Gauging the spectator equations

A. N. Kvinikhidze* and B. Blankleider
Department of Physics, The Flinders University of South Australia, Bedford Park, SA 5042, Australia
(Received 30 June 1997)

We show how to derive relativistic, unitary, gauge-invariant, and charge-conserving three-dimensional scattering equations for a system of hadrons interacting with an electromagnetic field. In the method proposed, the spectator equations describing the strong interactions of the hadrons are gauged using our recently introduced gauging of equations method. A key ingredient in our model is the on-mass-shell particle propagator. We discuss how to gauge this on-mass-shell propagator so that both the Ward-Takahashi and Ward identities are satisfied. We then demonstrate our gauging procedure by deriving the gauge-invariant three-dimensional expression for the deuteron photodisintegration amplitude within the spectator approach.
[S0556-2813(97)00512-8]

PACS number(s): 21.45.+v, 24.10.Jv, 25.20.−x, 25.30.Bf

I. INTRODUCTION

Recently we have shown how to describe the interaction of an electromagnetic probe with a hadronic system described by four-dimensional integral equations [1]. Our method is based on the idea of gauging the integral equations themselves and, in this way, incorporates the electromagnetic interaction into the hadronic description without the need for any perturbation expansion. As a result, the external photon becomes attached to all possible places in every contributing Feynman diagram of the theory, so that gauge invariance and charge conservation are implemented in the theoretically correct fashion. In Ref. [1] we applied the gauging of equations method to the four-dimensional three-nucleon problem, thereby obtaining gauge-invariant expressions for the electromagnetic currents of all possible transitions between three-nucleon states induced by an external electromagnetic field. The power of the method was particularly evident in the formulation of the three-body bound-state current where a previously overlooked overcounting problem was solved automatically by the natural appearance of a subtraction term.

Combined with the integral equations describing the strong interactions, the gauging of equations method provides a consistent unified description of hadronic systems and their interactions with an external electromagnetic field. Since the starting point of Ref. [1] was relativistic quantum field theory, at this stage such a unified description is inherently four dimensional. In view of the technical difficulty in solving four-dimensional equations, the question naturally arises if there is a way to do a three-dimensional reduction of the unified description so that covariance, unitarity, gauge invariance, and charge conservation are all preserved. This paper is devoted to answering this question.

In the strong interaction sector, three-dimensional reductions of the Bethe-Salpeter (BS) equation have been developed over a number of years [2–5] and now provide a powerful approach for practical calculations in quantum field theory. All these reductions preserve covariance and unitarity, and in this respect give rise to the question of which reduction is to be preferred [6–8]. In Ref. [5], Gross showed that his reduction scheme has the important property of giving a three-dimensional two-body equation that approaches the correct one-body equation in the limit when one of the masses becomes very large. We find that the Gross reduction scheme is also appealing in that it easily lends itself to our gauging of equations method.

In the Gross approach, also called the "spectator approach," three-dimensional equations are derived by restricting some of the intermediate-state particles (typically the spectator particles) in the BS equation to their mass shell. Equivalently, the Feynman propagators d of these particles in the BS equation are replaced by the quantities δ containing a positive energy on-mass-shell δ function:

$$d(p)=\frac{i\Lambda(p)}{p^2-m^2+i\epsilon}\to\delta(p)=2\pi\Lambda(p)\delta^+(p^2-m^2), \quad (1)$$

where $\Lambda(p)=1$ or $\not{p}+m$ for scalar and spinor particles, respectively. We shall call $\delta(p)$ the "on-mass-shell particle propagator." Thus, in the two-body case, the propagator $G_0(P,p)=d_1(P-p)d_2(p)$ in the BS equation

$$T(P;k',k)=K(P;k',k)+\int\frac{d^4p}{(2\pi)^4}K(P;k',p)G_0(P,p)T(P;p,k) \quad (2)$$

is replaced by $\mathcal{G}_0(P,p)=d_1(P-p)\delta_2(p)$:

*On leave from the Mathematical Institute of Georgian Academy of Sciences, Tbilisi, Georgia.

$$\mathcal{G}_0(P,p)=\begin{cases}2\pi d_1(P-p)\,\delta^+(p^2-m_2^2) & \text{for scalar particle 2,}\\ 2\pi d_1(P-p)\,\delta^+(p^2-m_2^2)(\not p+m_2) & \text{for spinor particle 2.}\end{cases} \quad (3)$$

This replacement turns the BS equation into the four-dimensional equation

$$T(P;k',k)=K(P;k',k)+\int\frac{d^4p}{(2\pi)^4}K(P;k',p)\mathcal{G}_0(P,p)T(P;p,k), \quad (4)$$

which after a trivial integration over p_0 becomes the three-dimensional ''spectator equation'' for the t matrix [in this sense, we shall also refer to four-dimensional equations like Eq. (4) as being ''three dimensional'']. The significance of expressing the three-dimensional spectator equation in the four-dimensional form of Eq. (4) is that we can then apply our gauging of equations method directly to Eq. (4) in just the same way as was done for the BS case in Ref. [1].

Yet an immediate problem arises. As the gauging of an equation involves the gauging of all terms in the equation, we are faced with having to gauge the on-mass-shell one-body propagator $\delta(p)$ in Eq. (4). The resulting gauged on-mass-shell one-body propagator $\delta^\mu(p',p)$ needs to satisfy both the Ward-Takahashi identity and the Ward identity if the overall gauging procedure is to yield results that are gauge invariant and that obey charge conservation (as we shall see later, it is possible for a gauged on-mass-shell propagator to satisfy the Ward-Takahashi identity but not the Ward identity). How to gauge $\delta(p)$ so that both these identities are satisfied is therefore the key question that needs to be answered before a unified three-dimensional description can be given. The major part of this paper is devoted to answering this question. With this achieved, we then go on and demonstrate the gauging procedure by deriving the gauge-invariant three-dimensional expression for deuteron photodisintegration within the spectator approach. Application to the three-dimensional three-nucleon problem is given in a separate work [9]. Clearly, the gauging method we propose is directly applicable to any system of hadrons for which the strong interaction spectator equations can be written down.

