Covariant equations for the three-body bound state

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The covariant spectator (or Gross) equations for the bound state of three identical spin-1/2 particles, in which two of the three interacting particles are always on shell, are developed and reduced to a form suitable for numerical solution. The equations are first written in operator form and compared to the Bethe-Salpeter equation, then expanded into plane wave momentum states, and finally expanded into partial waves using the three-body helicity formalism first introduced by Wick. In order to solve the equations, the two-body scattering amplitudes must be boosted from the overall three-body rest frame to their individual two-body rest frames, and all effects which arise from these boosts, including the Wigner rotations and ρ-spin decomposition of the off-shell particle, are treated *exactly*. In their final form, the equations reduce to a coupled set of Faddeev-like double integral equations with additional channels arising from the negative ρ-spin states of the off-shell particle. [S0556-2813(97)02311-X]

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I. INTRODUCTION AND OVERVIEW

The three-body spectator (or Gross) equations were first introduced and applied to scalar particles in 1982 [1]. This original paper included a treatment of nonidentical particles and an introductory discussion of the definition and role of three-body forces in a relativistic context. Shortly afterward, in lectures given at the University of Hannover [2], the equations for three identical spin-1/2 particles were written down, but many details needed for a practical solution of the equations were never worked out. In this paper we complete the development by expanding the amplitudes into partial waves and reducing the equations to a compact form suitable for numerical solution. The development is carried out only for the case when the three-body scattering amplitude can be obtained by iterating successive two-body interactions, so that the three-body forces of relativistic origin discussed in the original paper [1] are neglected. However, because our covariant equations include the negative energy part of the Dirac propagator of the off-shell nucleon, many contributions are automatically included which would arise from three-body forces in a nonrelativistic context.

The bound state equations we present in this paper have already been solved numerically for a variety of cases, and some results have already been published [3,4]. From this experience we know that the general development presented here is a suitable basis for a practical solution of the covariant three-body problem.

In the remainder of this section we give a brief summary of the current status of nonrelativistic calculations of the binding energy of the three-nucleon bound state, and a review of previous work on the relativistic three-body problem. Then we will give a brief summary of the physics underlying our spectator equations, and present the final equations. The derivation of these results is found in the subsequent sections. In Sec. II we begin the development by writing the three-body equations in an operator form which is independent of the basis states used to describe the three-body system. In Sec. III we introduce basis states and write the equations in momentum space. In this representation the physical content of the equations is clear, but the equations are not in a form most convenient for numerical solution. To solve the equations numerically it is convenient to use a partial wave decomposition based on the helicity states originally introduced (in a three-body context) by Wick [5] and this is developed in detail in Sec. IV. The evaluation of the permutation operator, which interchanges particles between interactions and permits us to express the equation in terms of only one amplitude, is discussed in detail in Sec. V, and all of the results are collected together and the final equations given in Sec. VI. There are three appendices which discuss some points in detail.

A. Brief history of the three-body bound state problem

The first realistic nonrelativistic calculations of the triton binding energy were completed in the 1970s [6]. Later it was shown that different methods arrived at the same results, and that the binding energy could be calculated to a numerical accuracy of a few keV by considering all nucleon-nucleon (NN) partial waves up to $j=4$ [7]. Today, if three-body forces (3BF's) are not considered, a discrepancy of about 0.5–1.0 MeV remains between the experimentally observed value of -8.48 MeV and values obtained from realistic nonrelativistic NN potentials. Calculations of the contribution of the Δ resonance to the 3BF find that the net effect of the Δ is small [8,9]. State-of-the art calculations often include in addition also 3BF's based on meson-nucleon interaction processes other than Δ excitation [10]. When the strength of phenomenological 3BF's is adjusted to give the correct triton

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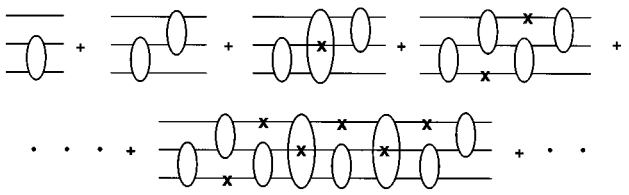

FIG. 1. Diagrams from the infinite class of successive two-body scatterings (represented by the ovals) which contribute to scattering of the three-body system. We use the convention that the initial state is on the right and the final state on the left in each diagram. The $\times$ labels internal spectators, which are put on shell in the spectator formalism. In this example, all of the diagrams but the first contribute to the subamplitude $\mathcal{T}^{13}$ where particle one (on the top) is the last spectator and particle three (on the bottom) is the first spectator. The first diagram contributes to the subamplitude $\mathcal{T}^{11}$.

binding energy, an excellent value is also obtained for the ^{4}He binding energy (and to a lesser extent other light nuclei up to $A \simeq 7$) [11].

However, relativistic effects should make a contribution to the binding energy at the level of several hundred keV. Using a mean momentum of about 200 MeV (consistent with nonrelativistic estimates) we expect to see corrections of the order of $(v/c)^2 \simeq (p/m)^2 \simeq 4\%$. If this is 4% of the binding energy, then it amounts to about 300 keV. However, if relativity has a greater effect on the attractive σ exchange part of the force (as it does in nuclear matter calculations using the Walecka model [12]) then we might obtain an effect 10 times larger.

Interest in relativistic three-body equations goes back to 1965, when Alessandrini and Omnes [13] used the Blankenbecler-Sugar equation [14] to describe the scattering of three particles, and Basdevant and Kreps [15] applied their ideas to a description of the three-pion system. Taylor [16] discussed the application of the Bethe-Salpeter equation [17] to three-body systems in 1966. In 1968 Aaron, Amado, and Young [18] introduced three-body scattering equations in which all the particles were on shell. Later, Garcilazo and his collaborators [19] treated three-body bound states using the Blankenbecler-Sugar equation, and Garcilazo [20] applied Wick's helicity formalism to the three-body problem, and used it to treat the πNN system relativistically [21]. Recently, the size of relativistic effects were estimated by Rupp and Tjon [22] using a separable kernel in the Bethe-Salpeter equation, by Sammarruca, Xu, and Machleidt [23] using minimal relativity and the Blankenbecler-Sugar equation, and by the Urbana group [24] using a relativisitc kinetic energy operator with boost corrections to the potential of order $(v/c)^2$. All of these calculations include some contributions coming from relativistic kinematics, but none treats the Dirac structure of the nucleons to all orders in $(v/c)^2$, or investigates effects which might arise from a realistic relativistic treatment of the *NN dynamics*.

B. The physics behind the spectator equations

In the absence of three-body forces, the three-body scattering amplitude (and the three-body bound state vertex functions) can be obtained by summing all successive two-body scatterings, as shown diagrammatically in Fig. 1. This summation can be organized into Faddeev-like equations, shown diagrammatically in Fig. 2. When the three particles are identical, the different Faddeev subamplitudes can be obtained from each other by interchange of variables, leading to a single equation for a single subamplitude represented diagrammatically in Fig. 3. It is necessary to know the two-body scattering amplitude before the equation shown in Fig. 3 can be solved. More specifically, the two-body scattering amplitude must be known in the rest frame of the *three-body system* (or any other frame independent of the internal variables).

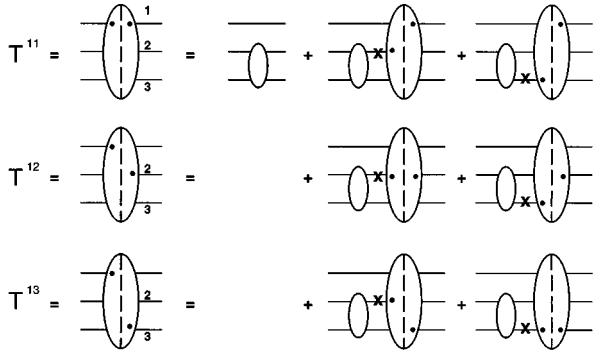

FIG. 2. Diagrammatic representation of the Faddeev equations for the amplitudes $\mathcal{T}^{1i}$. Note that the spectator is identified by the solid dot.

The two-body amplitude is usually calculated in its own rest frame, so it must be *boosted* to the three-body rest frame before it can be used in the Faddeev equations. The velocity of this boost depends on the momentum of the spectator, which is one of the dynamical variables of the problem, and hence the boost must be known for all velocities. In the nonrelativistic case this is trivial because the two-body amplitude is invariant under Galilean boosts. However, in the relativistic case this may present a problem, depending on the type of formalism used. Here, for the purposes of discussion, we distinguish two fundamentally different ways to approach relativistic calculations. In one approach, which will be referred to as Hamiltonian dynamics (including light-cone methods) [25], some of the Poincaré generators include the interactions, and therefore relativistic effects from either boosts or rotations are normally treated approximately (see, however, the paper by Glöckle, Lee, and Coester [25]). In a second method, which we will refer to as manifestly covariant dynamics [26], the generators are all kinematic, and all boosts and rotations can be done *exactly*. The spectator equations developed in this paper are an example of the latter method; we will reduce the three-body equations to a prac-

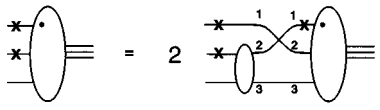

FIG. 3. Diagrammatic representation of the bound state spectator equation for three identical particles. Spectators are identified by the solid dots, and on-shell particles by the $\times$. Note the interchange of particles one and two.

tical form by *exploiting our ability to boost the two-body amplitudes to their rest frame*.

The Bethe-Salpeter (BS) formalism shares the property that the two-body amplitudes can be boosted exactly to their rest frame, and we will therefore compare the spectator equations with the corresponding BS equations. Both approaches conserve total four-momentum. This leaves an integration over all independent internal four-momenta, which are two for the three-body problem. The three-particle Bethe-Salpeter equation does not restrict any of these eight independent components, and after a partial wave decomposition there still remain four integrations, leading to coupled *four*-dimensional Faddeev equations. Furthermore, these equations contain singularities arising from the indefinite nature of the Minkowsky metric. In the spectator formalism the two time components of the internal four-momenta are eliminated (or, more precisely, expressed in terms of the other variables) by requiring that two of the three particles be fixed to their positive energy mass shell. This reduces the number of independent variables to only *six*, and after a partial wave decomposition one obtains coupled *two*-dimensional equations with a Faddeev structure. The three-body spectator equations therefore have the same structure as nonrelativistic equations, and this is one of their most significant advantages.

A particle is put on shell when it is a *spectator* to the interaction of two other particles. When this is done systematically, *two of the three* particles are always on shell. The particle which is off shell is the (unique) particle which has just interacted and is about to interact again (in a topological, not time-ordered, sense), as illustrated in Fig. 1.

It is natural to assume that restricting particles to their mass shell represents an approximation to the BS equation, but it can be shown that it is equivalent to a reorganization of the perturbation series of all ladder and crossed ladder diagrams which, in some cases, sums these diagrams more efficiently [1,27].

In summary, the spectator equation is used because (i) in some cases it sums the infinite series of all ladder and crossed ladder interactions efficiently, (ii) it reduces the number of independent variables to a minimum, making the covariant three-body problem tractable, and (iii) it permits us to boost the two-body amplitudes to their rest frame and calculate relativistic effects exactly.

Before we turn to the details of the derivation of the spectator equations, we present the equations in their final form in the next subsection.

C. Spectator equations for three spin-1/2 particles

In the absence of 3BF's the three-body scattering amplitude is obtained from a sum of all successive two-body scatterings. Because the three particles are identical, each two-body scattering differs from the others only by a permutation, and they can therefore all be summed by one operator equation of the form

$$|\Gamma^1\rangle = 2M^1 G^1 \mathcal{P}_{12} |\Gamma^1\rangle, \tag{1.1}$$

where $|\Gamma^1\rangle$ is a vertex function describing the contribution to the bound state from all processes in which the 23 pair was

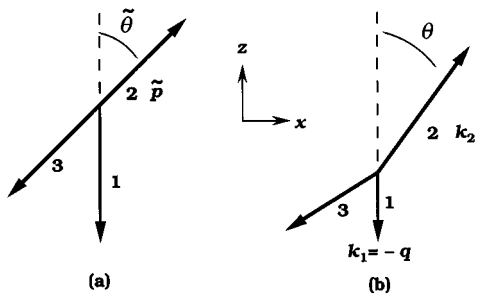

FIG. 4. Diagrams showing the momenta in the two-body rest frame (left panel) and in the three-body rest frame (right panel). We chose the momenta of particle one to be in the $-\hat{z}$ direction, so the boost from the two- to three-body rest frames is the $+\hat{z}$ direction.

the last to interact (with particle 1 a spectator), the two-body amplitude M^1 describes the scattering of the 23 pair, G^1 is the propagator for the 23 pair, and $\mathcal{P}_{12}$ is a permutation operator interchanging particles 1 and 2. These are labeled in Fig. 3. The factor of 2 comes from the contribution of $\mathcal{P}_{13}$ which equals the one of $\mathcal{P}_{12}$. The permutation operator rearranges the particles so that the same equation sums up the scattering of *all* pairs: 12, 23, and 13.

The three-body spectator equations have the same structure as Eq. (1.1), but incorporate the additional feature that the spectator is restricted to its positive energy mass shell in all intermediate states. With the conventions implied above, consistency also requires that particle 2 be on shell, so that two particles are always on shell. As already stated above, we think of these constraints as a reorganization of Eq. (1.1) which will, in some cases, improve its convergence. The constraints are manifestly covariant, and lead to the following equation:

$$|\Gamma_2^1\rangle = 2M_{22}^1 G_2^1 \mathcal{P}_{12} |\Gamma_2^1\rangle, \tag{1.2}$$

where the lower index labels the second on-shell particle. Hence only particle 3, the (unique) particle which has just left one interaction and is about to enter another one, is off shell in Eq. (1.2).

To prepare Eq. (1.2) for numerical evaluation, we take matrix elements of the operators using three-particle states. Both ρ-spin states (where $\rho = +$ is the u spinor positive energy state and $\rho = -$ is the v spinor negative energy state) of the off-shell particle must be treated. First we reduce the equation using states with definite particle helicities, similar to those defined by Wick [5]. These three-body states will be written in the abbreviated form $|J1(23)\rho\rangle$, where J is the total angular momentum of the state, ρ the ρ spin of the off-shell particle, $1 = \{q, \lambda_1\}$ (where q and λ_1 are the magnitude of the three-momentum and the helicity of the spectator in the three-body c.m.), and $(23) = \{\tilde{p}, j, m, \lambda_2, \lambda_3\}$ (where $\tilde{p}$ is the magnitude of the relative three-momentum of the 23 system, j and m are the angular momentum of the pair and its projection in the direction of $\mathbf{q}$, and λ_2 and λ_3 are the helicities of particles 2 and 3, *all defined in the rest frame of the 23 pair*). The momenta are defined in Fig. 4, which also shows the relation between the rest frames of the

two- and three-body systems. The three particles have mass m, and the total mass of the three-body bound state is denoted by M_t. (We use the symbol m to denote both the projection of the momentum and the particle mass, but the difference between them should be clear from the context.) Using this notation, and suppressing isospin, the final form of the three-body spectator equation for Γ^1 is given in Eq. (6.3). It can be written

$$\langle J1(23)\rho|\Gamma^1\rangle=\sum_{j'm'}\sum_{\substack{\lambda_2''\lambda_3''\rho''\\ \lambda_1'\lambda_2'\lambda_3'\rho'}}\int_0^{q_{\rm crit}}q'^2dq'\,\frac{m}{E_{q'}}\int_0^{\pi}d\chi\sin\chi\langle j(23)\rho|M^1|j(2''3'')\rho''\rangle$$

$$\times\frac{m}{E_{\tilde{p}''}}g^{\rho''}(q,\tilde{p}'')\mathcal{P}_{12}^{\rho''\rho'}[1(2''3''),1'(2'3')]\frac{m}{E_{\tilde{p}'}}\langle J'1'(2'3')\rho'|\Gamma^1\rangle, \tag{1.3}$$

where $\langle j(23)\rho|M^1|j(2''3'')\rho''\rangle$ is the jth two-body partial wave amplitude for the scattering of particles 2 and 3 in their own rest frame (precisely the amplitude obtained from the two body spectator theory as described in Ref. [28]), $\mathcal{P}_{12}^{\rho''\rho'}[1(2''3''),1'(2'3')]$ is the matrix element of the permutation operator, given in Eq. (1.6) below, and $g^{\rho}(q,\tilde{p})$ the propagator of the off-shell particle in different ρ-spin states

$$g^{+}(q,\tilde{p})=\frac{1}{2E_{\tilde{p}}-W_q},\quad g^{-}(q,\tilde{p})=-\frac{1}{W_{q'}} \tag{1.4}$$

where W_q is the mass of the 23 pair, and depends on q,

$$W_q^2=M_t^2+m^2-2M_tE_q, \tag{1.5}$$

with $E_q=\sqrt{m^2+\mathbf{q}^2}$. Note that Eq. (1.3) includes a sum over intermediate helicities and angular momentum quantum numbers, and an integration over the internal spectator momentum q' and the angle χ between the directions of $\mathbf{q}'$ and $\mathbf{q}$. The momenta $\tilde{p}'$ and $\tilde{p}''$ depend on q, q', and χ, as given in Eq. (5.17).

