this approach, one can control the precision more easily in calculation. To complete the summation, one can introduce a generating function

$$G(x,s)=2e^{-2s}\sum_{m=0}^{\infty}x^mI_m(2s), \quad G_q(x,s)=\frac{d^qG(x,s)}{dx^q}. \tag{5}$$

With this function, $f_q^{(\mathrm{stat})}$ can be rewritten as

$$f_q^{(\mathrm{stat})}=G_q(1,s)\equiv G_q(s). \tag{6}$$

Direct algebra shows that

$$G(x,s)=2e^{(x-2)s}\left[a_0^0+\sum_{i=1}^{\infty}a_i^0\frac{d^i}{dx^i}\frac{1-\exp(-xs)}{x}\right] \tag{7}$$

with $a_i^0=(-1)^is^{2i}/(i!)^2$ for $i=0,1,\ldots$, and that

$$f_q^{(\mathrm{stat})}=2e^{-s}\left[a_0^q+\sum_{i=1}^{\infty}a_i^q\sum_{j=i}^{\infty}\frac{(-s)^{j+1}}{(j+1)(j-i)!}\right], \tag{8}$$

where $a_i^q$ can be calculated by recurrence relation from $a_i^0$, $a_0^q=sa_0^{q-1}$, $a_1^q=sa_1^{q-1}$, $a_i^q=sa_i^{q-1}+a_{i-1}^{q-1}$ ($i\geq2$). Then one can get two specially important coefficients $a_0^q=s^q$, $a_1^q=-s^{q+2}$. The most important advantage of such calculations is that these formalisms facilitate analytical calculations for quantities in the range of very small bin size in which we are now interested. We will discuss it later in this paper.

Now, we begin to discuss the FMMD in second-order quark-hadron phase transition. We use the Ginzburg-Landau description to specify the probability that $s$ hadrons are created in the two-dimensional, such as $\delta\eta\delta\varphi$, area $\delta^2$. In this description, the distribution of multiplicity is no longer a Poisson one and that for multiplicity difference is given by [11]

$$qQ_m(\delta,\tau)=Z^{-1}\int\mathcal{D}\phi P_m(\delta^2\tau|\phi|^2)e^{-F[\phi]}, \tag{9}$$

where $\tau$ is an indication of lifetime of the whole parton system, $\mathcal{D}\phi=\pi d|\phi|^2$, $Z=\int\mathcal{D}\phi e^{-F[\phi]}$, and the free energy $F[\phi]=\int_{\delta^2}dz[a|\phi|^2+b|\phi|^4+c|\partial\phi/\partial z|^2]$.

As has been pointed out in [2,6] that for small bin the gradient term in $F[\phi]$ does not have any significant effect on the multiplicity fluctuation, so one can set $c=0$. This setting means that $\phi$ can be regarded as a constant over the area $\delta^2$. Of course, this is approximately true only when $\delta^2$ is very small.

Substituting $Q_m(\delta,\tau)$ into Eq. (1), one gets

$$f_q=\int_0^{\infty}duG_q(\tau xu)e^{xu-u^2}\Bigg/\int_0^{\infty}due^{xu-u^2} \tag{10}$$

with $x=|a|\delta/b$ related to the bin width $\delta$. Define [6]

$$J_q(\alpha)=\int_0^{\infty}duu^qe^{\alpha u-u^2} \tag{11}$$

which satisfies recurrence relation $J_q(\alpha)=(\alpha/2)J_{q-1}(\alpha)+[(q-1)/2]J_{q-2}(\alpha)$ and can be directly integrated for $q=0$ and 1, $J_0(\alpha)=(\sqrt{\pi}/2)e^{\alpha^2/4}[1+\mathrm{erf}(\alpha/2)]$, $J_1(\alpha)=\frac{1}{2}+(\alpha/2)J_0(\alpha)$. With $J_q(\alpha)$, $f_q$ can be expressed as

$$f_q=J_0^{-1}(x)\sum_{i=q}^{\infty}b_i^q(\tau x)^iJ_i[-(\tau-1)x] \tag{12}$$

with $b_i^q$ constants, $b_q^q=1$, especially. Notice that the second nonzero $b_i^q$ for fixed $q$ is for $i=q+4$. One can check this from the expression for $G(x,s)$ and the recurrence relations for $a_i^q$.

