2_1^+ and 2_2^+ states in collective nuclei as multiple Q-phonon excitations

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In the framework of the Q-phonon scheme and the interacting boson model a description of the 2_1^+ and 2_2^+ states is obtained. The wave vectors are written as an expansion in multiple Q-phonon excitations of the ground state. It is shown that the convergence of this expansion is fast and the main component already provides a good approximation of the wave vector. The accuracy of such a leading-order description exceeds 90% of the wave vector. By taking into account two Q-phonon configurations the accuracy of the description of the 2_1^+ and 2_2^+ states is increased to 98%. This is true for all collective nuclei with the exception of some rotors where the first excited $K=0$ band lies below or very close to the γ band and, due to band mixing, the wave vector of the 2_2^+ state contains higher Q-phonon excitations. An application to some decay transitions is given. [S0556-2813(98)04501-4]

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I. INTRODUCTION

Collective nuclei can be classified according to the pattern of their lowest excitations. In the geometrical model of Bohr and Mottelson [1] they can be described in terms of vibrators, deformed rotors, and transitional cases as, e.g., γ-soft nuclei [2]. The interpretation of the low-lying collective states in nuclei depends on the model applied. For instance, in rotor nuclei the 2_1^+ state is interpreted as the rotational state and the 2_2^+ state is identified with the $K=2$ γ vibration, while in γ-soft nuclei this state is the bandhead of the so-called ''quasi-γ band.'' In vibrators the 2_1^+ and 2_2^+ states are considered as one- and two-quadrupole phonon states. The different names for these 2^+ states reflect the application of different geometrical models. This is clearly justified because the properties of these states (and others) differ in the various geometrical models. On the other hand, the common properties of these states are hidden because of the different descriptions. To investigate the common properties of the states one can consider algebraic models which contain the geometrical models as limits.

An outstanding model which can serve this aim is the sd-interacting boson model (IBM) [3,4]. The IBM is formulated in terms of s and d boson creation and annihilation operators. An extensively used form of the IBM is the following four-parameter Hamiltonian:

$$H=\kappa\left(\frac{\epsilon}{\kappa}n_d+Q^\chi\cdot Q^\chi+\frac{\lambda}{\kappa}L\cdot L\right). \tag{1}$$

It contains a d boson energy term, a quadrupole-quadrupole interaction, and a rotational energy term. In the consistent Q formalism (CQF) [5] the quadrupole operator

$$Q^\chi=s^+\tilde{d}+d^+s+\chi(d^+\tilde{d})^{(2)} \tag{2}$$

is proportional to the $E2$ transition operator

$$T(E2)=e_B\,Q \tag{3}$$

(from now on we drop the superscript χ and denote the quadrupole operator by Q). We note that we use the extended consistent Q formalism (ECQF) with an n_d term in the Hamiltonian [6]. As a function of its parameters, the IBM Hamiltonian has three analytically solvable limits—the dynamical symmetries—which in the geometrical model correspond to simple cases of the (an)harmonic vibrator (finite ϵ, $\kappa=0$, $\chi=0$), to the symmetric rotor ($\epsilon=0$, $\chi=-\sqrt{7}/2$), and to γ-soft deformation ($\epsilon=0$, $\chi=0$). Transitional nuclei lying between the dynamical symmetries can be investigated by a variation of the parameters. The wave vectors of the ECQF Hamiltonian (1) depend on the two structural parameters ϵ/κ and χ only. The parameter κ determines the absolute energy scale and the rotational term $L\cdot L$ is diagonal and does not affect the wave vectors. The wave vectors can thus be investigated as a function of the two structural parameters ϵ/κ and χ.

In Refs. [7–9] it has been shown that the wave vectors of the low-spin yrast states can be described to good accuracy ($>90\%$ of the wave vector) by multiple Q-phonon excitations of the ground state for nearly the whole parameter space

$$|L_1^+\rangle\approx\mathcal{N}^{(L,n)}(\underbrace{Q\ldots Q}_{n})^{(L)}|0_1^+\rangle \quad \text{with} \quad n=\frac{L}{2}+\left[1-(-1)^L\right]\frac{3}{4}\,. \tag{4}$$

In the U(5) and O(6) dynamical symmetries Eq. (4) describes the eigenstates exactly for the lowest collective excitations [10,11]. In the SU(3) dynamical symmetry only the ground band is accessible. A detailed investigation for O(6) was done in Ref. [11] as well as for its fermionic version [12]. For the O(6) symmetry all eigenstates can be written as Q-phonon excitations of the ground state or of an appropriate excited 0^+ state. For the U(6/4) supersymmetry a similar description has been given in Ref. [13] in the framework of the interacting boson-fermion model [14]. Equation (4) represents an approximate description of the yrast states in the whole parameter space of the ECQF of the IBM. It is one of the aims of this work to obtain a similar universal expression

for the wave vector of the nonyrast 2_2^+ state. The expressions of the eigenstates in terms of multiple Q-phonon excitations are useful for an intuitive and simple understanding of the $E2$ decay [10,15] and the quadrupole moment [16] in the framework of the IBM.

II. Q-CONFIGURATIONS AND FIRST ORDER APPROXIMATION

For analytical calculations the wave vectors in the IBM are usually expressed in terms of the U(5) basis. When the wave vectors are expanded in the U(5) basis outside the U(5) symmetry, many components contribute and the expansion may have a slow convergence. The Q-phonon excitations can be used as an alternative basis with a faster convergence (few important terms). In the O(6) dynamical symmetry of the IBM, which describes γ-soft nuclei, all low-lying states with the O(6) quantum number $\sigma=N$ can be described as multiple Q-phonon excitations of the ground state [11]. The 2_1^+ and 2_2^+ states in the O(6) dynamical symmetry are represented by only one individual multiple Q-phonon excitation. In general, however, multiple Q-phonon excitations are not eigenstates of the Hamiltonian (1). The wave vectors of the lowest-lying eigenstates may then be approximated by series expansion in multiple Q-phonon excitations of the ground state

$$|L_i^+\rangle \approx \left[\sum_{k\geqslant L/2} a_{i,k}(Q_1 \ldots Q_k)^{(L)}\right]|0_1^+\rangle. \tag{5}$$

We will show that, for the Hamiltonian of Eq. (1), the convergence of this series is fast. This makes the Q-phonon scheme a useful concept for semianalytical descriptions outside the dynamical symmetries of the IBM.