The integration over q' has been limited to the finite interval $[0,q_{\rm crit}]$, where $q_{\rm crit}$ is the root of the equation $W_{q_{\rm crit}}=0$. At this critical spectator momentum (equal to $\simeq 4m/3\simeq 1200$ MeV), the two-body subsystem is recoiling at the speed of light and the relativistic effects are enormous. One consequence of this is that the solutions of the three-body equations go smoothly to zero as $q\to q_{\rm crit}$ (this is discussed in detail in Sec. VI). Contributions from $q'>q_{\rm crit}$, which come from two-body states with *spacelike* four-momenta, are suppressed both because of this zero and because the propagators for large q are small. Hence, even if the spacelike two-body scattering amplitude is not small, we expect spacelike contributions to the overall three-body amplitudes to be very much suppressed, and it seems sensible to simply neglect the region $q'\geqslant q_{\rm crit}$ and set the three-body amplitudes to zero there. This also removes the need to calculate two-body amplitudes for spacelike total four-momenta.

Exchanging particles 1 and 2 implies that particle 2 becomes the spectator and now its momentum and helicity must be expressed in the c.m. frame of the three-body system, while the variables of particles 1 and 3 must be expressed in the rest frame of the 13 pair. Boosting from one frame to another introduces Wigner rotations of both the single particle and two-body helicities. In the helicity basis, this exchange operator is

$$\mathcal{P}_{12}^{\rho''\rho'}[1(2''3''),1'(2'3')]=(-1)^{m-\lambda_1+\lambda_3'}\sqrt{2j+1}\sqrt{2j'+1}\,d^{(J)}_{m-\lambda_1,m'-\lambda_2'}(\chi)d^{(j)}_{m,\lambda_2''-\lambda_3''}(\tilde{\theta}'')d^{(j')}_{m',\lambda_1'-\lambda_3'}(\tilde{\theta}')$$

$$\times d^{(1/2)}_{\lambda_1\lambda_1'}(\beta_1)d^{(1/2)}_{\lambda_2''\lambda_2'}(-\beta_2)\mathcal{N}^{\rho''\rho'}_{\lambda_3''\lambda_3'}(q,q',\chi), \tag{1.6}$$

where the functions $d^{(1/2)}_{m_1,m_2}(\beta)$ are the Wigner rotation matrices, and $\mathcal{N}^{\rho''\rho'}_{\lambda_3''\lambda_3'}(q,q',\chi)$ describes *exactly* the Wigner rotations of the off-shell particle 3, as well as the nontrivial matrix elements between the different ρ spinors u and v of particle 3 as they appear in the rest frames of the 23 pair and the 13 pair. The matrix $\mathcal{N}$ is defined in Eq. (6.2), the angles $\tilde{\theta}'$ and $\tilde{\theta}''$ in Eq. (5.17), and the Wigner rotation angles β_1 and β_2 in Eqs. (5.24) and (5.27).

For practical calculations it is more convenient to express Eq. (1.3) in terms of states with definite isospin and parity. These states will be denoted $|Tj^r(m\lambda)\rho\rangle$, where we suppress reference to the total angular momentum and parity $J^{\Pi}=1/2^+$, $T=0$ or 1 and $r=\pm 1$ are the isospin and parity of the 23 pair, and $\lambda=\lambda_2-\lambda_3=1/2-\lambda_3$. As discussed in Sec. VI D, the states of good parity are superpositions of positive and negative helicity states [see Eq. (6.17)], so that the two-body subspace is fully described by adopting the convention $\lambda_2=+1/2$, and identifying the states by their parity r and helicity difference $\lambda=0$ or 1. In this basis Eq. (1.2) becomes

$$\langle Tj^r(m\lambda)\rho|\Gamma_T^1\rangle = \sum_{\substack{j'r'\\ m'T'}} \sum_{\substack{\lambda''\rho''\\ \lambda'\rho'}} \int_0^{q_{\rm crit}} q'^2 dq' \frac{m}{E_{q'}} \int_0^\pi d\chi \sin\chi \langle Tj^r(m\lambda)\rho|M^{1T}|Tj^r(m\lambda'')\rho''\rangle$$

$$\times \frac{m}{E_{\tilde{p}''}} g^{\rho''}(q,\tilde{p}'') \overline{\mathcal{P}}_{12}^{\rho''\rho'} [Tj^r(m\lambda'')\rho'', T'j'^{r'}(m'\lambda')\rho'] \frac{m}{E_{\tilde{p}'}} \langle T'j'^{r'}(m'\lambda')\rho'|\Gamma_T^1\rangle, \tag{1.7}$$

where the permutation operator $\overline{\mathcal{P}}_{12}^{\rho''\rho'}$ is given in Eqs. (6.27) and (6.30). Note that Eq. (1.7) includes a sum over the intermediate isospin T'.

This concludes our brief introduction; we now turn to a detailed derivation of the three-body equations (1.3) and (1.7) given above.

II. THREE-BODY EQUATIONS IN OPERATOR FORM

We start with a derivation of Faddeev-type Bethe-Salpeter equations and introduce the spectator equations afterwards by substituting a new propagator and repeating the derivation with all necessary modifications.

A. Bethe-Salpeter equations

The total scattering amplitude for the three-nucleon system $\mathcal{T}$ can be decomposed into three parts $\mathcal{T}^i$,

$$\mathcal{T} = \sum_{i=1}^{3} \mathcal{T}^i. \tag{2.1}$$

The partial amplitude $\mathcal{T}^i$ sums up all diagrams in which particle i is the spectator during the ''last'' interaction (in the sense of ''leftmost'' in the diagrams of Fig. 1). Each amplitude $\mathcal{T}^i$ is further split into subamplitudes $\mathcal{T}^{ij}$, this time according to which particle does not participate in the ''first'' (or ''rightmost'') two-body interaction,

$$\mathcal{T}^i = \sum_{j=1}^{3} \mathcal{T}^{ij}. \tag{2.2}$$

The amplitudes $\mathcal{T}^{ij}$ satisfy the integral equation

$$\mathcal{T}^{ij} = i\delta_{ij}\mathcal{M}^i G_i^{-1} - \mathcal{M}^i G_{\rm BS}^i \sum_{k\neq i} \mathcal{T}^{kj}, \tag{2.3}$$

where G_i is the propagator of a *single* off-shell particle i, $G_{\rm BS}^i = G_{\rm BS}^i \otimes 1_i = -iG_j \otimes G_k \otimes 1_i$ is the free two-body propagator for the $\{j,k\}$ pair, and $\mathcal{M}^i = \mathcal{M}^i \otimes 1_i$ is the two-body scattering operator acting in the two-body subspace of particles j and k, with 1_i the identity operator for the spectator particle i. In our notation G_i is real, and any overall factor of i which emerges when the operator expressions are represented by Feynman diagrams is included in the propagator $G_{\rm BS}^i$. If $V^i = V^i \otimes 1_i$ represents the sum of all irreducible diagrams describing the interaction of the two particles j and k with particle i a spectator, the Bethe-Salpeter equation

$$\mathcal{M}^i = V^i - V^i G_{\rm BS}^i \mathcal{M}^i \tag{2.4}$$

yields the scattering operator $\mathcal{M}^i$.

A bound state of the three-body system can be defined as the residue of a pole of the three-body scattering amplitude $\mathcal{T}$. For the triton we denote the position of the pole as $P^2 = M_t^2$, where $P = k_1 + k_2 + k_3$ is the total four-momentum of the system and the k_i are single-particle four-momenta. One can write $\mathcal{T}^{ij}$ as the sum of a pole term and a part $\mathcal{R}^{ij}$ regular at $P^2 = M_t^2$:

$$\mathcal{T}^{ij} = -\frac{|\Gamma^i\rangle\langle\Gamma^j|}{M_t^2 - P^2} + \mathcal{R}^{ij}, \tag{2.5}$$

where $|\Gamma^i\rangle$ are the partial vertex amplitudes for the bound state. Insertion into Eq. (2.3), multiplication by $(M_t^2 - P^2)$, and performing the limit $P^2 \to M_t^2$ yields

$$|\Gamma^i\rangle = -\mathcal{M}^i G_{\rm BS}^i \sum_{j\neq i} |\Gamma^j\rangle. \tag{2.6}$$

These are the Bethe-Salpeter equations for the partial bound state vertex amplitudes.

Up to this point the equations are very general and apply to systems of any three distinguishable particles. Now we want to specialize to the case of three identical particles. We define the transpositions $\mathcal{P}_{ij}$ of two particles i and j as follows:

$$\mathcal{P}_{12}|abc\rangle = |bac\rangle,$$

$$\mathcal{P}_{13}|abc\rangle = |cba\rangle. \tag{2.7}$$

Note that $\mathcal{P}_{ij}$ interchanges the quantum numbers of the particles in the ith and jth locations in the state ket. The symmetry of the scattering amplitude under particle interchange can be expressed as

$$\mathcal{P}_{ij} T = \zeta T,$$

$$T\mathcal{P}_{ij} = \zeta T, \tag{2.8}$$

where $\zeta = +1$ for bosons and -1 for fermions, and T is the symmetrized version of $\mathcal{T}$. If we introduce the combined amplitude

$$|\Gamma\rangle = \sum_{i=1}^{3} |\Gamma^i\rangle, \tag{2.9}$$

then the symmetry (2.8) of T carries over to $|\Gamma\rangle$, i.e.,

$$\mathcal{P}_{ij}|\Gamma\rangle = \zeta|\Gamma\rangle,$$

$$\langle\Gamma|\mathcal{P}_{ij}=\zeta\langle\Gamma|. \tag{2.10}$$

These relations can be used to derive the permutation properties of the individual vertex factors $|\Gamma^i\rangle$. If the particles are identical, then the two-body scattering operators and propagators acting in each two-body subspace are identical, and this is expressed formally by the relations

$$\mathcal{P}_{ij}M^i\mathcal{P}_{ij}=M^j,$$

$$\mathcal{P}_{ij}G^i_{\rm BS}\mathcal{P}_{ij}=G^j_{\rm BS}, \tag{2.11}$$

where M is a symmetrized version of $\mathcal{M}$. Using these, and the fact that $\mathcal{P}_{ij}^2=1$, we obtain

$$\begin{aligned}\mathcal{P}_{ij}|\Gamma^i\rangle&=-\mathcal{P}_{ij}M^i\mathcal{P}_{ij}\mathcal{P}_{ij}G^i_{\rm BS}\sum_{k\neq i}|\Gamma^k\rangle\\&=-M^jG^j_{\rm BS}\mathcal{P}_{ij}(|\Gamma\rangle-|\Gamma^i\rangle)\\&=-M^jG^j_{\rm BS}(\zeta|\Gamma\rangle-\mathcal{P}_{ij}|\Gamma^i\rangle).\end{aligned} \tag{2.12}$$

Comparing with

$$\zeta|\Gamma^j\rangle=-M^jG^j_{\rm BS}(\zeta|\Gamma\rangle-\zeta|\Gamma^j\rangle) \tag{2.13}$$

one obtains immediately

$$\mathcal{P}_{ij}|\Gamma^i\rangle=\zeta|\Gamma^j\rangle. \tag{2.14}$$

Thus the three-body equations for identical particles can be written

$$|\Gamma^i\rangle=-\zeta M^iG^i_{\rm BS}(\mathcal{P}_{ij}+\mathcal{P}_{ik})|\Gamma^i\rangle. \tag{2.15}$$

The three equations for the three possible choices of i are equivalent. It is therefore sufficient to solve Eq. (2.15) for, say, $i=1$, and calculate $|\Gamma^2\rangle$ and $|\Gamma^3\rangle$ by means of Eq. (2.14).

Equation (2.15) can be simplified further if we take into account the fact that the two-body amplitude M^1 is symmetric or antisymmetric under exchange of particles 2 and 3 for the case of identical bosons or fermions, respectively. Thus

$$\mathcal{P}_{23}M^1=M^1\mathcal{P}_{23}=\zeta M^1. \tag{2.16}$$

Using this relation, Eq. (2.15) (with $i=1$) can be written

$$\begin{aligned}\mathcal{P}_{23}|\Gamma^1\rangle&=-\zeta\mathcal{P}_{23}M^1G^1_{\rm BS}(\mathcal{P}_{12}+\mathcal{P}_{13})|\Gamma^1\rangle\\&=-\zeta^2M^1G^1_{\rm BS}(\mathcal{P}_{12}+\mathcal{P}_{13})|\Gamma^1\rangle=\zeta|\Gamma^1\rangle.\end{aligned} \tag{2.17}$$

Next, using the definitions Eq. (2.7) note that

$$\begin{aligned}\mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23}|abc\rangle&=\mathcal{P}_{23}\mathcal{P}_{12}|acb\rangle=\mathcal{P}_{23}|cab\rangle=|cba\rangle\\&=\mathcal{P}_{13}|abc\rangle.\end{aligned} \tag{2.18}$$

Hence, the operator $\mathcal{P}_{13}$ can be written

$$\mathcal{P}_{13}=\mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23}. \tag{2.19}$$

Using the relations (2.16)–(2.19) together with the fact that $G^1_{\rm BS}$ commutes with $\mathcal{P}_{23}$ we can write the Faddeev equations (2.15) in the following simple form:

$$\begin{aligned}|\Gamma^1\rangle&=-\zeta M^1G^1_{\rm BS}(\mathcal{P}_{12}+\mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23})|\Gamma^1\rangle\\&=-\zeta M^1G^1_{\rm BS}(1+\zeta\mathcal{P}_{23})\mathcal{P}_{12}|\Gamma^1\rangle\\&=-2\zeta M^1G^1_{\rm BS}\mathcal{P}_{12}|\Gamma^1\rangle.\end{aligned} \tag{2.20}$$

To reduce these equations to a practical form, it is sufficient to evaluate the permutation operator $\mathcal{P}_{12}$.

B. Spectator equations

Now we turn to the spectator equations. We begin by replacing the two-body propagator $G^i_{\rm BS}\otimes 1_i$, which describes the propagation of particles j and k (both not equal to i) in Eq. (2.6), by a new propagator,

$$G^i_{\rm BS}\otimes 1_i\rightarrow G_j\mathcal{Q}_k\otimes 1_i, \tag{2.21}$$

where $\mathcal{Q}_k$ is a *projection operator* which places particle k on the positive energy mass shell, and, as in the BS case, G_j is the propagator of a single off-shell particle j. Choosing particle k to be the spectator during the ''previous'' interaction gives the unclosed form of the spectator Faddeev equations

$$|\Gamma^i\rangle=-\sum_{k\neq i\neq j}M^iG_j\mathcal{Q}_k|\Gamma^k\rangle, \tag{2.22}$$

where the sum is over k and j with i fixed and no two indices equal. Explicitly, Eq. (2.22) is shorthand for the following three equations:

$$\begin{aligned}|\Gamma^1\rangle&=-\{M^1G_2\mathcal{Q}_3|\Gamma^3\rangle+M^1G_3\mathcal{Q}_2|\Gamma^2\rangle\},\\|\Gamma^2\rangle&=-\{M^2G_3\mathcal{Q}_1|\Gamma^1\rangle+M^2G_1\mathcal{Q}_3|\Gamma^3\rangle\},\\|\Gamma^3\rangle&=-\{M^3G_1\mathcal{Q}_2|\Gamma^2\rangle+M^3G_2\mathcal{Q}_1|\Gamma^1\rangle\}.\end{aligned} \tag{2.23}$$

Note that the projection operator $\mathcal{Q}_k$ ensures that particle k is on shell both as it leaves the partial amplitude $|\Gamma^k\rangle$ and as it enters the two-body scattering amplitude M^i.