The scaled FMMD $\mathcal{F}_q$ defined do contain contributions from statistical fluctuations, contrary to the usual scaled factorial ones. As a way to seek for the dynamical fluctuations, one can define the dynamical scaled FMMD as

$$\mathcal{F}_q^{(\mathrm{dyn})}=\frac{\mathcal{F}_q}{\mathcal{F}_q^{(\mathrm{stat})}}. \tag{13}$$

For the definition to make sense, one should ensure that the mean multiplicity is the same for all calculations of the moments concerned. In the Ginzburg-Landau model, the mean multiplicity is $\bar{s}=\tau xJ_1(x)/J_0(x)$. Then deviations of $\mathcal{F}_q^{(\mathrm{dyn})}$ from one should indicate the existence of dynamical fluctuations. The three classes of moments defined in this paper can all be calculated directly.

Up to now, all moments are expressed as infinite sums and are exact within the model. The infinite summation in Eq. (12) will hinder us from an explicit formalism for interesting quantities. Now we focus on the range of very small bin size. As has been shown, the smallness of the bin size $\delta$ is for the need of self-consistence of the Ginzburg-Landau model adopted in this paper, otherwise the gradient term plays a role and cannot be set to zero. Experimentally, the bin size $\delta$ can be chosen very small indeed. For example, experimental data [13] show that the number of total produced charged particles is about 70 within a rapidity range of about 7 in $200A$ GeV S+Em collisions. The rapidity resolution in EMU01 experiments can be high, up to 0.01. In a two-dimensional analysis as in this paper, the area $\delta^2$ considered can be so small that in that area the mean multiplicity satisfies $s\ll1$. The mean multiplicity in a single bin can still
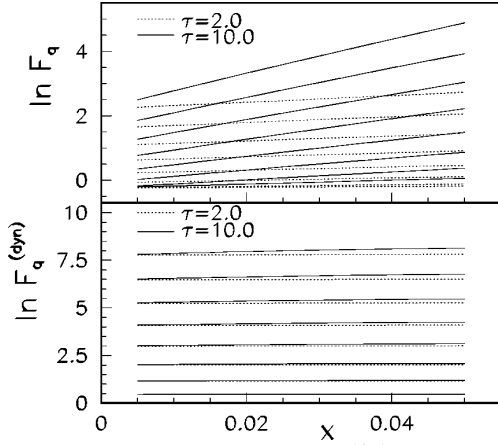
FIG. 1. Dependences of $\ln \mathcal{F}_q$ and $\ln \mathcal{F}_q^{(\mathrm{dyn})}$ on the bin width $x$ for $\tau = 10.0$ and $\tau = 2.0$.



FIG. 2. Scaling behaviors of $\ln \mathcal{F}_q$, $\ln \mathcal{F}_q^{(\mathrm{dyn})}$ vs $\ln \mathcal{F}_2$, $\ln \mathcal{F}_2^{(\mathrm{dyn})}$ for the same choices of lifetime as in Fig. 1.

be much less than 1 even for Pb-Pb collisions in which the number of produced particles can reach 1500 or more. Generally, if the multiplicity density is so low that $\rho \ll 1/r$, with $r$ the highest resolution achievable in the analysis, the condition $\bar{s} \ll 1$ can be satisfied. Because of such experimental facts, we can discuss only the cases with $s \ll 1$ in the following, and our results can be checked directly in experiments. At such low mean multiplicity it is still useful to study multiplicity difference correlators at $\bar{s}$, since the correlators can be used experimentally to study the correlations between different bins just as the usual correlators [4], whereas usual scaled factorial moments for single bins are very effective for the study of dynamical fluctuations inside single bins. One can see that the condition of low mean multiplicity will enable us to reach simple expressions for all the moments.

For the pure statistical fluctuation case, terms except the leading term in Eq. (8) can be neglected, and one can easily get

$$\ln \mathcal{F}_q^{(\mathrm{stat})} = (q-1)(\bar{s} - \ln 2). \tag{14}$$

One can check that the relative contribution from all non-leading terms is about 1% for $\bar{s} = 0.3$. From Eqs. (13) and (14), one can see that $\ln \mathcal{F}_q^{(\mathrm{dyn})}$ can be obtained by a one-step subtraction from $\ln \mathcal{F}_q$ if the mean multiplicity $\bar{s}$ is known.