To begin with, we first define the Q configurations that we will use in the following. It is much more convenient to work with orthonormalized configurations instead of the multiple Q-phonon excitations in Eq. (5). We thus define the orthonormal Q configurations

$$|2_Q^+\rangle = \mathcal{N}_Q Q|0_1^+\rangle = \sum_i \alpha_i |2_i^+\rangle \tag{6}$$

and

$$|2_{QQ}^+\rangle = \mathcal{N}_{QQ}[(QQ)^{(2)} - vQ]|0_1^+\rangle = \sum_i \beta_i |2_i^+\rangle. \tag{7}$$

From the orthogonality condition

$$\langle 2_Q^+|2_{QQ}^+\rangle = 0 \tag{8}$$

we obtain the constant v

$$v = \frac{\langle 0_1^+|(QQQ)^{(0)}|0_1^+\rangle}{\langle 0_1^+|(QQ)^{(0)}|0_1^+\rangle}. \tag{9}$$

The (positively chosen) constants $\mathcal{N}_Q$ and $\mathcal{N}_{QQ}$ are obtained from the normalizations of the Q configurations

$$\langle 2_Q^+|2_Q^+\rangle = 1 = \langle 2_{QQ}^+|2_{QQ}^+\rangle \tag{10}$$

yielding the expressions

$$\frac{1}{\mathcal{N}_Q} = \sqrt{\langle 0_1^+|(QQ)^{(0)}|0_1^+\rangle} \tag{11}$$

and

$$\frac{1}{\mathcal{N}_{QQ}} = \sqrt{\frac{1}{\sqrt{5}}\left[\langle 0_1^+|((QQ)^{(2)}(QQ)^{(2)})^{(0)}|0_1^+\rangle - \frac{\langle 0_1^+|(QQQ)^{(0)}|0_1^+\rangle^2}{\langle 0_1^+|(QQ)^{(0)}|0_1^+\rangle}\right]}. \tag{12}$$

The normalization constants are functions of the expectation values of multiples of the quadrupole operator $\langle 0_1^+|Q^n|0_1^+\rangle$ up to $n=4$ which are considered ''shape invariants'' [17,18]. Those shape invariants can approximately be calculated from a few $B(E2)$ values between low-lying states [19].

The amplitudes α_i and β_i from Eqs. (6),(7) represent the overlaps of the eigenstates 2_i^+ with the Q configurations and can be given explicitly in terms of matrix elements of the quadrupole operator

$$\alpha_i = \langle 2_i^+|2_Q^+\rangle = \frac{\langle 2_i^+|Q|0_1^+\rangle}{\sqrt{\langle 0_1^+|(QQ)^{(0)}|0_1^+\rangle}} = \frac{\langle 2_i^+|T(E2)|0_1^+\rangle}{\sqrt{\Sigma_j |\langle 2_j^+|T(E2)|0_1^+\rangle|^2}} \tag{13}$$

and

$$\beta_i = \langle 2_i^+|2_{QQ}^+\rangle \tag{14}$$

$$= \mathcal{N}_{QQ}\left[\sum_j \left(\langle 2_i^+|Q|2_j^+\rangle\langle 2_j^+|Q|0_1^+\rangle - \frac{\Sigma_k \langle 0_1^+|Q|2_j^+\rangle\langle 2_j^+|Q|2_k^+\rangle\langle 2_k^+|Q|0_1^+\rangle}{\Sigma_k \langle 0_1^+|Q|2_k^+\rangle^2}\langle 2_i^+|Q|0_1^+\rangle\right)\right], \tag{15}$$

where the $E2$ matrix elements are calculated by the same Q operator as was used in the construction of the Q configurations in the Eqs. (6),(7). Summations over magnetic quantum numbers m are implied.

The 2_1^+ state is in first order approximated [7] by the 1-Q-phonon configuration $|2_Q^+\rangle$. The 2_2^+ state will in first order be approximated by the 2-Q-phonon configuration $|2_{QQ}^+\rangle$. In order to describe details of the wave functions such as the

relatively weak $E2$ decay transition strength $B(E2,2_2^+ \rightarrow 0_1^+)$ or the quadrupole moment $Q(2_1^+)$ [16], it will be necessary to improve the Q-phonon description by going to the second-order approximations

$$|2_1^+\rangle_2^Q = \frac{1}{\sqrt{\alpha_1^2+\beta_1^2}}(\alpha_1|2_Q^+\rangle + \beta_1|2_{QQ}^+\rangle) \tag{16}$$

and

$$|2_2^+\rangle_2^Q = \frac{1}{\sqrt{\alpha_2^2+\beta_2^2}}(\alpha_2|2_Q^+\rangle + \beta_2|2_{QQ}^+\rangle), \tag{17}$$

where the amplitudes involved are given by Eqs. (13)–(15). The second-order approximations $|2_1^+\rangle_2^Q$ and $|2_2^+\rangle_2^Q$ represent those normalized linear combinations of the Q configurations $|2_Q^+\rangle$ and $|2_{QQ}^+\rangle$ that have the largest possible overlaps with the exact eigenstates $|2_1^+\rangle$ and $|2_2^+\rangle$. It is obvious from Eqs. (16),(17) how the approximations can be improved to higher order in Q-phonon excitations including 3-Q-, 4-Q-, ... excitations. In the following we will deal at most with the second-order approximations.

Within the IBM Eqs. (16),(17) are the most natural definitions of the second-order approximations in terms of Q-phonon excitations. We will show below that they approximate the wave vectors of the 2_1^+ and the 2_2^+ eigenstates to more than 97% in a large range of the structural parameters. The overlap of the second-order approximations ${}_2^Q\langle 2_1^+|2_2^+\rangle_2^Q$ is thus less than 3%. But in general they are not exactly orthogonal because of the presence of small components of higher order in Q phonons. For comparison to experimental data, however, it is more useful to require orthogonality in order to put an additional constraint on the amplitudes involved in Eqs. (16),(17). We will later use second Q-phonon approximations for the 2_1^+ and the 2_2^+ eigenstates which are orthonormalized to each other

$$|2_1^+\rangle_2^{Q_\perp} = \frac{1}{\sqrt{\alpha_1^2+\alpha_2^2}}[\alpha_1|2_Q^+\rangle + \mathrm{sgn}(v\alpha_1\alpha_2)\alpha_2|2_{QQ}^+\rangle] \tag{18}$$

and

$$|2_2^+\rangle_2^{Q_\perp} = \frac{1}{\sqrt{\alpha_1^2+\alpha_2^2}}[\alpha_2|2_Q^+\rangle - \mathrm{sgn}(v\alpha_1\alpha_2)\alpha_1|2_{QQ}^+\rangle] \tag{19}$$

with v and α_i from Eqs. (9) and (13), respectively. The approximations (18),(19) have recently been used in Ref. [16] for the prediction of the static quadrupole moment of the 2_1^+ state $Q(2_1^+)$ from other $B(E2)$ values. The differences between the configurations from Eqs. (18),(19) and Eqs. (16),(17) are quite small as will become evident below. We will use expressions (18),(19) later in order to obtain approximate two-body creation operators for the 2_1^+ and the 2_2^+ states.

