Quasiparticle random-phase approximation and β-decay physics: Higher-order approximations in a boson formalism

M. Sambataro[1,*] and J. Suhonen[2,†]
[1]*Istituto Nazionale di Fisica Nucleare, Sezione di Catania Corso Italia 57, I-95129 Catania, Italy*
[2]*Department of Physics, University of Jyväskylä, Post Office Box 35, SF-40351 Jyväskylä, Finland*
(Received 4 March 1997)

The quasiparticle random-phase approximation (QRPA) is reviewed and higher-order approximations are discussed with reference to β-decay physics. The approach is fully developed in a boson formalism. Working within a schematic model, we first illustrate a fermion-boson mapping procedure and apply it to construct boson images of the fermion Hamiltonian at different levels of approximation. The quality of these images is tested through a comparison between approximate and exact spectra. Standard QRPA equations are derived in correspondence with the quasi-boson limit of the first-order boson Hamiltonian. The use of higher-order Hamiltonians is seen to improve considerably the stability of the approximate solutions. The mapping procedure is also applied to Fermi β operators: exact and approximate transition amplitudes are discussed together with the Ikeda sum rule. The range of applicabilty of the QRPA formalism is analyzed.
[S0556-2813(97)01208-9]

PACS number(s): 21.60.Jz, 23.40.−s

I. INTRODUCTION

Within a microscopic approach to nuclear structure, the random-phase approximation (RPA) plays a prominent role in the analysis of all those cases which are unaccessible to shell-model calculations. Currently, in fact, it provides the simplest theory of excited states of the nucleus which admits the possibility of including correlations in the nuclear ground state [1].

An aspect of the RPA which holds an important role within this theory is represented by its recourse to the so-called ''quasi-boson approximation'' (QBA) [2]. This approximation, which consists in replacing the correlated RPA ground state with the uncorrelated Hartree-Fock (HF) state, causes the RPA excitation operators (pairs of particle-hole creation-annihilation operators in the standard theory) to behave as bosons. Obviously the consequences of this approximation become more serious the more important the correlations are, which, therefore, implies severe restrictions on the range of applicability of the theory itself.

A field in which the limitations of the RPA [or, better, of its variation the quasiparticle RPA (QRPA)] have been clearly pointed out in recent years is that of β-decay and double-beta ($\beta\beta$)-decay physics, [3–16]. In this case, the QRPA provides the most frequently followed approach to deal with the nuclear structure aspects of these processes. An important result which has emerged from the calculations performed so far is that matrix elements associated with the $\beta\beta$ processes, at least in the two-neutrino ($2\nu\beta\beta$) decay modes, are highly sensitive to the particle-particle component of the residual interaction, [3–9]. On the other hand, the physical value of this force is usually close to a point in which QRPA calculations ''collapse.'' This makes the evaluation of the results rather difficult and encourages then every effort devoted to a further development of the theory.

Several attempts have been made so far to improve $\beta\beta$-decay results working, first of all, on the stability of the QRPA solutions. We quote, as examples, those based on the inclusion of proton-neutron pairing [13], particle number projection [11,12] and higher-order corrections [14]. However, none of these approaches has proved to be fully satisfactory. Interesting results have been obtained instead within the renormalized QRPA (RQRPA) [17–20]. This method searches to go beyond the QBA by never resorting to the use of the HF state and can be traced back to some old works of Hara [21], Rowe [22], and da Providencia [23]. Further elaborated, it has also been applied to metallic cluster physics [24]. Similar attempts to overcome QBA have been made within the self-consistent RPA approach [25,26] of which, however, we are not aware of any application to β-decay physics.

The approach which is discussed in this paper and which aims at improving the standard QRPA theory is fully developed in a boson formalism. The boson space is that built on the images of the QRPA raising operators, and a mapping procedure relates fermion operators with their images in this boson space. Dealing with boson operators has the merit that the QBA can be avoided. On the other hand, one has to deal with all natural problems inherent in the mapping mechanism.

In order to simplify the discussion of the method, allowing at the same time a comparison between approximate and exact results, we will work within a schematic model. This model has been recently formulated in connection with β- and $\beta\beta$-decay physics [27–29] and used as a test for QRPA and RQRPA calculations [30]. We will begin illustrating the mapping procedure and apply it to construct boson images of the model Hamiltonian at different levels of approximation. The degree of reliabilty of these images will be tested by comparing approximate and exact spectra. Standard QRPA

*Electronic address: samba@ct.infn.it
†Electronic address: suhonen@jyfl.jyu.fi

 © 1997 The American Physical Society

equations will be derived in correspondence with the QBA limit of the first-order boson Hamiltonian. The use of higher-order images will provide a natural way to go beyond the standard QRPA.

Calculations will be performed in which exact and QRPA excitation energies, at different levels of approximation, will be compared. The quality of the approximate wave functions will also be tested by examining exact and approximate expectation values of the quasiparticle operator. Finally, we will discuss Fermi β transition amplitudes and the related Ikeda sum rule.

The paper is organized as follows. In Sec. II, we describe the model and, in Sec. III, the mapping procedure. In Sec. IV, we will review the QRPA in the boson formalism and illustrate the higher-order approximations. In Sec. V, we will present the results and, finally, in Sec. VI, we will summarize the results and draw the conclusions.

II. THE MODEL

The model Hamiltonian for our calculations has the form

$$H_F=\epsilon C+\lambda_1 A^\dagger A+\lambda_2(A^\dagger A^\dagger+AA), \tag{1}$$

where the operators $\{A,A^\dagger,C\}$ satisfy the SU(2) Lie Algebra

$$[A,A^\dagger]=1-\frac{C}{2\Omega},\quad [C,A^\dagger]=2A^\dagger, \tag{2}$$

with Ω being a parameter which will be specified in the following. H_F resembles the Hamiltonian of the Lipkin model [31] but with, in addition, the λ_1 term.

A model Hamiltonian of this form is not new to applications in the context of RPA calculations. We quote, as an example, the work of Beaumel and Chomaz [32]. These authors relate the above Hamiltonian to a system of two Ω-fold levels filled by Ω identical fermions. In this case, ϵ is the Hartree-Fock energy between the two levels and the operators $A^\dagger$ and C are expressed in terms of operators which create a hole in the Fermi sea or a particle in the upper level.