It is also important to realize that although we concentrate our efforts in this paper on the electromagnetic interaction for which gauge invariance (or current conservation) is a major issue, the gauging of equations method itself is totally independent of the type of external field involved. Thus the procedure for obtaining three-dimensional equations for transition currents outlined in this paper is valid ''as is'' for the case of other interactions (e.g., weak) when the external field is that of a W or other gauge boson. Only gauged inputs like the nucleon vertex function Γ^μ would need to be changed.

II. GAUGED ON-MASS-SHELL PROPAGATOR

To discuss the gauging of the one-body on-mass-shell propagator $\delta(p)$, it is sufficient to consider a bound two-body system and its interaction with an external electromagnetic field. In the BS approach, two-body scattering is described by Eq. (2) and the two-body bound state is described by the equation

$$\Phi_P(k)=\int\frac{d^4p}{(2\pi)^4}K(P;k,p)G_0(P,p)\Phi_P(p), \quad (5)$$

where Φ_P is the bound-state vertex function. Interaction with an external electromagnetic field is then described by the bound-state current [10]

$$\begin{aligned}\langle P'|J^\mu(0)|P\rangle &= \int\frac{d^4p}{(2\pi)^4}\bar\Phi_{P'}(p')d_1(P-p)d_2^\mu(p',p)\Phi_P(p)\\ &+\int\frac{d^4p}{(2\pi)^4}\bar\Phi_{P'}(P-p)d_1^\mu(p',p)d_2(P-p)\Phi_P(P-p)\\ &+\int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\bar\Phi_{P'}(k)d_1(P'-k)\\ &\times d_2(k)K^\mu(P',k;P,p)d_1(P-p)d_2(p)\Phi_P(p),\end{aligned} \quad (6)$$

where $q=P'-P=p'-p$ is the four-momentum of the incoming photon, K^μ is the interaction current, and

$$d_i^\mu(p',p)=d_i(p')\Gamma_i^\mu(p',p)d_i(p) \quad (7)$$

is the gauged Feynman propagator for particle i, with $\Gamma_i^\mu(p',p)$ being the particle's electromagnetic vertex function. Equation (6) is illustrated in Fig. 1. A simple way to derive Eq. (6) is to gauge the BS equation for the two-body Green function [1]. As d_i is the propagator of a particle without dressing, consistency requires that $\Gamma_i^\mu(p',p)$ be the bare electromagnetic vertex, i.e., for a scalar or spinor particle of charge e_i, $\Gamma_i^\mu(p',p)=e_i(p'+p)^\mu$ or $e_i\gamma^\mu$, respectively. The case where dressing is included does not add to the essential discussion of this paper and is therefore relegated to the Appendix.

The three-dimensional reduction of Eqs. (5) and (6) by putting particle 2 on mass shell was discussed by Gross and

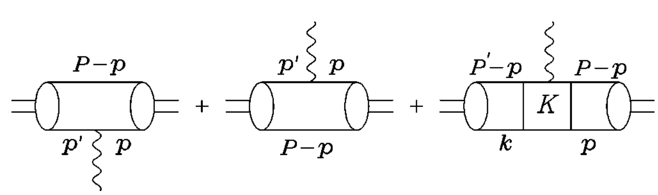

FIG. 1. The two-body bound-state current $\langle P'|J^\mu(0)|P\rangle$ as given by Eq. (6).

Riska (GR) [11]. Replacing d_2 by δ_2 in Eq. (5) gives the bound-state spectator equation

$$\Phi_P(k)=\int \frac{d^4p}{(2\pi)^4}K(P;k,p)\mathcal{G}_0(P,p)\Phi_P(p). \tag{8}$$

In Eq. (6), d_2 can be replaced by δ_2 in the second and third terms on the right-hand side (RHS) of the equation (second and third terms of Fig. 1), thus reducing the four-dimensional integrations to three-dimensional ones, and at the same time reducing the BS bound-state vertex functions to the quasipotential ones. Unfortunately it is impossible to do the same replacement for both propagators of $d_2^\mu(p',p)=d_2(p')\Gamma_2^\mu(p',p)d_2(p)$ in the first term on the RHS of Eq. (6) (first term of Fig. 1), as at the very least this would make the bound-state current diverge at zero momentum transfer. To avoid this problem, GR replaced the first term by a sum of two terms corresponding to particle 2 being on mass shell to the right and to the left of the photon. That is, their prescription is equivalent to the gauge-invariant replacement

$$d^\mu(p',p)\rightarrow\delta'^\mu(p',p)=\delta(p')\Gamma^\mu(p',p)d(p)+d(p')\Gamma^\mu(p',p)\delta(p). \tag{9}$$

Although this prescription has been used in a number of calculations [12–14], we shall see below that it leads to the breaking of charge conservation. For this reason, here we propose a different gauge-invariant replacement

$$d^\mu(p',p)\rightarrow\delta^\mu(p',p)=i\frac{\delta(p')\Gamma^\mu(p',p)\Lambda(p)-\Lambda(p')\Gamma^\mu(p',p)\delta(p)}{p^2-p'^2}, \tag{10}$$

which does lead to charge conservation. Equation (10) can also be written in the form

$$\delta^\mu(p',p)=2\pi i\Lambda(p')\Gamma^\mu(p',p)\Lambda(p)\times\frac{\delta^+(p'^2-m^2)-\delta^+(p^2-m^2)}{p^2-p'^2}, \tag{11}$$

showing that $\delta^\mu(p',p)$ is explicitly regular at $p^2-p'^2=0$. Using this replacement, together with that of Eq. (1), the bound-state current of Eq. (6) is reduced to the three-dimensional expression

$$\begin{aligned}\langle P'|J^\mu(0)|P\rangle &=\int \frac{d^4p}{(2\pi)^4}\overline{\Phi}_{P'}(p')d_1(P-p)\delta_2^\mu(p',p)\Phi_P(p)\\ &+\int \frac{d^4p}{(2\pi)^4}\overline{\Phi}_{P'}(P-p)d_1^\mu(p',p)\delta_2(P-p)\Phi_P(P-p)\\ &+\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\overline{\Phi}_{P'}(k)d_1(P'-k)\\ &\times\delta_2(k)K^\mu(P',k;P,p)d_1(P-p)\delta_2(p)\Phi_P(p).\end{aligned} \tag{12}$$

Just as Eq. (6) can be derived by gauging the BS equation for the two-body Green function [1], one can similarly show that Eq. (12) results from the gauging of the spectator equation for the two-body Green function, with $\delta^\mu(p',p)$ being the result of gauging $\delta(p)$. Thus Eq. (10) [or Eq. (11)] constitutes our answer to the question of how to gauge the on-mass-shell particle propagator.

