Comparison of the Harris and *ab* expressions for the description of nuclear superdeformed rotational bands

Z. X. Hu and J. Y. Zeng
Department of Physics, Peking University, Beijing 100871, China
(Received 30 May 1997)

The nuclear superdeformed rotational bands in the $A\sim190$ region were systematically analyzed by using the Harris two-parameter formula and the *ab* expression. Similar to the situation in normally deformed nuclei, there exists obvious and systematic deviation of the Harris formula from the experiment. In contrast, the prediction of the ab formula is very close to the experiment, and may be conveniently used for the description of both normally deformed and superdeformed bands.
[S0556-2813(97)01711-1]

PACS number(s): 21.10.Re, 21.60.Ev, 27.70.+q, 27.80.+w

Since the discovery of the nuclear superdeformed (SD) band ^{152}Dy [1], a large number of SD bands have been observed in the $A\sim190$, 150, 130, and 80 mass regions. For the description of normally deformed (ND) bands, some useful expressions were presented. Based on the symmetry consideration, Bohr and Mottelson [2] pointed out that, under the adiabatic approximation, the rotational energy of an axially symmetric nucleus may be expanded as (for $K=0$ band)

$$E(\xi)=A\xi^2+B\xi^4+C\xi^6+D\xi^8+\cdots,\quad [\xi^2=I(I+1)]. \tag{1}$$

The expression for the $K\neq0$ band takes a form similar to Eq. (1), but includes a bandhead energy and ξ^2 is replaced by $I(I+1)-K^2$. It was well established that extensive ND bands can be described rather well by Eq. (1). Systematic analyses of a large number of ND bands in the rare-earth and actinide nuclei showed [2,3] that $|B/A|\sim10^{-3}$, $|C/A|\sim10^{-6}$, $|D/A|\sim10^{-9}$, etc.; i.e., the convergence of the $I(I+1)$ expansion is satisfactory. For SD bands, the convergence is even better [4], ($|B/A|\sim10^{-4}$, $|C/A|\sim10^{-8}$).

Another useful expression for nuclear rotational spectra is the Harris ω^2 expression [5] [$\omega=(1/\hbar)(dE/d\xi)$]

$$E(\omega)=\alpha\omega^2+\beta\omega^4+\gamma\omega^6+\delta\omega^8+\cdots, \tag{2}$$

whose convergence is believed [2] to be superior to the $I(I+1)$ expansion (1), and particularly, the Harris two-parameter expansion

$$E(\omega^2)=a\omega^2+\beta\omega^4 \tag{3}$$

was shown [6] to be equivalent to the variable moment of inertia mode [7], and was widely used in the high-spin nuclear physics. Bohr and Mottelson pointed out [2] that, if the Harris two-parameter expression (3) holds, then only two coefficients in the expansion (1) are independent, and the following relations may be derived:

$$C/4(B^2/A)=1,\quad D/24(B^3/A^2)=1. \tag{4}$$

In [8], the ab formula,

$$E(I)=a[\sqrt{1+bI(I+1)}-1], \tag{5}$$

was derived from the Bohr Hamiltonian for a well-deformed nucleus with small axial asymmetry ($\sin^4 3\gamma\ll1$). This expression had been suggested empirically by Holmberg and Lipas [9]. It was found that an extensive amount of ND bands can be described very well by this simple expression. Particularly, if Eq. (5) holds, also only two coefficients in the $I(I+1)$ expansion (1) are independent, but instead of Eq. (4), one has

$$C/4(B^2/A)=1/2,\quad D/24(B^3/A^2)=5/24. \tag{6}$$

Systematic analyses of experimental ND bands in the rare-earth and actinide nuclei showed [3] that the coefficients determined by the least-squares fitting of Eq. (1) with observed ND bands definitely deviate from relation Eq. (4), but are close to Eq. (6).

