Energy-weighted $M1$ sum rule in deformed nuclei: A self-consistent approach

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An energy-weighted sum rule covering the full $M1$ scissorslike spectrum is computed in the context of the random phase approximation (RPA) using a separable Hamiltonian. The deformed mean field is derived in Hartree approximation from the quadrupole-quadrupole interaction which is accordingly modified so as to avoid further distortions of the mean field. The main effect of this modification consists in the complete cancellation of the contribution to the sum rule coming from the one-body potential. The resulting sum is almost entirely exhausted once the contribution from the observed low-lying $M1$ transitions is implemented with the one coming from energy and strength of the high-lying $M1$ mode, computed in schematic RPA. It also emerges naturally that the $M1$ transitions of both modes are strictly correlated with the quadrupole collectivity of the ground state. [S0556-2813(98)05502-2]

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I. INTRODUCTION

Out of the many properties characterizing the low-lying magnetic dipole excitations observed in deformed nuclei [1,2], known as scissors modes [3], the intimate connection between the $M1$ transitions and the quadrupole collectivity is of the utmost importance for assessing the real nature of the mode. It has been found in Sm [4] and Nd [5] isotopic chains that the summed strength of the $M1$ transitions below 4 MeV undergoes a sudden increase in going from spherical to deformed nuclei and then grows quadratically with deformation. Such a deformation law, which reflects the similar saturation properties of the summed $M1$ strength and the ground rotational $E2$ transition probability [6], has been found to hold for all rare earth nuclei [7].

Theoretical studies of this problem have been carried out in the Tamm-Dancoff approximation (TDA) [8] and, more extensively, in the random phase approximation (RPA) [9–11]. All these calculations are approximately free of the uncertainties induced by the occurrence of spurious rotational admixtures [12]. Such a redundant mode was removed either by a method developed by Pyatov *et al.* [13] which leads to a modification of the quadrupole-quadrupole interaction [12] or by a Schmidt orthogonalization of the basis states [14] or by formulating the RPA eigenvalue problem directly in the laboratory frame [11]. Another way of solving the problem is through the use of a self-consistent basis [15,16]. In a recent paper [17] it was shown by an analytical procedure carried out in schematic RPA that the use of a mean field deduced from a quadrupole-quadrupole interaction in the Hartree approximation removes completely the redundant mode. It was also shown that the method is equivalent to the Pyatov approach and, therefore, is effective in more general, realistic, contexts.

Global properties such as the deformation law have been described, with different degrees of accuracy, in all phenomenological and schematic models adopted in the past to study the mode [18–32]. Sum rule techniques have been exploited in some of these models. This has been done in IBM-2 [8,21–25], in the realm of shell model [26] and in a purely phenomenological analysis [29]. These studies were confined to the low-lying $M1$ excitations. On the other hand, schematic [33] as well as realistic [34–38] RPA calculations suggest the existence of a high-energy mode of scissors nature. This would be just the $K^{\pi}=1^{+}$ component of the isovector quadrupole giant resonance.

When framed in particle-hole space, a sum rule, by its own nature, necessarily accounts for the contribution of both modes unless low- and high-energy states get decoupled by some special ansatz as in the pioneering work by Lipparini and Stringari [39]. The same can be said for the shell model sum rule derived in [26]. A shell model treatment of both high and energy modes, however, is necessarily confined to light nuclei. This has been done [40] and, as we shall briefly see in the following sections, interesting and surprising results have been obtained.

In heavy nuclei, the two modes were explicitly treated in a sum rule approach based on the use of a one-body Hamiltonian only [27]. A quadratic deformation law was found to hold for both transitions. The approximation of neglecting the two-body interaction was seriously questioned [41]. Indeed, the calculation is to be viewed as a phenomenological tool for describing such a peculiar property of the mode [42].

In this paper we will compute, within the framework of RPA, a $M1$ sum rule which involves the full $M1$ spectrum. Contributions coming from one- and two-body potentials are taken into account. Spurious rotational admixtures are avoided by following the strategy adopted in Ref. [17], which consists in generating the deformed mean field in Hartree approximation from a rotational invariant quadrupole-quadrupole potential. The latter interaction is then modified by the use of doubly stretched coordinates [43–45] so as to avoid any further distortion of the mean field.

In Sec. II the Hartree treatment developed in Ref. [17] is briefly reviewed and doubly stretched coordinates are introduced. In Sec. III $M1$ and $E2$ transition amplitudes and strengths are derived in proton-neutron schematic RPA. Use of the two strengths will be made for a numerical analysis of the scissors sum rule. This is derived in Sec. IV for a schematic Hamiltonian and in Sec. V for a more realistic separable Hamiltonian. A qualitative and quantitative analysis of the sum rule is carried out in Sec. VI. Some concluding remarks are drawn in Sec. VII.

II. DEFORMED HARTREE MEAN FIELD AND DOUBLY STRETCHED COORDINATES

We assume first that our nuclear system, composed of Z protons and N neutrons, is described by an Hamiltonian of the rotational invariant form

$$H=H_0+\frac{1}{2}\chi(Q_2^{(p)^\dagger}\cdot Q_2^{(p)}+Q_2^{(n)^\dagger}\cdot Q_2^{(n)})+\frac{1}{2}\chi_{pn}(Q_2^{(p)^\dagger}\cdot Q_2^{(n)}+Q_2^{(n)^\dagger}\cdot Q_2^{(p)}), \tag{2.1}$$

where H_0 is an isotropic harmonic oscillator (HO) Hamiltonian of frequency ω_0 and

$$Q_{2\mu}^{(\tau)}=\sum_i q_{2\mu}^{(\tau)}(i)=\sum_i r_i^{(\tau)^2}Y_{2\mu}^{(\tau)}(i) \tag{2.2}$$

are proton ($\tau=p$) and neutron ($\tau=n$) quadrupole fields. The above two-body potential can also be written in the isospin form, adopted for instance in Ref. [46], as the sum of an isoscalar and an isovector quadrupole-quadrupole interaction with respective coupling constants

$$\chi(T=0)=\frac{1}{2}(\chi+\chi_{pn}),\quad \chi(T=1)=\frac{1}{2}(\chi-\chi_{pn}). \tag{2.3}$$

A Hartree treatment [17] yields for the ith nucleon the deformed one-body potential

$$V_\beta^{(\tau)}=\frac{1}{2}m\omega_0^2 r^{(\tau)^2}-\beta_\tau m\omega_0^2 q_{20}^{(\tau)}, \tag{2.4}$$

where the deformation parameters β_τ are defined through

$$\beta_p m\omega_0^2=-(\chi\langle Q_{20}^{(p)}\rangle+\chi_{pn}\langle Q_{20}^{(n)}\rangle),$$
$$\beta_n m\omega_0^2=-(\chi\langle Q_{20}^{(n)}\rangle+\chi_{pn}\langle Q_{20}^{(p)}\rangle). \tag{2.5}$$

These equations, which can be called the Hartree self-consistent conditions, ensure the separation of the intrinsic $M1$ states from the rotational mode [17].