In the present work, wishing to work in a quasiparticle formalism, we follow the approach of Hirsch *et al.* [30]. These authors express $A^\dagger$ and C in terms of proton (neutron) quasiparticle creation $\alpha_p^\dagger$ ($\alpha_n^\dagger$) and annihilation operators:

$$A^\dagger=[\alpha_p^\dagger\alpha_n^\dagger]^{J=0}, \tag{3}$$

$$C=\sum_m \alpha_{pm}^\dagger\alpha_{pm}+\sum_m \alpha_{nm}^\dagger\alpha_{nm}. \tag{4}$$

These quasiparticle operators are associated with a system of protons and neutrons occupying both a single j shell. If $a_{pm}^\dagger$ is the operator which creates a proton with angular momentum j_p and projection m, $\tilde{a}_{pm}^\dagger=(-1)^{j-m}a_{p-m}^\dagger$ is its time reversal, and $a_{nm}^\dagger$, $\tilde{a}_{nm}^\dagger$ are the equivalent operators for neutrons, a Bogoliubov transformation [1] relates these operators to the above quasiparticle operators:

$$\alpha_{pm}^\dagger=u_p a_{pm}^\dagger-v_p\tilde{a}_{pm}, \tag{5}$$

$$\tilde{\alpha}_{pm}=v_p a_{pm}^\dagger+u_p\tilde{a}_{pm}, \tag{6}$$

(where $u_i^2+v_i^2=1,\ i=p,n$). It can be shown [30] that the Hamiltonian (1), with $A^\dagger$, C expressed as in Eqs. (3), (4), can be obtained by means of this Bogoliubov transformation performed separately for protons and neutrons starting from the Hamiltonian

$$H=H_p+H_n+H_{\rm res}, \tag{7}$$

where

$$H_p=e_p\sum_m a_{pm}^\dagger a_{pm}-G_p S_p^\dagger S_p, \tag{8}$$

$$H_n=e_n\sum_m a_{nm}^\dagger a_{nm}-G_n S_n^\dagger S_n, \tag{9}$$

$$H_{\rm res}=2\chi:\beta^-\beta^+:-2k:P^-P^+:, \tag{10}$$

$$S_p^\dagger=\frac{1}{2}\sum_m a_{pm}^\dagger\tilde{a}_{pm}^\dagger,\quad S_n^\dagger=\frac{1}{2}\sum_m a_{nm}^\dagger\tilde{a}_{nm}^\dagger, \tag{11}$$

$$\beta^-=\sum_m a_{pm}^\dagger a_{nm},\quad \beta^+=(\beta^-)^\dagger, \tag{12}$$

$$P^-=\sum_m a_{pm}^\dagger\tilde{a}_{nm}^\dagger,\quad P^+=(P^-)^\dagger, \tag{13}$$

under the hypothesis that $j_p=j_n\equiv j$, $G_p=G_n\equiv G$ and by neglecting the so-called scattering terms $B^\dagger=[\alpha_p^\dagger\tilde{\alpha}_n]^{J=0}$ and B. In this operation, also the single-particle terms are neglected since their contribution is irrelevant. One obtains

$$\epsilon=\frac{\Omega}{2}G, \tag{14}$$

$$\lambda_1=4\Omega[\chi(u_p^2v_n^2+v_p^2u_n^2)-k(u_p^2u_n^2+v_p^2v_n^2)], \tag{15}$$

$$\lambda_2=4\Omega(\chi+k)u_pv_pu_nv_n, \tag{16}$$

where

$$v_i=\sqrt{\frac{N_i}{2\Omega}},\quad i=p,n, \tag{17}$$

$$u_i=\sqrt{1-\frac{N_i}{2\Omega}},\quad i=p,n, \tag{18}$$

and $\Omega=j+\frac{1}{2}$. A detailed derivation of the coefficients u and v can be found in Ref. [2].

In choosing the parameters λ_1, λ_2 of Eq. (1) for our calculations, we will keep the dependence Eqs. (15), (16) on the parameters k,χ of the Hamiltonian (7). In particular, calculations will be performed for two different values of χ and for k ranging in a given interval, as specified in Sec. V. The reason for this choice is that, as it has already been observed in Ref. [30], calculated quantities like excitation energies and β transition amplitudes will exhibit a dependence on these parameters similar to that observed for the realistic quantities in terms of the particle-particle ($g_{\rm pp}$) and particle-hole ($g_{\rm ph}$) strengths [3–5]. In this way, the model Hamiltonian

(1), although not meant to reproduce actual nuclear properties, will provide at least some qualitative features of a realistic pn-QRPA calculation.

In the following, then, whenever talking about exact energies and eigenstates, we will always mean those obtained from the diagonalization of Eq. (1) in the space of states

$$\{|n\rangle=(A^{\dagger})^{n}|0\rangle, \quad 0\leqslant n\leqslant 2\Omega\}, \tag{19}$$

where $|0\rangle$ is the vacuum of the quasiparticle operators. These eigenvalues are, of course, not those of the Hamiltonian (7) since, as just noticed, H_F (1) only provides an approximation of Eq. (7). However, aiming in this work at a relative comparison among RPA-type calculations at different orders and providing Hamiltonian (1), an appropriate tool for these calculations, an examination of the spectra of Eq. (7) goes beyond the goals of the present paper.

To conclude this section, we notice that for the diagonalization of Eq. (1) we have made use of the following expressions which can be easily derived by means of the commutation relations (2):

$$\langle 0|A^{N}H_F(A^{\dagger})^{N}|0\rangle=\left[2\epsilon N+\lambda_1\left(N-\binom{N+1}{2}\frac{1}{\Omega}+\frac{N}{\Omega}\right)\right]\times\langle 0|A^{N}(A^{\dagger})^{N}|0\rangle, \tag{20}$$

$$\langle 0|A^{N}H_F(A^{\dagger})^{N-2}|0\rangle=\lambda_2\langle 0|A^{N}(A^{\dagger})^{N}|0\rangle, \tag{21}$$

with

$$\langle 0|A^{N}(A^{\dagger})^{N}|0\rangle=\left(N-\binom{N}{2}\frac{1}{\Omega}\right)\langle 0|A^{N-1}(A^{\dagger})^{N-1}|0\rangle, \tag{22}$$

and where $\binom{n}{m}=n!/(n-m)!m!$.

III. THE MAPPING PROCEDURE

That of fermion-boson mapping is a subject widely treated in literature and several techniques are known [33]. The one which we have employed in this paper follows the main lines of a procedure which has had several applications in the past both in the fermion-boson correspondence [34] and in the fermion-fermion one [35,36]. We refer to Ref. [36] for a general discussion of the method. In this case, however, due to the ''essential'' structure of the fermion states (19), the procedure takes a very simple form.