In this paper the SD bands in the $A\sim190$ region were analyzed. Though the spins of most SD bands have not been determined experimentally, several approaches to assign the spins of SD bands were presented [4,10–13]. It is noted that for SD bands in the $A\sim190$ region, the spins assigned by these approaches are consistent with each other (except for a very few cases). It is encouraging to note that recently the spins of the SD bands ^{194}Hg(1) [14] and ^{194}Pb(1) [15] have been determined by experiment, which are in agreement with these spin assignments. Therefore, we believe that these spin assignments of the SD bands in the $A\sim190$ region are reliable. Based on these spin assignments, the SD bands in the $A\sim190$ region were analyzed by using the $I(I+1)$ expansion (1). Because the coefficient D is extremely small for SD bands and rather difficult to be determined accurately, we use the three-parameter (ABC) expansion to fit the experimental γ transition energies $E_\gamma(I\rightarrow I-2)$, and then extract the ratio

$$R=AC/4B^2. \tag{7}$$

The results are plotted in Fig. 1. It is obviously seen that, from the statistical point of view, the extracted R ratios are close to the prediction of the ab formula ($R=1/2$), but systematically deviate from that expected for the Harris $\alpha\beta$ expression ($R=1$). It should be noted that the values of C extracted by the least-squares fitting are very small for SD

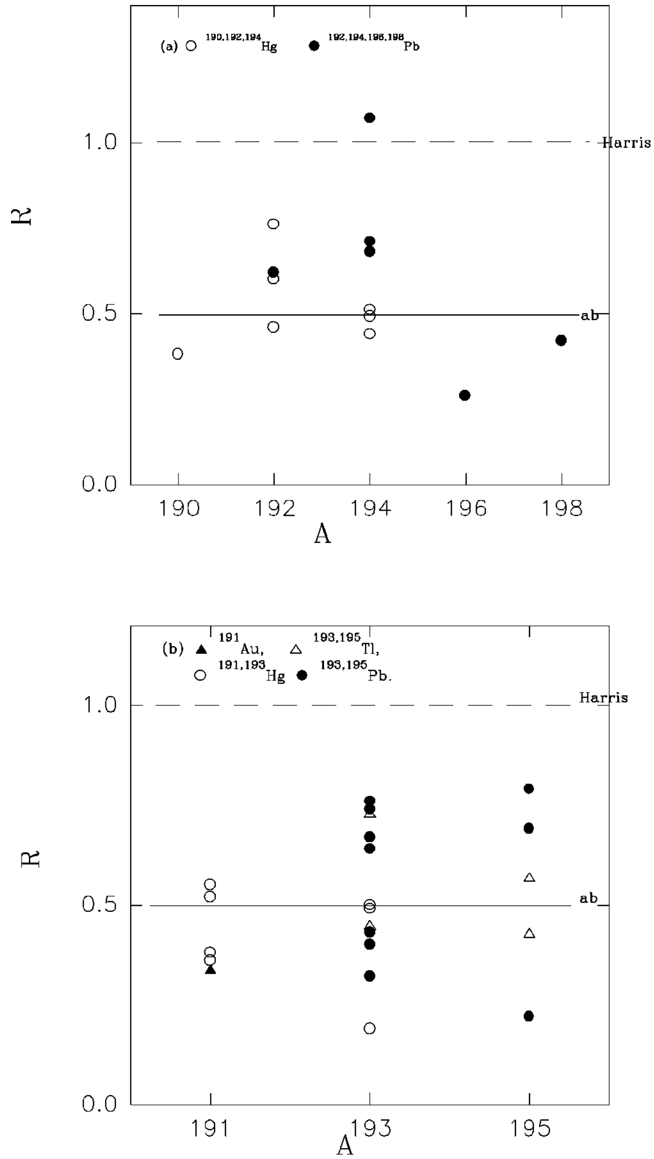

FIG. 1. The R ratio for the SD bands in the $A\sim 190$ region, $R=AC/4B^2$, A, B, and C are the coefficients of the $I(I+1)$ expansion (1). For the Harris two-parameter formula (3), $R=1$. For the ab formula (5), $R=1/2$.

bands and depend on the accuracy of the experimental E_γ's. Hence, the fluctuation of the extracted R's around 1/2 seems understandable.