For the moments with the onset of phase transition, it is a little complicated because of the integration in Eq. (10) over the whole range of $s$. But, one can see that the leading term in $G_q(s)$ plays a dominant role. One needs to notice that an integrating $u^n$ term is associated with the product of two factors $(\tau x)^n$ and $\exp(-\tau' xu - u^2)$, $\tau' = \tau - 1$. For small $x\tau$ the first factor strongly suppresses the contribution. For larger $x\tau$ the term $\exp(-\tau' xu)$ overdepresses the contribution from the former. In fact, numerical results show that $(x\tau)^4 J_{q+4}(-\tau' x)/J_q(-\tau' x)$ is always of the order $10^{-4}$ for $x\tau \lesssim 0.5$, corresponding to $\bar{s} \simeq 0.3$ for $\tau = 10.0$, so that the results will not be affected practically if only the leading terms are kept for the calculation of the moments in small $x$ region. Then to a good approximation,

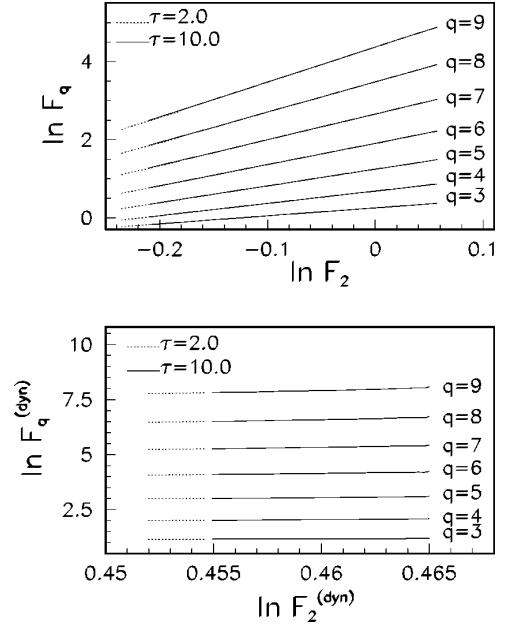$$\ln \mathcal{F}_q = (q-1)\ln \frac{J_0(x)}{2} + \ln J_q(-\tau' x) - q \ln J_1(-\tau' x), \tag{15}$$

$$\ln \mathcal{F}_q^{(\mathrm{dyn})} = (q-1)\ln \frac{J_0(x)}{\exp(\bar{s})} + \ln J_q(-\tau' x)$$

$$- q \ln J_1(-\tau' x). \tag{16}$$

The behaviors of $\ln \mathcal{F}_q$ and $\ln \mathcal{F}_q^{(\mathrm{dyn})}$ as functions of $x$ from 0.005 to 0.05 are shown in Fig. 1 for $\tau = 2.0$ and 10.0. The $x$ range is chosen from the requirement $\bar{s} \lesssim 0.3$ for $\tau = 10.0$. One can see that both $\ln \mathcal{F}_q$ and $\ln \mathcal{F}_q^{(\mathrm{dyn})}$ have linear dependence on bin size $x$. This dependence is completely different from the usual intermittency behaviors. This result can also be seen directly from last expressions for the moments if one substitutes $J_q(\alpha)$ with $\frac{1}{2}[\Gamma((q+1)/2) + \alpha\Gamma((q+2)/2)]$ for very small $\alpha$. In small $x$ approximation,

$$\ln \mathcal{F}_q = \mathrm{const} + \left[ (q-1)\frac{\Gamma(1)}{\Gamma(\frac{1}{2})} - \tau'\left( \frac{\Gamma(q+2/2)}{\Gamma(q+1/2)} - q\frac{\Gamma(\frac{3}{2})}{\Gamma(1)} \right) \right] x$$

$$+ O(x^2), \tag{17}$$

$$\ln \mathcal{F}_q^{(\mathrm{dyn})} = \mathrm{const} + \tau'\left[ q\frac{\Gamma(\frac{3}{2})}{\Gamma(1)} - \frac{\Gamma(q+2/2)}{\Gamma(q+1/2)} - (q-1)\frac{\Gamma(1)}{\Gamma(\frac{1}{2})} \right] x$$