III. PROPERTIES OF THE GAUGED ON-MASS-SHELL PROPAGATOR

A. Gauge invariance

In order to prove that the bound-state current of Eq. (12) satisfies current conservation, all we need to do is follow the corresponding proof for the bound-state current of Eq. (6) in the original four-dimensional BS approach of Ref. [1]. Indeed, to keep the correspondence with the four-dimensional BS approach, we will use Eq. (1) for the propagator $\delta(p)$ and Eq. (10) for the gauged propagator $\delta^\mu(p',p)$, but we will *not* get rid of the relative energy integration in Eq. (12) (with the help of the δ functions contained in δ and δ^μ). Thus our derivation will look identical to the one in the four-dimensional BS approach, except that particle 2 will have the propagator $\delta(p)$ instead of the usual one $d(p)$.

Following this strategy, there is no need to repeat the proof of current conservation here, except to note that a necessary ingredient in the proof of the BS case is the Ward-Takahashi identity for the propagator $d(p)$. Thus to prove current conservation for the three-dimensional expression of Eq. (12), it is sufficient to show that the on-mass-shell propagator $\delta(p)$ likewise satisfies the Ward-Takahashi identity

$$(p'_\mu-p_\mu)\delta^\mu(p',p)=ie[\delta(p)-\delta(p')]. \tag{13}$$

To prove Eq. (13), all that is required is a simple evaluation of $\delta^\mu(p',p)$ as given by Eq. (10). In the case of a spinor particle, $\Gamma^\mu=e\gamma^\mu$, and one part of Eq. (10) gives

$$\begin{aligned}(p'_\mu-p_\mu)\delta(p')\Gamma^\mu(p',p)\Lambda(p) &=2\pi ie\delta^+(p'^2-m^2)(\not p'+m)(\not p'-\not p)(\not p+m)\\ &=2\pi ie\delta^+(p'^2-m^2)[(p'^2-m^2)(\not p+m)\\ &\quad-(\not p'+m)(p^2-m^2)]\\ &=-ie\delta(p')(p^2-p'^2).\end{aligned} \tag{14}$$

Similarly the other part of Eq. (10) gives

$$(p'_\mu-p_\mu)\Lambda(p')\Gamma^\mu(p',p)\delta(p)=-ie\delta(p)(p^2-p'^2). \tag{15}$$

The Ward-Takahashi identity of Eq. (13) follows immediately. In the case of a scalar particle the algebra showing Eq. (13) is even simpler.

B. Charge conservation

Although current conservation, according to Noether's theorem, implies charge conservation, in the currently used terminology ‘‘charge conservation’’ means that the expression for the bound-state electromagnetic current of Eq. (6) should give the charge of the composite system at zero momentum transfer if one uses the relativistic normalization condition for the bound-state vertex function [15,16]. Here we shall show that this is the case also for the three-dimensional expression of Eq. (12) if one uses our choice for the gauged on-mass-shell propagator, Eq. (10).

It is convenient at this stage to introduce a symbolic notation for some of our equations. For example, we write the bound-state BS equation, Eq. (5), symbolically as

$$\Phi_P = KG_0\Phi_P, \tag{16}$$

where $G_0 = d_1 d_2$, and the corresponding equation for the bound-state current, Eq. (6), as

$$j^\mu(P',P) = \langle P'|J^\mu(0)|P\rangle = \bar{\Phi}_{P'}(G_0^\mu + G_0 K^\mu G_0)\Phi_P. \tag{17}$$

Here we have also used the fact that the gauged two-particle propagator is given by [1]

$$G_0^\mu = (d_1 d_2)^\mu = d_1^\mu d_2 + d_1 d_2^\mu. \tag{18}$$

The spectator version of the above three equations is obtained by making the replacement $G_0 \to \mathcal{G}_0$ (which implies that $d_2 \to \delta_2$). Below we shall occasionally use such symbolic notation without further explanation.

To prove charge conservation for the three-dimensional spectator approach, we use the philosophy outlined above; namely, we follow the proof of the four-dimensional BS case only replacing the Feynman propagator of particle 2 by our on-mass-shell version. The proof of the BS case relies on the fact that the Feynman propagator $d(p)$ satisfies the Ward identity

$$d^\mu(p,p) = -ie\frac{\partial d(p)}{\partial p_\mu}. \tag{19}$$

Similarly, because the interaction current K^μ is an input to the expression of Eq. (17), it too must be constructed to satisfy the Ward identity, which in the two-particle case reads

$$K^\mu(P,k;P,p) = -i\left[e_2\frac{\partial K(P,k,p)}{\partial k_\mu} + \frac{\partial K(P,k,p)}{\partial p_\mu}e_2 + (e_1+e_2)\frac{\partial K(P,k,p)}{\partial P_\mu}\right]. \tag{20}$$

Combining the last two equations with the relativistic normalization condition for the bound-state vertex function,

$$-i\bar{\Phi}_P\left(\frac{\partial G_0}{\partial P_\mu} + G_0\frac{\partial K}{\partial P_\mu}G_0\right)\Phi_P = 2P_\mu, \tag{21}$$

one then obtains the charge conservation condition

$$\langle P|J^\mu(0)|P\rangle = \bar{\Phi}_P(G_0^\mu + G_0K^\mu G_0)\Phi_P = 2QP_\mu, \tag{22}$$

where Q is the total charge of the two-body system.