Let $b^{\dagger},b$ be creation, annihilation $J=0$ boson operators, $|0)$ the corresponding vacuum and let us define the states

$$|n)=\frac{1}{\sqrt{n!}}(b^{\dagger})^{n}|0), \quad 0\leqslant n\leqslant 2\Omega. \tag{23}$$

We can establish a one-to-one correspondence between the fermion states (19) and the boson ones (23). Let us call F the space built in terms of the states (19) and B that built in terms of the states (23). We define the boson image of the fermion operator O_F in B a boson operator O_B such that all the eigenvalues of O_F in F are also eigenvalues of O_B in B. Due to the orthonormality of the states (23), this requirement simply amounts to a search for a boson operator O_B such that

$$\frac{1}{\sqrt{N_n}}\frac{1}{\sqrt{N_{n'}}}\langle n|O_F|n'\rangle=(n|O_B|n') \quad 0\leqslant n,n'\leqslant 2\Omega, \tag{24}$$

where $N_n=\langle n|n\rangle$. This condition, which preserves matrix elements between corresponding states and, therefore, is of Marumori-type [2], defines the operator O_B.

An important aspect of the mapping procedure concerns the n-body structure of the boson image O_B. There are, in principle, infinite combinations of one-body plus two-body plus... up to 2Ω-body boson operators which can satisfy Eq. (24). However, wishing O_B to define a boson image of O_F at all orders, namely independently of the maximum dimension 2Ω of B, one has to proceed step by step. Let us define $F^{(1)}$ as the fermion space built in terms of the states containing up to one $A^{\dagger}$ operator, i.e.,

$$F^{(1)}=\{|0\rangle,A^{\dagger}|0\rangle\} \tag{25}$$

and similarly for bosons

$$B^{(1)}=\{|0),b^{\dagger}|0)\}. \tag{26}$$

The most general Hermitian boson operator acting within $B^{(1)}$ (we are assuming O_F Hermitian, for simplicity) has the form

$$O_{B,1}=\alpha+\beta(b^{\dagger}+b)+\gamma b^{\dagger}b. \tag{27}$$

In order for this operator to be the image of O_F in $B^{(1)}$, it must satisfy condition (24). This implies

$$\alpha=\langle 0|O_F|0\rangle, \tag{28a}$$

$$\beta=\langle 0|O_F A^{\dagger}|0\rangle, \tag{28b}$$

$$\gamma=\langle 0|AO_F A^{\dagger}|0\rangle-\langle 0|O_F|0\rangle. \tag{28c}$$

Let us now proceed one step further and consider the correspondence between

$$F^{(2)}=\{|0\rangle,A^{\dagger}|0\rangle,A^{\dagger}A^{\dagger}|0\rangle\} \tag{29}$$

and

$$B^{(2)}=\left\{|0),b^{\dagger}|0),\frac{1}{\sqrt{2}}b^{\dagger}b^{\dagger}|0)\right\}. \tag{30}$$

Wishing the image of O_F in $B^{(2)}$ to be at the same time image in $B^{(1)}$, the most general expression for this operator has to be written as

$$O_{B,2}=O_{B,1}+\phi b^{\dagger}b^{\dagger}bb+\epsilon(b^{\dagger}b^{\dagger}b+b^{\dagger}bb)+\delta(b^{\dagger}b^{\dagger}+bb). \tag{31}$$

The new coefficients δ,ϵ,ϕ can be evaluated making use of Eq. (24) as it has already been done for α,β,γ. The procedure can be extended in a similar way up to $O_{B,2\Omega}$ which is then the exact image of O_F in the correspondence $F\rightarrow B$.

Boson images:
The model Hamiltonian and β transition operators

In this section, we will show two applications of the mapping procedure just discussed: the first one will refer to the fermion Hamiltonian (1), while the second one will refer to the Fermi β transition operators (12). In the case of the Hamiltonian (1), the boson operator which one constructs at the first two levels of approximation (by ordering these according to the maximum number of the $b^\dagger, b$ operators contained in each term) reads

$$H_B^{(1)} = \alpha b^\dagger b + \beta(b^\dagger b^\dagger + bb), \tag{32}$$

for the first-order approximation, and

$$H_B^{(2)} = H_B^{(1)} + \gamma b^\dagger b^\dagger bb + \delta(b^\dagger b^\dagger b^\dagger b + b^\dagger bbb) \tag{33}$$

with

$$\alpha = 2\epsilon + \lambda_1, \tag{34a}$$

$$\beta = \lambda_2 \sqrt{1 - \frac{1}{2\Omega}}, \tag{34b}$$

$$\gamma = -\frac{\lambda_1}{2\Omega}, \tag{34c}$$

$$\delta = \lambda_2 \left(\sqrt{\left(1 - \frac{1}{3\Omega}\right)\left(1 - \frac{1}{2\Omega}\right)} - \sqrt{1 - \frac{1}{2\Omega}} \right), \tag{34d}$$

for the second-order approximation. As it emerges from the above equations, all the coefficients except α depend on j. This j dependence closely reflects the structure of the fermion states involved [through the condition (24)] into the definition of the coefficients. In the limit $j \to \infty$, which is equivalent to transforming the fermion operators into bosons, $\beta \to \lambda_2$, while $\gamma, \delta \to 0$. Moreover, still in this limit, no contributions arise from next order approximations so that the boson Hamiltonian reduces to

$$H_B^{(0)} = \alpha b^\dagger b + \lambda_2(b^\dagger b^\dagger + bb). \tag{35}$$

This boson Hamiltonian, where no memory is kept of the internal structure of the quasiparticle pairs $A^\dagger$ due to the $j \to \infty$ limit, will be called in the following as the ''zeroth-order'' boson approximation of H_F, i.e., $H_B^{(0)}$. It represents the QBA limit of $H_B^{(1)}$. In the next section, we will test numerically the ''quality'' of $H_B^{(0)}$, $H_B^{(1)}$, $H_B^{(2)}$ as boson images of H_F.

For what concerns the Fermi β transition operators (12), in the quasiparticle formalism, one has

$$\beta^- = \sqrt{2\Omega}(u_p v_n A^\dagger + v_p u_n A - u_p u_n B^\dagger + v_p v_n B). \tag{36}$$

By proceeding as for the boson Hamiltonian, but taking into account that the β operators (36) are not Hermitian, we obtain at the first two levels of approximation

$$(\beta^-)_B^{(1)} = \sqrt{2\Omega}(u_p v_n b^\dagger + v_p u_n b) \tag{37}$$

and

$$(\beta^-)_B^{(2)} = (\beta^-)_B^{(1)} + c_1 b^\dagger b^\dagger b + c_2 b^\dagger bb, \tag{38}$$

with

$$c_1 = \sqrt{2\Omega} u_p v_n \left(\sqrt{1 - \frac{1}{2\Omega}} - 1 \right), \tag{39a}$$

$$c_2 = \sqrt{2\Omega} v_p u_n \left(\sqrt{1 - \frac{1}{2\Omega}} - 1 \right). \tag{39b}$$

It is also

$$(\beta^+)_B^{(i)} = ((\beta^-)_B^{(i)})^\dagger, \quad i = 1,2. \tag{40}$$

One should notice that the scattering terms $B^\dagger$, B of Eq. (36) do not contribute to matrix elements of this operator between states (19) so that they do not play any role in the mapping $F \to B$. In Sec. V, we will compare calculations performed both with $(\beta^-)_B^{(1)}$ and $(\beta^-)_B^{(2)}$.