To make a closed set of equations from Eq. (2.22), it is necessary to place the final spectator particle i on shell, which then also forces one of the two interacting particles in the final state (denoted by k') to be on shell. The spectator scattering equations are shown diagrammatically in Fig. 2. The final bound state equations can be written algebraically in the following form:

$$\mathcal{Q}_i\mathcal{Q}_{k'}|\Gamma^i\rangle=-\sum_{k\neq i\neq j}\mathcal{Q}_{k'}M^i\mathcal{Q}_kG_j\mathcal{Q}_k\mathcal{Q}_i|\Gamma^k\rangle, \tag{2.24}$$

where no summation over the index i is implied, and we used the projection property $\mathcal{Q}_k\mathcal{Q}_k=\mathcal{Q}_k$ and $G_j\mathcal{Q}_k=\mathcal{Q}_kG_j$.

Alternatively, we may introduce the notation

$$|\Gamma^i_j\rangle = \mathcal{Q}_i \mathcal{Q}_j |\Gamma^i\rangle,$$

$$M^i_{k'k} = \mathcal{Q}_{k'} M^i \mathcal{Q}_k \otimes 1_i,$$

$$G^i_k = G_j \mathcal{Q}_k \otimes 1_i, \tag{2.25}$$

where the indices i, j, and k are all different, and the lower indices on M and Γ label which particles, apart from the spectator, are on mass shell. In this notation Eq. (2.24) becomes

$$|\Gamma^i_{k'}\rangle = -\sum_{k\neq i} M^i_{k'k} G^i_k |\Gamma^k_i\rangle. \tag{2.26}$$

We will use this notation in most of the remainder of this section, but will return to the definitions (2.25) later in the paper. As an example, consider the case $i=1$ and $k'=2$:

$$|\Gamma^1_2\rangle = -M^1_{22} G^1_2 |\Gamma^2_1\rangle - M^1_{23} G^1_3 |\Gamma^3_1\rangle. \tag{2.27}$$

As discussed in Ref. [1], for distinguishable particles without three-body forces Eq. (2.26) becomes a coupled set of *six* equations for the six amplitudes $|\Gamma^i_j\rangle$, instead of only three equations for three $|\Gamma^i\rangle$, as in the Bethe-Salpeter case.

We emphasize that an important difference between the spectator subamplitudes $|\Gamma^i_j\rangle$ and the Bethe-Salpeter subamplitudes $|\Gamma^i\rangle$ is that it is no longer possible to add the spectator subamplitudes together in order to construct a total amplitude, as we did in Eq. (2.9). This is because the amplitude $|\Gamma^1_2\rangle$, for example, restricts particles 1 and 2 to the mass shell, while the amplitude $|\Gamma^1_3\rangle$ restricts particles 1 and 3 to the mass shell, and hence they are defined for different regions of phase space. Only operators or amplitudes which satisfy identical constraints, such as $|\Gamma^1_2\rangle$ and $|\Gamma^2_1\rangle$, for example, can be combined. *No total three-body amplitude exists in the spectator formalism.*

For identical particles, Eqs. (2.26) can be further reduced by using permutation operators. Using the fact that the operator M^1 is symmetric under particle interchange, Eq. (2.16), and the relation

$$\mathcal{P}_{32} \mathcal{Q}_2 = \mathcal{Q}_3 \mathcal{P}_{32}, \tag{2.28}$$

we obtain

$$\mathcal{P}_{32} M^1_{22} = \zeta M^1_{32} = M^1_{33} \mathcal{P}_{32},$$

$$\mathcal{P}_{23} M^1_{33} = \zeta M^1_{23} = M^1_{22} \mathcal{P}_{23}, \tag{2.29}$$

where $\zeta = +1$ for bosons and -1 for fermions, as before. (These are the operator forms of the symmetry relations discussed in Ref. [28]; note that, in the spectator formalism, the exchange operator does not relate an amplitude to itself, but to another amplitude with a different particle off shell.) We will find it convenient to exploit the fact that $\mathcal{P}_{jk} = \mathcal{P}_{kj}$, and always write relations like those above so that the initial and final indices on both sides of the equation match. The spectator and on shell interacting particle can also be interchanged, leading to the following relations for the operators G^i_j:

$$\mathcal{P}_{12} G^1_2 \mathcal{P}_{21} = G^2_1,$$

$$\mathcal{P}_{23} G^1_3 \mathcal{P}_{32} = G^1_2. \tag{2.30}$$

The two-body amplitudes exhibit a similar symmetry

$$\mathcal{P}_{12} M^1_{22} \mathcal{P}_{21} = M^2_{11},$$

$$\mathcal{P}_{23} M^1_{33} \mathcal{P}_{32} = M^1_{22}. \tag{2.31}$$

Further relations can be found by combining relations (2.29) and (2.31). One relation we will use below is

$$\mathcal{P}_{12} M^1_{23} \mathcal{P}_{32} \mathcal{P}_{21} = \zeta M^2_{11} = M^2_{13} \mathcal{P}_{31}. \tag{2.32}$$

It is now easy to derive the effect of permutations on the spectator subamplitudes. For example, under the interchange of two particles in the interacting pair,

$$\begin{aligned} \mathcal{P}_{32}|\Gamma^1_2\rangle &= -\mathcal{P}_{32} M^1_{22} G^1_2 |\Gamma^2_1\rangle - \mathcal{P}_{32} M^1_{23} G^1_3 |\Gamma^3_1\rangle \\ &= -\zeta M^1_{32} G^1_2 |\Gamma^2_1\rangle - \zeta M^1_{33} G^1_3 |\Gamma^3_1\rangle = \zeta |\Gamma^1_3\rangle. \end{aligned} \tag{2.33}$$

Using this, the interchange of the spectator with the on-shell particle in the interacting pair is

$$\begin{aligned} \mathcal{P}_{12}|\Gamma^1_2\rangle &= -\mathcal{P}_{12} M^1_{22} \mathcal{P}_{21} \mathcal{P}_{12} G^1_2 \mathcal{P}_{21} \mathcal{P}_{21} |\Gamma^2_1\rangle - \mathcal{P}_{12} M^1_{23} \mathcal{P}_{32} \mathcal{P}_{21} \mathcal{P}_{12} \mathcal{P}_{23} G^1_3 |\Gamma^3_1\rangle = -M^2_{11} G^2_1 \mathcal{P}_{21} |\Gamma^2_1\rangle - M^2_{13} \mathcal{P}_{31} \mathcal{P}_{12} G^1_2 \mathcal{P}_{23} |\Gamma^3_1\rangle \\ &= -M^2_{11} G^2_1 \mathcal{P}_{21} |\Gamma^2_1\rangle - M^2_{13} G^2_3 \mathcal{P}_{31} \mathcal{P}_{12} \mathcal{P}_{23} |\Gamma^3_1\rangle = -M^2_{11} G^2_1 \mathcal{P}_{21} |\Gamma^2_1\rangle - M^2_{13} G^2_3 \mathcal{P}_{21} |\Gamma^3_1\rangle \\ &= -M^2_{11} G^2_1 \mathcal{P}_{21} |\Gamma^2_1\rangle - \zeta M^2_{13} G^2_3 |\Gamma^3_2\rangle, \end{aligned} \tag{2.34}$$

where Eq. (2.19) (with $1 \leftrightarrow 3$) was used in the next to last step. Comparison with the equation for $|\Gamma^2_1\rangle$,

$$\zeta |\Gamma^2_1\rangle = -\zeta M^2_{11} G^2_1 |\Gamma^1_2\rangle - \zeta M^2_{13} G^2_3 |\Gamma^3_2\rangle, \tag{2.35}$$

implies

$$\mathcal{P}_{21} |\Gamma^2_1\rangle = \zeta |\Gamma^1_2\rangle. \tag{2.36}$$

Using these relations, we can obtain a single equation for $|\Gamma^1_2\rangle$

$$\begin{aligned} |\Gamma^1_2\rangle &= -\zeta M^1_{22} G^1_2 \mathcal{P}_{21} |\Gamma^1_2\rangle - \zeta^2 M^1_{23} G^1_3 \mathcal{P}_{13} \mathcal{P}_{32} |\Gamma^1_2\rangle \\ &= -\zeta M^1_{22} G^1_2 \mathcal{P}_{21} |\Gamma^1_2\rangle - \zeta^2 M^1_{23} \mathcal{P}_{32} \mathcal{P}_{23} G^1_3 \mathcal{P}_{32} \mathcal{P}_{21} |\Gamma^1_2\rangle \\ &= -2\zeta M^1_{22} G^1_2 \mathcal{P}_{21} |\Gamma^1_2\rangle, \end{aligned} \tag{2.37}$$

where Eq. (2.19) was used in the next to last step. From now on we will only consider fermions, so that the three-body equation for the vertex function Γ_2^1, which singles out particle 1 as spectator in the ''last'' interaction and particle 2 as the interacting particle to be put on mass shell, becomes

$$|\Gamma_2^1\rangle = 2M_{22}^1 G_2^1 \mathcal{P}_{21} |\Gamma_2^1\rangle. \tag{2.38}$$

This equation is illustrated diagrammatically in Fig. 3.

A more explicit form of Eq. (2.38), which expresses M_{22}^1 and G_2^1 as operators, and $|\Gamma_2^1\rangle$ as a vector in Dirac space, is

$$|\Gamma_2^1\rangle_{\alpha\beta\gamma} = 2[M_{22}^1]_{\beta\beta_1,\gamma\gamma_1}[G_2^1]_{\beta_1\beta_2,\gamma_1\gamma_2}[\mathcal{P}_{12}|\Gamma_2^1\rangle]_{\alpha\beta_2\gamma_2}, \tag{2.39}$$

where α, β, and γ are Dirac indices for particles 1, 2, and 3 respectively, and summation over repeated Dirac indices is implied. Note that M and G operate on a two-body space only; the third particle (the spectator) is unaffected by these operators.

In the next section we will give a momentum space representation of these equations.

III. MOMENTUM SPACE REPRESENTATION

We specialize to three identical particles with mass m, spin 1/2, and four-momenta k_1, k_2, and k_3. The total momentum

$$P = k_1 + k_2 + k_3 \tag{3.1}$$

is conserved. The Gross equation restricts two of the three particles to be on mass shell, which for the choice (2.38) are particles 1 and 2, with particle 3 off mass shell. In the three-body c.m. system

$$P = (M_t, \mathbf{0}), \tag{3.2}$$

where M_t is the mass of the three-body system, the momenta are

$$k_1 = (E_{k_1}, \mathbf{k}_1),$$

$$k_2 = (E_{k_2}, \mathbf{k}_2),$$

$$k_3 = (k_{30}, \mathbf{k}_3) = (M_t - E_{k_1} - E_{k_2}, -\mathbf{k}_1 - \mathbf{k}_2). \tag{3.3}$$

In Eq. (3.3) the four-momentum of the off-shell particle, k_3, is fixed by four-momentum conservation. It is obvious that the problem has only six independent momentum variables, just as in the nonrelativistic case.

The three-body basis states are direct products of a single-particle state and a two-particle state,

$$|k_1(k_2\overline{k_3})\rangle = |k_1\rangle \otimes |k_2\overline{k_3}\rangle, \tag{3.4}$$

where, by convention, the off-shell particle has a bar over its momentum.

Completeness and orthogonality relations are

$$\langle k_1(k_2\overline{k_3})|k_1'(k_2'\overline{k_3'})\rangle = 2E_{k_1}\delta^3(k_1-k_1')2E_{k_2}\delta^3(k_2-k_2')\delta^4(P-P'), \tag{3.5}$$

$$1 = \int \frac{d^3k_1}{2E_{k_1}} \frac{d^3k_2}{2E_{k_2}} d^4P |k_1(k_2\overline{k_3})\rangle\langle k_1(k_2\overline{k_3})|. \tag{3.6}$$

Next, we specify the matrix elements of all operators in this momentum space basis. The propagator is

$$\langle k_1(k_2\overline{k_3})|[G_2^1]_{\beta\beta',\gamma\gamma'}|k_1'(k_2'\overline{k_3'})\rangle = 2E_{k_1}\delta^3(k_1-k_1')2E_{k_2}\delta^3(k_2-k_2')\delta^4(P-P')2m[\Lambda_+(k_2)]_{\beta\beta'}\frac{(m+\not{k}_3)_{\gamma\gamma'}}{m^2-k_3^2-i\epsilon}, \tag{3.7}$$

where $\Lambda_\pm(k) = (m \pm \not{k})/2m$ are the positive and negative energy projection operators. The two-body M matrix is

$$\langle k_1(k_2\overline{k_3})|[M_{22}^1]_{\beta\gamma,\beta'\gamma'}|k_1'(k_2'\overline{k_3'})\rangle = 2E_{k_1}\delta^3(k_1-k_1')\delta^4(P-P')M_{\beta\beta',\gamma\gamma'}(k_{23},k_{23}';P-k_1), \tag{3.8}$$

where $P-k_1$ is the total two-body four-momentum, and the relative momenta in the two-body space are denoted by

$$k_{ij} = \tfrac{1}{2}(k_i - k_j). \tag{3.9}$$

Note that $k_{ij} = -k_{ji}$. The two-body amplitudes in Eq. (3.8) are identical to those discussed in Sec. II A of Ref. [28]. The partial vertex amplitudes will be written

$$\langle k_1(k_2\overline{k_3})|\Gamma_2^1\rangle_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma}(k_1,k_2,k_3), \tag{3.10}$$

where, by convention, it is understood that the last momentum is the one which is off shell. Therefore

$$\langle k_1(k_2\overline{k_3})|\mathcal{P}_{12}|\Gamma_2^1\rangle_{\alpha\beta\gamma} = -\langle k_2(k_1\overline{k_3})|\Gamma_2^1\rangle_{\beta\alpha\gamma} = -\Gamma_{\beta\alpha\gamma}(k_2,k_1,k_3). \tag{3.11}$$

We can now obtain the momentum space representation of Eq. (2.38). Inserting the completeness relation (3.6) gives

$$\langle k_1(k_2\overline{k_3})|\Gamma_2^1\rangle_{\alpha\beta\gamma}=2\int\frac{d^3k_1'}{2E_{k_1'}}\frac{d^3k_2'}{2E_{k_2'}}d^4P'\frac{d^3k_1''}{2E_{k_1''}}\frac{d^3k_2''}{2E_{k_2''}}d^4P''\langle k_1(k_2\overline{k_3})|[M_{22}^1]_{\beta\beta',\gamma\gamma'}|k_1'(k_2'\overline{k_3'})\rangle$$
$$\times\langle k_1'(k_2'\overline{k_3'})|[G_2^1]_{\beta,\beta'',\gamma'\gamma''}|k_1''(k_2''\overline{k_3''})\rangle\langle k_1''(k_2''\overline{k_3''})|\mathcal{P}_{12}|\Gamma_2^1\rangle_{\alpha\beta''\gamma''}. \tag{3.12}$$

Inserting the above expressions for M and G, and carrying out all integrals gives

$$\Gamma_{\alpha\beta\gamma}(k_1,k_2,k_3)=-2\int d^3k_2'\frac{m}{E_{k_2'}}M_{\beta\beta',\gamma\gamma'}(k_{23},k_{23}';P-k_1)[\Lambda_+(k_2')]_{\beta'\beta''}\frac{(m+\not{k}_3')_{\gamma'\gamma''}}{m^2-k_3'^2-i\epsilon}\Gamma_{\beta''\alpha\gamma''}(k_2',k_1,k_3'), \tag{3.13}$$

where $k_3'=P-k_2'-k_1$. This equation is manifestly covariant.