To show charge conservation in the three-dimensional case, we see that it is sufficient to prove the Ward identity for our on-mass-shell propagator:

$$\delta^\mu(p,p) = -ie\frac{\partial\delta(p)}{\partial p_\mu}. \tag{23}$$

The rest of the proof is the same as above, but with $\mathcal{G}_0$ everywhere replacing G_0.

We shall prove Eq. (23) for the spinor particle case by again using a direct evaluation of our expression for $\delta^\mu(p,p)$:

$$\begin{aligned}
\delta^\mu(p,p) &= -2\pi i\Lambda(p)\gamma^\mu\Lambda(p)\frac{\partial\delta^+(p^2-m^2)}{\partial p^2}\\
&= -2\pi ie(\not p+m)\gamma^\mu(\not p+m)\frac{\partial\delta^+(p^2-m^2)}{\partial p^2} = -2\pi ie[2p^\mu(\not p+m)-\gamma^\mu(p^2-m^2)]\frac{\partial\delta^+(p^2-m^2)}{\partial p^2}\\
&= -2\pi ie\left[(\not p+m)\frac{\partial\delta^+(p^2-m^2)}{\partial p_\mu} - \gamma^\mu\frac{\partial(p^2-m^2)\delta^+(p^2-m^2)}{\partial p^2} + \gamma^\mu\frac{\partial(p^2-m^2)}{\partial p^2}\delta^+(p^2-m^2)\right]\\
&= -2\pi ie\left[(\not p+m)\frac{\partial\delta^+(p^2-m^2)}{\partial p_\mu} + \gamma^\mu\delta^+(p^2-m^2)\right] = -2\pi ie\frac{\partial(\not p+m)\delta^+(p^2-m^2)}{\partial p_\mu} = -ie\frac{\partial\delta(p)}{\partial p_\mu}.
\end{aligned}$$

It should be emphasised that we did not try to obtain the Ward identity of Eq. (23) from the Ward-Takahashi (WT) identity of Eq. (13) as there is an ambiguity in extracting the value of $\delta^\mu(p,p)$ in this way. Indeed a good example of this ambiguity is the GR expression for the gauged on-mass-shell propagator, Eq. (9), which satisfies the WT identity of Eq. (13) as well, but does not satisfy the Ward identity of Eq. (23) since it differs from Eq. (10) by the term

$$\begin{aligned}
&\delta^\mu(p',p) - \delta'^\mu(p',p)\\
&\quad = -\delta(p')\Gamma^\mu(p',p)\delta(p)\\
&\quad = -4\pi^2\Lambda(p')\Gamma^\mu(p',p)\Lambda(p)\\
&\qquad \times\delta^+(p'^2-m^2)\delta^+(p^2-m^2),
\end{aligned} \tag{24}$$

which does not vanish at zero momentum transfer. Equation

(24) is derived by paying careful attention to the $i\epsilon$ terms present in the one-particle propagators in $\delta'^{\mu}(p',p)$ and using the fact that, for $q^2<4m^2$,

$$\delta^-(p'^2-m^2)\delta^+(p^2-m^2)=\delta^+(p'^2-m^2)\delta^-(p^2-m^2)=0.$$

This means that the use of Eq. (9) does not lead to charge conservation (in contrast to what is claimed in Ref. [16]). The ambiguity of extracting δ^{μ} from the WT identity can be seen explicitly from the fact that

$$\begin{aligned} q_{\mu}[\delta^{\mu}(p',p)-\delta'^{\mu}(p',p)] &= -4\pi^2(p'_{\mu}-p_{\mu})\Lambda(p')\Gamma^{\mu}(p',p)\Lambda(p) \\ &\quad\times\delta^+(p'^2-m^2)\delta^+(p^2-m^2) \\ &= -4\pi^2(\not{p}'+m)(\not{p}'-\not{p})(\not{p}+m)\delta^+(p'^2-m^2) \\ &\quad\times\delta^+(p^2-m^2)=0, \end{aligned}$$

while

$$\delta^{\mu}(p,p)-\delta'^{\mu}(p,p)\neq 0.$$

C. Comparison of the two prescriptions

In the previous discussion of charge conservation we found a significant difference between our prescription for the gauged on-mass-shell propagator and the one of GR at the point $q=0$. Here we would like to compare the two prescriptions also for $q\neq 0$.

The first thing to note is that there is no difference between the two prescriptions for $q^2>0$ as well as for $q^2=0$ (but $q\neq 0$), since the product of the two δ functions in Eq. (24) will always be zero under these conditions. Thus our prescription will not change the results of Ref. [14] where pion photoproduction off a nucleon was calculated using the GR prescription. On the other hand, for $q^2<0$, which includes the case of electron scattering, the contribution of Eq. (24) is not zero. We would therefore like to investigate this difference between the two prescriptions when applied to the two-body bound-state current in the case where $q^2<0$. Writing the bound-state current symbolically as in Eq. (17), the difference in using the two prescriptions in Eq. (17) is clearly given by

$$\Delta j^{\mu}(P',P)=\overline{\Phi}_{P'}(\mathcal{G}'^{\mu}_0-\mathcal{G}^{\mu}_0)\Phi_P. \tag{25}$$

Using Eq. (24), numerically we have that

$$\begin{aligned} \Delta j^{\mu}(P',P) &= \int\frac{d^4p}{(2\pi)^4}\overline{\Phi}_{P'}(p')d_1(P-p)\delta_2(p')\Gamma^{\mu}_2(p',p)\delta_2(p)\Phi_P(p) \\ &= \int\frac{d^3p}{(2\pi)^2}\frac{1}{2\sqrt{\mathbf{p}^2+m^2}}\overline{\Phi}_{P'}(p')d_1(P-p)\Lambda_2(p')\delta^+(p'^2-m^2)\Gamma^{\mu}_2(p',p)\Lambda_2(p)\Phi_P(p) \\ &= \int\frac{d^3p}{(2\pi)^2}\frac{1}{2\sqrt{\mathbf{p}^2+m^2}}\overline{\Psi}_{P'}(\mathbf{p}')\delta^+(p'^2-m^2)\Gamma^{\mu}_2(p',p)d_1^{-1}(P-p)\Psi_P(\mathbf{p}), \end{aligned} \tag{26}$$

where we have introduced the wave function $\Psi_P(\mathbf{p})$ defined by

$$\Psi_P(\mathbf{p})=d_1(P-p)\Lambda_2(p)\Phi_P(p)|_{p^0=\sqrt{\mathbf{p}^2+m^2}}. \tag{27}$$