Another approach to test the validity of the Harris expression Eq. (3) for the description of rotational bands was suggested by Peker *et al.* [16], who used each pair of γ-transition energies in a ND band, $E_\gamma(I+2\rightarrow I)$ and $E_\gamma(I\rightarrow I-2)$ $(I=2,4,6,\ldots)$, to extract the values of α and β in Eq. (3). If Eq. (3) is able to describe rotational bands well, the α's and β's thus extracted should be independent of the angular momentum (or angular frequency). However, the extracted α's and β's are by no means constant; in particular, the extracted moment of inertia parameter $J_0=2\alpha$ [see Eq. (8)] changes significantly with increasing ω, which implies that Eq. (3) is not able to describe the high-spin states of ND bands well. In this paper, following the approach of Peker, we use the Harris formula (3) and the ab expression (5) to fit each pair of γ-transition energies, $E_\gamma(I+2\rightarrow I)$ and $E_\gamma(I\rightarrow I-2)$, $(I=I_0+2, I=I_0+4,\ldots,$ I_0 is the spin of the lowest level observed in a SD band), and extract the values of α,β and a,b to investigate their ω (or I) dependence. The kinematic and dynamic moments of inertia for the Harris expression (3) are

$$J^{(1)}/\hbar^2=2\alpha+\frac{4}{3}\beta\omega^2, \tag{8}$$

$$J^{(2)}/\hbar^2=2\alpha+4\beta\omega^2. \tag{9}$$

$J_0/\hbar^2=2\alpha$ corresponds to the "bandhead" moment of inertia. If Eq. (3) holds well, 2α should be ω independent. Similarly, the kinematic and dynamic moments of inertia associated with the ab formula (5) are

$$J^{(1)}=\frac{\hbar^2}{ab}[1+bI(I+1)]^{1/2}=\frac{\hbar^2}{ab}\left[1-\frac{\hbar^2\omega^2}{a^2b}\right]^{-1/2}, \tag{10}$$

$$J^{(2)}=\frac{\hbar^2}{ab}[1+bI(I+1)]^{3/2}=\frac{\hbar^2}{ab}\left[1-\frac{\hbar^2\omega^2}{a^2b}\right]^{-3/2}. \tag{11}$$

Thus, the parameter corresponding to $J_0/\hbar^2=2\alpha$ in the Harris expression is $J_0/\hbar^2=1/ab$ in the ab formula. The analysis of some typical SD bands is plotted in Fig. 2. It is seen that, as in the situation in the ND bands [16], while 2α fluctuates significantly with ω, the corresponding moment of inertia parameter $1/ab$ almost remains independent of angular momentum.

TABLE I. Comparison between the calculated and experimental $E2$ γ transition energies of the yrast SD bands ^{194}Hg(1) and ^{194}Pb(1).

	^{194}Hg(1) $E_\gamma(I+2\rightarrow I)$, keV		^{194}Pb(1) $E_\gamma(I+2\rightarrow I)$, keV	
I	Expt. [14]	Calc[a]	Expt. [15]	Calc[b]
6			169.6	169.5
8			213.1	213.3
10	254.3	254.3	256.4	256.4
12	296.2	296.4	298.8	298.6
14	337.7	337.6	339.7	339.8
16	377.8	377.8	380.0	380.1
18	417.1	416.9	419.1	419.4
20	455.2	455.1	458.4	457.8
22	492.3	492.2	495.6	495.4
24	528.3	528.2	531.9	532.1
26	563.6	563.3	567.9	568.0
28	597.3	597.4	603.3	603.3
30	630.5	630.6		
32	662.4	662.9		
34	693.8	694.3		
36	725.4	725.0		
38	754.6	754.9		
40	783.9	784.1		
42	812.9	812.7		
44	841.0	840.7		

[a] $a=1.086\times 10^4$ keV, $b=0.5796\times 10^{-3}$, $c=2.4983$ keV.
[b] $a=0.4252\times 10^4$ keV, $b=1.027\times 10^{-3}$, $c=3.5261$ keV.