We will first investigate to what extent the 2_2^+ state is approximated by the 2-Q configuration $|2_{QQ}^+\rangle$. By definition (8) this configuration does not decay to the ground state by an $E2$ transition

$$\langle 0_1^+|Q|2_{QQ}^+\rangle = 0. \tag{20}$$

The Q configuration $|2_{QQ}^+\rangle$ is an exact eigenstate of the IBM Hamiltonian for the U(5) and the O(6) dynamical symmetries but it cannot be an exact eigenstate over the whole symmetry triangle. However, the exact eigenstate $|2_2^+\rangle$ can be written as a sum of the Q configuration (7) and a rest term which is orthogonal to it

$$|2_2^+\rangle = \beta_2|2_{QQ}^+\rangle + |r\rangle \quad \text{with} \quad \langle 2_{QQ}^+|r\rangle = 0. \tag{21}$$

If the overlap of the $|2_2^+\rangle$ state with $|2_{QQ}^+\rangle$ is close to unity, i.e., if the norm of the rest wave vector nearly vanishes $R_1^Q(2_2^+) = \langle r|r\rangle \approx 0$, then $|2_{QQ}^+\rangle$ approximately represents the wave vector of the $|2_2^+\rangle$ state. To check the accuracy we have considered the norm $R_1^Q(2_2^+)$ of the rest wave vector which can be calculated for the whole parameter space of the ECQF of the IBM by numerical diagonalization of the Hamiltonian (1) and by the calculation of all $E2$ matrix elements between the 0^+ and the 2^+ states. The norm of the rest wave vector from Eq. (21) is given by

$$R_1^Q(2_2^+) = 1 - \beta_2^2. \tag{22}$$

For boson numbers $N=7,10,12$ we have calculated $R_1^Q(2_2^+)$ gridwise for the whole parameter space of the ECQF of the IBM. The results of the calculation with boson number $N=10$ are shown in Fig. 1. The results for the boson numbers $N=7$ and $N=12$ are qualitatively the same. As can be seen from Fig. 1, the norm of the rest wave vector $R_1^Q(2_2^+)$ is less than 10% for a wide range of the structure parameters in particular for $|\chi| \leqslant 0.7$ and arbitrary ϵ/κ.

The parameter range with $|\chi| \leqslant 0.7$ is particularly important for the description of actual nuclei with the exception of a few rotors where the 2^+ state of the first excited $K=0$ band lies below the 2^+ band head of the γ band. For these parameters $|\chi| \leqslant 0.7$, the Q configuration $|2_{QQ}^+\rangle$ represents an approximate, parameter-free, analytical expression for the wave vector of the 2_2^+ state. We denote this parameter region of the ECQF below the thick dashed line in Fig. 1, where $R_1^Q(2_2^+)$ is less than 10%, by *region I*.

Let us consider the region in the parameter space where the Q configuration $|2_{QQ}^+\rangle$ does not describe the $|2_2^+\rangle$ state. In Fig. 1 this parameter region is denoted by *region II*. This region corresponds to very large values of $|\chi|$ and finite values of $|\epsilon/\kappa|$ appropriate for the description of some rotors where the $K=0$ band lies below or very close to the γ band. To clarify this point we consider a transition path between region I and region II. We chose $\chi = -1.2 = \text{const}$ close to the SU(3) value $\chi = -\sqrt{7}/2 \approx -1.32$ and vary ϵ/κ from 0 to -8. We denote the 2^+ state which has the largest $B(E2,2_i^+ \rightarrow 0_2^+)$ value by ''$2_{K=0}^+$'' and we denote that 2^+ state which has the largest $B(E2,2_i^+ \rightarrow 3_1^+)$ value by ''2_γ^+.'' Figure 2 shows the level ordering of the $2_{K=0}^+$ and the 2_γ^+

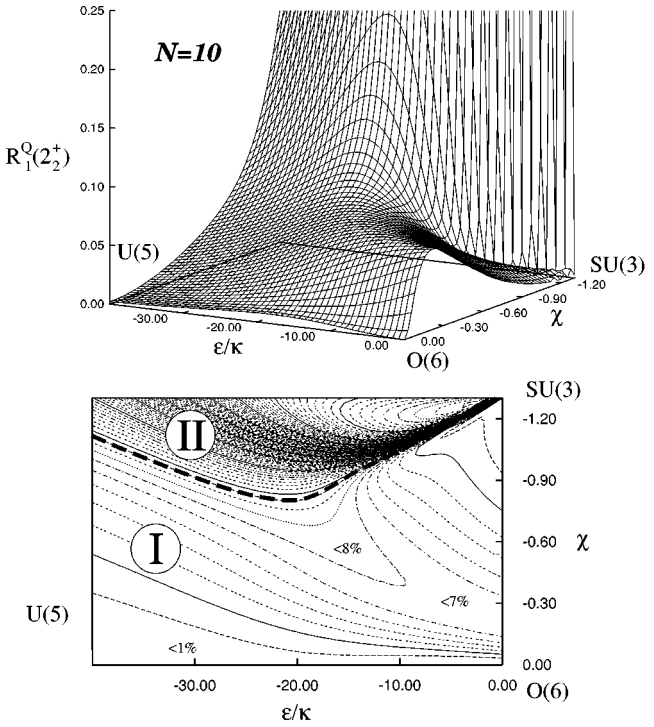

FIG. 1. The squared norm $R_1^Q(2_2^+)$ of the rest wave vector $|r\rangle$ defined in Eq. (21) is given for a boson number $N=10$. Almost the whole parameter space of the ECQF Hamiltonian (1) is surveyed for the structural parameters ϵ/κ and χ. In the physically important parameter range $|\chi|<1$ the wave vector of the 2_2^+ state is approximated by the Q configuration of Eq. (7) by more than 90%. In the following we denote the parameter region below the thick dashed line where $R_1^Q(2_2^+)$ is less than 10% with *region I*. We denote the parameter region above the thick dashed line by *region II*.