These equations may be further reduced by multiplying the M matrix and the three-body vertex functions Γ by the on-shell spinors u (for on-shell particles in the initial state) and $\overline{u}$ (for on-shell particles in the final state):

$$\Gamma_{\lambda_1\lambda_2\gamma}(k_1,k_2,k_3)=\overline{u}_\alpha(\mathbf{k_1},\lambda_1)\,\overline{u}_\beta(\mathbf{k_2},\lambda_2)\Gamma_{\alpha\beta\gamma}(k_1,k_2,k_3),$$

$$M_{\lambda_2\lambda_2',\gamma\gamma'}(k_{23},k_{23}';P-k_1)=\overline{u}_\beta(\mathbf{k_2},\lambda_2)M_{\beta\beta',\gamma\gamma'}(k_{23},k_{23}';P-k_1)u_{\beta'}(\mathbf{k_2'},\lambda_2'), \tag{3.14}$$

where $u_\alpha(\mathbf{k_1},\lambda_1)$ is an on-shell Dirac spinor with three-momentum $\mathbf{k_1}$ and helicity λ_1. This gives us quantities with ''mixed indices;'' a Dirac index on a matrix element is replaced by a helicity index when it is contracted with a u spinor of that helicity and with matching momentum. These amplitudes are still covariant, and simpler because the four-dimensional Dirac space is replaced by a two-dimensional helicity space. If we then replace the on-shell projection operator by a sum over on-shell u spinors

$$[\Lambda_+(k_2)]_{\beta\beta'}=\sum_{\lambda_2}u(\mathbf{k_2},\lambda_2)\,\overline{u}(\mathbf{k_2},\lambda_2), \tag{3.15}$$

and multiply Eq. (3.13) from the left by $\overline{u}_\alpha(\mathbf{k_1},\lambda_1)\,\overline{u}_\beta(\mathbf{k_2},\lambda_2)$ we get

$$\Gamma_{\lambda_1\lambda_2\gamma}(k_1,k_2,k_3)=-2\sum_{\lambda_2'}\int d^3k_2'\frac{m}{E_{k_2'}}M_{\lambda_2\lambda_2',\gamma\gamma'}(k_{23},k_{23}';P-k_1)\frac{(m+\not{k}_3')_{\gamma'\gamma''}}{m^2-k_3'^2-i\epsilon}\Gamma_{\lambda_2'\lambda_1\gamma''}(k_2',k_1,k_3'). \tag{3.16}$$

Equation (3.16) is still manifestly covariant, but is not suitable for a numerical solution. The main reason is that the two-body M matrix is given as a partial wave expansion in the *two-body rest frame*, and not the three-body c.m. system, as needed in the above equation. A related problem is that the propagator for particle 3 depends on the angle between the vectors $\mathbf{k_1}$ and $\mathbf{k_2'}$ and is therefore not diagonal with respect to all angular momenta after a partial wave decomposition.

In the nonrelativistic case, the first problem does not occur because the partial wave expansion is invariant under a Galilean boost, and the second is solved by introducing Jacobi coordinates. Because of the different energy-momentum relations in special relativity, neither of these problems can be handled so simply here.

However, we can eliminate these problems here by exploiting the covariance of the formalism, and by explicitly boosting the two-body subsystem to its rest frame. To prepare the way, introduce the total four-momentum of the two-body subsystem,

$$P_{23}=k_2+k_3=P-k_1=P+q, \tag{3.17}$$

where here it is convenient to introduce the momentum $q=-k_1$. Next, the boost operator Λ_{k_1} is defined by the requirement

$$\Lambda_{k_1}P_{23}=\widetilde{P}_{23}=(W_q,\mathbf{0}). \tag{3.18}$$

The square of the mass of the (23) pair is then

$$W_q^2=\widetilde{P}_{23}^2=P_{23}^2=(M_t-E_q)^2-\mathbf{q}^2$$
$$=M_t^2+m^2-2M_tE_q. \tag{3.19}$$

A tilde on top of a variable always indicates that it is defined in the two-body rest frame. We have, e.g.,

$$\widetilde{k}_2=\Lambda_{k_1}k_2,$$

$$\widetilde{k}_3=\Lambda_{k_1}k_3. \tag{3.20}$$

We now define the relative momentum $\widetilde{p}$ through

$$\tilde{k}_2=\tfrac{1}{2}\tilde{P}_{23}+\tilde{p}=(E_{\tilde{p}},\tilde{\mathbf{p}}),$$

$$\tilde{k}_3=\tfrac{1}{2}\tilde{P}_{23}-\tilde{p}=(W_q-E_{\tilde{p}},-\tilde{\mathbf{p}}), \tag{3.21}$$

and therefore

$$\tilde{p}=\tilde{k}_{23}=\tfrac{1}{2}(\tilde{k}_2-\tilde{k}_3)=\left(E_{\tilde{p}}-\tfrac{1}{2}W_q,\tilde{\mathbf{p}}\right). \tag{3.22}$$

Next, we introduce the representation on the Dirac space, $S(\Lambda)$, of a Lorentz boost Λ. These transform Dirac matrices and spinors according to the following rules:

$$S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda)=\Lambda^{\mu}{}_{\nu}\gamma^{\nu}, \tag{3.23}$$

$$S(\Lambda)u(\mathbf{k},\lambda)=\sum_{\mu}\mathcal{D}^{(1/2)}_{\mu\lambda}(R_{\Lambda k})u(\Lambda\mathbf{k},\mu), \tag{3.24}$$

$$\overline{u}(\mathbf{k},\lambda)S^{-1}(\Lambda)=\sum_{\mu}\overline{u}(\Lambda\mathbf{k},\mu)\mathcal{D}^{(1/2)*}_{\lambda\mu}(R_{\Lambda k}), \tag{3.25}$$

where $R_{\Lambda k}$ is the Wigner rotation accompanying the boost that connects the momenta $\mathbf{k}$ and $\Lambda\mathbf{k}$ (see Appendix C), and the Dirac indices have been suppressed. The propagator of the off-shell particle 3 in the three-body rest frame can therefore be expressed in terms of its form in the two-body rest frame using (3.23)

$$\frac{(m+\not{k}_3)}{m^2-k_3^2-i\epsilon}=S^{-1}(\Lambda_{k_1})\frac{(m+\not{\tilde{k}}_3)}{m^2-\tilde{k}_3^2-i\epsilon}S(\Lambda_{k_1}). \tag{3.26}$$

Similarly, the full two-body M matrix in the three-body system can be written

$$M_{2,3}(k_{23},k'_{23};P_{23})=S_2^{-1}(\Lambda_{k_1})S_3^{-1}(\Lambda_{k_1})M_{2,3}(\tilde{p},\tilde{p}';\tilde{P}_{23})S_2(\Lambda_{k_1})S_3(\Lambda_{k_1}), \tag{3.27}$$

where the subscripts 2 and 3 are shorthand for pairs of Dirac indices on particle 2 ($\beta\beta'$, etc.) and on particle 3 ($\gamma\gamma'$, etc.), and the two-body scattering amplitude $M(\tilde{p},\tilde{p}';\tilde{P}_{23})$ is a solution of the two-body Gross equations in the two-body c.m. frame. (Do not confuse $M_{2,3}$ with amplitudes like M^1_{23} used in the last subsection; here the subscripts refer to the Dirac indices, and in the previous subsection they referred to which of the interacting particles was on shell. From now on we have made the choice that particle 2 is on shell, and in the language of the previous subsection, all two-body amplitudes are M^1_{22}.) Using Eqs. (3.24) and (3.25), we obtain the following expression for the mixed index M matrix:

$$M_{\lambda_2\lambda'_2,3}(k_{23},k'_{23};P_{23})=S_3^{-1}(\Lambda_{k_1})\mathcal{D}^{(1/2)*}_{\lambda_2\mu_2}(R_{\Lambda_{k_1}k_2})M_{\mu_2\mu'_2,3}(\tilde{p},\tilde{p}';\tilde{P}_{23})\mathcal{D}^{(1/2)}_{\mu'_2\lambda'_2}(R_{\Lambda_{k_1}k'_2})S_3(\Lambda_{k_1}), \tag{3.28}$$

where summation over all repeated indices (including helicities) is implied. Substituting these relations into Eq. (3.16) gives

$$\Gamma_{\lambda_1\lambda_2\gamma}(k_1,k_2,k_3)=-2\int d^3k'_2\frac{m}{E_{k'_2}}S^{-1}_{\gamma\gamma_1}(\Lambda_{k_1})\mathcal{D}^{(1/2)*}_{\lambda_2\mu_2}(R_{\Lambda_{k_1}k_2})M_{\mu_2\mu'_2,\gamma_1\gamma_2}(\tilde{p},\tilde{p}';\tilde{P}_{23})\mathcal{D}^{(1/2)}_{\mu'_2\lambda'_2}(R_{\Lambda_{k_1}k'_2})\frac{(m+\not{\tilde{k}}'_3)_{\gamma_2\gamma_3}}{m'^2-\tilde{k}'^2_3-i\epsilon}$$
$$\times S_{\gamma_3\gamma'}(\Lambda_{k_1})\Gamma_{\lambda'_2\lambda_1\gamma'}(k'_2,k_1,k'_3). \tag{3.29}$$

This equation can be further reduced if we decompose of the propagator of the off-shell particle 3 into positive and negative energy parts

$$\frac{(m+\not{\tilde{k}}_3)_{\gamma\gamma'}}{m^2-\tilde{k}_3^2-i\epsilon}=\frac{m}{E_{\tilde{p}}}\sum_{\lambda_3}\left[\frac{u_{\gamma}(\tilde{\mathbf{k}}_3,\lambda_3)\overline{u}_{\gamma'}(\tilde{\mathbf{k}}_3,\lambda_3)}{2E_{\tilde{p}}-W_q-i\epsilon}-\frac{v_{\gamma}(-\tilde{\mathbf{k}}_3,\lambda_3)\overline{v}_{\gamma'}(-\tilde{\mathbf{k}}_3,\lambda_3)}{W_q-i\epsilon}\right]=\frac{m}{E_{\tilde{p}}}\sum_{\lambda_3}\left[\frac{u_{\gamma}(-\tilde{\mathbf{p}},\lambda_3)\overline{u}_{\gamma'}(-\tilde{\mathbf{p}},\lambda_3)}{2E_{\tilde{p}}-W_q-i\epsilon}\right.$$
$$\left.-\frac{v_{\gamma}(\tilde{\mathbf{p}},\lambda_3)\overline{v}_{\gamma'}(\tilde{\mathbf{p}},\lambda_3)}{W_q-i\epsilon}\right], \tag{3.30}$$

where the second expression can be obtained from the first using the fact that the spinors depend only on the three-momentum and $\tilde{\mathbf{k}}_3=-\tilde{\mathbf{p}}$. At this point it is convenient to introduce ρ spin by letting

$$u^{\rho}(\mathbf{p},\lambda)=\begin{cases}u(\mathbf{p},\lambda) & \text{if } \rho=+,\\ v(-\mathbf{p},\lambda) & \text{if } \rho=-.\end{cases} \tag{3.31}$$

Then the decomposition (3.30) becomes

$$\frac{(m+\not{\tilde{k}}_3)_{\gamma\gamma'}}{m^2-\tilde{k}_3^2-i\epsilon}=\frac{m}{E_{\tilde{p}}}u^{\rho}_{\gamma}(-\tilde{\mathbf{p}},\lambda_3)g^{\rho}(q,\tilde{p})\overline{u^{\rho}}_{\gamma'}(-\tilde{\mathbf{p}},\lambda_3), \tag{3.32}$$

where summation over ρ and λ_3 is implied, and

$$g^+(q,\tilde{p}) = \frac{1}{2E_{\tilde{p}} - W_q - i\epsilon},$$

$$g^-(q,\tilde{p}) = -\frac{1}{W_q - i\epsilon}. \tag{3.33}$$

Substituting Eq. (3.32) into Eq. (3.25), multiplying from the left by $\overline{u^\rho}(\mathbf{k_3},\lambda_3)$, and using Eqs. (3.24) and (3.25), give the following reduced three-body equations:

$$\Gamma^\rho_{\lambda_1\lambda_2\lambda_3}(k_1,k_2,k_3) = -2\int d^3\tilde{k}_2' \left(\frac{m}{E_{\tilde{k}_2'}}\right)^2 \mathcal{D}^{(1/2)*}_{\lambda_2\mu_2}(R_{\Lambda_{k_1}k_2}) \mathcal{D}^{(1/2)*}_{\lambda_3\mu_3}(R^\rho_{\Lambda_{k_1}k_3}) M^{\rho\rho'}_{\mu_2\mu_2',\mu_3\mu_3'}(\tilde{p},\tilde{p}';W_q) \mathcal{D}^{(1/2)}_{\mu_2'\lambda_2'}(R_{\Lambda_{k_1}k_2'})$$
$$\times \mathcal{D}^{(1/2)}_{\mu_3'\lambda_3'}(R^{\rho'}_{\Lambda_{k_1}k_3'}) g^{\rho'}(\tilde{p}',q) \Gamma^{\rho'}_{\lambda_2'\lambda_1'\lambda_3'}(k_2',k_1,k_3'), \tag{3.34}$$

where $R^\rho_{\Lambda k}$ is the Wigner rotation for the spinor u^ρ, and

$$\Gamma^\rho_{\lambda_1\lambda_2\lambda_3}(k_1,k_2,k_3) = \overline{u^\rho_\gamma}(\mathbf{k_3},\lambda_3)\Gamma_{\lambda_1\lambda_2\gamma}(k_1,k_2,k_3),$$

$$M^{\rho\rho'}_{\mu_2\mu_2',\mu_3\mu_3'}(\tilde{p},\tilde{p}';W_q)$$
$$= \overline{u^\rho_\gamma}(\tilde{\mathbf{k}}_\mathbf{3},\mu_3) M_{\mu_2\mu_2',\gamma\gamma'}(\tilde{p},\tilde{p}';W_q) u^{\rho'}_{\gamma'}(\tilde{\mathbf{k}}_\mathbf{3}',\mu_3'). \tag{3.35}$$

We have reduced the three-body equations to six-dimensional integral equations for the coupled set of $2^4=16$ amplitudes $\Gamma^\rho_{\lambda_1\lambda_2\lambda_3}$, which can be written

$$\Gamma^\rho_{\lambda_1\lambda_2\lambda_3}(k_1,k_2,k_3) = \langle k_1\lambda_1(k_2\lambda_2\overline{k_3}\lambda_3)\rho|\Gamma\rangle. \tag{3.36}$$

The new states $|k_1\lambda_1(k_2\lambda_2\overline{k_3}\lambda_3)\rho\rangle$ have simple completeness and orthogonality relations (developed in the next section) which make them a useful starting point for further discussion.

This form (3.34) for the three-body equations displays the Wigner rotations which appear when the two-body scattering amplitude is boosted from the overall three-body rest frame to its two-body rest frame. For practical calculations the equations will be further reduced by decomposing the amplitudes into partial waves, which will be discussed in the next section.

IV. ANGULAR MOMENTUM STATES

In this section we follow the conventions of Wick [5] and define a basis of three-body partial wave helicity states. Completeness and orthogonality relations are defined and the matrix elements of the propagator and two-body scattering amplitude are obtained. Using these states, the operator equations (2.38) are written directly in terms of the partial wave states. To obtain the final equations, the matrix elements of the permutation operator must be evaluated, and this is done in the following section.

A. Construction of the states

The three-body states are constructed in three stages. First, we construct the state of particles 2 and 3 in its rest system, choosing the momenta so that $\mathbf{k}_2$ lies in the xz plane with k_{2x} positive, as shown in Fig. 4. By convention, particle three is off shell, and requires both u ($\rho=+$) and v ($\rho=-$) spinors to describe its Dirac structure. This degree of freedom is referred to as the ''ρ spin'' of the off-shell particle. Next, we boost the (23) system to a frame with three-momentum $\mathbf{q}=-\mathbf{k}_1$ in the positive z direction, and take the direct product of this state with the state of particle one with its three-momentum $\mathbf{k}_1$ in the negative z direction. Finally, we obtain the partial wave states by an angular average over the Euler angles $\{\Phi,\Theta,\phi\}$, as defined below. In shorthand, this three-body state is denoted $|1(23)\rangle$, to remind us that particles 2 and 3 are the pair which was boosted from their rest system.

Begin with the construction of the state for particle 2 with momentum $|\tilde{\mathbf{k}}_2| = \tilde{p}$ pointing in the positive z direction, and with helicity λ_2. This state will be denoted by $|(\tilde{p},0,0),\lambda_2\rangle$, where the second two arguments in the parentheses are the polar and azimuthal angles of the momentum. The state with momentum pointing in an arbitrary direction can be obtained by applying a rotation operator $R_{\phi,\tilde{\theta},\gamma} = e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\gamma J_z}$ through Euler angles ϕ, θ, and γ. For vectors without internal structure, we need only two angles, and following Wick characterize the states by the polar angles $\tilde{\theta}$ and ϕ, and represent the rotations by $R_{\phi,\tilde{\theta},0}$, so that

$$|(\tilde{p},\tilde{\theta},\phi),\lambda_2\rangle = R_{\phi,\tilde{\theta},0}|(\tilde{p},0,0),\lambda_2\rangle. \tag{4.1}$$

Note that this differs by a phase from the convention adopted in Jacob and Wick [29] and used in Ref. [28], where the rotation was defined to be $R_{\phi,\tilde{\theta},-\phi}$ instead of $R_{\phi,\tilde{\theta},0}$. The phase difference is

$$R_{\phi,\tilde{\theta},-\phi}|(\tilde{p},0,0),\lambda_2\rangle = e^{i\phi\lambda_2} R_{\phi,\tilde{\theta},0}|(\tilde{p},0,0),\lambda_2\rangle. \tag{4.2}$$

As discussed in Wick [5], the new phase convention turns out to have significant advantages for the treatment of the three-body system, and *gives identical results if* $\phi=0$, where the two-body states were previously defined [28].