For the scalar particle case in the Breit reference frame where $q_0=0$ and $\mathbf{P}'=-\mathbf{P}=\mathbf{q}/2$, we have that

$$\begin{aligned} \Delta j^{\mu}(P',P) &= -i\int\frac{d^3p}{(2\pi)^2}\frac{1}{2\sqrt{\mathbf{p}^2+m^2}}\overline{\Psi}_{P'}(\mathbf{p}+\mathbf{q}) \\ &\quad\times\Gamma^{\mu}_2(p',p)\Psi_P(\mathbf{p})|\mathbf{q}|^{-1}\delta(2p_z+|\mathbf{q}|) \\ &\quad\times\left(M^2-\sqrt{\mathbf{q}^2+4M^2}\sqrt{\mathbf{p}^2+m^2}+\frac{\mathbf{q}^2}{2}\right), \end{aligned} \tag{28}$$

where we have chosen the z axis along $\mathbf{q}$ to write

$$\delta(2\mathbf{p}\cdot\mathbf{q}+\mathbf{q}^2)=|\mathbf{q}|^{-1}\delta(2p_z+|\mathbf{q}|), \tag{29}$$

which is valid for $\mathbf{q}\neq 0$. From Eq. (28) it is clearly seen that $\Delta j^{\mu}(P',P)$ diverges as $\mathbf{q}\to 0$.

To estimate the significance of $\Delta j^{\mu}(P',P)$ at values of $\mathbf{q}$ away from zero, we may compare Eq. (28) with the second term on the RHS of Eq. (12), which describes the contribution to the bound-state current of the particle 1 gauged (Feynman) propagator:

$$\int\frac{d^3p}{(2\pi)^2}\frac{1}{2\sqrt{\mathbf{p}^2+m^2}}\overline{\Psi}_{P'}(\mathbf{p})\Gamma^{\mu}_1(P'-p,P-p)\Psi_P(\mathbf{p}). \tag{30}$$

It can be seen that these two contributions are roughly of comparable size.

IV. DERIVATION

Having established the validity of our expression of Eq. (10) for the gauged on-mass-shell particle propagator, in this section we would like present two ‘‘derivations’’ of this expression that can give a better insight into the origin of this particular form.

A. Connection with the four-dimensional approach

Here we show that our gauged on-mass-shell particle propagator corresponds to the contribution of the positive energy propagator poles of the corresponding term in the four-dimensional BS expression for the bound-state current. The relevant term is the first term on the RHS of Eq. (6):

$$A=\int \frac{d^4p}{(2\pi)^4}\bar{\Phi}_{P'}(p')d_1(P-p)d_2(p')\times\Gamma_2^\mu(p',p)d_2(p)\Phi_P(p).$$

Ignoring all poles in the complex p_0 plane except those contained in the two d_2 propagators, we may close the p_0 integration contour in the bottom half plane to obtain that

$$A=-\int \frac{d^4p}{(2\pi)^4}\frac{\bar{\Phi}_{P'}(p')d_1(P-p)\Lambda_2(p')\Gamma_2^\mu(p',p)\Lambda_2(p)\Phi_P(p)}{[(p_0+q_0)^2-\omega'^2+i\epsilon](p_0^2-\omega^2+i\epsilon)}$$

$$\approx 2\pi i\int \frac{d^4p}{(2\pi)^4}\bar{\Phi}_{P'}(p')d_1(P-p)\Lambda_2(p')\left[\frac{\delta_2^+(p'^2-m^2)}{p^2-m^2\pm i\epsilon}+\frac{\delta_2^+(p^2-m^2)}{p'^2-m^2\mp i\epsilon}\right]\Gamma_2^\mu(p',p)\Lambda_2(p)\Phi_P(p)$$

$$=\int \frac{d^4p}{(2\pi)^4}\bar{\Phi}_{P'}(p')d_1(P-p)[\delta_2(p')\Gamma_2^\mu(p',p)d_2(p)+d_2^-(p')\Gamma_2^\mu(p',p)\delta_2(p)]\Phi_P(p), \tag{31}$$

$$=\int \frac{d^4p}{(2\pi)^4}\bar{\Phi}_{P'}(p')d_1(P-p)[\delta_2(p')\Gamma_2^\mu(p',p)d_2^-(p)+d_2(p')\Gamma_2^\mu(p',p)\delta_2(p)]\Phi_P(p), \tag{32}$$

where it is important to notice that

$$d^-(p)=\frac{i\Lambda(p)}{p^2-m^2-i\epsilon}=d(p)-2\pi\Lambda(p)\delta(p^2-m^2) \tag{33}$$

differs from the Feynman propagator $d(p)$ in the sign of the $i\epsilon$ term. We can use either of the forms Eq. (31) or Eq. (32) to extract the gauged on-mass-shell propagator since they both give the same result. We can choose, for example,

$$\delta^\mu(p',p)=\delta(p')\Gamma^\mu(p',p)d(p)+d^-(p')\Gamma^\mu(p',p)\delta(p) \tag{34}$$

$$=i\frac{\delta(p')\Gamma^\mu(p',p)\Lambda(p)-\Lambda(p')\Gamma^\mu(p',p)\delta(p)}{p^2-p'^2+i\epsilon}. \tag{35}$$

Noticing that the latter expression is regular at $p^2-p'^2=0$, it becomes clear that the $i\epsilon$ term may be dropped from the denominator, in this way giving our expression of Eq. (10).

Note that Eq. (34) is particularly useful for a comparison with the prescription of GR given by Eq. (9). The difference lies in the sign of the $i\epsilon$ term in $d^-(p')$. As shown above, this difference is crucial for charge conservation.