The Hartree potential can be put in the Nilsson form of an anisotropic HO potential with frequencies

$$\omega_1(\tau)=\omega_0\sqrt{1+\frac{2}{3}\delta_\tau}\simeq\omega_0\left(1+\frac{1}{3}\delta_\tau\right),$$
$$\omega_3(\tau)=\omega_0\sqrt{1-\frac{4}{3}\delta_\tau}\simeq\omega_0\left(1-\frac{2}{3}\delta_\tau\right), \tag{2.6}$$

where new and old deformation parameters are connected by

$$\delta_\tau=\sqrt{\frac{45}{16\pi}}\,\beta_\tau. \tag{2.7}$$

Once generated, the Hartree mean field must not be distorted further by the two-body force. To this purpose it is appropriate to modify the quadrupole-quadrupole potential by using doubly stretched coordinates $\tilde{x}_i=(\omega_i/\omega_0)x_i$ in the quadrupole fields [43–45]. Indeed, with these new variables we have

$$\langle\tilde{Q}_{2\mu}^{(\tau)}\rangle=\langle Q_{2\mu}^{(\tau)}(\tilde{x}_i)\rangle=0 \tag{2.8}$$

as long as the nuclear self-consistent conditions

$$\omega_1(\tau)\sum_{n_1^{(\tau)}}\left[n_1^{(\tau)}+\frac{1}{2}\right]=\omega_2(\tau)\sum_{n_2^{(\tau)}}\left[n_2^{(\tau)}+\frac{1}{2}\right]=\omega_3(\tau)\sum_{n_3^{(\tau)}}\left[n_3^{(\tau)}+\frac{1}{2}\right] \tag{2.9}$$

are enforced. The explicit form of the new quadrupole fields is

$$\tilde{Q}_{2\pm1}=\frac{\omega_1\omega_3}{\omega_0^2}Q_{2\pm1},$$
$$\tilde{Q}_{2\pm2}=\frac{\omega_1^2}{\omega_0^2}Q_{2\pm2},$$
$$\tilde{Q}_{20}=\frac{1}{3\omega_0^2}(\omega_1^2+2\omega_3^2)Q_{20}-\frac{\sqrt{5}}{3\omega_0^2}(\omega_1^2-\omega_3^2)Q_{00}, \tag{2.10}$$

where $Q_{00}=r^2Y_{00}$ is a monopole term. The new $(\tilde{Q}\tilde{Q})$ potential is therefore composed of pure quadrupole-quadrupole plus monopole-quadrupole and monopole-monopole terms.

It is well known that the Hartree conditions (2.5) fix the isoscalar coupling constant. To this purpose we first compute the mean value of the quadrupole field using the asymptotic HO basis. After exploiting the nuclear self-consistent conditions (2.9), and Eqs. (2.6), we obtain

$$\langle Q_{20}^{(p)}\rangle=\sqrt{\frac{5}{16\pi}}\,\langle Q_0^{(p)}\rangle=\sqrt{\frac{5}{16\pi}}\frac{4}{3}Z\langle r^2\rangle\delta^{(p)}\left(1+\frac{2}{3}\delta^{(p)}\right), \tag{2.11}$$

and similarly for neutrons. From summing both sides of Eq. (2.5) it is now easy to deduce

$$\chi(0)=\frac{1}{2}(\chi+\chi_{pn})=-\frac{4\pi}{5}\frac{m\omega_0^2}{A\langle r^2\rangle}\left(1-\frac{2}{3}\delta\right), \tag{2.12}$$

where $\delta=(\delta_p+\delta_n)/2$. The lowest order piece of $\chi(0)$ is the well-known expression derived in Ref. [47].

As Eqs. (2.3) show, the isovector coupling constant $\chi(1)$ is nonvanishing only if $\chi\neq\chi_{pn}$. There is no agreement on the value to be assigned to such a constant. This is usually deduced from the symmetry energy mass formula [47] and results to be related to $\chi(0)$ by the ratio $b=-\chi(1)/\chi(0)\simeq3.5$. This estimate has been questioned in recent analyses [37,48], both pointing at a smaller value. In particular, shell model calculations in light nuclei [48] have shown that a large value of b, while needed to give a correct energy splitting between isoscalar and isovector giant quadrupole resonances, leads to the collapse of states which should be well above the ground state. Given these uncertainties, we will consider the ratio $b=-\chi(1)/\chi(0)$ as a free parameter.

III. $M1$ AND $E2$ TRANSITIONS IN PROTON-NEUTRON SCHEMATIC RPA

The Hamiltonian resulting from the self-consistent approach illustrated in the previous section is composed of the Hartree deformed field (2.4) plus the $(\widetilde{Q}\widetilde{Q})$ interaction. This Hamiltonian is adopted here to derive explicit expressions for energies and strengths in proton-neutron schematic RPA by adopting the formalism developed in Ref. [17]. Use of these expressions will be made in the numerical analysis of the sum rule which will be derived in the next section.

In the proton-neutron formalism the RPA energies ω are the roots of the eigenvalue equation

$$[1-2\chi S^{(p)}(\omega)][1-2\chi S^{(n)}(\omega)]=4\chi_{pn}^2 S^{(p)}(\omega)S^{(n)}(\omega), \tag{3.1}$$

where

$$S^{(\tau)}(\omega)=\sum_{ph}\frac{E_{ph}^{(\tau)}}{\omega^2-E_{ph}^{(\tau)2}}|(\widetilde{Q}_{21}^{(\tau)})_{ph}|^2(uv)_{ph}^{(+)^2}. \tag{3.2}$$

Here $(uv)_{ph}^{(\pm)}=v_p^{(\tau)}u_h^{(\tau)}\pm u_p^{(\tau)}v_h^{(\tau)}$ are coefficients produced by the Bogoliubov-Valatin transformation, $(\widetilde{Q}_{2\mu}^{(\tau)})_{ph}=\langle p|\widetilde{Q}_{2\mu}^{(\tau)}|h\rangle$ are single particle matrix elements of the quadrupole field, and E_{ph} denotes the two quasiparticle energies. In our case the space is spanned by proton and neutron unperturbed excited states of energies $E_0^{(\tau)}=\sqrt{\epsilon_0^{(\tau)^2}+(2\Delta^{(\tau)})^2}$ and $E_2^{(\tau)}=\epsilon_2^{(\tau)}$, where $\epsilon_0^{(\tau)}=\omega_1(\tau)-\omega_3(\tau)\simeq\delta_\tau\omega_0$ and $\epsilon_2^{(\tau)}=\omega_1(\tau)+\omega_3(\tau)$ are the HO particle-hole (ph) energies and $\Delta^{(\tau)}$ is the pairing gap.

Using the key relation

$$\frac{1}{\epsilon_2^{(\tau)}}\sum_{ph\in\epsilon_2}|\langle ph|Q_{21}^{(\tau)}|\rangle|^2=\frac{1}{\epsilon_0^{(\tau)}}\sum_{ph\in\epsilon_0}|\langle ph|Q_{21}^{(\tau)}|\rangle|^2 \tag{3.3}$$

one obtains [17], from the request that the eigenvalue equation yields a vanishing root, quasiparticle self-consistent relations equivalent to the Hartree conditions (2.5). It follows from them that $(uv)_{ph}^{(+)}=1$ for both low- and high-energy modes, $(uv)_{ph}^{(-)}=1$ for the high-energy level, and $(uv)_{ph}^{(-)}=\epsilon_0/E_0$ for the low-lying one [17].