IV. QRPA AND HIGHER-ORDER APPROXIMATIONS

One of the standard approaches to RPA is that proposed by Rowe with his equation of motion method [1] and is fully developed in the fermion space. To briefly resume this approach with reference to our model, we introduce the phonon creation operator

$$Q^\dagger = XA^\dagger - YA \tag{41}$$

and we define the excited and ground states by the conditions

$$|Q\rangle = Q^\dagger|\tilde{0}\rangle, \quad Q|\tilde{0}\rangle = 0. \tag{42}$$

Minimization of the one-phonon energy $E = \langle Q|H_F|Q\rangle / \langle Q|Q\rangle$ with respect to the amplitudes X and Y leads, together with Eq. (42), to

$$\langle\tilde{0}|[\delta Q, [H_F, Q^\dagger]]|\tilde{0}\rangle = \mathcal{E}\langle\tilde{0}|[\delta Q, Q^\dagger]|\tilde{0}\rangle, \tag{43}$$

where $\mathcal{E}$ is the excitation energy of the system. This expression is evaluated in the two cases $\delta Q = A^\dagger, A$ under the approximation $|\tilde{0}\rangle \to |0\rangle$, namely within QBA, and gives rise to the well-known system of equations

$$\mathcal{A}X + \mathcal{B}Y = \mathcal{E}_{\mathrm{QRPA}}X, \quad -\mathcal{B}X - \mathcal{A}Y = \mathcal{E}_{\mathrm{QRPA}}Y, \tag{44}$$

where

$$\mathcal{A} = \langle 0|[A, [H_F, A^\dagger]]|0\rangle = 2\epsilon + \lambda_1,$$

$$\mathcal{B} = -\langle 0|[A, [H_F, A]]|0\rangle = 2\lambda_2,$$

and $\mathcal{E}_{\mathrm{QRPA}} = [(2\epsilon + \lambda_1)^2 - 4\lambda_2^2]^{1/2}$.

Working in our boson formalism, we define, in full analogy with Eqs. (41), (42), the phonon creation operator

$$q^\dagger = Xb^\dagger - Yb \tag{45}$$

and the corresponding excited and ground states by the conditions

TABLE I. Comparison of the spectra of the Hamiltonian H_F (1) in the basis (19), for three values of the parameter k', with the corresponding spectra of the boson Hamiltonians $H_B^{(0)}$ (35), $H_B^{(1)}$ (32), and $H_B^{(2)}$ (33) in the basis (23). Spectra of H_F and $H_B^{(2)}$ coincide within 0.2% so that they have been reported in the same column, for simplicity. Calculations refer to the choice $\chi'=0.5$. No significant changes are observed for $\chi=0.0$.

	$k'=0.5$			$k'=1.0$			$k'=1.5$	
$H_B^{(0)}$	$H_B^{(1)}$	$H_B^{(2)},H_F$	$H_B^{(0)}$	$H_B^{(1)}$	$H_B^{(2)},H_F$	$H_B^{(0)}$	$H_B^{(1)}$	$H_B^{(2)},H_F$
-0.120	-0.107	-0.104	-0.471	-0.401	-0.311	-4.374	-3.799	-0.823
1.680	1.718	1.795	0.221	0.397	0.910	-3.773	-3.274	-0.276
3.480	3.544	3.800	1.020	1.260	2.594	-1.042	-0.883	1.396
5.287	5.373	5.873	2.278	2.480	4.506	-0.251	-0.080	3.179
7.098	7.206	7.987	3.498	3.693	6.579	0.852	0.962	5.256
9.061	9.159	10.116	5.756	5.846	8.759	3.383	3.419	7.507
10.982	11.080	12.240	7.413	7.476	11.004	4.879	4.877	9.877
13.826	13.822	14.333	11.147	11.055	13.272	9.196	9.020	12.313
16.034	16.006	16.370	13.226	13.092	15.524	11.175	10.942	14.760
21.145	20.928	18.323	19.598	19.222	17.712	18.444	17.914	17.156
23.725	23.471	20.148	22.113	21.680	19.771	20.909	20.301	19.413

$$|q)=q^\dagger|\tilde{0}), \quad q|\tilde{0})=0. \tag{46}$$

The one-phonon energy which is derived from the minimization of $E_B=(q|H_B|q)/(q|q)$ is, of course, subordinated to the choice of the boson Hamiltonian. Minimizing $(q|H_B^{(0)}|q)/(q|q)$, with $H_B^{(0)}$ given by Eq. (35), leads straight to Eqs. (44). Therefore, if $E_B^{(0)}$ is the excitation energy resulting from this minimization, $E_B^{(0)}\equiv\mathcal{E}_{\rm QRPA}$. In other words, the effect of the QBA, which in Rowe's approach is introduced in the evaluation of Eq. (43) via the replacement $|\tilde{0}\rangle\rightarrow|0\rangle$, is now equivalently incorporated in the boson Hamiltonian (35). But we have seen in the previous section that $H_B^{(0)}$ is only a special case of the first-order boson image which we can construct with our mapping procedure. Therefore, the use of higher-order approximations such as $H_B^{(1)}$ (32) and $H_B^{(2)}$ (33) is expected to correspond to an improvement of the approximated energies and wave functions.

$H_B^{(1)}$ differs from $H_B^{(0)}$ only by having the coefficient β Eq. (34b) in place of λ_2. Correspondingly, minimizing $(q|H_B^{(1)}|q)/(q|q)$ leads to the excitation energy

$$E_B^{(1)}=\sqrt{(2\epsilon+\lambda_1)^2-4\lambda_2^2\left(1-\frac{1}{2\Omega}\right)}, \tag{47}$$

which differs from $E_B^{(0)}$ by the factor $(1-1/2\Omega)$. As one could expect, this factor becomes more significant the smaller the j value.