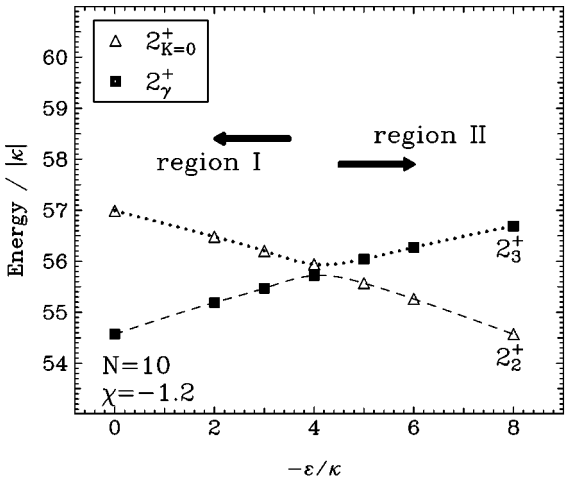

FIG. 2. Level ordering of the 2_γ^+ state and the $2_{K=0}^+$ state in a transition path from parameter region I to region II. For the boson number $N=10$ we kept $\chi=-1.2$ constant and we varied ϵ/κ in the ECQF Hamiltonian. We assign $K=0$ to that 2^+ state which has the largest $B(E2,2_i^+\rightarrow 0_2^+)$ value. We denote that 2^+ state by 2_γ^+ which has the largest $B(E2,2_i^+\rightarrow 3_1^+)$ value to the first 3^+ state. In this particular transition path between region I and region II 2_γ^+ state and the $2_{K=0}^+$ state cross close to the parameter value $\epsilon/\kappa=-4$.

states for this transition path. For small values of $|\epsilon/\kappa|$ the 2_γ^+ state is the 2_2^+ state. With increasing $|\epsilon/\kappa|$ the 2_γ^+ state and the $2_{K=0}^+$ state approach each other and they start to mix. For the parameter value $\chi=-1.2$ the inversion of the states 2_γ^+ and $2_{K=0}^+$ occurs close to $\epsilon/\kappa=-4$. For values of the parameter χ which are closer to the SU(3) value the inversion point occurs for smaller values of $|\epsilon/\kappa|$ until this inversion point is reached for $|\epsilon/\kappa|=0$ in the exact SU(3) dynamical symmetry where the $2_{K=0}^+$ state and the 2_γ^+ are degenerate. There, of course, the $2_{K=0}^+$ state and the 2_γ^+ with $K=2$ do not mix because K is a good quantum number. The SU(3) dynamical symmetry is a singularity for the Q-phonon description of states which do not belong to the ground-state band because for $\chi=-\sqrt{7}/2$ the CQF quadrupole operator is a generator of the SU(3) group and thus it does not induce any $E2$ transition between the ground-state band and other excited bands. Therefore, in the SU(3) dynamical symmetry the normalization constant $\mathcal{N}_{QQ}$ from Eq. (12) becomes infinite. In the SU(3) dynamical symmetry the 2^+ states of the $K=0$ band and the γ band are degenerate. Close to the SU(3) limit the value of $|\chi|<\sqrt{7}/2$ tends to lower the γ band versus the $K=0$ band, while the parameter ϵ increases the mixing between these bands until the inversion point is reached. Thus for $\epsilon=0$, the Q configuration $|2_{QQ}^+\rangle$ describes the wave vector of the 2_2^+ state, which is here the bandhead of the γ band, up to the SU(3) singularity. For $|\epsilon/\kappa|\gg 1$ a considerably smaller value of $|\chi|$ is necessary to avoid an inversion or at least a strong mixing of the $K=0$ band and the γ band.

Consequently, in the region II the 2_2^+ state belongs to the $K=0$ band and contains higher Q-phonon excitations necessary to describe this excitation and thus the simple description of the 2_2^+ state by the 2-Q configuration $|2_{QQ}^+\rangle$ fails. It is interesting to clarify the following question: how many eigenstates of the Hamiltonian (1) have a significant overlap with the Q configuration $|2_{QQ}^+\rangle$? Some results of the calculations of the squares of the scalar products $\langle 2_{QQ}^+|2_i^+\rangle$ with $i=1,2,3,4$ are presented in Table I for the boson number $N=10$. It is seen that in region I the Q configuration $|2_{QQ}^+\rangle$ is dominated by the 2_2^+ state and in region II it is fragmented mainly into two 2^+ eigenstates: $|2_2^+\rangle$ and $|2_3^+\rangle$. For some structure parameters the mixing of these two states is up to 50% but their sum is close to 1. This is an interesting result. It means that even if it is impossible to describe the eigenstates of the IBM Hamiltonian in region II by a single Q configuration, two will do. From the discussion above it is clear why the Q configuration $|2_{QQ}^+\rangle$ alone does not describe the 2_2^+ state in the parameter region II: there additionally a Q-phonon description of the excited $K=0$ band based on the 0_2^+ state is necessary.

In the following we exclude the region II where the Q-phonon description of the 2_2^+ state fails. Thus we exclude from our discussion the parameter range which may describe some rotors, e.g., in the $A\approx 150$ region where the 2^+ state of the $K=0$ band lies below or close to the 2_γ^+ band head. We emphasize that the failure of the simple description of the 2_2^+ state by Eq. (21) reflects the interchange of the order of the low-lying $K=0$ and $K=2$ rotational bands or the mixing between them. In all other cases the wave vector of the 2_2^+ state in collective even-A nuclei can analytically be written

TABLE I. Squared amplitudes of the 2^+ eigenstates to the Q configuration $|2^+_{QQ}\rangle$ from Eq. (7) calculated for some ECQF parameter combinations from the regions I and II outside of the dynamical symmetries.