The transition amplitude of a time even $(+)$ and odd $(-)$ operator $W_\mu^{(\pm)}$ from the ground to an excited RPA state $|\mu\rangle$ is given by

$$\langle\mu|W_\mu^{(\pm)}|0\rangle=\sum_{\tau,ph}[Y_{ph}^*(\tau)\pm Z_{ph}(\tau)]\langle p|W_\mu^{(\pm)}|h\rangle(uv)_{ph}^{(\pm)}, \tag{3.4}$$

where

$$Y_{ph}(\tau)=N_\tau\frac{(\widetilde{Q}_{2\mu}^{(\tau)})_{ph}(uv)_{ph}^{(+)}}{\omega-E_{ph}^{(\tau)}},$$

$$Z_{ph}(\tau)=-N_\tau\frac{(\widetilde{Q}_{2\mu}^{(\tau)*})_{ph}(uv)_{ph}^{(+)}}{\omega+E_{ph}^{(\tau)}}. \tag{3.5}$$

The constants N_τ are fixed by the normalization condition

$$\sum_{\tau,ph}(|Y_{ph}(\tau)|^2-|Z_{ph}(\tau)|^2)=1 \tag{3.6}$$

and by the ratio

$$\frac{N_p}{N_n}=\frac{2\chi_{pn}S^{(n)}(\omega)}{1-2\chi S^{(p)}(\omega)}=\frac{1-2\chi S^{(n)}(\omega)}{2\chi_{pn}S^{(p)}(\omega)}. \tag{3.7}$$

In order to compute the transition strength of the time-even $E2$ operator

$$\mathcal{M}(E2,\mu)=eQ_{2\mu}^{(p)}, \tag{3.8}$$

we make use of the expression of N_τ, obtained by solving Eqs. (3.6) and (3.7), exploit the key relation (3.3), and express the unperturbed strength of $Q_{21}^{(\tau)}$ in terms of the ground state mean value of the quadrupole field through

$$\sum_{ph\in\epsilon_0}|\langle ph|Q_{21}^{(\tau)}|\rangle|^2=\frac{3}{4}\sqrt{\frac{5}{16\pi}}\frac{1}{m\omega_0}\langle Q_{20}^{(\tau)}\rangle. \tag{3.9}$$

The above equation is derived by making use of the asymptotic HO basis. After all these steps, we obtain for the $E2$ strength the expression

$$B(E2)\uparrow=\frac{15}{16\pi}\frac{1}{m\omega_0}\frac{E_0}{\omega}\langle Q_0^{(p)}\rangle\mathcal{N}^2\left[1+\frac{\epsilon_2^2}{\epsilon_0E_0}\frac{\omega^2-E_0^2}{\omega^2-\epsilon_2^2}\right]^2e^2. \tag{3.10}$$

The constant $\mathcal{N}$ is given by

$$\mathcal{N}^{-2}=\left[1+\frac{\epsilon_2^2}{\epsilon_0E_0}\left(\frac{\omega^2-E_0^2}{\omega^2-\epsilon_2^2}\right)^2\right]\frac{\omega^2-E_0^2(1-\mathcal{R}_\omega^{(n)})}{\omega^2-E_0^2(1-\mathcal{R}_\omega^{(p)}-\mathcal{R}_\omega^{(n)})}, \tag{3.11}$$

where

$$\mathcal{R}_\omega^{(\tau)}=\frac{15}{32\pi}\frac{1}{m\omega_0}\chi\langle Q_0^{(\tau)}\rangle\left[1+\frac{\epsilon_2^2}{\epsilon_0E_0}\frac{\omega^2-E_0^2}{\omega^2-\epsilon_2^2}\right]. \tag{3.12}$$

The time odd $M1$ operator can be written

$$\mathcal{M}(M1,\mu)=\sqrt{\frac{3}{4\pi}}L_\mu^{(p)}\mu_N\Rightarrow\mathcal{M}_{(sc)}(M1,\mu)$$
$$=\sqrt{\frac{3}{16\pi}}(L_\mu^{(p)}-L_\mu^{(n)})\mu_N, \tag{3.13}$$

where the arrow points out the scissors $M1$ component, the only one contributing to the transition strength. The other term, being proportional to the total angular momentum, gives a vanishing contribution in virtue of the Hartree self-consistent conditions [17]. In computing the strength of this operator we need to exploit first the relation

$$\langle ph|L^{(\tau)}_{\mu=1}|\rangle=\sqrt{\frac{4\pi}{15}}\,m\,\frac{\omega_1^{(\tau)^2}-\omega_3^{(\tau)^2}}{\epsilon^{(\tau)}_{ph}}\langle ph|Q^{(\tau)}_{21}|\rangle, \tag{3.14}$$

where $\epsilon^{(\tau)}_{ph}=\epsilon^{(\tau)}_0$ or $\epsilon^{(\tau)}_{ph}=\epsilon^{(\tau)}_2$ depending on whether the ph states span the $\Delta N=0$ or $\Delta N=2$ HO space. We then proceed as for the $E2$ transition and obtain

$$B(M1)\uparrow=\frac{3}{16\pi}m\left(\frac{\epsilon_0}{E_0}\right)^2\frac{\omega}{E_0}\frac{\epsilon_2^2}{\omega_0}\langle Q_0^{(p)}\rangle\mathcal{N}^2 \times\left[1+\frac{E_0}{\epsilon_0}\frac{\omega^2-E_0^2}{\omega^2-\epsilon_2^2}\right]^2\mu_N^2. \tag{3.15}$$

From the comparison of the expressions of the two strengths we get

$$B(M1)\uparrow=\frac{1}{2}m^2\omega^2\delta^2\left(1+\frac{1}{3}\delta\right)\times\left[\frac{(1+\epsilon_0/E_0)\omega^2-(E_0^{(p)^2}+\epsilon_0/E_0\epsilon_2^{(p)^2})}{(1+\epsilon_0E_0/\epsilon_2^{(p)^2})\omega^2-E_0^{(p)^2}(1+\epsilon_0/E_0)}\right]^2\times B(E2)\uparrow\frac{\mu_N^2}{e^2}. \tag{3.16}$$

This equation correlates the $E2$ transitions to the low and high scissors modes with the $M1$ transitions to the same states. Such a link, not to be confused with the well-known $M1$-$E2$ relation found experimentally and to be discussed in the subsequent sections, needs to be tested in more realistic RPA approaches. If valid in this more general context, it could be used to deduce the $M1$ strength of a given mode once the corresponding $E2$ strength is known experimentally and vice versa.

It is worth mentioning that the $M1$-$E2$ relation just derived yields a vanishing $M1$ strength when ω takes the value

$$\omega=\sqrt{\frac{E_0^{(p)^2}+\epsilon_0/E_0\epsilon_2^{(p)^2}}{1+\epsilon_0/E_0}}. \tag{3.17}$$

It can be checked, by direct substitution, that this is an exact root of the eigenvalue equation (3.1) as long as the quasiparticle self-consistent conditions are fulfilled. Such a root is just the eigenvalue of the isoscalar quadrupole giant resonance mode.