The state for particle 3, which *in the rest system of the pair* has a momentum of the same magnitude but opposite in direction, is defined

$$|(\tilde{p},\tilde{\theta},\phi),\lambda_3\rho\rangle = R_{\phi,\tilde{\theta},0}|(\tilde{p},\pi,\pi),\lambda_3\rho\rangle, \tag{4.3}$$

where $\rho=\pm$ is the ρ-spin of the state (for more details, see the discussion below), and the following phase convention is incorporated into the definition of the state $|(\tilde{p},\pi,\pi),\lambda_3\rho\rangle$:

$$|(\tilde{p},\pi,\pi),\lambda_3\rho\rangle = e^{-i\pi s_3}R_{\pi,\pi,0}|(\tilde{p},0,0),\lambda_3\rho\rangle, \tag{4.4}$$

where s_3 is the spin of particle 3 (in our case $s_3=1/2$). The phase factor $e^{-i\pi s_3}$ is precisely what is needed for the definition Eq. (4.3) to agree with the phase convention of Jacob and Wick [29] for ''particle 2,'' which was used previously in Ref. [28]. To see this, recall that

$$R_{\pi,\pi,0}=e^{-i\pi J_z}e^{-i\pi J_y}=e^{-i\pi J_y}e^{i\pi J_z} \tag{4.5}$$

and hence

$$\begin{aligned} e^{-i\pi s_3}R_{\pi,\pi,0}|(\tilde{p},0,0),\lambda_3\rho\rangle &= e^{-i\pi s_3}e^{-i\pi J_y}e^{i\pi J_z}|(\tilde{p},0,0),\lambda_3\rho\rangle \\ &= e^{-i\pi(s_3-\lambda_3)}R_{0,\pi,0}|(\tilde{p},0,0),\lambda_3\rho\rangle \\ &= (-1)^{s_3-\lambda_3}R_{0,\pi,0}|(\tilde{p},0,0),\lambda_3\rho\rangle, \end{aligned} \tag{4.6}$$

as used in Ref. [28]. (According to our phase conventions, the value of J_z is independent of ρ; see Eq. (A9) of Ref. [28].) The two-particle state (in its rest system) is now written

$$\begin{aligned} |(\tilde{p},\tilde{\theta},\phi),\lambda_2\lambda_3\rho\rangle &= R_{\phi,\tilde{\theta},0}|(\tilde{p},0,0),\lambda_2\lambda_3\rho\rangle \\ &= R_{\phi,\tilde{\theta},0}(|(\tilde{p},0,0),\lambda_2\rangle\otimes|(\tilde{p},\pi,\pi),\lambda_3\rho\rangle), \end{aligned} \tag{4.7}$$

where we emphasize that the phase $e^{-i\pi s_3}$ is included in the definition of $|(\tilde{p},\pi,\pi),\lambda_3\rho\rangle$, as given in Eq. (4.4). Two particle states of definite total angular momentum and total helicity can be projected from these general two-particle states by integrating over the polar and azimuthal angles

$$|\tilde{p}jm,\lambda_2\lambda_3\rho\rangle = \eta_j\int_0^{2\pi}d\phi\int_0^{\pi}d\tilde{\theta}\sin\tilde{\theta}\,\mathcal{D}^{(j)*}_{m,\lambda_2-\lambda_3}(\phi,\tilde{\theta},0)R_{\phi,\tilde{\theta},0}|(\tilde{p},0,0),\lambda_2\lambda_3\rho\rangle, \tag{4.8}$$

where we use the abbreviation

$$\eta_j=\left(\frac{2j+1}{4\pi}\right)^{1/2}. \tag{4.9}$$

The next step is to boost the two-particle state in the direction of the positive z axis such that its total three-momentum becomes q. The required boost operator will be denoted Z_q (it is equal to Λ_q^{-1} of the last section), and the (23) pair can be treated like an elementary particle with momentum q in the positive z direction, with ''spin'' j and ''helicity'' m, and a mass W_q given by $W_q^2=(P-k_1)^2$. The boosted state of the pair is no longer an eigenstate of the single-particle helicities, because a boost which is not in the direction of a particle's momentum mixes helicities. The three-body helicity state is then constructed by taking a direct product of the boosted (23) pair and the state of the single particle 1 with a momentum of magnitude q in the negative z direction. For consistency, the same phase convention is used to define the state of particle 1 that was used before to define particle 3, i.e.,

$$|(\tilde{q},\pi,\pi),\lambda_1\rangle = e^{-i\pi s_1}R_{\pi,\pi,0}|(q,0,0),\lambda_1\rangle. \tag{4.10}$$

This gives a three-body helicity state with total three-momentum zero and the momentum of the pair in the positive z direction:

$$|(q,0,0),\tilde{p}jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle = |(q,0,0),\lambda_1\rangle\otimes|\tilde{p}jm,\lambda_2\lambda_3\rho\rangle. \tag{4.11}$$

Following the convention for rotation of states first introduced in Eq. (4.1), the state in which the momentum of the pair is in an arbitrary direction is obtained from Eq. (4.11) by applying the rotation $R_{\Phi,\Theta,0}$

$$|(q,\Theta,\Phi),\tilde{p}jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle = R_{\Phi,\Theta,0}|(q,0,0),\tilde{p}jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle. \tag{4.12}$$

Finally, the three-body helicity states with fixed total angular momentum J and projection M are obtained from the states (4.12) by the angular average

$$|qJM,\widetilde{p}jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle = \eta_J\int_0^{2\pi}d\Phi\int_0^{\pi}d\Theta\sin\Theta\,\mathcal{D}^{(J)*}_{M,m-\lambda_1}(\Phi,\Theta,0)|(q,\Theta,\Phi),\widetilde{p},jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle$$

$$= \eta_J\eta_j\int_0^{2\pi}d\Phi\int_0^{\pi}d\Theta\sin\Theta\int_0^{2\pi}d\phi\int_0^{\pi}d\widetilde{\theta}\sin\widetilde{\theta}\,\mathcal{D}^{(J)*}_{M,m-\lambda_1}(\Phi,\Theta,0)\mathcal{D}^{(j)*}_{m,\lambda_2-\lambda_3}(\phi,\widetilde{\theta},0)$$

$$\times R_{\Phi,\Theta,0}\{|(q,\pi,\pi),\lambda_1\rangle\otimes Z_q R_{\phi,\widetilde{\theta},0}|(\widetilde{p},0,0),\lambda_2\lambda_3\rho\rangle\}, \tag{4.13}$$

where η_J is obtained from Eq. (4.9) by replacing j with J . Note that this expression contains the two-body partial wave states (4.8), and if we denote the rotation $R_{\Phi,\Theta,0}$ by R_U, and

$$\int dU = \int_0^{2\pi}d\Phi\int_0^{\pi}d\Theta\sin\Theta,$$

$$\mathcal{D}^{(J)*}_{M,m-\lambda_1}(\Phi,\Theta,0) = \mathcal{D}^{(J)*}_{M,m-\lambda_1}(U),$$

we have

$$|qJM,\widetilde{p}jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle = \eta_J\int dU\mathcal{D}^{(J)*}_{M,m-\lambda_1}(U)R_U\{|(q,\pi,\pi),\lambda_1\rangle\otimes Z_q|\widetilde{p}jm,\lambda_2\lambda_3\rho\rangle\}. \tag{4.14}$$

Another useful form of Eq. (4.13) is obtained by exploiting the fact that a rotation about the z axis commutes with a boost in z direction, so that the operation of the rotations on the (23) pair can be written

$$R_{\Phi,\Theta,0}Z_qR_{\phi,\widetilde{\theta},0} = R_{\Phi,\Theta,0}R_{0,0,\phi}Z_qR_{0,\widetilde{\theta},0} = R_{\Phi,\Theta,\phi}Z_qR_{0,\widetilde{\theta},0}. \tag{4.15}$$

On the other hand, the rotation of particle 1 can be written

$$R_{\Phi,\Theta,0} = R_{\Phi,\Theta,\phi}R_{0,0,-\phi}\rightarrow R_{\Phi,\Theta,\phi}e^{-i\phi\lambda_1}, \tag{4.16}$$

where the last step is obtained by letting $R_{0,0,-\phi}$ operate on $|(q,\pi,\pi),\lambda_1\rangle$, and recalling that this state is an eigenstate of J_z with projection $-\lambda_1$. Finally, noting that

$$\mathcal{D}^{(J)*}_{M,m-\lambda_1}(\Phi,\Theta,0)\mathcal{D}^{(j)*}_{m,\lambda_2-\lambda_3}(\phi,\widetilde{\theta},0) = e^{i\lambda_1\phi}\mathcal{D}^{(J)*}_{M,m-\lambda_1}(\Phi,\Theta,\phi)\mathcal{D}^{(j)*}_{m,\lambda_2-\lambda_3}(0,\widetilde{\theta},0) = e^{i\lambda_1\phi}\mathcal{D}^{(J)*}_{M,m-\lambda_1}(\Phi,\Theta,\phi)d^{j}_{m,\lambda_2-\lambda_3}(\widetilde{\theta}) \tag{4.17}$$

shows that the factors of $e^{i\lambda_1\phi}$ cancel, and that Eq. (4.13) can be written

$$|qJM,\widetilde{p}jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle = \eta_J\eta_j\int dS\mathcal{D}^{(J)*}_{M,m-\lambda_1}(S)\int_0^{\pi}d\widetilde{\theta}\sin\widetilde{\theta}\,d^{(j)}_{m,\lambda_2-\lambda_3}(\widetilde{\theta})R_S|k_1^o\lambda_1(k_2^o\lambda_2\overline{k_3^o}\lambda_3)\rho\rangle, \tag{4.18}$$

where

$$|k_1^0\lambda_1(k_2^0\lambda_2\overline{k_3^0}\lambda_3)\rho\rangle = |(q,\pi,\pi),\lambda_1\rangle\otimes Z_qR_{0,\widetilde{\theta},0}|(\widetilde{p},0,0),\lambda_2\lambda_3\rho\rangle \tag{4.19}$$

is the three-body state in its *canonical configuration* in the xz plane with special four-momenta k_1^0,k_2^0, and k_3^0, as shown in Fig. 4, $R_S = R_{\Phi,\Theta,\phi}$ is the rotation which carries the three-body system from its canonical configuration to the most general orientation described by Euler angles Φ,Θ, and ϕ, and

$$\int dS = \int_0^{2\pi}d\Phi\int_0^{\pi}d\Theta\sin\Theta\int_0^{2\pi}d\phi,$$

$$\mathcal{D}^{(J)*}_{M,m-\lambda_1}(S) = \mathcal{D}^{(J)*}_{M,m-\lambda_1}(\Phi,\Theta,\phi). \tag{4.20}$$

Equation (4.19) shows that the canonical three-body configuration is constructed by starting from a two-body state in the two-body rest frame where the relative momentum of the two particles is restricted to the xz plane with polar angle $\widetilde{\theta}$, then boosting this state in the positive z direction, and finally adding the spectator (particle 1) with momentum along the negative

z axis. Since the most general rotation R_S is performed *after* the boost Z_q, the Wigner rotations that accompany the boost are all rotations about the y axis, which greatly simplifies the calculation.

The results (4.14) and (4.18) are equivalent, and either may be used to evaluate matrix elements.

B. Representation of the states

In the previous subsection we showed how the states

$$|k_1\lambda_1(k_2\lambda_2\overline{k_3}\lambda_3)\rho\rangle = R_S|k_1^0\lambda_1(k_2^0\lambda_2\overline{k_3^0}\lambda_3)\rho\rangle, \tag{4.21}$$

introduced abstractly in Eq. (3.36), are to be constructed. These states can also be written as a direct product of the momentum space plane wave states introduced in Eq. (3.4) and Dirac helicity spinors:

$$|k_1\lambda_1(k_2\lambda_2\overline{k_3}\lambda_3)\rho\rangle = e^{-i\pi(s_1+s_3)}|k_1(k_2\overline{k_3})\rangle\otimes R_S([R_{\pi,\pi,0}\,u(q,\lambda_1)]_\alpha\, Z_q R_{0,\tilde{\theta},0}\{u_\beta(\tilde{p},\lambda_2)[R_{\pi,\pi,0}u^\rho(\tilde{p},\lambda_3)]_\gamma\}), \tag{4.22}$$

where α, β, and γ are the Dirac indices of particles 1, 2, and 3, respectively, and all rotations are displayed explicitly, so that all spinor states in Eq. (4.22) are ''particle 1'' states (in the sense of Jacob and Wick). Explicitly

$$u(p,\lambda)=\begin{pmatrix}\cosh(\eta/2)\\ 2\lambda\sinh(\eta/2)\end{pmatrix}\chi(\lambda),\quad v(p,\lambda)=\begin{pmatrix}-2\lambda\sinh(\eta/2)\\ \cosh(\eta/2)\end{pmatrix}\chi(\lambda), \tag{4.23}$$

with

$$\chi(\tfrac{1}{2})=\begin{pmatrix}1\\0\end{pmatrix},\quad \chi(-\tfrac{1}{2})=\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.24}$$

and

$$\tanh\eta=\frac{p}{E_p}. \tag{4.25}$$

Since the helicity spinors depend only on the *magnitude* of the three momentum $\mathbf{p}$, we have used the notation $v(p,\lambda)=v(-\mathbf{p},\lambda)$, so that the correspondence given in Eq. (3.31) now becomes

$$u^\rho(p,\lambda)=\begin{cases}u(p,\lambda) & \text{if } \rho=+,\\ v(p,\lambda) & \text{if } \rho=-.\end{cases} \tag{4.26}$$

All of these conventions are consistent with our previous work [check Eq. (A9) of Ref. [28] with $i=1$].