The use of $H_B^{(2)}$ presents itself more complicated. In fact,

$$\begin{aligned}\mathcal{A}&\equiv(\tilde{0}|[b,[H_B^{(2)},b^\dagger]]|\tilde{0})\\&=(\tilde{0}|\alpha+4\gamma b^\dagger b+3\delta(b^\dagger b^\dagger+bb)|\tilde{0})\end{aligned} \tag{48}$$

and

$$\begin{aligned}\mathcal{B}&\equiv-(\tilde{0}|[b,[H_B^{(2)},b]]|\tilde{0})\\&=(\tilde{0}|2\beta+2\gamma bb+6\delta b^\dagger b|\tilde{0})\end{aligned} \tag{49}$$

so that, with respect to the case $H_B^{(1)}$ (where $\gamma=\delta=0$), one has now to evaluate extra matrix elements of the type $(\tilde{0}|b^\dagger b|\tilde{0})$ and $(\tilde{0}|b^\dagger b^\dagger|\tilde{0})$. This can be done by inverting expression (45) (and H.c.) so that

$$b^\dagger=Xq^\dagger+Yq, \tag{50}$$

which leads to the new expressions

$$\mathcal{A}=\alpha+4\gamma Y^2+6\delta XY, \tag{51}$$

$$\mathcal{B}=2\beta+2\gamma XY+6\delta Y^2. \tag{52}$$

Due to this dependence of $\mathcal{A}$ and $\mathcal{B}$ on the X,Y amplitudes, the minimization of $(q|H_B^{(2)}|q)/(q|q)$ leads to a set of nonlinear equations for $E_B^{(2)}$, X, Y which is difficult to solve analytically as for $E_B^{(0)}$ and $E_B^{(1)}$. In the next section, however, we will see numerically how $E_B^{(2)}$ compares with the lower-order approximations $E_B^{(0)}$, $E_B^{(1)}$ in the cases we have studied.

V. RESULTS

Following the indications of Ref. [30], we have made the choice

$$j=9/2, \quad Z=4, \quad N=6, \quad \epsilon=1 \ \text{MeV}. \tag{53}$$

In order to avoid dealing with small numbers, we have also redefined the two parameters k and χ as follows

$$k\rightarrow k'\equiv 2\Omega k, \quad \chi\rightarrow\chi'\equiv 2\Omega\chi. \tag{54}$$

Calculations have been performed in the two cases $\chi'=0,0.5$ while k' has been kept as a variable in the interval (0,2).

A. The boson Hamiltonians

Before examining QRPA and higher-order calculations we will concentrate on the analysis of the boson Hamilto-

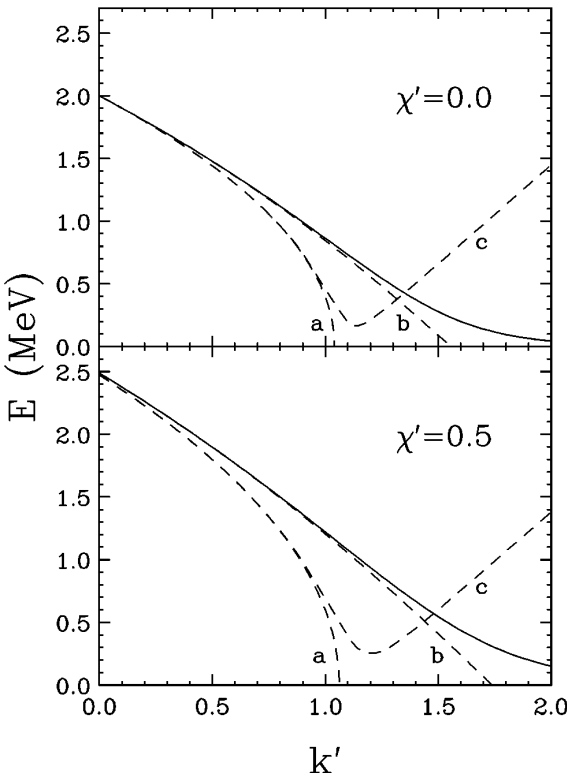

FIG. 1. Comparison of the lowest excitation energies resulting from the diagonalization of H_F (1) in the basis (19) (solid line) with the standard (a) and second-order (b) QRPA values. Also shown are the values obtained from the diagonalization of the boson Hamiltonian $H_B^{(0)}$ (35) in the basis (23) (c).

nians constructed in the previous section. In Table I, we show the energies resulting from the diagonalization of the three boson Hamiltonians $H_B^{(0)}$, $H_B^{(1)}$, $H_B^{(2)}$ in the space B Eq. (23) and compare them with those of H_F in the space F Eq. (19). Calculations refer to three different values of k' (0.5, 1.0, 1.5), while $\chi'=0.5$ ($\chi'=0.0$ spectra do not show significant differences). One notices that (a) spectra of $H_B^{(0)}$ and $H_B^{(1)}$, for this choice of the parameters, are close to each other but $H_B^{(1)}$ better approximate H_F, as expected; (b) the quality of the $H_B^{(0)}$ and $H_B^{(1)}$ spectra rapidly deteriorates for increasing k' so that, even though these spectra are rather good at $k'=0.5$, they become definitely bad at $k'=1.5$. Spectra of $H_B^{(2)}$ reproduce almost perfectly the corresponding ones of H_F for all values of k' in the interval (0,2), differences remaining within 0.2%. Due to these minor differences, we have shown the spectra of $H_B^{(2)}$ and H_F in the same column.

In conclusion, then, the second-order boson Hamiltonian $H_B^{(2)}$ emerges as an excellent boson image of H_F in the whole range of k', while the quality of $H_B^{(0)}$ and $H_B^{(1)}$ strictly depends on the variable k' and remains acceptable only for small values of it.

B. Excitation energies

In Fig. 1, we compare the excitation energies $E_B^{(0)}$ (a) and $E_B^{(2)}$ (b) with the values resulting from the diagonalization of H_F (1) (solid line). We do not show for simplicity the $E_B^{(1)}$ values since, in agreement with the already observed similarity between the spectra of $H_B^{(0)}$ and $H_B^{(1)}$, it is $E_B^{(0)} \sim E_B^{(1)}$. We show, instead, the energies which result from the diagonalization of $H_B^{(0)}$ (c). These exhibit a minimum for k' in the interval (1.1)-(1.2). It turns out that the standard (or zeroth-order, in our notation) QRPA energies are quite good approximations of these energies, deviating from them only for values of the variable k' approaching their minimum. Second-order QRPA energies $E_B^{(2)}$ reproduce quite well those of H_F particularly in the region of the collapse point of the standard QRPA. There occurs also in this case a collapse but at larger k' so that, globally, the new energies appear to have gained stability if compared to the standard QRPA ones.