χ	ϵ/κ	Region	β_1^2	β_2^2	β_3^2	β_4^2	$R_1^Q(2_2^+)$
−0.05	0	I	0.020	0.979	0.000	0.001	0.021
−0.2	0	I	0.058	0.932	0.000	0.010	0.068
−0.5	0	I	0.029	0.960	0.003	0.001	0.040
−1.0	0	I	0.004	0.992	0.002	0.002	0.008
−1.3	0	I	0.000	0.997	0.003	0.000	0.003
−0.05	−8	I	0.008	0.988	0.000	0.000	0.012
−0.2	−8	I	0.047	0.942	0.000	0.001	0.058
−0.5	−8	I	0.036	0.937	0.001	0.014	0.063
−1.0	−8	I	0.007	0.976	0.003	0.010	0.024
−1.3	−8	II	0.000	0.144	0.851	0.002	0.856
−0.05	−24	I	0.001	0.992	0.000	0.000	0.008
−0.2	−24	I	0.011	0.981	0.000	0.000	0.019
−0.5	−24	I	0.033	0.946	0.003	0.011	0.054
−1.0	−24	II	0.017	0.782	0.139	0.058	0.218
−1.3	−24	II	0.003	0.300	0.616	0.074	0.700

as in Eq. (21) with $\beta_2 \geq 0.95$. For the 2_2^+ state the Q configuration $|2^+_{QQ}\rangle$ represents a first-order approximation in terms of Q phonons which we denote by $|2_2^+\rangle_1^Q = |2^+_{QQ}\rangle$. Here, the subscript $|\rangle_i$ indicates the approximation order.

Analogously, the 4_2^+ and the 6_2^+ states can in first order be approximated by the Q configurations:

$$|4^+_{QQQ}\rangle = \mathcal{N}_{QQQ}\Bigg[[Q(QQ)^{(4)}]^{(4)} - \frac{\langle 0_1^+|(QQ(QQQ)^{(4)})^{(0)}|0_1^+\rangle}{\langle 0_1^+|(QQ(QQ)^{(4)})^{(0)}|0_1^+\rangle}(QQ)^{(4)}\Bigg]|0_1^+\rangle = |4_2^+\rangle_1^Q \tag{23}$$

and

$$|6^+_{QQQQ}\rangle = \mathcal{N}_{QQQQ}\Bigg[[Q(QQQ)^{(6)}]^{(6)} - \frac{\langle 0_1^+|(QQQ(QQQQ)^{(6)})^{(0)}|0_1^+\rangle}{\langle 0_1^+|(QQQ(QQQ)^{(6)})^{(0)}|0_1^+\rangle}(QQQ)^{(6)}\Bigg] \times |0_1^+\rangle = |6_2^+\rangle_1^Q. \tag{24}$$

Expanding the Q configurations in the eigenstates we obtain the norms of the corresponding rest wave vectors $R_1^Q(4_2^+)$ and $R_1^Q(6_2^+)$. These were also calculated for $N=10$ bosons. They are shown in the Fig. 3. In the parameter region I that we discuss here Eqs. (4), (7), (23), and (24) provide approximate analytic expressions for the lowest members of the ''ground'' and ''γ band.''

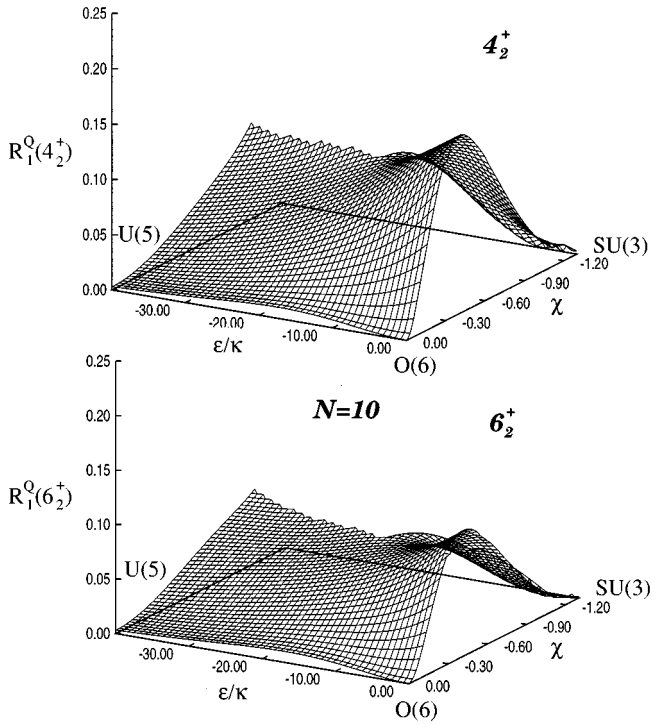

FIG. 3. Squared norms of the rest wave vectors for the 4_2^+ state and the 6_2^+ state calculated for a boson number $N=10$ as in Fig. 1. In the physically important parameter region I (e.g., $|\chi|<1$) the wave vectors of these states are approximated by the Q configurations of Eqs. (23),(24) by more than 90%.

III. SECOND-ORDER APPROXIMATION

In the preceding section we have found an analytical expression which is a first-order approximation to the 2_2^+ state vector. The exact eigenstates of the lowest 2^+ states can thus be written in terms of the Q configuration $|2_Q^+\rangle$ and $|2^+_{QQ}\rangle$. The approximations by these Q configurations are better than 90% in squared amplitude and account for the dominant features of the 2_1^+ and 2_2^+ states. However, the detailed properties of the states may be influenced by the missing 10% of the wave vector. For instance, these 10% are responsible for the ground-state decay of the 2_2^+ state.

We can get an improved expression for the 2^+ states by including the next important term of the Q-phonon expansion of Eq. (5). We thus write the exact eigenstates in terms of the second-order approximations from Eqs. (16),(17)

$$|2_1^+\rangle = y_1|2_1^+\rangle_2^Q + |r_1'\rangle = \alpha_1|2_Q^+\rangle + \beta_1|2^+_{QQ}\rangle + |r_1'\rangle,$$

$$R_2^Q(2_1^+) = \langle r_1'|r_1'\rangle = 1-\alpha_1^2-\beta_1^2, \tag{25}$$

$$|2_2^+\rangle = y_2|2_2^+\rangle_2^Q + |r_2'\rangle = \alpha_2|2_Q^+\rangle + \beta_2|2^+_{QQ}\rangle + |r_2'\rangle,$$

$$R_2^Q(2_2^+) = \langle r_2'|r_2'\rangle = 1-\alpha_2^2-\beta_2^2. \tag{26}$$

The rest wave vectors $|r_i'\rangle$ are again defined to be orthogonal to the 1- and 2-Q configurations $\langle r_i'|2_Q^+\rangle = 0$ and $\langle r_i'|2^+_{QQ}\rangle = 0$ which yield $y_1 = \sqrt{\alpha_1^2+\beta_1^2}$ and $y_2 = \sqrt{\alpha_2^2+\beta_2^2}$. The am-