Using the representation (4.22), and the orthogonality relations (3.5) we obtain *generalized* orthogonality relations for the three-particle helicity states (4.21)

$$\begin{aligned}\langle 1'(2'3')\rho'|1(23)\rho\rangle &= \langle k_1'\lambda_1'(k_2'\lambda_2'\overline{k_3'}\lambda_3')\rho'|k_1\lambda_1(k_2\lambda_2\overline{k_3}\lambda_3)\rho\rangle = \langle k_1'(k_2'\overline{k_3'})|k_1(k_2\overline{k_3})\rangle\\ &\times[\overline{u}(q',\lambda_1')R^{-1}_{\pi,\pi,0}R_{S'}^{-1}R_SR_{\pi,\pi,0}u(q,\lambda_1)][\overline{u}(\tilde{p}',\lambda_2')R^{-1}_{0,\tilde{\theta}',0}Z_{q'}^{-1}R_{S'}^{-1}R_SZ_qR_{0,\tilde{\theta},0}u(\tilde{p},\lambda_2)]\\ &\times[\overline{u^\rho}'(\tilde{p}',\lambda_3')R^{-1}_{\pi,\pi,0}R^{-1}_{0,\tilde{\theta}',0}Z_{q'}^{-1}R_{S'}^{-1}R_SZ_qR_{0,\tilde{\theta},0}R_{\pi,\pi,0}u^\rho(\tilde{p},\lambda_3)]\\ &= 2E_{k_1}\delta^3(k_1-k_1')2E_{k_2}\delta^3(k_2-k_2')\delta^4(P-P')\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}[\overline{u^\rho}'(\tilde{p},\lambda_3')u^\rho(\tilde{p},\lambda_3)].\end{aligned} \tag{4.27}$$

Note that the states are *not* orthogonal in ρ space. Using Eq. (4.23) gives

$$[\overline{u^\rho}'(\tilde{p},\lambda_3')u^\rho(\tilde{p},\lambda_3)]=\delta_{\lambda_3'\lambda_3}O_{\rho'\rho}(\tilde{p},\lambda_3), \tag{4.28}$$

where, if $\rho=+$ is the first column and $\rho=-$ the second, the matrix representation of O is

$$O_{\rho'\rho}(p,\lambda)=\begin{pmatrix}1 & -2\lambda\sinh\eta\\ -2\lambda\sinh\eta & -1\end{pmatrix}=(\tau_3)_{\rho'\rho}-2\lambda\sinh\eta(\tau_1)_{\rho'\rho}, \tag{4.29}$$

where $\sinh\eta=p/m$. It is also useful to express the covariant product $2E_{k_2}\delta^3(k_2-k_2')$ in the rest frame of the two-body subsystem, where $\mathbf{k}_2=\tilde{\mathbf{p}}$ depends on the polar angles $\tilde{\theta}$ and ϕ. Hence the generalized orthogonality relation will be written

$$\langle 1'(2'3')\rho'|1(23)\rho\rangle=\langle k_1'\lambda_1'(k_2'\lambda_2'\overline{k_3'}\lambda_3')\rho'|k_1\lambda_1(k_2\lambda_2\overline{k}_3\lambda_3)\rho\rangle=\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\ \delta_{\lambda_3'\lambda_3}O_{\rho'\rho}(\tilde{p},\lambda_3)2E_{k_1}\delta^3(k_1-k_1')$$
$$\times 2E_{\tilde{p}}\delta^3(\tilde{p}'-\tilde{p})\,\delta^4(P'-P). \tag{4.30}$$

Even though the states (4.21) are not orthogonal, the matrix $O(\tilde{p},\lambda)$ has a simple property which will enable us to carry out the calculation much as it was. From Eq. (4.29) we obtain

$$[O(p,\lambda)O(p,\lambda)]_{\rho'\rho}=\delta_{\rho'\rho}\cosh^2\eta=\delta_{\rho'\rho}\frac{E_p^2}{m^2}. \tag{4.31}$$

Using this we can show that the completeness relation for the states can be written

$$\mathbf{1}\equiv\mathcal{Q}_{\alpha'\alpha}\mathcal{Q}_{\beta'\beta}\delta_{\gamma'\gamma}=\int\frac{d^3k_1}{2E_{k_1}}\frac{d^3\tilde{p}}{2E_{\tilde{p}}}\frac{m^2}{E_{\tilde{p}}^2}d^4P\sum_{\lambda_1\lambda_2\lambda_3\rho'\rho}|1(23)\rho'\rangle O_{\rho'\rho}(\tilde{p},\lambda_3)\langle 1(23)\rho|$$
$$=\int\frac{d^3k_1}{2E_{k_1}}\frac{d^3\tilde{p}}{2E_{\tilde{p}}}\frac{m^2}{E_{\tilde{p}}^2}d^4P\sum_{\lambda_1\lambda_2\lambda_3\rho'\rho}|k_1\lambda_1(k_2\lambda_2\overline{k}_3\lambda_3)\rho'\rangle O_{\rho'\rho}(\tilde{p},\lambda_3)\langle k_1\lambda_1(k_2\lambda_2\overline{k}_3\lambda_3)\rho|, \tag{4.32}$$

where $\mathcal{Q}_1=\mathcal{Q}_{\alpha'\alpha}$ is the positive energy projection operator for particle 1 (with Dirac indices α and α') introduced in Sec. II B. Equation (4.32) tells us that the states span only the positive energy sectors of particles 1 and 2 (which is sufficient) but they span the entire four-dimensional Dirac space for the off-shell particle 3.

We will only describe the emergence of the factor $\delta_{\gamma'\gamma}$ in the derivation of the completeness relation (4.32). To see how this factor emerges, evaluate the sum over λ_3, ρ, and ρ' explicitly using Eq. (4.23):

$$\frac{m^2}{E_{\tilde{p}}^2}\sum_{\lambda_3\rho'\rho}u_{\gamma'}^{\rho'}(\tilde{p},\lambda_3)O_{\rho'\rho}(\tilde{p},\lambda_3)\overline{u}_{\gamma}^{\rho}(\tilde{p},\lambda_3)=\frac{m^2}{E_{\tilde{p}}^2}\left\{\sum_{\lambda_3\rho}\rho\ u_{\gamma'}^{\rho}(\tilde{p},\lambda_3)\overline{u}_{\gamma}^{\rho}(\tilde{p},\lambda_3)-\sinh\tilde{\eta}\sum_{\lambda_3,\rho\neq\rho'}2\lambda_3u_{\gamma'}^{\rho'}(\tilde{p},\lambda_3)\overline{u}_{\gamma}^{\rho}(\tilde{p},\lambda_3)\right\}$$
$$=\frac{m^2}{E_{\tilde{p}}^2}\begin{bmatrix}1 & -\tau_3\sinh\tilde{\eta}\\ \tau_3\sinh\tilde{\eta} & 1\end{bmatrix}_{\gamma'\gamma}+\frac{m^2\sinh\tilde{\eta}}{E_{\tilde{p}}^2}\begin{bmatrix}\sinh\tilde{\eta} & \tau_3\\ -\tau_3 & \sinh\tilde{\eta}\end{bmatrix}_{\gamma'\gamma}=\delta_{\gamma'\gamma}\frac{m^2}{E_{\tilde{p}}^2}\cosh^2\tilde{\eta}$$
$$=\delta_{\gamma'\gamma}. \tag{4.33}$$

Subsequent operations by the rotations and boosts leave this factor invariant. In the same way, the sums over λ_1 and λ_2 give the projection operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

We will now use these relations to work out the generalized orthogonality and completeness relations for the partial wave amplitudes (4.18).

C. Generalized orthogonality and completeness relations

Using the generalized orthogonality relations (4.30), the definition (4.21), and the notation

$$|J'1''(2''3'')\rho\rangle=|q''J'M',\tilde{p}''j'm';\lambda_1''(\lambda_2''\lambda_3'')\rho\rangle, \tag{4.34}$$

we obtain

$$\langle J'1'(2'3')\rho'|J1(23)\rho\rangle=\langle q'J'M',\tilde{p}'j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|qJM,\tilde{p}jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle=\eta_{J'}\eta_{j'}\eta_J\eta_j\int dS'\mathcal{D}^{(J')}_{M',m'-\lambda_1'}(S')$$
$$\times\int dS\mathcal{D}^{(J)*}_{M,m-\lambda_1}(S)\int_0^\pi d\tilde{\theta}'\sin\tilde{\theta}'d^{(j')}_{m',\lambda_2'-\lambda_3'}(\tilde{\theta}')\int_0^\pi d\tilde{\theta}\sin\tilde{\theta}d^{(j)}_{m,\lambda_2-\lambda_3}(\tilde{\theta})$$
$$\times\langle 1'(2'3')\rho'|1(23)\rho\rangle=\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\ \delta_{\lambda_3'\lambda_3}\eta_{J'}\eta_{j'}\eta_J\eta_j\int dS'\mathcal{D}^{(J')}_{M',m'-\lambda_1}(S')$$
$$\times\int dS\mathcal{D}^{(J)*}_{M,m-\lambda_1}(S)\int_0^\pi d\tilde{\theta}'\sin\tilde{\theta}'d^{(j')}_{m',\lambda_2-\lambda_3}(\tilde{\theta}')\int_0^\pi d\tilde{\theta}\sin\tilde{\theta}d^{(j)}_{m,\lambda_2-\lambda_3}(\tilde{\theta})$$
$$\times O_{\rho'\rho}(\tilde{p},\lambda_3)2E_{k_1}\delta^3(k_1-k_1')2E_{\tilde{p}}\delta^3(\tilde{p}'-\tilde{p})\,\delta^4(P'-P). \tag{4.35}$$

Writing the δ^3 functions in polar coordinates

$$2E_{k_1}\delta^3(k_1-k_1')2E_{\tilde{p}}\delta^3(\tilde{p}'-\tilde{p})=2E_q\frac{\delta(q'-q)}{q^2}\delta(\cos\Theta'-\cos\Theta)\,\delta(\Phi'-\Phi)2E_{\tilde{p}}\frac{\delta(\tilde{p}'-\tilde{p})}{\tilde{p}^2}\delta(\cos\tilde{\theta}'-\cos\tilde{\theta})\,\delta(\phi'-\phi), \tag{4.36}$$

allows us to integrate easily over dS' and $d\tilde{\theta}'$, giving $S'=S$ and $\tilde{\theta}'=\tilde{\theta}$. The remaining integrals over dS and $d\tilde{\theta}$ can then be easily done using the orthogonality properties of the $\mathcal{D}$ and d functions. We obtain

$$\langle J'1'(2'3')\rho'|J1(23)\rho\rangle=\delta_{J'J}\delta_{M'M}\delta_{j'j}\delta_{m'm}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\delta_{\lambda_3'\lambda_3}O_{\rho'\rho}(\tilde{p},\lambda_3)2E_q\frac{\delta(q-q')}{q^2}2E_{\tilde{p}}\frac{\delta(\tilde{p}-\tilde{p}')}{\tilde{p}^2}\delta^4(P'-P). \tag{4.37}$$

Using Eq. (4.32) and the orthogonality of the $\mathcal{D}$ and d functions, the completeness relation for the partial wave states can be derived

$$\mathbf{1}\equiv\mathcal{Q}_{\alpha'\alpha}\mathcal{Q}_{\beta'\beta}\delta_{\gamma'\gamma}=\int\frac{q^2dq}{2E_q}\frac{\tilde{p}^2d\tilde{p}}{2E_{\tilde{p}}}\frac{m^2}{E_{\tilde{p}}^2}d^4P\sum_{\substack{JMjm\\ \lambda_1\lambda_2\lambda_3\rho'\rho}}|J1(23)\rho'\rangle O_{\rho'\rho}(\tilde{p},\lambda_3)\langle J1(23)\rho|$$

$$=\int\frac{q^2dq}{2E_q}\frac{\tilde{p}^2d\tilde{p}}{2E_{\tilde{p}}}\frac{m^2}{E_{\tilde{p}}^2}d^4P\sum_{\substack{JMjm\\ \lambda_1\lambda_2\lambda_3\rho'\rho}}|qJM,\tilde{p}jm;\lambda_1(\lambda_2\lambda_3)\rho'\rangle O_{\rho'\rho}(\tilde{p},\lambda_3)\langle qJM,\tilde{p}jm;\lambda_1(\lambda_2\lambda_3)\rho|. \tag{4.38}$$

Note that this is consistent with Eq. (4.37).

D. Reduction of the equations

The partial wave expanded three-body equations can now be obtained directly from the operator equation (2.38). Restoring the projection operators $\mathcal{Q}$, this equation is

$$\mathcal{Q}_1\mathcal{Q}_2\Gamma^1=2\mathcal{Q}_2M^1G_3\mathcal{Q}_2[\mathcal{Q}_1\mathcal{Q}_2]\mathcal{P}_{12}[\mathcal{Q}_1\mathcal{Q}_2]\mathcal{Q}_1\mathcal{Q}_2\Gamma^1, \tag{4.39}$$

where we have made frequent use of the fact that $\mathcal{Q}_i$ operates only in the space of particle i, and hence commutes with all operators which operate only on particles $j\neq i$, and the property $\mathcal{Q}_i\mathcal{Q}_i=\mathcal{Q}_i$. Replacing the two terms $[\mathcal{Q}_1\mathcal{Q}_2]\otimes\mathbf{1}$ with the completeness relation (4.38), and using the relation

$$\mathcal{Q}_1\mathcal{Q}_2|J1(23)\rho\rangle=|J1(23)\rho\rangle, \tag{4.40}$$

we obtain

$$\langle J1(23)\rho|\Gamma^1\rangle=2\sum_{J'M'j'm'}\sum_{\substack{\lambda_1''\lambda_2''\lambda_3''\rho_4\rho_3\\ \lambda_1'\lambda_2'\lambda_3'\rho_2\rho_1}}\int\frac{q''^2dq''}{2E_{q''}}\frac{\tilde{p}''^2d\tilde{p}''}{2E_{\tilde{p}''}}\frac{m^2}{E_{\tilde{p}''}^2}\int\frac{q'^2dq'}{2E_{q'}}\frac{\tilde{p}'^2d\tilde{p}'}{2E_{\tilde{p}'}}\frac{m^2}{E_{\tilde{p}'}^2}$$
$$\times\langle J1(23)\rho|M^1G_3|J1''(2''3'')\rho_4\rangle O_{\rho_4\rho_3}(\tilde{p}'',\lambda_3'')\langle J1''(2''3'')\rho_3|\mathcal{P}_{12}|J'1'(2'3')\rho_2\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')$$
$$\times\langle J'1'(2'3')\rho_1|\Gamma^1\rangle. \tag{4.41}$$

This result anticipates the overall conservation of the total four-momentum, P, and the conservation of the quantum numbers J, M, j, and m by the operator M^1G_3.

The matrix elements of the operator M^1G_3 are most easily evaluated using Eq. (4.14). Recalling that M^1G_3 operates only in the two-body subspace (23), the matrix element is

$$\langle J1(23)\rho|M^1G_3|J'1'(2'3')\rho_2\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')=\eta_J\eta_{J'}\int dU\int dU'\mathcal{D}^{(J)}_{M,m-\lambda_1}(U)\mathcal{D}^{(J')*}_{M',m'-\lambda_1'}(U')$$
$$\times\langle(q,\pi,\pi),\lambda_1|R_U^{-1}R_{U'}|(q,\pi,\pi),\lambda_1\rangle$$
$$\times\langle\tilde{p}jm,\lambda_2\lambda_3\rho|R_U^{-1}Z_q^{-1}M^1G_3Z_{q'}R_{U'}|\tilde{p}'j'm',\lambda_2'\lambda_3'\rho_2\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3'). \tag{4.42}$$

The orthogonality relation for single-particle states is

$$\langle (q,\pi,\pi),\lambda_1|R_U^{-1}R_{U'}|(q,\pi,\pi),\lambda_1\rangle=\langle (q,\Theta,\Phi),\lambda_1|(q',\Theta',\Phi'),\lambda_1'\rangle=\delta_{\lambda_1\lambda_1'}2E_q\delta^{(3)}(q-q')$$

$$=\delta_{\lambda_1\lambda_1'}2E_q\frac{\delta(q-q')}{q^2}\,\delta(\cos\Theta-\cos\Theta')\,\delta(\Phi-\Phi'). \tag{4.43}$$

Inserting this, integrating over dU' (which fixes $U'=U$), noting that the boost operator Z_q and the rotation operator R_U commute with M^1G_3, and carrying out the U integration using the orthogonality of the $\mathcal{D}$ functions gives

$$\langle J1(23)\rho|M^1G_3|J'1'(2'3')\rho_2\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')=\delta_{JJ'}\delta_{MM'}\delta_{jj'}\delta_{mm'}\delta_{\lambda_1\lambda_1'}2E_q\frac{\delta(q-q')}{q^2}\langle j(23)\rho|M^1G_3|j(2'3')\rho_2\rangle$$

$$\times O_{\rho_2\rho_1}(\tilde{p}',\lambda_3'), \tag{4.44}$$

where we have used the following shortened notation:

$$|j(2'3')\rho_2\rangle=|\tilde{p}'jm,\lambda_2'\lambda_3'\rho_2\rangle \tag{4.45}$$

and removed from the two-body matrix element of M^1G_3 the factor of $\delta_{jj'}\delta_{mm'}$. Now, including the projection operators, the propagator G_2^3 in Dirac space is

$$G_2^3=G_3\mathcal{Q}_2\to(m+\not{k}_2)\frac{(m+\not{k}_3)}{m^2-k_3^2-i\epsilon}\to 2m\,\mathcal{Q}_2\,\frac{m}{E_{\tilde{p}'}}[u^{\rho'}(\tilde{p}',\lambda_3'')g^{\rho'}(q,\tilde{p}')\,\overline{u^{\rho}}'(\tilde{p}',\lambda_3'')], \tag{4.46}$$

where we inserted the decomposition (3.32) with $g^{\rho}(q,\tilde{p})$ given in Eq. (3.33). The only dependence of the two-body helicity states $|j(2'3')\rho_2\rangle$ on ρ_2 and λ_3' comes from the factor $u^{\rho_2}(\tilde{p}',\lambda_3')$, which leads to the following property (where there is no sum over ρ'):

$$[u^{\rho'}(\tilde{p}',\lambda_3'')\,\overline{u^{\rho}}'(\tilde{p}',\lambda_3'')]|j(2'3')\rho_2\rangle=|j(2'3'')\rho'\rangle[\,\overline{u^{\rho}}'(\tilde{p}',\lambda_3'')u^{\rho_2}(\tilde{p}',\lambda_3')]. \tag{4.47}$$