Exact eigenvalues of the high- and low-energy (scissors) modes can also be deduced. A good approximation to them is given by

$$\omega^{(+)}\simeq\epsilon_2\sqrt{1+b/2},\quad \omega^{(-)}\simeq E_0\sqrt{1+b_{\rm eff}}, \tag{3.18}$$

where $b_{\rm eff}=(b/2)/(1+b/2)$.

As for the strengths, we can drop out 1 in the square brackets appearing in Eqs. (3.11) and (3.15). Under this approximation, Eq. (3.15) yields, through ϵ_0 and $\langle Q_0^{(p)}\rangle$, $B(M1)\uparrow\propto\delta^2$. The same quadratic dependence is found when the $M1$ strength is computed exactly.

In the case of the low-energy mode we can approximate the square brackets in Eqs. (3.11) and (3.15) with 1. The behavior of the resulting $M1$ strength is not as simple as in the case of the high-energy mode. It goes approximately as $B(M1)\uparrow\propto\delta^3$ for vanishingly small deformations and becomes linear in δ for superdeformed nuclei. This transition to a rigid-body regime is in agreement with the result of more realistic RPA calculations [36]. Between these two extreme cases, the behavior is rather involved. That schematic RPA does not yield a δ^2 law for the low-energy mode is a consequence of the inadequacy of this approximation scheme, when used to study these low-lying magnetic transitions.

IV. ENERGY-WEIGHTED SCISSORS SUM RULE

As discussed in Ref. [49] the operator responsible for the $M1$ transition to the scissors mode is

$$\mathcal{M}_{\rm sc}(M1,\mu)=\sqrt{\frac{3}{16\pi}}\,S_\mu g_r\mu_N=\sqrt{\frac{3}{16\pi}}\,(J_\mu^{(p)}-J_\mu^{(n)})\times(g_p-g_n)\mu_N, \tag{4.1}$$

where S_μ is the generator of the scissorslike oscillations of protons versus neutrons. Consistently with RPA calculations, we will use for the gyromagnetic factors the values $g_p=1$ and $g_n=0$ so that $g_r=1$. If $J_\mu^{(\tau)}$ are purely orbital we gain the $M1$ scissors operator adopted in schematic RPA.

For a $M1$ operator of the scissors form the following energy weighted sum rule holds [39]:

$$\sum_n\omega_nB_n^{(sc)}(M1)\uparrow=\frac{3}{16\pi}\sum_{n,\mu}\omega_n|\langle n\mu|S_\mu|0\rangle|^2\mu_N^2 =\frac{3}{32\pi}\sum_{\mu=\pm1}\langle 0|[S_\mu^\dagger,[H,S_\mu]]|0\rangle\mu_N^2 =S_{\rm EW}^{(\rm sc)}(M1). \tag{4.2}$$

The double commutator is computed using a Hamiltonian composed of a one-body term containing the Hartree deformed field (2.4) and of a quadrupole-quadrupole interaction ($\tilde{Q}\tilde{Q}$) expressed in doubly stretched coordinates. The resulting value can be decomposed into one- and two-body pieces:

$$S_{\rm EW}^{(\rm sc)}(M1)=\frac{3}{16\pi}(S_0^{(\rm sc)}+S_2^{(\rm sc)})\mu_N^2. \tag{4.3}$$

The one-body contribution comes from the Hartree field and is given by

$$S_0^{(\rm sc)}=\frac{1}{2}\sum_{\mu=\pm1}\langle 0|[S_\mu^\dagger,[H_0,S_\mu]]|0\rangle =3m\omega_0^2(\beta_p\langle Q_{20}^{(p)}\rangle+\beta_n\langle Q_{20}^{(n)}\rangle). \tag{4.4}$$

Assuming equal deformation for protons and neutrons, we get

$$S_0^{(\mathrm{sc})} \simeq 3 m \omega_0^2 \beta \langle Q_{20} \rangle \simeq \frac{4}{3} \delta^2 m \omega_0^2 A \langle r^2 \rangle, \tag{4.5}$$

which is the result derived in Ref. [27].

The two-body term can be decomposed into two parts

$$S_2^{(\mathrm{sc})} = S_{20}^{(\mathrm{sc})} + S_{22}^{(\mathrm{sc})}. \tag{4.6}$$

The first one comes from the component of the two-body interaction which involves the monopole operators and is given by

$$S_{20}^{(\mathrm{sc})} = \frac{\sqrt{5}}{3} \frac{1}{\omega_0^4} (\omega_1^{(p)^2} - \omega_3^{(p)^2}) [\chi (\omega_1^{(p)^2} + 2\omega_3^{(p)^2}) \langle Q_{00}^{(p)} Q_{20}^{(p)} \rangle + \chi_{pn} (\omega_1^{(n)^2} + 2\omega_3^{(n)^2}) \langle Q_{00}^{(p)} Q_{20}^{(n)} \rangle] + (p \leftrightarrow n). \tag{4.7}$$

To compute this quantity we use closure. Having in mind that the states which couple strongly to the ground state through the quadrupole field, couple weakly (if at all) through the monopole operator and vice versa, we put

$$\langle Q_{20}^{(\tau)} Q_{00}^{(\tau')} \rangle \simeq \langle Q_{20}^{(\tau)} \rangle \langle Q_{00}^{(\tau')} \rangle. \tag{4.8}$$

Under this approximation and the assumption of equal frequencies for protons and neutrons, we obtain from Eqs. (4.7), upon exploitation of the Hartree conditions (2.5),

$$S_{20}^{(\mathrm{sc})} \simeq -m(\omega_1^2 + 2\omega_3^2)[\beta_p \langle Q_{20}^{(p)} \rangle + \beta_n \langle Q_{20}^{(n)} \rangle]. \tag{4.9}$$

This, in virtue of the explicit expressions (2.6) of the frequencies, can be written in the form

$$S_{20}^{(\mathrm{sc})} \simeq -3 m \omega_0^2 \left(1 - \frac{2}{3}\delta\right) (\beta_p \langle Q_{20}^{(p)} \rangle + \beta_n \langle Q_{20}^{(n)} \rangle). \tag{4.10}$$

The lowest order term is exactly equal and opposite to the one-body contribution (4.4).