C. Expectation values of the quasiparticle number operator

A comparison between relative energies, as the one discussed in the previous subsection, does not provide by itself a decisive element to judge the quality of different approximations. In this subsection, we will examine two quantities which will help us to shed further light upon this quality. These are the expectation values of half the quasiparticle number operator, $C/2$ (4), in the ground state and in the first excited state.

The image of $C/2$ in the boson space B is the operator $N_B=b^\dagger b$. This is an exact boson image in the mapping $F \to B$ since it exactly fulfills the condition (24) for every n. It is easy to see [making use of the transformation (50)] that

$$N_0 \equiv (\tilde{0}|N_B|\tilde{0}) = Y^2. \tag{55}$$

It is also true that

$$(\tilde{0}|qN_Bq^\dagger|\tilde{0}) = (\tilde{0}|[q,[N_B,q^\dagger]]|\tilde{0}) + (\tilde{0}|N_B|\tilde{0}) \tag{56}$$

so that

$$\Delta N \equiv (\tilde{0}|qN_Bq^\dagger|\tilde{0}) - (\tilde{0}|N_B|\tilde{0}) = X^2 + Y^2 = 1 + 2Y^2. \tag{57}$$

This quantity is therefore predicted to be always larger than 1 (unless $Y=0$) and it should increase proportionally to Y^2 similarly to N_0. In Figs. 2 and 3, we show N_0 and ΔN in the zeroth- (a) and second-order (b) QRPA calculations and in the exact cases (solid line). As for the excitation energies, zeroth- and first-order results are quite similar and we have omitted the last ones for simplicity. Looking at the exact quantities it is evident that there is a clear disagreement between their behavior and the above predictions. Only up to k' slightly larger than 1, ΔN indeed grows as also N_0 does. But for larger values of k' these two quantities start behaving oppositely, N_0 continuing its growth, while ΔN going toward zero. Above the threshold $k' \sim 1$, it becomes therefore impossible to find any X,Y compatible with these two quantities. This fact reveals a clear failure of the QRPA approach beyond this point, no matter which order of approximation we are considering, and so provides a severe constraint on the range of applicability of this theory. Remaining in the interval $0 \leqslant k' \leqslant 1$, in the case of ΔN, the second-order QRPA results show a much better agreement with the exact

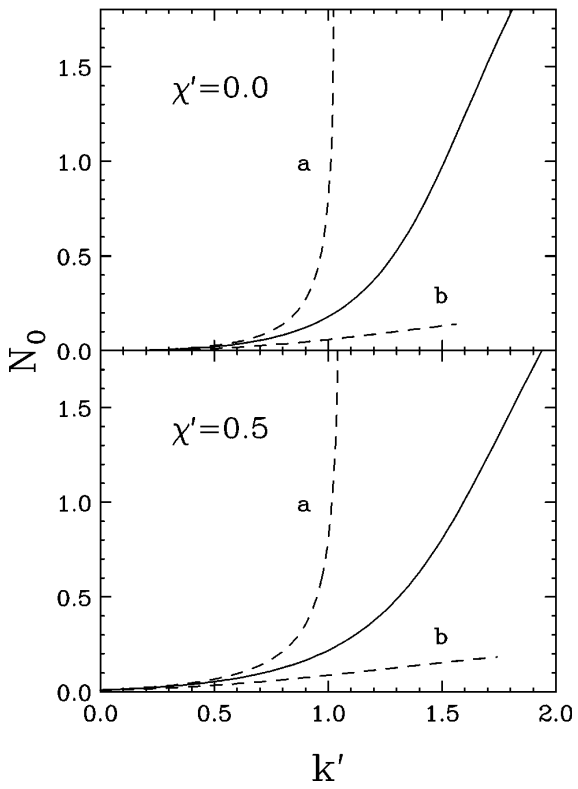

FIG. 2. Comparison of the expectation values of (half) the quasiparticle number operator, $C/2$ (4), in the ground state of the Hamiltonian H_F (1) with the values obtained in the standard (a) and second-order (b) QRPA.

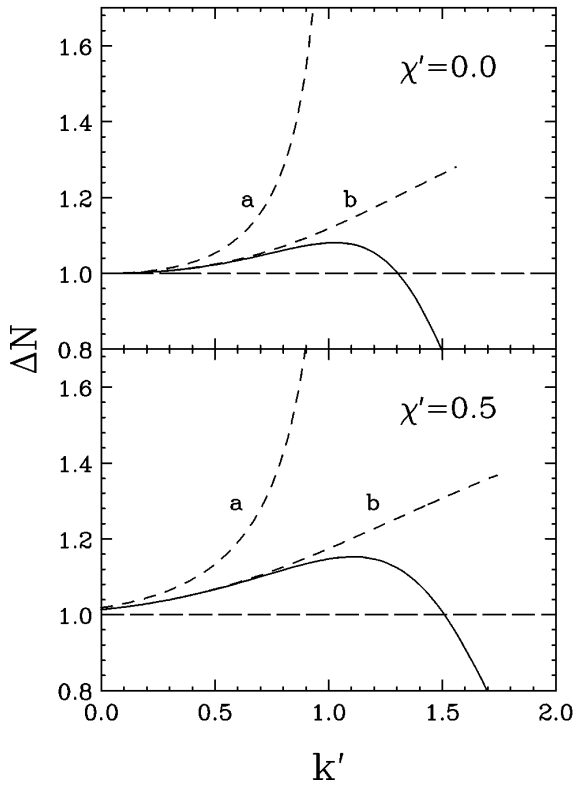

FIG. 3. Differences between the expectation values of (half) the quasiparticle number operator, $C/2$ (4), in the first excited state and in the ground state. Solid line refers to the eigenstates of the Hamiltonian H_F (1) while lines (a),(b) are the standard and second-order QRPA results, respectively.

ones if compared with standard QRPA. In the case of N_0, one notices a clear difference between zeroth- and second-order approximations: in the first case, there appears the well-known overestimation of the ground-state correlations while, in the second one, there occurs an underestimation of the same quantities. In this case, second-order QRPA results do not reach the same qualitative level as in ΔN.