Using Eqs. (4.46) and (4.47), and recalling Eq. (4.31), the two-body matrix element in Eq. (4.44) is reduced as follows:

$$\sum_{\rho_2}\langle j(23)\rho|M^1G_3|j(2'3')\rho_2\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')$$

$$=2m\frac{m}{E_{\tilde{p}'}}\sum_{\rho'\rho_2\lambda_3''}\langle j(23)\rho|M^1[u^{\rho'}(\tilde{p}',\lambda_3'')g^{\rho'}(q,\tilde{p}')\,\overline{u^{\rho}}'(\tilde{p}',\lambda_3'')]|j(2'3')\rho_2\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')$$

$$=2m\frac{m}{E_{\tilde{p}'}}\sum_{\rho'\rho_2\lambda_3''}\langle j(23)\rho|M^1|j(2'3'')\rho'\rangle g^{\rho'}(q,\tilde{p}')[u^{\rho'}(\tilde{p}',\lambda_3'')\,\overline{u^{\rho_2}}(\tilde{p}',\lambda_3')]O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')$$

$$=2m\frac{m}{E_{\tilde{p}'}}\sum_{\rho'\rho_2\lambda_3''}\langle j(23)\rho|M^1|j(2'3'')\rho'\rangle g^{\rho'}(q,\tilde{p}')\delta_{\lambda_3''\lambda_3'}O_{\rho'\rho_2}(\tilde{p}',\lambda_3')O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')$$

$$=2m\frac{E_{\tilde{p}'}}{m}\langle j(23)\rho|M^1|j(2'3')\rho_1\rangle g^{\rho_1}(q,\tilde{p}'). \tag{4.48}$$

Inserting this result into Eq. (4.41) gives finally

$$\langle J1(23)\rho|\Gamma^1\rangle=2\sum_{J'M'j'm'}\sum_{\substack{\lambda_2''\lambda_3''\rho_3\\ \lambda_1'\lambda_2'\lambda_3'\rho_2\rho_1}}\int q'^2dq'\,\frac{m}{E_{q'}}\int\frac{\tilde{p}''^2d\tilde{p}''}{2E_{\tilde{p}''}}\frac{m}{E_{\tilde{p}''}}\int\frac{\tilde{p}'^2d\tilde{p}'}{2E_{\tilde{p}'}}\frac{m^2}{E_{\tilde{p}'}^2}\langle j(23)\rho|M^1|j(2''3'')\rho_3\rangle g^{\rho_3}(q,\tilde{p}'')$$

$$\times\langle J1(2''3'')\rho_3|\mathcal{P}_{12}|J'1'(2'3')\rho_2\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')\langle J'1'(2'3')\rho_1|\Gamma^1\rangle. \tag{4.49}$$

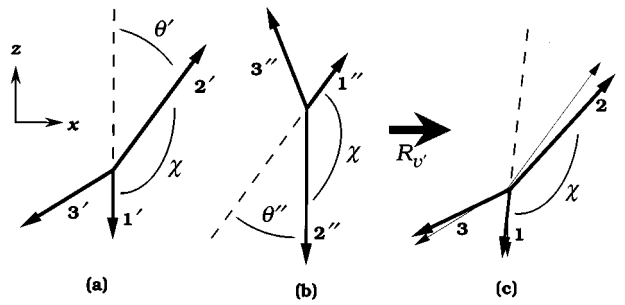

FIG. 5. (a) The momenta $k_1'^0, k_2'^0, k_3'^0$ in their canonical configuration. (b) The canonical configuration of the momenta $k_1''^0, k_2''^0, k_3''^0$ which result from the interchange of particles one and two. (c) Figure showing how $k_1''^0, k_2''^0, k_3''^0$ are rotated into k_1, k_2, k_3 by the rotation $R_{V'}$ which equals R_V only when k_1, k_2, k_3 line up precisely with $k_1'^0, k_2'^0, k_3'^0$.

In Appendix A we show that

$$\langle j(23)\rho|M^1|j(2''3'')\rho_3\rangle = \frac{E_{\tilde{p}}E_{\tilde{p}''}}{(2\pi)^3 m^2} M^{\rho\rho_3 j}_{\lambda_2\lambda_2'',\lambda_3\lambda_3''}(\tilde{p},\tilde{p}'';P_{23}), \tag{4.50}$$

where $M^{\rho\rho_3 j}_{\lambda_2\lambda_2'',\lambda_3\lambda_3''}(\tilde{p},\tilde{p}'';P_{23})$ are the *two-body amplitudes previously determined* from Ref. [28], Eq. (2.88).

To obtain the final three-body equations, the matrix elements of the permutation operator must be evaluated, which will be done in the following section.

V. THE PERMUTATION OPERATOR $\mathcal{P}_{12}$

In this section we will derive the matrix elements of the operator $\mathcal{P}_{12}$ which interchanges the states of particles 1 and 2. Using the shorthand notation given in Eq. (4.34), the action of the permutation operator is

$$\mathcal{P}_{12}|J1(23)\rho\rangle = |J2(13)\rho\rangle, \tag{5.1}$$

where the relation of the momenta of the individual particles to the relative momenta q and $\tilde{p}$, and to the quantum numbers j and m, is unambiguously determined by the order in which the single-particle names are written. For example, the state $|J2(13)\rho\rangle$ is one in which the second particle is the spectator with momentum q, and particles 1 and 3 are the pair with angular momentum j and m, and particle 1 has c.m. momentum variables $\tilde{p}$ and $\tilde{\theta}$. More precisely, from the result (4.18), the state $|J1(23)\rho\rangle$ is obtained by averaging over rotations $R_{S'}$ of a state with the canonical configuration $k_1'^0, k_2'^0, k_3'^0$, shown in Fig. 5(a), and the state $|J2(13)\rho\rangle$ is obtained by averaging over rotations $R_{S''}$ of a state with the canonical configuration $k_1''^0, k_2''^0, k_3''^0$, shown in Fig. 5(b). Note that the two configurations are related by interchanging particles 1 and 2, but the definition of the states requires that $\mathbf{k}_1'^0$ and $\mathbf{k}_2''^0$ both be in the negative z direction, and that $\mathbf{k}_2'^0$ and $\mathbf{k}_1''^0$ both lie in the xz plane with a positive x component.

Any three-body configuration with canonical orientations in the xz plane can be completely characterized by three variables. For the vectors in Fig. 5(a), these variables can be chosen to be $|\mathbf{k}_1'| = q'$, $|\tilde{\mathbf{k}}_2'| = \tilde{p}'$, and the angle χ between $\mathbf{k}_1'$ and $\mathbf{k}_2'$ (recall that $\tilde{\mathbf{k}}_2'$ is the vector $\mathbf{k}_2'$ in the c.m. of the pair). For the configuration in Fig. 5(b), the corresponding variables are $|\mathbf{k}_2''| = q''$, $|\tilde{\mathbf{k}}_1''| = \tilde{p}''$, and the same angle χ. If the total c.m. energy of the three-body system is fixed, then there is a constraint between these three variables, leaving only two independent. If q' and $\tilde{p}'$ are specified, the angle χ can be determined, and this was the approach taken by one of us previously [2]. However, the final equations are more tractable if q' and χ are specified, and $\tilde{p}'$ is determined by the constraints, and this is the approach we will take below.

Examination of the two configurations shown in Figs. 5(a) and 5(b) shows that the rotation $R_V = R_{\pi,\chi,0}$ [not to be confused with the rotation $R_{V'}$ discussed below which carries the configuration shown in Fig. 5(b) into Fig. 5(c)] will bring them into alignment, or

$$\begin{aligned} R_V k_1''^0 &= k_1'^0, \\ R_V k_2''^0 &= k_2'^0, \\ R_V k_3''^0 &= k_3''^0. \end{aligned} \tag{5.2}$$

Furthermore, since the final momenta k_1, k_2, and k_3 can be obtained either by rotating the (k'^0)'s through $R_{S'}$, or the (k''^0)'s through $R_{S''}$ (because they are equal), we have the relation

$$k_1 = R_{S'} k_1^0 = R_{S''} k_1''^0 = R_{S''} R_V k_1'^0 \tag{5.3}$$

which implies

$$R_{S'} = R_{S''} R_V. \tag{5.4}$$

This rotation R_V will eventually emerge from the derivation below.

A. Initial reduction of the matrix element

We now turn to the details of the evaluation of the matrix element. Using the form (4.18) for the three-body state, the matrix element of $\mathcal{P}_{12}$ can be written

$$\begin{aligned}\langle J'1'(2'3')\rho'|\mathcal{P}_{12}|J1(23)\rho\rangle &= \langle q'J'M',\widetilde{p}'j'm',\lambda_1'(\lambda_2'\lambda_3')\rho'|\mathcal{P}_{12}|qJM,\widetilde{p}jm,\lambda_1(\lambda_2\lambda_3)\rho\rangle \\ &= \langle q'J'M',\widetilde{p}'j'm',\lambda_1'(\lambda_2'\lambda_3')\rho'|q''JM,\widetilde{p}''jm,\lambda_2(\lambda_1\lambda_3)\rho\rangle \\ &= \eta_{J'}\eta_J\eta_{j'}\eta_j\int dS'\int dS''\mathcal{D}^{(J')}_{M',m'-\lambda_1'}(S')\mathcal{D}^{(J)*}_{M,m-\lambda_2}(S'') \\ &\quad\times\int_0^\pi d\widetilde{\theta}'\sin\widetilde{\theta}'\int_0^\pi d\widetilde{\theta}''\sin\widetilde{\theta}''d^{(j')}_{m',\lambda_2'-\lambda_3'}(\widetilde{\theta}')d^{(j)}_{m,\lambda_1-\lambda_3}(\widetilde{\theta}'') \\ &\quad\times\langle k_1'^0\lambda_1'(k_2'^0\lambda_2'\overline{k_3'^0}\lambda_3')\rho'|R_{S'}^{-1}R_{S''}|k_2''^0\lambda_2(k_1''^0\lambda_1\overline{k_3''^0}\lambda_3)\rho\rangle. \end{aligned}\tag{5.5}$$

Hence the matrix element depends only on the rotation $R^{-1}_{S'S''}$. This rotation will be equal to R_V after the constraints imposed by the evaluation of the $\langle k'^0|k''^0\rangle$ matrix element have been realized, but until then this rotation will be denoted $R_{V'}=R_{\alpha,\chi',\beta}$. Hence $R_{S''}=R_{S'}R^{-1}_{S'}R_{S''}=R_{S'}R_{V'}$, and using the group properties of the rotation matrices

$$\mathcal{D}^{(J)*}_{M,m-\lambda_2}(S'')=\sum_\Lambda\mathcal{D}^{(J)*}_{M,\Lambda}(S')\mathcal{D}^{(J)*}_{\Lambda,m-\lambda_2}(V').\tag{5.6}$$

The invariance of the group integration ensures that

$$\int dS''=\int dV',\tag{5.7}$$

and the orthogonality relation for the $\mathcal{D}$ functions,

$$\int dS'\mathcal{D}^{(J')}_{M',m'-\lambda_1'}(S')\mathcal{D}^{(J)*}_{M,\Lambda}(S')=\delta_{J'J}\delta_{M'M}\delta_{m'-\lambda_1',\Lambda}\frac{2\pi}{\eta_J^2},\tag{5.8}$$

allows the reduction of Eq. (5.5) to

$$\begin{aligned}\langle J'1'(2'3')\rho'|\mathcal{P}_{12}|J1(23)\rho\rangle &= 2\pi\delta_{J'J}\delta_{M'M}\eta_{j'}\eta_j\int dV'\mathcal{D}^{(J)*}_{m'-\lambda_1',m-\lambda_2}(V')\int_0^\pi d\widetilde{\theta}'\sin\widetilde{\theta}'\int_0^\pi d\widetilde{\theta}''\sin\widetilde{\theta}''d^{(j')}_{m',\lambda_2'-\lambda_3'}(\widetilde{\theta}') \\ &\quad\times d^{(j)}_{m,\lambda_1-\lambda_3}(\widetilde{\theta}'')\langle k_1'^0\lambda_1'(k_2'^0\lambda_2'\overline{k_3'^0}\lambda_3')\rho'|R_{V'}|k_2''^0\lambda_2(k_1''^0\lambda_1\overline{k_3''^0}\lambda_3)\rho\rangle. \end{aligned}\tag{5.9}$$

We now define the vectors

$$\begin{aligned}R_{V'}k_1''^0&=k_1,\\ R_{V'}k_2''^0&=k_2,\\ R_{V'}k_3''^0&=k_3\end{aligned}\tag{5.10}$$

(where k_1 is not lined up along the negative z axis and equal to $k_1'^0$ until $R_{V'}=R_V$). This rotation of the vectors k_i'' into the vectors k_i is represented in Fig. 5(c). Guided by the discussion leading up to Eq. (5.2) and the representation (4.22) for the three-body states, the matrix element involving $R_{V'}$ is a product of a plane wave momentum space matrix element and Dirac space matrix elements

$$\begin{aligned}&\langle k_1'^0\lambda_1'(k_2'^0\lambda_2'\overline{k_3'^0}\lambda_3')\rho'|R_{V'}|k_2''^0\lambda_2(k_1''^0\lambda_1\overline{k_3''^0}\lambda_3)\rho\rangle \\ &\quad=\langle k_1'^0(k_2'^0\overline{k_3'^0})|R_{V'}|k_2''^0(k_1''^0\overline{k_3''^0})\rangle\times\mathcal{U}=2E_{k_1}\delta^{(3)}(k_1'^0-k_1)2E_{k_2}\delta^{(3)}(k_2'^0-k_2)\delta^{(4)}(P'-P)\mathcal{U},\end{aligned}\tag{5.11}$$

where

$$\begin{aligned}\mathcal{U}&=[e^{i\pi s_1}\overline{u(q',\lambda_1')}R^{-1}_{\pi,\pi,0}R_{V'}Z_qR_{0,\widetilde{\theta},0}u(\widetilde{p},\lambda_1)][e^{-i\pi s_2}\overline{u(\widetilde{p}',\lambda_2')}R^{-1}_{0,\widetilde{\theta}',0}Z^{-1}_{q'}R_{V'}R_{\pi,\pi,0}u(q,\lambda_2)] \\ &\quad\times[\overline{u^\rho}'(\widetilde{p}',\lambda_3')R^{-1}_{\pi,\pi,0}R^{-1}_{0,\widetilde{\theta}',0}Z^{-1}_{q'}R_{V'}Z_qR_{0,\widetilde{\theta},0}R_{\pi,\pi,0}u^\rho(\widetilde{p},\lambda_3)]=\mathcal{U}^{(1)}_{\lambda_1'\lambda_1}\mathcal{U}^{(2)}_{\lambda_2'\lambda_2}\mathcal{U}^{(3)}_{\lambda_3'\rho',\lambda_3\rho}=\langle\langle\widetilde{p}',\lambda_1'(\lambda_2'\lambda_3')\rho'|\widetilde{p},\lambda_2(\lambda_1\lambda_3)\rho\rangle\rangle\end{aligned}\tag{5.12}$$

will be referred to as the *reduced* matrix element of the permutation operator. Note that $q=|\mathbf{k}_1|$ and $q'=|\mathbf{k}_1'|$ (as above). The matrix element (5.12) contains Wigner rotations which result from the fact that the helicities $\{\lambda_i'\}$ and $\{\lambda_i\}$ are defined in different frames.

We first turn to the evaluation of the δ functions on the right-hand side of Eq. (5.11).

B. Evaluation of the δ functions

In Appendix B it is shown that the two δ functions can be written

$$2E_{k_1}\delta^{(3)}(k_1'^0-k_1)2E_{k_2}\delta^{(3)}(k_2'^0-k_2)=4E_{\tilde{p}_0}E_{\tilde{p}_0'}\delta(\beta)\delta(\alpha-\pi)\frac{\delta(\tilde{p}-\tilde{p}_0)}{\tilde{p}_0^2}\,\frac{\delta(\tilde{p}'-\tilde{p}_0)}{\tilde{p}_0'^2}\,\delta(\cos\tilde{\theta}-\cos\tilde{\theta}_0)\,\delta(\cos\tilde{\theta}'-\cos\tilde{\theta}_0), \tag{5.13}$$

with

$$\tilde{p}_0=\tilde{p}_0(q,q',\chi),\quad \tilde{p}_0'=\tilde{p}_0(q',q,\chi),$$

$$\tilde{\theta}_0=\tilde{\theta}_0(q,q',\chi),\quad \tilde{\theta}_0'=\tilde{\theta}_0(q',q,\chi), \tag{5.14}$$

and

$$\tilde{p}_0(q,q',\chi)=\sqrt{\left[\frac{(M_t-E_q)E_{q'}+qq'\cos\chi}{W_q}\right]^2-m^2},$$

$$\cos\{\tilde{\theta}_0(q,q',\chi)\}=\frac{W_qE_{q'}-(M_t-E_q)E_{\tilde{p}_0(q,q',\chi)}}{q\tilde{p}_0(q,q',\chi)}. \tag{5.15}$$

The first two δ functions insure that the rotation $R_{V'}=R_{\alpha,\chi',\beta}$ is now $R_{\pi,\chi',0}$, and the δ functions in $\tilde{p}$ and $\tilde{p}'$ fix the angle χ' to χ. The angle χ will remain a variable, since we prefer to express the ''allowed'' magnitudes of the momenta $\tilde{p}$ and $\tilde{p}'$ as functions of χ rather than the other way around.