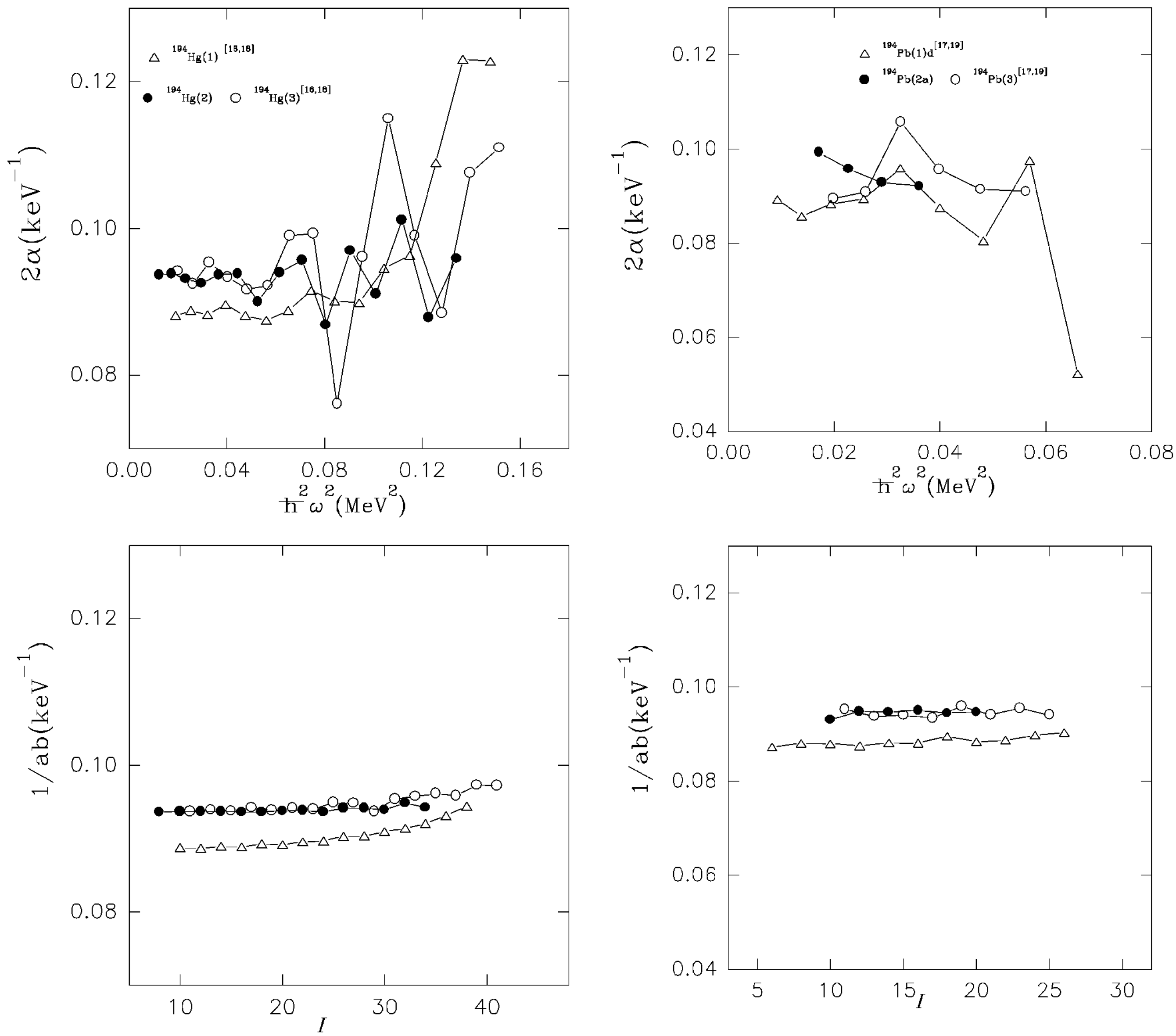

FIG. 2. The ω (or I) dependence of the moment of inertia of SD bands in the $A\sim190$ region. For high-spin states [16], $\omega\approx(1/\hbar)\times(\Delta E/\Delta I)\approx(1/2\hbar)\,E_\gamma(I+2\rightarrow I)$. If the Harris formula holds, $J_0=2\alpha$ should be ω independent. If the ab formula holds, $J_0=\hbar^2/ab$ should be I independent see Refs. [17–20].