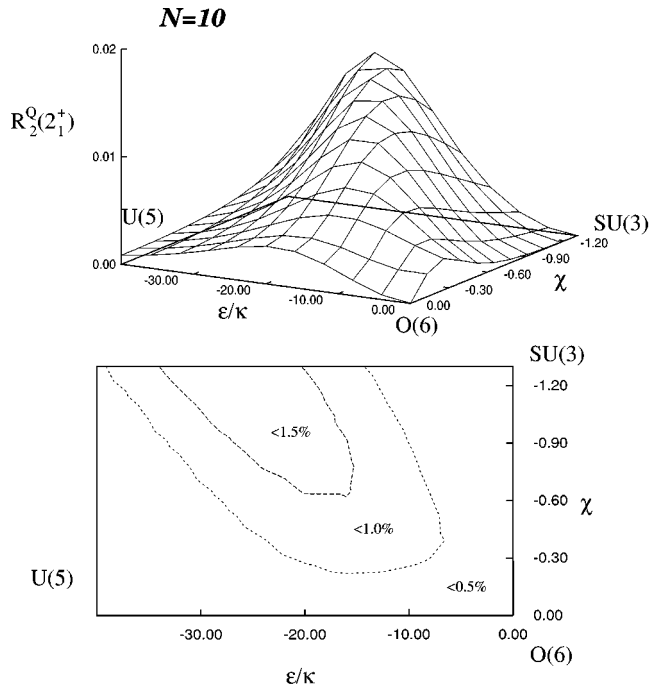

FIG. 4. Squared norm $R_2^Q(2_1^+)$ of the rest wave vector $|r_1'\rangle$ defined in Eq. (25) for a boson number $N=10$. For all parameters the wave vector of the 2_1^+ state is approximated by the combination of the Q configurations $|2_Q^+\rangle$ and $|2_{QQ}^+\rangle$ from Eq. (25) by more than 98%.

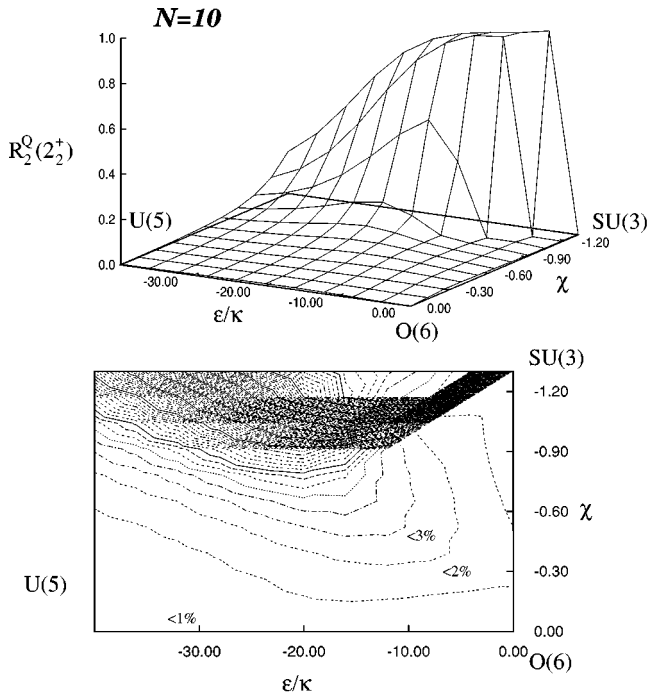

FIG. 5. Squared norm $R_2^Q(2_2^+)$ of the rest wave vector $|r_2'\rangle$ defined in Eq. (26) for a boson number $N=10$. In the physically important parameter region I (e.g., $|\chi|<1$, $|\epsilon/\kappa|<10$) the wave vector of the 2_2^+ state is approximated by the Q configurations $|2_Q^+\rangle$ and $|2_{QQ}^+\rangle$ from Eq. (26) by more than 97%.

plitudes $\alpha_{1,2}$ and $\beta_{1,2}$ are given by Eqs. (13)–(15). The calculated norms of the rest wave vectors $R_2^Q(2_1^+)$ and $R_2^Q(2_2^+)$ are shown in Figs. 4 and 5. For all parameter combinations shown in Fig. 4 the 2_1^+ state wave vector has a squared overlap $1-R_2^Q(2_1^+)$ of more than 98%. The 2_1^+ state is thus described by the second-order approximation $|2_1^+\rangle_2^Q$ within an accuracy of 98% for all parameter combinations analytically.

Similarly in Fig. 5 for the 2_2^+ state, in the parameter region I (practically with $|\chi|<1$, $|\epsilon/\kappa|<10$) the squared overlap of the second-order approximation for the 2_2^+ state in terms of Q-phonon excitations is $1-R_2^Q(2_2^+)\geqslant 97\%$. The quadratic norm of the rest term decreases by about an order of magnitude by going from the first-order to the second-order approximation.

For some applications it may be helpful to have an explicit formulation of the creation operators of the 2_1^+ and the 2_2^+ states from the ground state. We derive approximate creation operators in this paragraph. In parameter region I the 1- and 2-Q configurations $|2_Q^+\rangle$ and $|2_{QQ}^+\rangle$ exhaust the subspace formed by the exact wave vectors of the 2_1^+ and 2_2^+ eigenstates to more than 97%. An exact overlapping of the subspaces would require the relation $\beta_1^2=\alpha_2^2$. In Fig. 6 it is shown that this relation is fulfilled to a good accuracy. For the construction of the approximate creation operators we now assume exact overlapping of these subspaces based on the wave vectors of the 2_1^+ and 2_2^+ eigenstates, on the one hand, and based on the 1- and 2-Q configurations $|2_Q^+\rangle$ and $|2_{QQ}^+\rangle$, on the other hand. In addition we use the sign relation

$$\mathrm{sgn}(\alpha_1\beta_1)=-\mathrm{sgn}(\alpha_2\beta_2)=\mathrm{sgn}(v)=\mathrm{sgn}(\chi). \qquad (27)$$