$$+ O(x^2). \tag{18}$$

Numerical results show nontrivial scaling behaviors for $\ln \mathcal{F}_q$ vs $\ln \mathcal{F}_2$ and $\ln \mathcal{F}_q^{(\mathrm{dyn})}$ vs $\ln \mathcal{F}_2^{(\mathrm{dyn})}$ in Fig. 2, which follow trivially from Fig. 1. Though $\ln \mathcal{F}_q$ have different ranges of values for different $\tau$, the scaling behaviors seem independent of the lifetime of the system. For our choice of $x$ range, the range of $\ln \mathcal{F}_2^{(\mathrm{dyn})}$ is quite small. Considering the scaling between $\ln \mathcal{F}_q$ vs $\ln \mathcal{F}_2$ with wider range of $x$ shown in [11] and Eq. (14) in this paper, one can expect the scaling between $\ln \mathcal{F}_q^{(\mathrm{dyn})}$ and $\ln \mathcal{F}_2^{(\mathrm{dyn})}$ is still true for a wider range of $x$ and thus of $\ln \mathcal{F}_2^{(\mathrm{dyn})}$. One can see weak dependence on $\tau$ for $\beta_q$ from last equations. $\beta_q^{(\mathrm{dyn})}$ do not depend on any parameter in the model because the $\tau$ dependencies in the
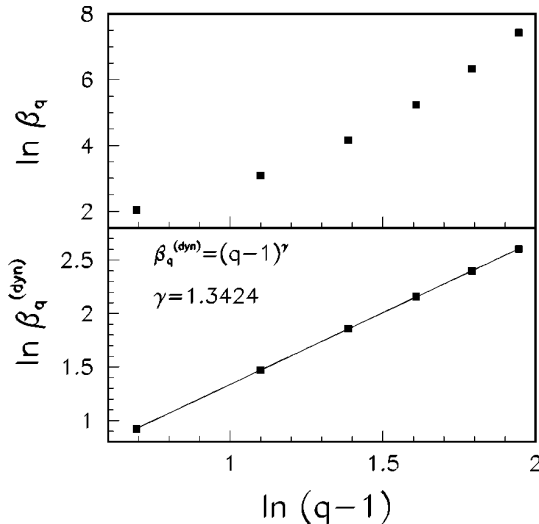
FIG. 3. Scaling exponent $\ln\beta_q$ vs $\ln(q-1)$.

local slopes are cancelled miraculously with each other in the small $x$ limit. More interestingly, $\beta_q^{(\text{dyn})}$ can be well fitted by

$$\beta_q^{(\text{dyn})}=(q-1)^\gamma \qquad (19)$$

with $\gamma=1.3424$, as shown in Fig. 3. But $\beta_q$ do not obey the same scaling law, as shown in Fig. 3 for the case with $\tau=10.0$. The universal exponent $\gamma$ is different from that in [11]. But the difference does not mean any contradiction between the present paper and [11], because they correspond to different quantities. The difference also comes from the different $x$ regions discussed since $\beta_q$ depend on the fitting range. In this paper, the exponent $\gamma$ is completely determined by the general features but does not depend on any parameter of the Ginzburg-Landau model used to describe the phase transition. The exponent $\gamma$ given here is very close to the exponent $\nu$ given in former studies on multiplicity fluctuations for second-order phase transition. In fact, from Eq. (4.4) in the first paper in [6] the usual scaled factorial moments $\ln F_q$ can be expressed, in small $\alpha$ or small $x$ limit, as

$$\ln F_q=\text{const}+2\alpha\left[\frac{\Gamma(q+2/2)}{\Gamma(q+1/2)}-q\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}+(q-1)\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}\right]$$
$$+O(\alpha^2).$$

Thus $\beta_q$ for the scaling between $F_q$ and between $\mathcal{F}_q$ is the same in the small $x$ limit; so are $\gamma$ and $\nu$. The slight difference between them comes from the different regions concerned. As shown in Fig. 3 of the first paper in [6], $\beta_q$ take minima at $\alpha\equiv(x/2)^{0.5}\simeq1$ and increase with the decrease of $\alpha$. Our result corresponds to $\alpha\simeq0$. Physically, since both $\nu$ and $\gamma$ describe dynamical fluctuations in small bins in the same phase transition, they should be equal in the small $x$ limit, though they correspond to different quantities. For experiments with $\bar{s}$ in a single bin a little larger than 0.3 the experimentally obtained $\gamma$ should be close to but less than 1.3424. Thus if experiments observe a scaling exponent $\gamma$ about 1.34 in a high resolution analysis, the onset of a second-order quark-hadron phase transition can be pronounced.

It should be pointed out that even without dropping off nonleading terms, the exponent $\gamma$ in this paper will not be changed, because all those terms are related to higher orders of $x$ and have no contribution to $\gamma$ which is connected with properties of the moments in the limit $x\rightarrow0$. In this sense, the exponent $\gamma$ given here is exact and truly universal.

In summary, scaled FMMD are studied analytically within the Ginzburg-Landau model in a kinetical region with mean multiplicity in a single bin less than 0.3 for second-order quark-hadron phase transition. The dynamical fluctuations in FMMD are extracted, which give the same physical contents as the usual scaled factorial moments. Scaling behaviors between scaled FMMD are shown, and a truly universal exponent is given.

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