B. Derivation by minimal substitution

It is well known that gauging a momentum-dependent quantity by minimal substitution $p^\mu\to p^\mu+eA^\mu(x)$ guarantees not only gauge invariance but charge conservation as well. For this reason it would be interesting to see if we can derive our form for $\delta^\mu(p',p)$ by implementing the minimal substitution procedure in the on-mass-shell propagator $\delta(p)$. The way that this can be done is by expressing $\delta(p)$ in terms of the difference of Feynman propagators:

$$\Delta(p)\equiv d(p)-d^-(p)=\frac{i\Lambda(p)}{p^2-m^2+i\epsilon}-\frac{i\Lambda(p)}{p^2-m^2-i\epsilon}=2\pi\Lambda(p)\delta(p^2-m^2). \tag{36}$$

Thus $\delta(p)=\theta(p_0)\Delta(p)$. Now by implementing minimal substitution in Eq. (36) we will clearly obtain that

$$\begin{aligned}\Delta^\mu(p',p)&=d(p')\Gamma^\mu(p',p)d(p)-d^-(p')\Gamma^\mu(p',p)d^-(p)\\&=[d(p')-d^-(p')]\Gamma^\mu(p',p)d(p)\\&\quad+d^-(p')\Gamma^\mu(p',p)[d(p)-d^-(p)]\\&=\Delta(p')\Gamma^\mu(p',p)d(p)+d^-(p')\Gamma^\mu(p',p)\Delta(p).\end{aligned} \tag{37}$$

If we now drop the negative energy δ functions in the Δ's, we derive the expression for the gauged on-mass-shell propagator,

$$\delta^\mu(p',p)=\delta(p')\Gamma^\mu(p',p)d(p)+d^-(p')\Gamma^\mu(p',p)\delta(p), \tag{38}$$

which is the same result as Eq. (34).

V. APPLICATION TO DEUTERON PHOTODISINTEGRATION

With the gauged on-mass-shell propagator specified, we now have all that is needed to derive gauge-invariant three-dimensional expressions within the spectator approach for any system of hadrons interacting with an external electromagnetic field. Here we would like to demonstrate our gauging procedure by calculating the amplitude for deuteron photodisintegration.

As the hadronic system of interest here consists of two identical nucleons, some of the previous expressions given for the distinguishable particle case need to be slightly modified. In particular, the bound-state spectator equation for identical nucleons is given by ($\Phi \equiv \Phi_P$)

$$\Phi = \frac{1}{2} K \mathcal{G}_0 \Phi, \tag{39}$$

where the kernel K is the sum of all possible irreducible diagrams for identical particles and is therefore antisymmetric under the exchange of nucleon labels.

In the four-dimensional approach of Ref. [1], the $d \to NN$ transition current j_0^μ is given by

$$j_0^\mu = G_0^{-1} [G_0 \Phi]^\mu = \Phi^\mu + G_0^{-1} G_0^\mu \Phi, \tag{40}$$

where Φ^μ is the gauged vertex function to be discussed shortly. The last equality in Eq. (40) was obtained by using the rule for gauging products [1]. To turn this BS expression into a three-dimensional one using the spectator approach, all we need to do is replace the BS version of Φ by the one that satisfies the spectator equation, Eq. (39). But we do not replace G_0 by $\mathcal{G}_0$ in Eq. (40) as this would introduce an unphysical δ-function behavior into the photoproduction amplitude. To obtain an expression for Φ^μ we gauge Eq. (39),

$$\Phi^\mu = \frac{1}{2} (K^\mu \mathcal{G}_0 \Phi + K \mathcal{G}_0^\mu \Phi + K \mathcal{G}_0 \Phi^\mu), \tag{41}$$

which may be solved for Φ^μ, giving

$$\Phi^\mu = \frac{1}{2} \left(1 - \frac{1}{2} K \mathcal{G}_0 \right)^{-1} (K^\mu \mathcal{G}_0 \Phi + K \mathcal{G}_0^\mu \Phi). \tag{42}$$

To simplify this expression we use the equations for the two-nucleon t matrix T. In the spectator approximation they are given by

$$T = K + \frac{1}{2} K \mathcal{G}_0 T = K + \frac{1}{2} T \mathcal{G}_0 K, \tag{43}$$

from which the relation

$$\left(1 - \frac{1}{2} K \mathcal{G}_0 \right)^{-1} = 1 + \frac{1}{2} T \mathcal{G}_0 \tag{44}$$

follows. Using this in Eq. (42) we obtain that

$$\Phi^\mu = \frac{1}{2} \left(1 + \frac{1}{2} T \mathcal{G}_0 \right) K^\mu \mathcal{G}_0 \Phi + \frac{1}{2} T \mathcal{G}_0^\mu \Phi, \tag{45}$$

where

$$\mathcal{G}_0^\mu = (d_1 \delta_2)^\mu = d_1^\mu \delta_2 + d_1 \delta_2^\mu. \tag{46}$$

The $d \to NN$ transition current is therefore given by

$$j_0^\mu = \left[\Gamma_1^\mu d_1 + \Gamma_2^\mu d_2 + \frac{1}{2} \left(1 + \frac{1}{2} T d_1 \delta_2 \right) K^\mu d_1 \delta_2 + \frac{1}{2} T (d_1 \delta_2^\mu + d_1^\mu \delta_2) \right] \Phi, \tag{47}$$

where we have used Eq. (7). This expression can be used to calculate the deuteron photodisintegration amplitude by contracting Eq. (47) with the photon polarization vector ε_μ.

An interesting aspect of Eq. (47) is the appearance of the Feynman propagator d_2 in the second term on the RHS, while in all other parts of the equation [including the equation for T, Eq. (43)] the on-mass-shell propagator δ_2 is used. This is, of course, a consequence of us having used G_0 instead of $\mathcal{G}_0$ in Eq. (40). Using d_2 here is reasonable since it is not inconsistent with the spectator approach, and it avoids the unphysical behavior of amplitudes that would result if δ_2 were used instead. On the other hand, it is not entirely clear if this singular use of d_2 will affect the gauge invariance of the electromagnetic transition current j_0^μ. We shall therefore show explicitly that the expression for j_0^μ given by Eq. (40) does indeed satisfy gauge invariance despite the use of G_0 in this equation.