A long and tedious calculation yields for the two-body quadrupole part

$$\begin{aligned} S_{22}^{(\mathrm{sc})} = {} & \frac{1}{3} \chi \frac{1}{\omega_0^4} (\omega_1^2 - \omega_3^2)(4\omega_3^2 - \omega_1^2) \langle Q_{20}^{(p)^\dagger} Q_{20}^{(p)} + Q_{20}^{(n)\dagger} Q_{20}^{(n)} \rangle \\ & + \frac{1}{3} \chi \frac{1}{\omega_0^4} (\omega_1^2 - \omega_3^2)(7\omega_1^2 - 4\omega_3^2) \\ & \times \langle Q_{21}^{(p)^\dagger} Q_{21}^{(p)} + Q_{21}^{(n)\dagger} Q_{21}^{(n)} \rangle - 2\chi \frac{1}{\omega_0^4} \omega_1^2 (\omega_1^2 - \omega_3^2) \\ & \times \langle Q_{22}^{(p)^\dagger} Q_{22}^{(p)} + Q_{22}^{(n)\dagger} Q_{22}^{(n)} \rangle - \frac{1}{3} \chi_{pn} \frac{1}{\omega_0^4} (\omega_1^4 + 4\omega_3^4 \\ & + 13\omega_1^2 \omega_3^2) \langle Q_{20}^{(p)^\dagger} Q_{20}^{(n)} + Q_{20}^{(n)\dagger} Q_{20}^{(p)} \rangle - \frac{1}{3} \chi_{pn} \frac{1}{\omega_0^4} \\ & \times (7\omega_1^4 + 4\omega_3^4 + 25\omega_1^2 \omega_3^2) \langle Q_{21}^{(p)^\dagger} Q_{21}^{(n)} + Q_{21}^{(n)\dagger} Q_{21}^{(p)} \rangle \\ & - 2\chi_{pn} \frac{1}{\omega_0^4} \omega_1^2 (5\omega_1^2 + \omega_3^2) \langle Q_{22}^{(p)^\dagger} Q_{22}^{(n)} + Q_{22}^{(n)\dagger} Q_{22}^{(p)} \rangle. \end{aligned} \tag{4.11}$$

By making use of the explicit expressions (2.6) of the frequencies, we can put the above quantity in the more transparent form

$$S_{22}^{(\mathrm{sc})} = S_{22}(0) + S_{22}(1)\delta + S_{22}(2)\delta^2. \tag{4.12}$$

The term in δ^2 has been found to be negligible and will not be discussed. The zeroth order term is

$$S_{22}(0) = -6\chi_{pn} \sum_{\mu} \langle Q_{2\mu}^{(p)^\dagger} Q_{2\mu}^{(n)} + Q_{2\mu}^{(n)\dagger} Q_{2\mu}^{(p)} \rangle. \tag{4.13}$$

Inserted into Eq. (4.3), it yields the shell model sum rule derived in Refs. [8,26].

The first order coefficient is

$$\begin{aligned} S_{22}(1) = {} & 2\chi [\langle Q_{20}^{(p)^\dagger} Q_{20}^{(p)} + Q_{20}^{(n)\dagger} Q_{20}^{(n)} \rangle \\ & + \langle Q_{21}^{(p)^\dagger} Q_{21}^{(p)} + Q_{21}^{(n)\dagger} Q_{21}^{(n)} \rangle \\ & - 2 \langle Q_{22}^{(p)^\dagger} Q_{22}^{(p)} + Q_{22}^{(n)\dagger} Q_{22}^{(n)} \rangle] \\ & + 6\chi_{pn} [\langle Q_{20}^{(p)^\dagger} Q_{20}^{(n)} + Q_{20}^{(n)\dagger} Q_{20}^{(p)} \rangle \\ & + \langle Q_{21}^{(p)^\dagger} Q_{21}^{(n)} + Q_{21}^{(n)\dagger} Q_{21}^{(p)} \rangle \\ & - 2 \langle Q_{22}^{(p)^\dagger} Q_{22}^{(n)} + Q_{22}^{(n)\dagger} Q_{22}^{(p)} \rangle]. \end{aligned} \tag{4.14}$$

Using closure, we obtain a static part and terms describing fluctuations of the quadrupole field. These latter pieces can be neglected. Indeed, aside from the ground state, the set of intermediate $|n\mu\rangle$ states include the low-energy isoscalar β, γ vibrations and the isovector scissors mode as well as the $K^\pi = 0^+, 1^+, 2^+$ components of both isoscalar and isovector quadrupole excitations. For a given resonance, the difference among the strengths of the different μ components $Q_{2\mu}^{(\tau)}$ arise only once deformation is switched on. It follows that the pieces in the square brackets mutually cancel to a large extent. What remains is a small contribution which can be neglected. A partial cancellation is achieved also for the contribution, in any case negligible, from β, γ, and low-energy $M1$ excitations. We therefore retain only the static term and put

$$S_{22}(1) \simeq 2\chi [\langle Q_{20}^{(p)} \rangle \langle Q_{20}^{(p)} \rangle + \langle Q_{20}^{(n)} \rangle \langle Q_{20}^{(n)} \rangle] + 6\chi_{pn} [\langle Q_{20}^{(p)} \rangle \times \langle Q_{20}^{(n)} \rangle + \langle Q_{20}^{(n)} \rangle \langle Q_{20}^{(p)} \rangle]. \tag{4.15}$$

Upon exploitation of the Hartree conditions (2.5), it is possible to perform the following decomposition:

$$S_{22}(1) \simeq -2 m \omega_0^2 (\beta_p \langle Q_{20}^{(p)} \rangle + \beta_n \langle Q_{20}^{(n)} \rangle) + 8\chi_{pn} \langle Q_{20}^{(p)} \rangle \times \langle Q_{20}^{(n)} \rangle. \tag{4.16}$$

The one-body piece contained in $S_{22}(1)$, when multiplied by δ, is exactly equal and opposite to the term in δ coming from the monopole-quadrupole interaction and appearing in Eq. (4.10). After this cancellation no trace either of the one-body or of the monopole terms remains. Only the purely quadrupole contribution survives. The total sum rule (4.3) assumes therefore the final simple form

$$\sum_n \omega_n B_n^{(\mathrm{sc})}(M1)\uparrow \simeq -\frac{9}{8\pi}\chi_{pn}\left(\sum_\mu \langle Q_{2\mu}^{(p)\dagger}Q_{2\mu}^{(n)}+Q_{2\mu}^{(n)\dagger}Q_{2\mu}^{(p)}\rangle - \frac{4}{3}\delta\langle Q_{20}^{(p)}\rangle\langle Q_{20}^{(n)}\rangle\right). \quad (4.17)$$

This, to zeroth order in δ, coincides with the shell model sum rule [26]. Before making any use of this formula, we intend to investigate if and how the energy-weighted sum rule is modified as we move from the schematic to a more realistic Hamiltonian.