This sign relation is invariant under a change of phases of the states involved. It was checked numerically in the IBM with the ECQF Hamiltonian from Eq. (1) with $N=9$ and $N=10$ bosons to hold true for the parameter range with $|\chi|<1$ which we are interested in here. The sign relation (27) has been used for the definition of the orthonormalized second-order approximations $|2_i^+\rangle_2^{Q_\perp}$ from Eqs. (18),(19). Thus the two-body operator

$$\Gamma^+=(QQ)^{(2)}-\mathrm{sgn}(v)\left(|v|+x\frac{\mathcal{N}_Q}{\mathcal{N}_{QQ}}\right)Q \qquad (28)$$

creates the 2_2^+ state to a high accuracy of $\geqslant 97\%$ for the physically relevant parameter region I:

$$|2_2^+\rangle_2^{Q_\perp}=-\mathrm{sgn}(v\,\alpha_2)\,\frac{N_{QQ}}{\sqrt{1+x^2}}\,\Gamma^+|0_1^+\rangle\approx|2_2^+\rangle. \qquad (29)$$

Here the abbreviations v from Eq. (9) and $x=\sqrt{B(E2;0_1^+\rightarrow 2_2^+)/B(E2;0_1^+\rightarrow 2_1^+)}$ have been used.

Similarly the second-order approximation to the 2_1^+ state

$$|2_1^+\rangle_2^{Q_\perp}=\mathrm{sgn}(\alpha_1)\mathcal{N}_Q\frac{1-x|v|\mathcal{N}_{QQ}/\mathcal{N}_Q}{\sqrt{1+x^2}}\,\varrho^+|0_1^+\rangle\approx|2_1^+\rangle \qquad (30)$$

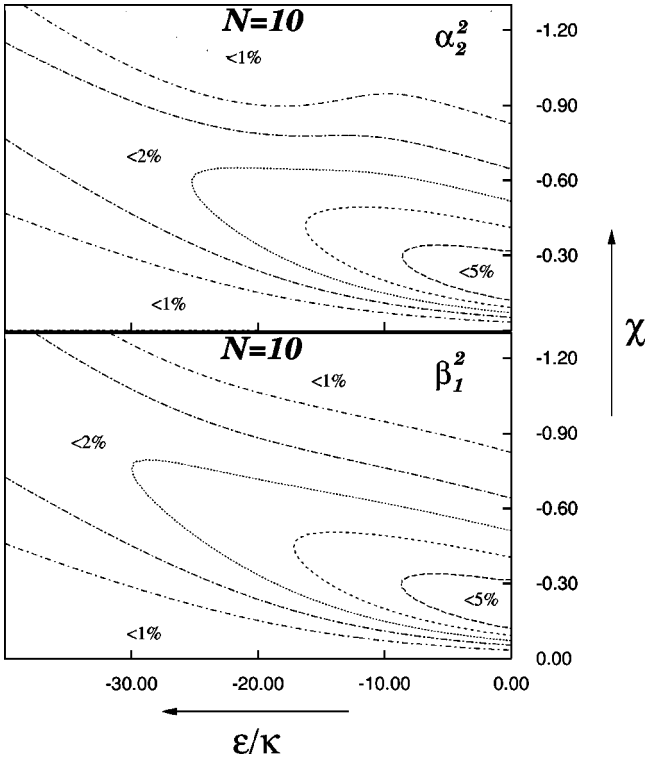

FIG. 6. Comparison of the squared amplitudes of the second important Q configuration in the wave vectors of the exact 2_1^+ state and 2_2^+ state. Their absolute values are approximately equal.

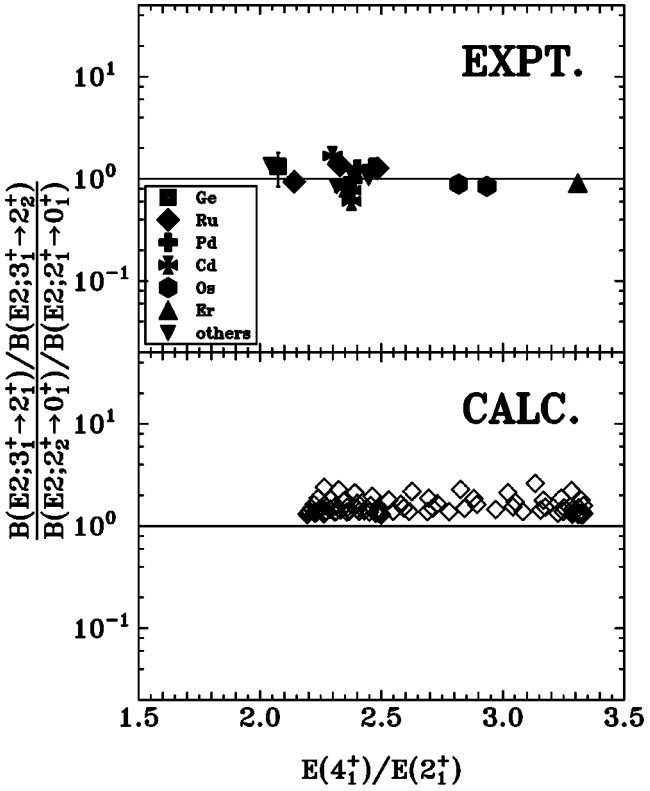

FIG. 7. Test of the approximate relation (34) in the physically important parameter region I. Experimental (top) and exactly calculated (bottom) data points are plotted against the excitation energy ratio $E(4_1^+)/E(2_1^+)$ which is appropriate to scale a transition between vibrators $[E(4_1^+)/E(2_1^+)=2.0]$ and rotors $[E(4_1^+)/E(2_1^+)=3.3]$. While the $B(E2)$ ratios from Eqs. (32),(33) can vary by an order of magnitude their ratio is constant within a factor of 2. The experimental $B(E2;0_1^+\to 2_{1,2}^+)$ data have been taken from Refs. [20, 21]. The decay branching ratios of the 3_1^+ states have been taken from the Nuclear Data Sheets assuming pure $E2$ transitions.

is created by the operator

$$\varrho^+ = Q + \mathrm{sgn}(v)\frac{x}{(\mathcal{N}_Q/\mathcal{N}_{QQ}) - x|v|}(QQ)^{(2)}. \qquad (31)$$

Equation (30) approximates the wave vector of the 2_1^+ state with an accuracy of $>98\%$. The expressions from Eqs. (28)–(31) involve—besides the quadrupole operator—the $B(E2)$ values from the ground state to the 2_1^+ and the 2_2^+ states and the three shape invariants $\langle 0_1^+|(QQ)^{(0)}|0_1^+\rangle$, $\langle 0_1^+|(QQQ)^{(0)}|0_1^+\rangle$, $\langle 0_1^+|[(QQ)^{(2)}(QQ)^{(2)}]^{(0)}|0_1^+\rangle$.