Equation (40) is a symbolic equation whose numerical form simplifies down to

$$j_0^\mu(k_1, k_2; P) = \Phi_P^\mu(k_1, k_2) + d^{-1}(k_1) d^\mu(k_1, k_1 - q) \Phi_P(k_1 - q, k_2) + d^{-1}(k_2) d^\mu(k_2, k_2 - q) \Phi_P(k_1, k_2 - q), \tag{48}$$

where we show the momenta of both particles explicitly, and it is understood that $k_1 + k_2 = P + q$ where q is the momentum of the incoming photon. By construction, the input quantities K^μ, d^μ, and δ^μ satisfy the WT identities

$$-i q_\mu K^\mu(p_1' p_2'; p_1 p_2) = e_1 K(p_1' - q, p_2'; p_1 p_2) - K(p_1' p_2'; p_1 + q, p_2) e_1 + e_2 K(p_1', p_2' - q; p_1 p_2) - K(p_1' p_2'; p_1, p_2 + q) e_2, \tag{49}$$

$$-i q_\mu d^\mu(p', p) = e d(p) - d(p') e, \tag{50}$$

$$-i q_\mu \delta^\mu(p', p) = e \delta(p) - \delta(p') e, \tag{51}$$

respectively. In Eq. (49) we again use a notation where the momentum of each particle is shown explicitly and where $p_1' + p_2' = p_1 + p_2 + q$. In Eqs. (50) and (51) we similarly have that $p' = p + q$. Using these relations it is easy to show that the WT identity for Φ^μ is given by

$$-i q_\mu \Phi_P^\mu(k_1, k_2) = e_1 \Phi_P(k_1 - q, k_2) + e_2 \Phi_P(k_1, k_2 - q), \tag{52}$$

where $k_1+k_2=P+q$. Then using the WT identities for d^μ and Φ^μ in calculating the divergence of Eq. (48), we obtain that

$$q_\mu j_0^\mu(k_1,k_2;P)=ie_1 d^{-1}(k_1)d(k_1-q)\Phi_P(k_1-q,k_2)+ie_2 d^{-1}(k_2)d(k_2-q)\Phi_P(k_1,k_2-q), \tag{53}$$

which is zero for on-mass-shell nucleons ($k_1^2=k_2^2=m^2$).

VI. SUMMARY

In this work we have shown how to construct three-dimensional integral equations that describe a system of hadrons and their interaction with an external electromagnetic field. The equations are relativistic (covariant), unitary, gauge invariant, and conserve charge. Our method is based upon a recent work where we show how four-dimensional integral equations of quantum field theory can be gauged so that an external photon is coupled to all possible places in the underlying strong interaction perturbation graphs, without the need to do a perturbation expansion [1].

The starting point of our construction is a set of four-dimensional integral equations of relativistic quantum field theory describing the system of hadrons in questions. For example, for the two-nucleon system below pion production threshold the starting point would be the Bethe-Salpeter equation, while above pion production threshold the equations of Ref. [17] would be appropriate. We do not gauge these equations at this stage, but instead convert them to the spectator equations of Gross [5] by the introduction of the ''on-mass-shell propagator'' δ. The modified four-dimensional equations are then gauged just in the same way as was done for the four-dimensional equation of field theory.

The three-dimensional reduction then rests on the construction of a gauged on-mass-shell propagator δ^μ. A δ^μ that satisfies both the Ward-Takahashi and Ward identities is necessary for the gauge invariance and charge conservation of the final equations. We have shown how such a gauged on-mass-shell propagator can be constructed and compared our results with what was proposed in the literature [11]. With δ^μ specified, we then demonstrated our gauging procedure by constructing the amplitude for deuteron photodisintegration within the spectator approach.

Our gauging procedure can be easily applied to more complicated systems. For example, in Ref. [9] we have used it to derive gauge-invariant three-dimensional expressions for the gauged three-nucleon system. It also does not depend on the nature of the external gauge field. Thus it can equally well be used to describe the weak interactions of hadronic systems.

ACKNOWLEDGMENTS

The authors would like to thank the Australian Research Council for their financial support.

APPENDIX

In the above discussion our particles were assumed to be structureless. Here we show one way to include electromagnetic form factors that preserves gauge invariance and charge conservation. Our approach is close in spirit to the one used by Gross and Riska [11].

As before, the three-dimensional reduction is effected by the replacement

$$d(p)=\frac{i\Lambda(p)}{p^2-m^2+i\epsilon}\to\delta(p)=2\pi\Lambda(p)\delta^+(p^2-m^2), \tag{A1}$$

but where now (for spinor particles) $\Lambda(p)\neq \not p+m$ because of dressing included in $d(p)$. Nevertheless, the on-mass-shell particle propagator $\delta(p)$ is not affected by dressing except for an overall renormalization constant Z:

$$\delta(p)=2\pi\Lambda(p)\delta^+(p^2-m^2)=2\pi Z(\not p+m)\delta^+(p^2-m^2). \tag{A2}$$

The latter result follows from the spectral decomposition of the dressed Feynman propagator

$$d(p)=\frac{Z(\not p+m)}{p^2-m^2+i\epsilon}+R(p), \tag{A3}$$

where $R(p)$ is a function that is regular at $p^2=m^2$. Although the on-mass-shell particle propagator of Eq. (A2) cannot have dressing in the usual sense, one can nevertheless introduce an electromagnetic form factor into the gauged on-mass-shell propagator through the definition

$$\delta^\mu(p',p)=i\,\frac{\delta(p')\Gamma^\mu(p',p)\Lambda(p)-\Lambda(p')\Gamma^\mu(p',p)\delta(p)}{p^2-p'^2} \tag{A4}$$

or in the explicitly regular form

$$\delta^\mu(p',p)=2\pi i\Lambda(p')\Gamma^\mu(p',p)\Lambda(p)\times\frac{\delta^+(p'^2-m^2)-\delta^+(p^2-m^2)}{p^2-p'^2}, \tag{A5}$$

where now the electromagnetic vertex function $\Gamma^\mu(p',p)\neq e\gamma^\mu$, but does satisfy the Ward-Takahashi identity

$$(p'-p)_\mu\Gamma^\mu(p',p)=ie[d^{-1}(p')-d^{-1}(p)]. \tag{A6}$$

On the mass shell Γ^μ takes on the usual form

$$\Gamma^\mu(p',p)|_{p'^2=p^2=m^2}=e\left[F_1(q^2)\gamma^\mu+i\frac{\sigma^{\mu\nu}}{2m}q_\nu F_2(q^2)\right]. \tag{A7}$$