V. SCISSORS SUM RULE FOR A REALISTIC SEPARABLE HAMILTONIAN

Let us consider the following separable Hamiltonian:

$$H = H_{\mathrm{Nil}} + V_P + V_{\tilde{Q}\tilde{Q}} + V_{\sigma\sigma}. \quad (5.1)$$

H_{Nil} is the full spherical Nilsson Hamiltonian including the spin orbit term with coupling constant ξ plus the l^2 piece, V_P a proton-proton and a neutron-neutron pairing, $V_{\tilde{Q}\tilde{Q}}$ the $(\tilde{Q}\tilde{Q})$ interaction already considered here, and $V_{\sigma\sigma}$ a spin-spin interaction of the form

$$V_{\sigma\sigma} = \frac{1}{2}\chi^{(\sigma)}(\vec{\mathcal{S}}^{(p)}\cdot\vec{\mathcal{S}}^{(p)} + \vec{\mathcal{S}}^{(n)}\cdot\vec{\mathcal{S}}^{(n)}) + \frac{1}{2}\chi_{pn}^{(\sigma)}(\vec{\mathcal{S}}^{(p)}\cdot\vec{\mathcal{S}}^{(n)} + \vec{\mathcal{S}}^{(n)}\cdot\vec{\mathcal{S}}^{(p)}), \quad (5.2)$$

where $(\tau = p,n)$

$$\vec{\mathcal{S}}^{(\tau)} = \sum_k \vec{s}^{(\tau)}(k). \quad (5.3)$$

In computing the energy-weighted sum rule we have now to use the full $M1$ shell model operator. This can be decomposed into a rotational, a scissors and a spin component

$$\vec{\mathcal{M}}(M1) = \vec{\mathcal{M}}_J(M1) + \vec{\mathcal{M}}_{\mathrm{sc}}(M1) + \vec{\mathcal{M}}_\sigma(M1). \quad (5.4)$$

The scissors operator $\vec{\mathcal{M}}_{\mathrm{sc}}(M1)$ is still given by Eq. (4.1), but with proton and neutron angular momenta composed of orbital and spin pieces; namely,

$$\vec{J}^{(\tau)} = \vec{L}^{(\tau)} + \vec{\mathcal{S}}^{(\tau)}. \quad (5.5)$$

The spin term $\vec{\mathcal{M}}_\sigma(M1)$ is given by

$$\vec{\mathcal{M}}_\sigma(M1) = \sqrt{\frac{3}{4\pi}}\,\vec{\mu} = \sqrt{\frac{3}{4\pi}}\,[(g_s^{(p)}-1)\vec{\mathcal{S}}^{(p)} + g_s^{(n)}\vec{\mathcal{S}}^{(n)}]\mu_N. \quad (5.6)$$

The contribution from the double commutator can be decomposed into a scissors plus a spin-orbit and a spin piece:

$$\sum_n \omega_n B_n(M1)\uparrow = S_{\mathrm{EW}}^{(\mathrm{sc})}(M1) + S_{\mathrm{EW}}^{(\mathrm{ls})}(M1) + S_{\mathrm{EW}}^{(\sigma)}(M1). \quad (5.7)$$

For the scissors operator we obtain

$$\sum_n \omega_n B_n^{(\mathrm{sc})}(M1)\uparrow = S_{\mathrm{EW}}^{(\mathrm{sc})}(M1) - \frac{3}{16\pi}\chi_{pn}^{(\sigma)}\langle 0|\vec{s}^{(p)}\cdot\vec{s}^{(n)}|0\rangle, \quad (5.8)$$

where $S_{\mathrm{EW}}^{(\mathrm{sc})}(M1)$ is just the energy-weighted sum (4.17) obtained when only the Q-Q interaction is used. The second term in the right-hand side of the equation comes from the proton-neutron spin-spin interaction and, as we shall argue later on, is negligible. We are therefore left with

$$\sum_n \omega_n B_n^{(\mathrm{sc})}(M1)\uparrow \simeq S_{\mathrm{EW}}^{(\mathrm{sc})}(M1), \quad (5.9)$$

where the right-hand side term is given exactly by Eq. (4.17) as in the schematic case. We then conclude that the use of a separable Hamiltonian of the type adopted in realistic RPA calculations does not change the scissors energy-weighted sum rule obtained using only a quadrupole-quadrupole interaction as in schematic RPA. It is worth pointing out that such an important result is obtained only if the angular momenta entering into the scissors operator (4.1) contain both orbital and spin pieces [Eq. (5.5)]. The use of purely orbital angular momenta would have produced a contribution coming from the spin-orbit potential (which is canceled by the opposite contribution due to the spin component).

The spin orbit term has the form

$$S_{\mathrm{EW}}^{(ls)}(M1) = -\frac{3}{4\pi}\xi\left[(g_s^{(p)}-1)\langle 0|\sum_k \vec{l}_k^{(p)}\cdot\vec{s}_k^{(p)}|0\rangle + g_s^{(n)}\langle 0|\sum_k \vec{l}_k^{(n)}\cdot\vec{s}_k^{(n)}|0\rangle\right]. \quad (5.10)$$

This is nothing but the Kurath sum rule [50] valid also for spherical nuclei. It is remarkable that the spin-orbit term does not affect at all the scissors energy-weighted sum. The spin contribution comes entirely from the proton-neutron spin-spin interaction and is given by

$$S_{\mathrm{EW}}^{(\sigma)}(M1) = -\frac{3}{4\pi}\chi_{pn}^{(\sigma)}\langle 0|\vec{s}^{(p)}\cdot\vec{s}^{(n)}|0\rangle, \quad (5.11)$$

where the spin term appearing in the scissors sum given by Eq. (5.8) is also included. The above equation may be written in the form

$$S_{\mathrm{EW}}^{(\sigma)}(M1) = -\frac{3}{16\pi}\chi_{pn}^{(\sigma)}\left(\sum_n |\langle n|\vec{s}^{(+)}|0\rangle|^2 - \sum_n |\langle n|\vec{s}^{(-)}|0\rangle|^2\right)\mathrm{MeV}\,\mu_N^2, \quad (5.12)$$

where $\vec{s}^{(\pm)} = \vec{s}^{(p)} \pm \vec{s}^{(n)}$. Spin excitations with a double-hump structure have been discovered recently [51,52]. No unique explanation of such a structure has been given. The different interpretations are correlated with the different values attached to the the p-n spin-spin coupling constant. In the analyses using a vanishing strength $(\chi_{pn}^{(\sigma)}=0)$ [11,53], the two bumps correspond to separate neutron and proton excitations. In those using a nonvanishing value [54] the bumps are explained as isoscalar and isovector excitations.

Also the spin contribution to the energy weighted sum depends clearly on such a constant. It vanishes or not according that $\chi_{pn}^{(\sigma)}=0$ or $\chi_{pn}^{(\sigma)}\neq 0$. Its value, being the net result of a cancellation between two comparable terms, the isoscalar and the isovector summed spin transition strengths, is likely to be small in any case. This should be specially true for the share pertaining to the scissors sum, which is 1/4 of the total piece given by Eq. (5.12).

VI. QUALITATIVE AND QUANTITATIVE ANALYSIS

Having ascertained the validity of the scissors sum rule for a separable interaction of general form, we will attempt a phenomenological study of Eq. (4.17). To this purpose we write the sum appearing on the left-hand side as

$$\sum_n \omega_n B_n^{(\mathrm{sc})}(M1)\uparrow = \sum_n \omega_n^{(-)} B_n^{(-)}(M1)\uparrow + \sum_n \omega_n^{(+)} B_n^{(+)}(M1)\uparrow \simeq \omega^{(-)}\sum_n B_n^{(-)}(M1)\uparrow + \omega^{(+)}\sum_n B_n^{(+)}(M1)\uparrow. \qquad (6.1)$$

Such a decomposition is suggested by the schematic RPA calculation illustrated here as well as by the results obtained in realistic RPA [34–38].