In Eqs. (29),(30) there appear also the signs of the amplitudes α_1 and α_2 which coincide with the signs of the $E2$ matrix elements between the ground state and the 2_1^+ and 2_2^+ states. The amplitudes α_1 and α_2 can always be chosen to be positive by a convention for the phases of the 2_1^+ and 2_2^+ states. The sign of v depends on the sign of the structural parameter χ according to Eq. (27).

IV. ELECTRIC QUADRUPOLE TRANSITIONS

The ground-state decay of the 2_2^+ state is due to the admixture of the 1-Q configuration $|2_Q^+\rangle$ in Eq. (26). The relative $E2$ strength distribution from the ground state to the lowest 2^+ states equals the squared amplitudes of the 1-Q configuration due to Eq. (13)

$$\frac{B(E2;0_1^+\to 2_2^+)}{B(E2;0_1^+\to 2_1^+)} = \frac{\alpha_2^2}{\alpha_1^2}. \qquad (32)$$

An alternative measure of this ratio is the $E2$ decay branching ratio of the 3_1^+ state (which can be approximated by a 3-Q configuration $|3_1^+\rangle \approx \mathcal{N}_3(QQQ)^{(3)}|0_1^+\rangle$ [8]) to the lower lying 2^+ states

$$\frac{B(E2;3_1^+\to 2_1^+)}{B(E2;3_1^+\to 2_2^+)} \approx \frac{\beta_1^2}{\beta_2^2} \approx \frac{\alpha_2^2}{\alpha_1^2} \qquad (33)$$

if one assumes that an $E2$ transition between the 3-Q configuration from Eq. (4) and the 1-Q configuration is forbidden and that possible $E2$ decays from higher lying 3^+ states are negligible. The second approximation in Eq. (33) is valid due to the fact that $\alpha_1^2 \approx \beta_2^2$ and $\alpha_2^2 \approx \beta_1^2$ as has been shown in Fig. 6. In [8] it was already demonstrated that the different $E2$ branching ratios (32),(33) are correlated. From Eqs. (32),(33) we can deduce a formula which allows us to estimate the absolute $B(E2)$ value for the decay of the 2_2^+ state from an $E2$ branching ratio and the absolute $B(E2)$ value of

the 2_1^+ state, both of which are often known. According to Eqs. (32) and (33) the $B(E2)$ value for the decay of the 2_2^+ state to the ground state can be written as

$$B(E2;2_2^+ \rightarrow 0_1^+) \approx \frac{B(E2;3_1^+ \rightarrow 2_1^+)}{B(E2;3_1^+ \rightarrow 2_2^+)} B(E2;2_1^+ \rightarrow 0_1^+). \tag{34}$$

This relation allows the prediction of the $E2$ excitation strength of the 2_2^+ state from the knowledge of the following two observables: the $E2$ branching ratio of the 3_1^+ state and the lifetime of the first excited state. Equation (34) holds exactly in all three dynamical symmetries of the IBM for which the left-hand side (lhs) and the right-hand side (rhs) vanish. As can be seen just below, exact numerical calculations show that outside of the dynamical symmetries the rhs of Eq. (34) slightly overpredicts the lhs systematically.

It is interesting to compare Eq. (34) with the data. Figure 7 shows the ratio of the lhs over the rhs of Eq. (34) versus the energy ratio of the lowest yrast states. The abscissa is appropriate to scale a transition from vibrators to well deformed rotors. The $B(E2;2_{1,2}^+ \rightarrow 0_1^+)$ values have been taken from [20] and [21], respectively. $B(E2;3_1^+ \rightarrow 2_1^+)/B(E2;3_1^+ \rightarrow 2_2^+)$ have been compiled from the relative decay intensities given in Nuclear Data Sheets. In most cases the $E2/M1$ mixing ratios δ for the $3_1^+ \rightarrow 2_{1,2}^+$ transitions are not known. We thus assumed pure $E2$ transitions. While the decay branching ratio of the 3_1^+ state and the ratio of the $E2$ excitation strengths of the ground state vary in these nuclei by an order of magnitude, Eq. (34) is valid within a factor of about 2. Equation (34) describes the data for all collective nuclei considered: vibrators, rotors, γ-soft nuclei, and different types of transitional nuclei. In rotor nuclei the relative decay intensity for the $3_1^+ \rightarrow 2_\gamma^+$ transition is rarely known experimentally due to the low transition energy. But the data from ^{168}Er, which is one of the best studied rotors, support the validity of Eq. (34) in rotors. Equation (34) is an exact relation in the rotor model of Bohr and Mottelson [1]. The data and the numerical test presented above show that this relation is approximately valid for all quadrupole collective nuclei discussed here.

V. SUMMARY

We derived analytic expressions for the $2_2^+, 4_2^+, 6_2^+, \ldots$ states in terms of multiple Q-phonon excitations. In the IBM these expressions are approximately valid also in the regions outside the dynamical symmetries where the exact wave vectors can be obtained only by numerical diagonalization of the Hamiltonian. Together with the even-spin and odd-spin yrast states [7,8] a complete approximate description of the gs band and the (quasi-)γ band in terms of Q configurations is thus given. With the ECQF Hamiltonian in Eq. (1) this is true for $|\chi| \leq 0.7$ and arbitrary ϵ/κ values. In some rotational nuclei where the 2^+ state of the $K=0$ band lies below or close to the γ-band head the Q-phonon description of the 2_2^+ fails. Aside from this case, we demonstrated for the 2_1^+ and 2_2^+ states that the inclusion of the second-order term of the Q-phonon excitations increases the accuracy of the wave vector to about 98%. We derived a relation to estimate the $B(E2)$ value for the ground-state decay of the 2_2^+ state from the $E2$ branching ratio of the 3_1^+ to the lower lying 2^+ states and the excitation strength of the 2_1^+ state. This relation is valid within a factor of 2 for the collective nuclei considered. Such global relations can be found because the Q-phonon description provides an universal description for the low-lying quadrupole collective states. Considering the change of the structure of the 3_1^+ state among vibrators, rotors, γ-unstable nuclei as well as transitional nuclei, the universal relation in Eq. (34) holds remarkably well which indicates the validity of the Q-phonon construction of the lowest two 2^+ states.

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