From the systematic analysis of the observed SD bands in the $A\sim190$ region by using the Harris $\alpha\beta$ expression (3) and the ab formula (5), it was drawn that, while there exists obvious deviation of the Harris expression (3) from the experimental results, the prediction of the ab expression (5) is very close to the experiment. Because in Eq. (5) the rotational energy is expressed in terms of the angular momentum, rather than the angular frequency, the ab expression (5) can be very conveniently used for the description of observed nuclear rotational spectra including high-spin states. Moreover, in the cranked shell model calculation, one may use the equivalent expressions

$$E(\omega)=a\left[\left(1-\frac{\hbar^2\omega^2}{a^2b}\right)^{-1/2}-1\right]\quad\text{(lab. frame)},\qquad(12)$$

$$E'(\omega)=a\left[\left(1-\frac{\hbar^2\omega^2}{a^2b}\right)^{1/2}-1\right]\quad\text{(rotating frame)}.\qquad(13)$$

The main reason why the ab formula works better than the others is that in the derivation of the ab formula [8] *no adiabatic approximation was made* for treating the eigenvalue problem of the Bohr Hamiltonian; i.e., the vibrational and rotational degrees of freedom were simultaneously treated and the influence of the vibration on the rotational motion was considered to a certain extent. Hence, the nuclear deformation and moment of inertia in a given rotational band no longer remain unchanged. In fact, the observed angular momentum dependence of the kinematic and dynamic moments of inertia of SD bands in the $A\sim190$ region can be reproduced rather well [10] by $J^{(1)}=J_0[1+bI(I+1)]^{1/2}$ and