D. Fermi β transition amplitudes

Similarly to the expectation values of the quasiparticle number operator, the comparison of matrix elements of the β operators (Sec. III) can add further elements of valuation upon the quality of the approximate wave functions. In Fig. 4, we show the transition amplitudes of these operators between ground and first excited states. In each figure, upper lines concern β^- transitions while the lower ones β^+ transitions. Solid lines have been calculated by using the eigenstates of the fermion Hamiltonian (1) and the operators (36) (and H.c.), while lines (a),(b) refer to matrix elements of the images $(\beta^{\pm})_B^{(1)}$ (37),(40) in the zeroth- and second-order QRPA calculations, respectively (we omit also in this case first-order results). These matrix elements are given by

$$(\widetilde{0}|q(\beta^+)_B^{(1)}|\widetilde{0}) = \sqrt{2\Omega}(v_p u_n X + u_p v_n Y), \tag{58}$$

$$(\widetilde{0}|q(\beta^-)_B^{(1)}|\widetilde{0}) = \sqrt{2\Omega}(u_p v_n X + v_p u_n Y). \tag{59}$$

Only the relative sign of these matrix elements can be fixed unambiguously. In these figures, we have taken the β^- matrix elements to be positive. Similarly to what has been observed in the case of N_0, zeroth- and second-order approximations show different behaviors, the first ones underestimating the exact quantities and the second ones overestimating them. For what concerns the β^+ matrix elements, however, it is worthy noticing that the crossing of zero exhibited by the exact values at $k' \sim 1.4, 1.5$ is not reproduced by the second-order QRPA results. This crossing is shown both by the zeroth- and first-order QRPA calculations although in a region of k' quite close to their collapse points. Evidence for this crossing is widely discussed in literature [3–10].

Differently from the quasiparticle number operator case, where $N_B = b^\dagger b$ is an *exact* image of $C/2$ in the mapping $F \to B$, in the case of the β operators, we have already seen in Sec. III that $(\beta^{\pm})_B^{(1)}$ are only the first-order images of the corresponding fermion operators $\beta^{\pm}$. To better understand the reasons of the bad behavior exhibited by the second-order QRPA results for $k' > 1$, the same calculations have been repeated by making use of the $(\beta^{\pm})_B^{(2)}$ operators (38),(40). In these cases the matrix elements become

$$(\widetilde{0}|q(\beta^+)_B^{(2)}|\widetilde{0}) = \sqrt{2\Omega} v_p u_n X(1-3GY^2) + \sqrt{2\Omega} u_p v_n Y(1-3GY^2-G), \tag{60}$$

$$(\widetilde{0}|q(\beta^-)_B^{(2)}|\widetilde{0}) = \sqrt{2\Omega} u_p v_n X(1-3GY^2) + \sqrt{2\Omega} v_p u_n Y(1-3GY^2-G), \tag{61}$$

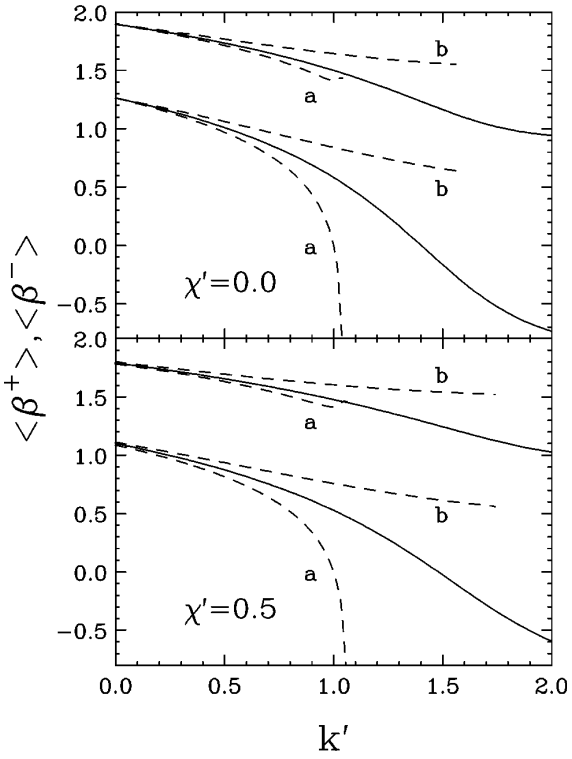

FIG. 4. Fermi β transition amplitudes between the ground and first excited states. In each plot, upper curves refer to the β^- transitions, while lower curves refer to the β^+ ones. Solid lines are obtained with the operators (12) and the eigenstates of the Hamiltonian H_F (1) while lines (a),(b) are the standard and second-order QRPA results, respectively [see Eqs. (58), (59)].

where

$$G \equiv 1 - \sqrt{1 - \frac{1}{2\Omega}}. \tag{62}$$

No significant changes ($\lesssim 5\%$) have emerged, however, from the new calculations. Therefore, the bad behavior of the β^+ matrix elements in the second-order QRPA calculations cannot be ascribed to a bad quality of the image operators $(\beta^{\pm})_B^{(1)}$ but rather confirms the conclusions obtained so far of an unreliability of the whole theory when k' approaches the second-order QRPA collapse point.

E. Ikeda sum rule

To conclude this section, we discuss, within our model, a peculiar aspect of the β transition amplitudes: the Ikeda sum rule. By defining the total $\beta^{\pm}$ strengths

$$S^{\pm} = \sum_{\nu} |\langle \nu | \beta^{\pm} | gs \rangle|^2, \tag{63}$$

where the summation extends over a complete set of states, $|gs\rangle$ is the ground state of the system and $\beta^{\pm}$ are the operators (12), it is seen that

$$S^- - S^+ = N - Z, \tag{64}$$

where $N(P)$ is the total number of neutrons (protons) of the system. The derivation of Eq. (64) simply results from the completeness of the states $\{|\nu\rangle\}$ and from the commutator

$$[\beta^{\dagger}, \beta^-] = \hat{N} - \hat{Z}. \tag{65}$$

As it is well known, an interesting aspect of the QRPA is that it fulfills exactly this sum rule. In our boson formalism, due to the fact, stressed in the previous subsection, that the boson images $(\beta^{\pm})_B$ can be constructed at different levels of approximation, one is forced to distinguish different cases. If one assumes the operators $(\beta^{\pm})_B^{(1)}$ (37), (40) as the images of $\beta^{\pm}$, it is easy to verify that Eq. (64) is fulfilled exactly and so irrespective of the order of the QRPA calculations and of the magnitude of the ground-state correlations. By assuming instead $(\beta^{\pm})_B^{(2)}$ (38),(40) as images of the operators $\beta^{\pm}$, things become more complicated. One finds now

$$S^- - S^+ = (N - Z)[1 - (4G + G^2)Y^2 + 3G^2Y^4], \tag{66}$$

where the factor G is given in Eq. (62). Therefore, the sum rule is not exactly fulfilled depending on the Y amplitude and so on the order of the QRPA calculation. With reference to our second-order results, however, due also to the smallness of the factor G, we find that the deviation of Eq. (66) from the value $(N-Z)$ remains within few percent. This further confirms what was stated in Sec. V, namely, that the use of the $(\beta^{\pm})_B^{(2)}$ operators in place of $(\beta^{\pm})_B^{(1)}$ introduces only minor changes in the β transition amplitudes.