Thus we have at the same time a structureless on-mass-shell propagator, Eq. (A2), together with a gauged on-mass-shell propagator that does have structure, Eq. (A5). We will now show that together they nevertheless satisfy the WT identity. This means that the structure described by Eq. (A5) does not contribute to the WT identity.

1. Gauge invariance

As in the structureless case, to prove gauge invariance of the theory it is sufficient to show that the on-mass-shell propagator δ satisfies the WT identity. We show this by explicitly evaluating δ^μ; however, as an explicit form for Γ^μ is no longer available, we make use of Eq. (A6) instead:

$$(p'-p)_\mu \delta^\mu(p',p) = -2\pi e \Lambda(p')[d^{-1}(p') - d^{-1}(p)]\Lambda(p) \frac{\delta^+(p'^2-m^2)-\delta^+(p^2-m^2)}{p^2-p'^2}$$
$$= 2\pi e \Lambda(p')\left[d^{-1}(p')\Lambda(p)\frac{\delta^+(p^2-m^2)}{m^2-p'^2} + d^{-1}(p)\Lambda(p)\frac{\delta^+(p'^2-m^2)}{p^2-m^2}\right]$$
$$= 2\pi i e[\Lambda(p)\delta^+(p^2-m^2) - \Lambda(p')\delta^+(p'^2-m^2)] = ie[\delta(p)-\delta(p')], \tag{A8}$$

where we used that

$$d^{-1}(p')\delta^+(p'^2-m^2) = d^{-1}(p)\delta^+(p^2-m^2) = 0. \tag{A9}$$

2. Charge conservation

To prove charge conservation, it is again sufficient to show that the on-mass-sell propagator δ satisfies the Ward identity. As in the structureless case, the Ward identity cannot be deduced unambiguously from the WT identity of Eq. (A8) and, therefore, must be shown explicitly. This we do by using Eq. (A5) in the limit of zero momentum transfer:

$$\delta^\mu(p,p) = -2\pi i \Lambda(p)\Gamma^\mu(p,p)\Lambda(p)\frac{\partial \delta^+(p^2-m^2)}{\partial p^2}. \tag{A10}$$

To evaluate $\Gamma^\mu(p,p)$ we use its Ward identity,

$$\Gamma^\mu(p,p) = ie\frac{\partial d^{-1}(p)}{\partial p_\mu}. \tag{A11}$$

Then

$$\frac{1}{2\pi e}\delta^\mu(p,p) = \Lambda(p)\frac{\partial d^{-1}(p)}{\partial p_\mu}\Lambda(p)\frac{\partial \delta^+(p^2-m^2)}{\partial p^2}$$
$$= \frac{\partial}{\partial p^2}\left[\Lambda(p)\frac{\partial d^{-1}(p)}{\partial p_\mu}\Lambda(p)\delta^+(p^2-m^2)\right]$$
$$-\frac{\partial}{\partial p^2}\left[\Lambda(p)\frac{\partial d^{-1}(p)}{\partial p_\mu}\Lambda(p)\right]\delta^+(p^2-m^2). \tag{A12}$$

Now

$$\Lambda(p)\frac{\partial d^{-1}(p)}{\partial p_\mu}\Lambda(p)$$
$$= \Lambda(p)\left[\frac{\partial d^{-1}(p)\Lambda(p)}{\partial p_\mu} - d^{-1}(p)\frac{\partial \Lambda(p)}{\partial p_\mu}\right]$$
$$= -i\left[\Lambda(p)\frac{\partial (p^2-m^2)}{\partial p_\mu} - (p^2-m^2)\frac{\partial \Lambda(p)}{\partial p_\mu}\right]$$
$$= -i\left[2p^\mu\Lambda(p) - (p^2-m^2)\frac{\partial \Lambda(p)}{\partial p_\mu}\right], \tag{A13}$$

so that

$$\Lambda(p)\frac{\partial d^{-1}(p)}{\partial p_\mu}\Lambda(p)\delta^+(p^2-m^2)$$
$$= -2ip^\mu\Lambda(p)\delta^+(p^2-m^2). \tag{A14}$$

This last relation is similar to the bound-state normalization condition and can also be deduced by taking the residues at $p^2=m^2$ in the identity

$$d(p)d^{-1}(p)d(p) = d(p).$$

Substituting the last two results into Eq. (A12), we obtain

$$\frac{i}{2\pi e}\delta^\mu(p,p) = \frac{\partial}{\partial p^2}[2p^\mu\Lambda(p)\delta^+(p^2-m^2)] - \frac{\partial}{\partial p^2}$$
$$\times\left[2p^\mu\Lambda(p) - (p^2-m^2)\frac{\partial \Lambda(p)}{\partial p_\mu}\right]$$
$$\times\delta^+(p^2-m^2)$$
$$= 2p^\mu\frac{\partial}{\partial p^2}[\Lambda(p)\delta^+(p^2-m^2)], \tag{A15}$$

where we used that

$$(p^2-m^2)\frac{\partial}{\partial p^2}\left[\frac{\partial \Lambda(p)}{\partial p_\mu}\right]\delta^+(p^2-m^2) = 0,$$

with all other terms canceling. We thus find that

$$\delta^\mu(p,p) = -2ip^\mu e\frac{\partial \delta(p)}{\partial p^2} = -ie\frac{\partial \delta(p)}{\partial p_\mu}, \tag{A16}$$

as required.

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