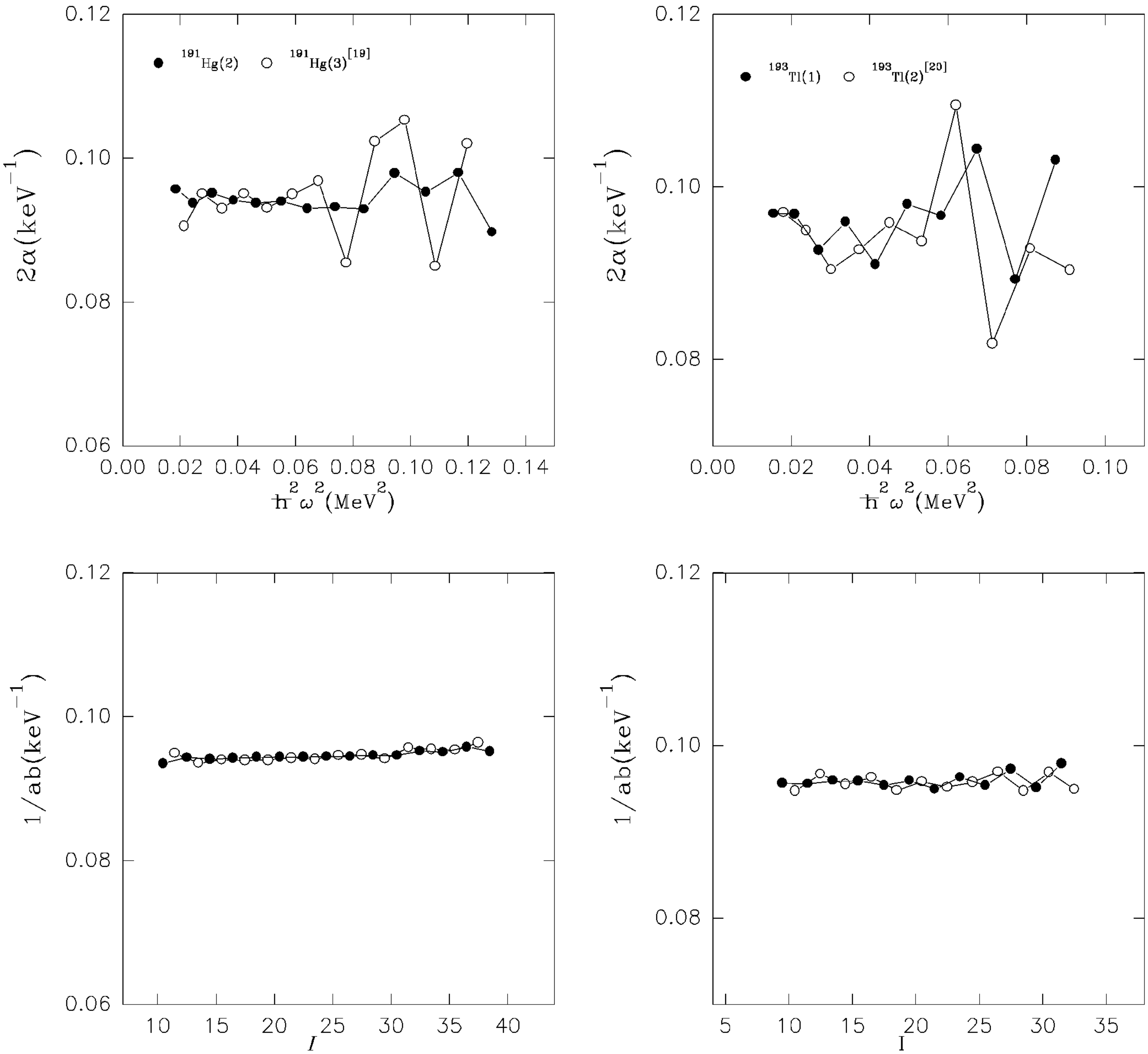

FIG. 2 (*Continued*).

$J^{(2)}=J_0[1+bI(I+1)]^{3/2}$, where $J_0/\hbar^2=1/ab$ is the ''band-head'' moment of inertia and b characterizes the nuclear softness ($b\sim10^{-4}$ for most SD bands in the $A\sim190$ region) and $b\rightarrow0$ (but ab keeping finite) corresponds to the rigid rotor limit. It should be noted that according to the ab formula (5), we have $[\omega=(1/\hbar)(dE/d\xi)]$, $\hbar\omega=ab\xi/\sqrt{1+b\xi^2}$, $\hbar^2\omega^2=a^2b^2\xi^2/(1+b\xi^2)$, so

$$\frac{\hbar^2\omega^2}{a^2b}=\frac{b\xi^2}{1+b\xi^2}<1, \qquad (14)$$

which implies that Eqs. (10)–(13) always can be expanded in powers of ω^2. The ω^2 expansion of Eqs. (10) and (11) are

$$J^{(1)}=J_0+J_1\omega^2+J_2\omega^4+\cdots, \qquad (15)$$

$$J^{(2)}=J_0+3J_1\omega^2+5J_2\omega^4+\cdots, \qquad (16)$$

$$J_0=\hbar^2/ab, \quad J_1=\frac{\hbar^4}{3a^3b^2}=\frac{J_0^3}{3\hbar^2}b,$$

$$J_2=\frac{3\hbar^6}{8a^5b^3}=\frac{3J_0^5}{8\hbar^4}b^2,\dots . \qquad (17)$$

On the other hand, the $I(I+1)$ expansions of Eqs. (5), (10), and (11) converge only when $bI(I+1)<1$, or $I<I_c\sim1/\sqrt{b}$; i.e., there exists a critical angular momentum I_c. From this one may understand why the ω^2 expansion of the rotational energy converges better than the $I(I+1)$ expansion [2].