VI. SUMMARY AND CONCLUSIONS

In this article the QRPA has been reviewed in a boson formalism and higher-order approximations have been discussed. A fermion-boson mapping procedure has been first illustrated within an exactly solvable model. Boson images of the model Hamiltonian have been constructed at the first two levels of approximation and the corresponding one-phonon excitation energies derived. Standard QRPA energies, resulting in correspondence with the QBA limit of the first-order boson Hamiltonian, have been compared with the first- and second-order energies. Only minor changes have been found in the first case while these changes have become much more evident for excitation energies calculated with the second-order Hamiltonian. In this case, the new energies have gained stability getting quite close to the exact solutions near the collapse point of the standard QRPA.

The quality of the solutions has been further tested by examining first of all the matrix elements of the quasiparticle number operator in the ground and first excited states. While evidencing an overall better fit of the exact values in the case of the second-order solutions, the comparison between exact and approximate results has pointed out a rapid deterioration of the quality of the second ones near the new collapse point. Furthermore, still in this region, it has been evidenced a breakdown of the whole QRPA formalism, independently of the level of approximation of the boson Hamiltonian employed. This result has also found confirmation in the analysis of the β transition amplitudes. β^+ amplitudes, in particular, have not been found to reproduce the crossing of zero exhibited by the exact values. This has been verified by mak-

ing use of both first- and second-order β^+ boson operators. Therefore, in correspondence with a better quality of the excitation energies found in second-order QRPA calculations, we have not observed a similar improved quality in the values of these observables.

Finally, we have examined the Ikeda sum rule showing that this is exactly fulfilled when the $\beta^\pm$ boson operators are calculated at the first order of approximation. The higher-order $\beta^\pm$ operators have introduced only negligible changes in the results and their use does not appear justified in the present calculations.

ACKNOWLEDGMENTS

M.S. wishes to acknowledge the hospitality of the Department of Physics of the University of Jyväskylä, where part of this work was done.

[1] D. J. Rowe, *Nuclear Collective Motion* (Methuen, London, 1970).
[2] P. Ring and P. Schuck, *The Nuclear Many-Body System* (Springer, New York, 1980).
[3] P. Vogel and M. R. Zirnbauer, Phys. Rev. Lett. **57**, 731 (1986).
[4] O. Civitarese, A. Faessler, and T. Tomoda, Phys. Lett. B **194**, 11 (1987).
[5] J. Engel, P. Vogel, and M. R. Zirnbauer, Phys. Rev. C **37**, 731 (1988).
[6] K. Muto and H. V. Klapdor, Phys. Lett. B **201**, 420 (1988).
[7] J. Suhonen, T. Taigel, and A. Faessler, Nucl. Phys. **A486**, 91 (1988).
[8] J. Hirsch and F. Krmpotić, Phys. Rev. C **41**, 792 (1990).
[9] F. Krmpotić, Phys. Rev. C **48**, 1452 (1993).
[10] O. Civitarese, J. Suhonen, and A. Faessler, Nucl. Phys. **A591**, 195 (1995).
[11] O. Civitarese, A. Faessler, J. Suhonen, and X. R. Wu, Nucl. Phys. **A524**, 404 (1991).
[12] F. Krmpotić, A. Mariano, T. T. S. Kuo, and K. Nakayama, Phys. Lett. B **319**, 393 (1993).
[13] M. K. Cheoun, A. Bobyk, A. Faessler, F. Šimkovic, and G. Teneva, Nucl. Phys. **A561**, 74 (1993); **A564**, 329 (1993).
[14] A. A. Raduta, A. Faessler, S. Stoica, and W. A. Kaminski, Phys. Lett. B **254**, 7 (1991).
[15] A. A. Raduta, A. Faessler, and S. Stoica, Nucl. Phys. **A534**, 149 (1991).
[16] A. A. Raduta and J. Suhonen, Phys. Rev. C **53**, 176 (1996).
[17] D. Karadjov, V. V. Voronov, and F. Catara, Phys. Lett. B **306**, 197 (1993).
[18] F. Catara, N. Dinh Dang, and M. Sambataro, Nucl. Phys. **A579**, 1 (1994).
[19] J. Toivanen and J. Suhonen, Phys. Rev. Lett. **75**, 410 (1995); Phys. Rev. C **55**, 2314 (1997).
[20] J. Schwieger, F. Šimkovic, and A. Faessler, Nucl. Phys. **A600**, 179 (1996).
[21] K. Hara, Prog. Theor. Phys. **32**, 88 (1964).
[22] D. J. Rowe, Rev. Mod. Phys. **40**, 153 (1968); Nucl. Phys. **A107**, 99 (1968); Phys. Rev. **175**, 1283 (1968).
[23] J. da Providencia, Nucl. Phys. **A108**, 589 (1968).
[24] F. Catara, G. Piccitto, M. Sambataro, and N. Van Giai, Phys. Rev. B **54**, 17 536 (1996).
[25] P. Schuck and S. Ethofer, Nucl. Phys. **A212**, 269 (1973).
[26] J. Dukelsky and P. Schuck, Nucl. Phys. **A512**, 466 (1990); Mod. Phys. Lett. A **26**, 2429 (1991); Phys. Lett. B **387**, 233 (1996).
[27] V. A. Kuz'min and V. G. Soloviev, Nucl. Phys. **A486**, 118 (1988).
[28] K. Muto, E. Bender, T. Oda, and H. V. Klapdor-Kleingrothaus, Z. Phys. A **341**, 407 (1992).
[29] O. Civitarese and J. Suhonen, J. Phys. G **20**, 1441 (1994); Nucl. Phys. **A578**, 62 (1994).
[30] J. G. Hirsch, P. O. Hess, and O. Civitarese, Phys. Rev. C **54**, 1976 (1996).
[31] H. J. Lipkin, N. Meshkov, and S. Glick, Nucl. Phys. **A62**, 188 (1965).
[32] D. Beaumel and Ph. Chomaz, Ann. Phys. (N.Y.) **2**, 405 (1992).
[33] A. Klein and E. R. Marshalek, Rev. Mod. Phys. **63**, 375 (1991).
[34] M. Sambataro, Phys. Rev. Lett. **57**, 1503 (1986); Phys. Rev. C **35**, 1530 (1987); **37**, 2186 (1988); **37**, 2199 (1988).
[35] F. Catara and M. Sambataro, Nucl. Phys. **A535**, 605 (1991); Phys. Rev. C **46**, 754 (1992).
[36] M. Sambataro, Phys. Rev. C **52**, 3378 (1995).