In order to estimate the right-hand side of Eq. (4.17) we single out the ground state contribution first. We then use Eq. (2.5) to express $\chi_{pn}\langle Q_{20}^{(n)}\rangle$ in terms of χ and of the proton quadrupole mean field $\langle Q_{20}^{(p)}\rangle$. This step ensures the consistency of the sum rule with the Hartree conditions and enables us to deal with a measurable quantity such as the proton quadrupole moment. We finally assume isospin as a good quantum number for the excited intrinsic states. In order to make connections with standard notations we put

$$\chi = \chi(0)(1-b), \qquad (6.2)$$

where $b=-\chi(1)/\chi(0)$ will be taken as a free parameter. Using Eqs. (2.12) and (2.11), we get the final result

$$\omega^{(-)}\sum_n B_n^{(-)}(M1)\uparrow + \omega^{(+)}\sum_n B_n^{(+)}(M1)\uparrow = \frac{9}{5}\,\frac{m\omega_0^2}{A\langle r^2\rangle}(1+b')\left(1-\frac{2}{3}\delta\right)\left[\frac{N}{Z}\left(1-\frac{2}{3}\delta\right)B_0(E2)\uparrow + \sum_{n\neq 0}\left[B_n^{(0)}(E2)\uparrow - B_n^{(1)}(E2)\uparrow\right]\right]\frac{\mu_N^2}{e^2} \qquad (6.3)$$

having put $b'=Z/Nb$. The $E2$ strengths appearing in the above equation are

$$B_0(E2)\uparrow = \langle Q_{20}^{(p)}\rangle^2 e^2,$$

$$B_{n\neq 0}^{(T)}(E2)\uparrow = \frac{e^2}{4}\sum_\mu |\langle n\mu|Q(T)|0\rangle|^2. \qquad (6.4)$$

Since the contribution coming from the ground state is by far the dominant one, we can neglect the other transitions in first approximation. Using Eq. (2.11) to compute $\langle Q_{20}^{(p)}\rangle$, we obtain

$$\omega^{(-)}\sum_n B_n^{(-)}(M1)\uparrow + \omega^{(+)}\sum_n B_n^{(+)}(M1)\uparrow \simeq \frac{1}{4\pi}m\omega_0^2\,\frac{4ZN}{A^2}A\langle r^2\rangle\delta^2(1+b')\,\frac{\mu_N^2}{e^2}. \qquad (6.5)$$

We now observe that for the high-energy mode we have $\omega^{(+)}\propto 2\omega_0$ and that, according to experiments [4–7], the centroid of the low-lying $M1$ excitations is constant ($\omega^{(-)}\sim 3$ MeV) throughout all nuclei of the rare-earth region. This can be qualitative explained in schematic RPA with the fact that $\omega^{(-)}\propto\sqrt{\epsilon_0^2+(2\Delta)^2}$. As the particle-hole energy $\epsilon_0=\delta\omega_0$ increases with deformation, the pairing gap Δ decreases [9] so as to leave the energy of the mode fairly insensitive to deformation. Since δ does not appear as leading term in the energies of either modes, we can conclude that Eq. (6.5) states that the summed strengths of both low- and high-energy $M1$ modes grow quadratically with deformation. Indeed, lower order terms in δ, if present in one or both strengths, should be positive. No mutual cancellation could therefore take place.

Apart from the assumption on the energies, the deformation law just derived is quite general. It comes indeed from the calculation of the double commutator of the Hamiltonian and, therefore, does not rely on the results of explicit RPA calculations. It is also of considerable relevance that, according to this calculation, the summed $M1$ strength is strictly proportional to the square of the Nilsson deformation parameter δ, which is the zero order term of the parameter directly deduced from the rotational $E2$ strength. This is consistent with the results of the first experimental work on the subject [4] and confirms the purely phenomenological analysis based on the TRM formula [29].

For a numerical estimate of the full sum $S_{\mathrm{EW}}^{(\mathrm{sc})}(M1)$ given by the right-hand side of Eq. (6.3), we need to have a complete experimental information on the $E2$ transitions. In ^{154}Sm, which is the nucleus considered here as an example, all the $E2$ strengths are available [55–57] except those of the low- and high-energy isovector quadrupole modes. We have computed in schematic RPA the $E2$ strengths of the $K^\pi=1^+$ (scissors) components using Eqs. (3.10) and (3.15) and then assumed the same value for the other K^π modes. The uncertainties induced by this simplifying assumption should be small and, in any case, of no practical effect on $S_{\mathrm{EW}}^{(\mathrm{sc})}(M1)$, which is dominated by the ground state rotational transition.

The sum $S_{\mathrm{EW}}^{(\mathrm{sc})}(M1)$, computed for different values of the free parameter b is shown in Fig. 1(a) (full line). The figure shows also that only a small part of the sum-rule is exhausted by the observed low-lying $M1$ transitions (dashed line). For $b=-\chi(1)/\chi(0)\geqslant 1.5$ the measured $M1$ strength accounts for less than 20%. This is a clear, model independent, indication that other, so far unobserved, $M1$ transitions contribute to the sum. In order to evaluate the contribution coming from the high-energy mode we compute energy and strength

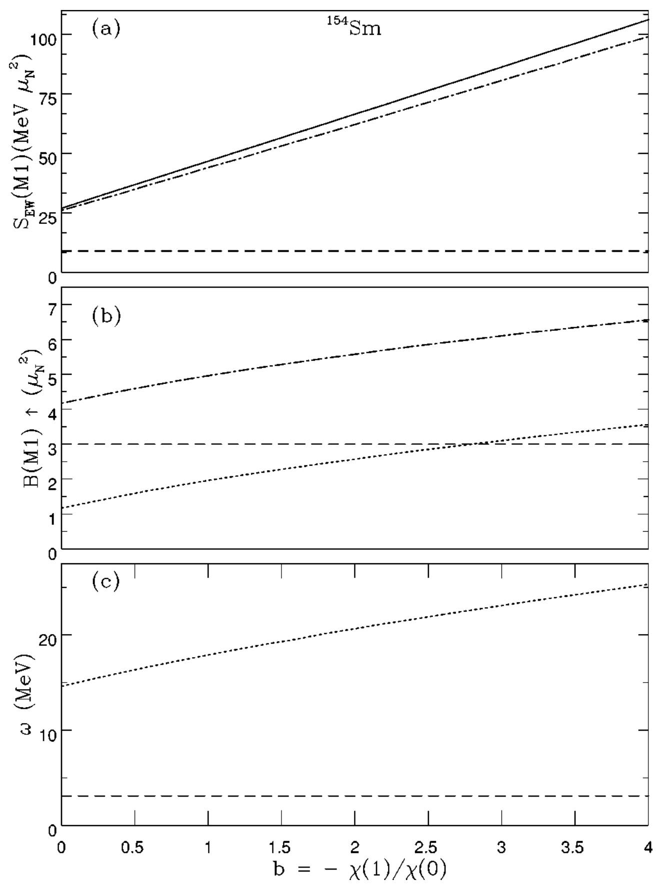

FIG. 1. Scissors energy-weighted $M1$ sum rule versus the ratio $b=-\chi(1)/\chi(0)$. In (a) the full line refers to $S_{\rm EW}^{\rm (sc)}(M1)$, the dashed line to $\omega_{\rm exp}^{(-)}\Sigma_n B_n^{(-)}(M1)_{\rm exp}\uparrow$ and the dot-dashed line to $\omega_{\rm exp}^{(-)}\Sigma_n B_n^{(-)} \times(M1)_{\rm exp}\uparrow+\omega_{\rm RPA}^{(+)}B_{\rm RPA}^{(+)}(M1)\uparrow$. In (b) and (c) the dashed lines give the experimental data while the dotted ones refer to the high-energy RPA mode. In (b) the sum $\Sigma_n B_n^{(-)}(M1)_{\rm exp}\uparrow+B_{\rm RPA}^{(+)}(M1)\uparrow$ is also given (dot-dashed line).