Careful observation shows that in Fig. 2 there still exists a small deviation of $1/ab$ from a constant near the top of each SD band, which implies that the ab formula may be improved further. In fact, if the higher-order (anharmonic) term $k\beta^4$ (β being the quadrupole deformation) of the potential energy in the Bohr Hamiltonian is taken into account, to the first order of perturbation, the ab formula may be replaced by the abc formula [8]

$$E=a[\sqrt{1+bI(I+1)}-1]+cI(I+1). \tag{18}$$

The last term in Eq. (18) is a small correction ($|c|\ll a$) and c may be positive or negative according to the sign of k. As an illustrative example, the comparison of the experimental γ transition energies of the SD bands ^{194}Hg(1) [14] and ^{194}Pb(1) [15] and the calculated ones using Eq. (18) is given in Table I. The root mean square deviation of the calculated γ transition energies from the experimental results is $\chi=0.491\times10^{-3}$ for ^{194}Hg(1) and $\chi=0.661\times10^{-3}$ for ^{194}Pb(1), which seem rather satisfactory. Therefore, the ab or abc formula may be satisfactorily and conveniently used to describe the experimental SD bands. The kinematic and dynamic moments of inertia corresponding to the abc formula (18) are

$$\hbar^2/J^{(1)}=ab[1+bI(I+1)]^{-1/2}+2c, \tag{19}$$

$$\hbar^2/J^{(2)}=ab[1+bI(I+1)]^{-3/2}+2c. \tag{20}$$

This work was supported by the Science Foundation of China and the Post-Doctoral Foundation of China.

[1] P. J. Twin *et al.*, Phys. Rev. Lett. **57**, 811 (1986).

[2] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, Reading, MA, 1975), Vol. II, Chap. 4.

[3] F. X. Xu, C. S. Wu, and J. Y. Zeng, Phys. Rev. C **39**, 1617 (1989).

[4] J. A. Becker *et al.*, Phys. Rev. C **46**, 889 (1992); J. E. Drapper *et al.*, *ibid.* **42**, R179 (1991).

[5] S. M. Harris, Phys. Rev. Lett. **13**, 663 (1964); Phys. Rev. **138**, 509 (1965).

[6] A. Klein, R. M. Dreizler, and T. K. Das, Phys. Lett. **31B**, 333 (1970).

[7] G. Scharff-Goldhaber, C. Dove, and A. L. Goodman, Annu. Rev. Nucl. Sci. **26**, 239 (1976).

[8] C. S. Wu and J. Y. Zeng, Commun. Theor. Phys. **8**, 51 (1987); H. X. Huang, C. S. Wu, and J. Y. Zeng, Phys. Rev. C **39**, 1617 (1989).

[9] P. Holmberg and P. O. Lipas, Nucl. Phys. **A117**, 552 (1968).

[10] J. Y. Zeng *et al.*, Phys. Rev. C **44**, R745 (1991); C. S. Wu *et al.*, *ibid.* **45**, 261 (1992).

[11] R. Piepenbring and K. V. Protasov, Z. Phys. A **345**, 7 (1993).

[12] F. Xu and J. Hu, Phys. Rev. C **49**, 1449 (1994).

[13] J. Y. Zeng, Y. A. Lei, W. Q. Wu, and E. G. Zhao, Commun. Theor. Phys. **24**, 125 (1995).

[14] T. L. Khoo *et al.*, Phys. Rev. Lett. **76**, 1583 (1996); C. W. Beausang *et al.*, Z. Phys. A **335**, 325 (1990).

[15] M. J. Brinkman *et al.*, Phys. Rev. C **53**, 1461 (1996); M. J. Brinkman *et al.*, Z. Phys. A **336**, 115 (1990).

[16] L. K. Peker, S. Pearlstein, and J. H. Hamilton, Phys. Lett. **100B**, 281 (1981).

[17] B. Cederwall *et al.*, Phys. Rev. Lett. **72**, 3180 (1994).

[18] J. R. Hughes *et al.*, Phys. Rev. C **50**, R1265 (1994).

[19] M. P. Carpenter *et al.*, Phys. Rev. C **51**, 2400 (1995).

[20] S. Bouneau *et al.*, Phys. Rev. C **53**, R9 (1996).