in schematic RPA using the proton-neutron formalism developed in Ref. [17] and briefly discussed here in Sec. III. The numerical results were obtained by using the standard formulas

$$\omega_0=41A^{-1/3}\ \text{MeV},\quad \langle r^2\rangle=\frac{3}{5}R^2,\quad R=1.2A^{1/3}\ \text{fm}, \tag{6.6}$$

and by taking the value $\Delta\simeq 0.7$ MeV, adopted in Ref. [9], for the pairing gap.

When the contribution of the high-energy mode is included [dash-dotted line in Fig. 1(a)], the energy-weighted sum rule is almost entirely exhausted. The discrepancy between $S_{\rm EW}^{\rm (sc)}(M1)$ and the sum $\omega_{\rm exp}^{(-)}\Sigma_n B_n^{(-)}(M1)_{\rm exp}\uparrow +\omega_{\rm RPA}^{(+)}B_{\rm RPA}^{(+)}(M1)\uparrow$ may be partly attributed to the approximate nature of the schematic RPA calculation. Even a small correction of the $M1$ strength of the high-lying mode would be amplified by the large weighting value of the excitation energy. It may be also possible that a small fraction of the low-energy scissors $M1$ strength has escaped observation. Indeed, the missing strength or part of it may be either distributed among very weak transitions forming part of the background or (and) may have been shifted above 4 MeV and got admixed with spin excitations. In any case, it is quite remarkable that the observed low-lying transitions and the RPA high-lying mode account almost entirely for the scissors sum rule. It implies, for instance, that the expected total strength of the low-lying $M1$ (scissors) transition amounts roughly to the one so far observed in the low-energy region and that only a small part of it might be present, if at all, in the region of spin excitations.

It is important to stress that the large contribution coming from the high-energy mode, already pointed out in Ref. [41], does not imply that this mode should be more collective. Even in the most favorable case in which the missing strength goes entirely to the high-energy mode, as depicted in our schematic RPA calculation, the $M1$ strength of the high-lying mode, though increasing with b, reaches a value which is at most comparable with the low-lying measured summed transition probability [dotted line in Fig. 1(b)]. This strength, however, is multiplied by a large weighting factor, namely the excitation energy of the mode, which increases as b increases [dotted line in Fig. 1(c)]. By contrary, the $M1$ strength of the low-lying transitions contributes with a much smaller weight. Hence the dominance of the high-energy mode in the sum rule.

We have computed for completeness also energy and strength of the low-energy mode in schematic RPA. The two quantities resulted to be practically insensitive to b. While, however, the energy is close to the observed centroid ($\omega_{\rm RPA}^{(-)}\simeq\omega_{\rm exp}^{(-)}\simeq 3$ MeV), the $M1$ strength is about twice the observed one [$B_{\rm RPA}^{(-)}(M1)\simeq 6\mu_N^2$], an additional indication of the inadequacy of schematic RPA for the study of the low-lying $M1$ excitations. Had we substituted the experimental with the schematic RPA values, the energy weighted sum $\omega_{\rm RPA}^{(-)}B_{\rm RPA}^{(-)}(M1)\uparrow+\omega_{\rm RPA}^{(+)}B_{\rm RPA}^{(+)}(M1)\uparrow$ would have resulted somewhat larger than $S_{\rm EW}^{\rm (sc)}(M1)$.

VII. CONCLUSION

A major result of our self-consistent treatment of the energy-weighted $M1$ sum rule is the complete cancellation of the contribution coming from the one-body potential. Indeed, the use of doubly stretched coordinates, required by the need of avoiding any further distortion of the Hartree mean field, represents an effective way of restoring the rotational invariance broken by the deformed field. Since only purely two-body effects survive, the sum rule, to lowest order in deformation, is formally the same as in spherical shell model.

The sum rule computed here refers to both low- and high-energy magnetic excitations. The inclusion of the two modes has posed the problem of disentangling the corresponding contributions. This has been done by using the experimental data for the low-lying mode and by computing the centroid and the $M1$ strength of the high energy mode in schematic RPA.

Independently of detailed calculations, it has emerged from inspecting the final outcome of the sum rule that the $M1$ strengths of both low- and high-energy modes are determined by the ground rotational $E2$ transition and, for this reason, are both quadratic in the (Nilsson) deformation parameter. Such a connection has been proved experimentally for the low-energy mode. It would be desirable, though not easy, to test the same law for the high-energy transitions.

As mentioned in the Introduction, also the sum rule derived in shell model has been shown to be valid not only in a small but also in a large space which includes the $\Delta N=0+\Delta N=2$ excitations [40]. The inclusion of these high-energy configurations causes a drastic increment of the energy-weighted $M1$ sum [left-hand side of Eq. (4.17)], but determines also a corresponding increase of the right-hand side part by enhancing the low-lying $E2$ transitions. According to this calculation, in light nuclei, such as Be isotopes, the approximation of retaining only the ground rotational $E2$ transition probability in the sum rule, necessary for deriving the δ^2 law expressed by Eq. (6.5), is no longer valid. In fact, the other $E2$ transitions, including the isovector ones, cannot be neglected. It follows that the full sum rule must be used to study the orbital motion with new and interesting consequences. It follows, for instance, that in singly closed shell nuclei, where isoscalar and isovector $E2$ transitions contribute equally, the orbital $M1$ strength vanishes even if the $E2$ transition probability is different from zero, a result confirmed by numerical calculations [58].

Coming back to the present work, the detailed analysis has shown that the sum rule is practically exhausted once the contribution from the high-lying mode, here estimated in schematic RPA, is added to the one from the low-lying observed $M1$ transition. Although providing a dominant contribution to the sum rule, the high-energy mode, if observed, should not be more collective. Indeed, its $M1$ strength should at most reach the value of the summed transition probability observed for the low-lying excitations.

Though initially embedded in a schematic RPA context, the approach has been proved to have a much more general valence. That the scissors sum rule remains practically unchanged when a more realistic separable Hamiltonian is adopted is a manifestation of its general validity. Moreover, hinging on the calculation of a double commutator, it does not rely on any assumption specific of schematic or realistic RPA. For this reason, the sum rule derived here may be of some help for consistency checks of RPA calculations using realistic separable Hamiltonians.

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