We now combine expressions (5.9), (5.11), and (5.13) and insert the result into the three-body equation (4.49). In doing this we must be careful to change the arguments of the matrix element (5.11), which is expressed in terms of $\langle p',q',\lambda_1'\lambda_2'\lambda_3'|p,q,\lambda_1\lambda_2\lambda_3\rangle$, to $\langle p'',q,\lambda_1\lambda_2''\lambda_3''|p',q',\lambda_1'\lambda_2'\lambda_3'\rangle$, so as to agree with the labeling used in Eq. (4.49). Carrying out the $d\tilde{p}'$ and $d\tilde{p}$ integrations then gives

$$\begin{aligned}\langle J1(23)\rho|\Gamma^1\rangle=\sum_{j'm'}\sqrt{2j+1}\sqrt{2j'+1}\sum_{\substack{\lambda_2''\lambda_3''\rho_3\\ \lambda_1'\lambda_2'\lambda_3'\rho_2\rho_1}}\int q'^2dq'\,\frac{m}{E_{q'}}\int_0^\pi d\chi\sin\chi\, d^{(J)}_{m-\lambda_1,m'-\lambda_2'}(\chi)\\ \times\langle j(23)\rho|M^1|j(2''3'')\rho_3\rangle g^{\rho_3}(q,\tilde{p}'')\;e^{i\pi(m-\lambda_1)}d^{(j)}_{m,\lambda_2''-\lambda_3''}(\tilde{\theta}'')d^{(j')}_{m',\lambda_1'-\lambda_3'}(\tilde{\theta}')\\ \times\frac{m}{E_{\tilde{p}''}}\frac{m^2}{E^2_{\tilde{p}'}}\langle\langle\tilde{p}'',\lambda_1(\lambda_2''\lambda_3'')\rho_3|\tilde{p}',\lambda_2'(\lambda_1'\lambda_3')\rho_2\rangle\rangle O_{\rho_2\rho_1}(\tilde{p}',\lambda_3')\langle J'1'(2'3')\rho_1|\Gamma^1\rangle,\end{aligned} \tag{5.16}$$

where

$$\tilde{p}'=\tilde{p}_0(q',q,\chi),\quad \tilde{p}''=\tilde{p}_0(q,q',\chi),$$

$$\tilde{\theta}'=\tilde{\theta}_0(q',q,\chi),\quad \tilde{\theta}''=\tilde{\theta}_0(q,q',\chi), \tag{5.17}$$

and $\langle\langle\tilde{p}'',\lambda_1(\lambda_2''\lambda_3'')\rho_3|\tilde{p}',\lambda_2'(\lambda_1'\lambda_3')\rho_2\rangle\rangle$ is the reduced matrix element defined in Eq. (5.12). This matrix element is calculated in the next subsection.

C. Wigner rotations and the reduced matrix element

It will be sufficient to define a Wigner rotation only for the special case when a spinor with helicity λ and three-momentum in the right half of the xz plane is boosted in the positive z direction, as shown in Fig. 6. The boost is denoted by Z_q [defined in Eq. (B4)], the initial three-momentum of the state by $\tilde{\mathbf{p}}$ (with magnitude $\tilde{p}$ and polar angle $\tilde{\theta}$), the final three-momentum by $\mathbf{p}$ (with magnitude $p=q'$ and polar angle $\theta=\pi-\chi$), so that the Wigner rotation $\mathcal{R}(q,q',\chi)$ is defined by the relation

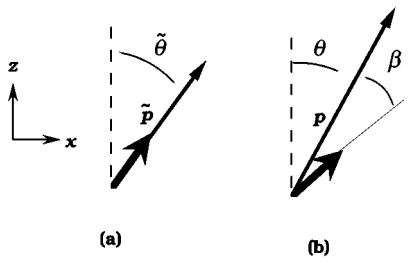

FIG. 6. (a) The canonical configuration of momentum for the calculation of the Wigner rotation. The helicity is $+\frac{1}{2}$ in this example. (b) The transformation of momentum and spin after the boost in the $+\hat{z}$ direction. The spin is now no longer aligned with the momentum, but rotated by angle β with respect to it.

$$Z_q u(\tilde{\mathbf{p}},\lambda)=Z_q R_{0,\tilde{\theta},0} L_{\tilde{p}} u(0,\lambda)=R_{0,\theta,0} L_p \mathcal{R}(q,q',\chi) u(0,\lambda), \tag{5.18}$$

where the representation of the pure boosts L_k (for $k=p$ or $\tilde{p}$) in four-dimensional spacetime is

$$L_k=\begin{pmatrix} \cosh\eta_k & & & \sinh\eta_k \\ & 1 & & \\ & & 1 & \\ \sinh\eta_k & & & \cosh\eta_k \end{pmatrix}, \tag{5.19}$$

with

$$\tanh\eta_k=\frac{k}{E_k}. \tag{5.20}$$

In Appendix C we show that $\mathcal{R}$ is a pure rotation about the y axis,

$$\mathcal{R}(q,q',\chi)=R_{0,\beta,0}, \tag{5.21}$$

and find the general equation for $\cos\beta$ as a function of q, q', and χ. Since $0\geqslant\beta\geqslant\pi$, β is uniquely determined by its cosine. Using the result (5.21), we have

$$Z_q u(\tilde{\mathbf{p}},\lambda)=\sum_\nu u(\mathbf{p},\nu) d^{(1/2)}_{\nu\lambda}(\beta). \tag{5.22}$$

We now are ready to evaluate each of the matrix elements in Eq. (5.12), but we will make the substitution

$$\langle p',q',\lambda_1'\lambda_2'\lambda_3'|p,q,\lambda_1\lambda_2\lambda_3\rangle \rightarrow\langle p'',q,\lambda_1\lambda_2''\lambda_3''|p',q',\lambda_1'\lambda_2'\lambda_3'\rangle$$

so as to agree with the labeling used in Eq. (5.16). Noting that $k_1=k_1'^0$ implies that $p'=q$ and $\theta'+\chi=\pi$ (see Fig. 5) the matrix element for particle 1 becomes

$$\mathcal{U}^{(1)}_{\lambda_1\lambda_1'}=[e^{i\pi s_1}\overline{u(q,\lambda_1)}R^{-1}_{\pi,\pi,0}R_V Z_{q'}R_{0,\tilde{\theta}',0}u(\tilde{p}',\lambda_1')]=e^{i\pi s_1}\sum_\nu\ [\overline{u(q,\lambda_1)}R^{-1}_{0,\pi,0}R_{0,\chi,0}R_{0,\theta',0}u(p',\nu)]d^{(1/2)}_{\nu\lambda_1'}(\beta_1)$$

$$=e^{i\pi s_1}\sum_\nu\ [\overline{u(q,\lambda_1)}u(q,\nu)]d^{(1/2)}_{\nu\lambda_1'}(\beta_1)=e^{i\pi s_1}d^{(1/2)}_{\lambda_1\lambda_1'}(\beta_1), \tag{5.23}$$

where, using the function β defined in Appendix C, Eq. (C14),

$$\beta_1=\beta(q',q,\chi). \tag{5.24}$$

Similarly, to evaluate the matrix element for particle 2 use $\theta''=\pi-\chi$, $p''=q'$, and

$$R_{\alpha,\pi,0}=R_{0,\pi,-\alpha},$$

$$R_{\pi,\alpha,0}=R_{0,-\alpha,\pi}, \tag{5.25}$$

which gives

$$\mathcal{U}^{(2)}_{\lambda_2''\lambda_2'}=[e^{-i\pi s_2}\overline{u(\tilde{p}'',\lambda_2'')}R^{-1}_{0,\tilde{\theta}'',0}Z_q^{-1}R_V R_{\pi,\pi,0}u(q',\lambda_2')]=e^{-i\pi s_2}\sum_\nu\ [\overline{u(q',\nu)}R^{-1}_{0,\theta'',0}R_{\pi,\chi,0}R_{\pi,\pi,0}u(q',\lambda_2')]d^{(1/2)}_{\nu\lambda_2''}(\beta_2)$$

$$=e^{-i\pi s_2}\sum_\nu\ [\overline{u(q',\nu)}R_{0,0,-2\pi}u(q',\lambda_2')]d^{(1/2)}_{\nu\lambda_2''}(\beta_2)=e^{-i\pi(s_2-2\lambda_2')}d^{(1/2)}_{\lambda_2'\lambda_2''}(\beta_2)=e^{-i\pi(s_2-2\lambda_2')}d^{(1/2)}_{\lambda_2''\lambda_2'}(-\beta_2), \tag{5.26}$$

where

$$\beta_2=\beta(q,q',\chi). \tag{5.27}$$

Calculation of the matrix element for particle 3 is complicated by the fact that its physical four-momentum is off shell, while the four-momenta used in the definition of the u^ρ spinors are on shell. However, as shown in Eq. (3.31), the four-momentum of the *negative energy* spinor is identical to the four-momentum of the on-shell particle in the interacting pair, and

an efficient way to proceed is to first express both of the u^{ρ} spinors in terms of u^{-}. Then it will turn out that the boosts of both ρ-spin states can be evaluated in terms of quantities related to the on-shell particle in the interacting pair.

To this end, note that u^{+} can be expanded in terms of u^{-} and $\gamma^5 u^{-}$

$$u^{+}(p,\lambda)=\gamma^5\left[\frac{E_p}{m}+2\lambda\frac{p}{m}\gamma^5\right]u^{-}(p,\lambda), \tag{5.28}$$

where the matrix

$$\gamma^5=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

commutes with all rotations and boosts. Then, using the fact that

$$R_{0,\pi,0}=e^{-i\pi\gamma^5\alpha_2/2}=-i\gamma^5\alpha_2, \tag{5.29}$$

where α_2 is the Dirac matrix, and using the explicit form of the spinors given in Eq. (4.23), we obtain

$$R_{\pi,\pi,0}u^{-}(p,\lambda)=e^{i\pi s_3}\gamma^5 u^{+}(p,-\lambda). \tag{5.30}$$

Combining this with Eq. (5.28) gives a simple formula for $R_{\pi,\pi,0}u^{+}(p,\lambda)$. These two relations will be summarized

$$R_{\pi,\pi,0}u^{\rho}(p,\lambda)=e^{i\pi s_3}\,N_{\rho}(\lambda)u(p,-\lambda), \tag{5.31}$$

where $u^{+}=u$ is implied and

$$N_{+}(p,\lambda)=\left[\frac{E_p}{m}+2\lambda\frac{p}{m}\gamma^5\right],\quad N_{-}(p,\lambda)=\gamma^5. \tag{5.32}$$

These relations can now be used to evaluate the matrix element $\mathcal{U}^{(3)}$

$$\begin{aligned}\mathcal{U}^{(3)}_{\lambda_3''\rho',\lambda_3'\rho}&=[\overline{u^{\rho}}'(\tilde{p}'',\lambda_3'')R^{-1}_{\pi,\pi,0}R^{-1}_{0,\tilde{\theta}'',0}Z_q^{-1}R_{\pi,\chi,0}Z_{q'}R_{0,\tilde{\theta}',0}R_{\pi,\pi,0}u^{\rho}(\tilde{p}',\lambda_3')]\\ &=[\overline{u}(\tilde{p}'',-\lambda_3'')\tilde{N}_{\rho'}(\tilde{p}'',\lambda_3'')R^{-1}_{0,\tilde{\theta}'',0}Z_q^{-1}R_{\pi,\chi,0}Z_{q'}R_{0,\tilde{\theta}',0}N_{\rho}(\tilde{p}',\lambda_3')u(\tilde{p}',-\lambda_3')]\\ &=\sum_{\nu\nu'}[\overline{u}(q',-\nu')R^{-1}_{0,\pi-\chi,0}\tilde{N}_{\rho'}(\tilde{p}'',\lambda_3'')N_{\rho}(\tilde{p}',\lambda_3')R_{\pi,\chi,0}R_{0,\pi-\chi,0}u(q,-\nu)]d^{(1/2)}_{-\nu',-\lambda_3''}(\beta_2)d^{(1/2)}_{-\nu,-\lambda_3'}(\beta_1),\end{aligned} \tag{5.33}$$

where $\tilde{N}_{\rho}=\gamma^0 N_{\rho}\,\gamma^0$ and, because we were able to write the states $R_{\pi,\pi,0}u^{\rho}$ in terms of the positive energy on-shell spinors u using Eq. (5.31), the matrix elements of particle 3 have been expressed in terms of the Wigner rotations which already appeared in the treatment of particles 1 and 2. Since the N factors commute with the rotations, the matrix element can be further simplified as follows:

$$\begin{aligned}\mathcal{U}^{(3)}_{\lambda_3''\rho',\lambda_3'\rho}&=-\sum_{\nu\nu'}e^{i\pi\nu'}d^{(1/2)}_{-\nu',-\lambda_3''}(\beta_2)d^{(1/2)}_{-\nu,-\lambda_3'}(\beta_1)[\overline{u}(q',-\nu')R_{0,-\chi,0}\tilde{N}_{\rho'}(\tilde{p}'',\lambda_3'')N_{\rho}(\tilde{p}',\lambda_3')u(q,-\nu)]\\ &=-\sum_{\nu\nu'}e^{i\pi\nu'}d^{(1/2)}_{-\nu',-\lambda_3''}(\beta_2)d^{(1/2)}_{-\nu,-\lambda_3'}(\beta_1)d^{(1/2)}_{-\nu',-\nu}(-\chi)[A_{\rho'\rho}d_1+B_{\rho'\rho}d_5],\end{aligned} \tag{5.34}$$

where the matrix elements d_1 and d_5 are

$$\begin{aligned}[\overline{u}(q',-\nu')R_{0,-\chi,0}\,u(q,-\nu)]&=d^{(1/2)}_{-\nu',-\nu}(-\chi)d_1=d^{(1/2)}_{-\nu',-\nu}(-\chi)(c'c-4\nu'\nu s's),\\ [\overline{u}(q',-\nu')R_{0,-\chi,0}\gamma^5\,u(q,-\nu)]&=d^{(1/2)}_{-\nu',-\nu}(-\chi)d_5=d^{(1/2)}_{-\nu',-\nu}(-\chi)(2\nu's'c-2\nu c's),\end{aligned} \tag{5.35}$$

with

$$\begin{aligned}c&=\cosh(\eta_q/2),\quad c'=\cosh(\eta_{q'}/2),\\ s&=\sinh(\eta_q/2),\quad s'=\sinh(\eta_{q'}/2),\end{aligned} \tag{5.36}$$

and

$$\sinh\eta_q=\frac{q}{m},\quad \sinh\eta_{q'}=\frac{q'}{m}\,. \tag{5.37}$$

From the explicit form of the N's, we obtain

$$A_{\rho'\rho}=\begin{pmatrix} \dfrac{E_{\tilde p''}E_{\tilde p'}}{m^2}-4\lambda_3''\lambda_3'\dfrac{\tilde p''\tilde p'}{m^2} & -2\lambda_3''\dfrac{\tilde p''}{m} \\ -2\lambda_3'\dfrac{\tilde p'}{m} & -1 \end{pmatrix},$$

$$B_{\rho'\rho}=\begin{pmatrix} 2\lambda_3'\dfrac{E_{\tilde p''}\tilde p'}{m^2}-2\lambda_3''\dfrac{E_{\tilde p'}\tilde p''}{m^2} & \dfrac{E_{\tilde p''}}{m} \\ -\dfrac{E_{\tilde p'}}{m} & 0 \end{pmatrix}. \tag{5.38}$$

The sum over ν and ν' in Eq. (5.34) can now be carried out if care is taken to remove all phases which depend on ν or ν'. There are four possibilities, all of which occur. We may write the ''standard'' sum in a compact form

$$\sum_{\nu\nu'}\mathcal{I}\equiv\sum_{\nu\nu'}e^{i\pi\nu'}d^{(1/2)}_{-\nu',-\lambda_3''}(\beta_2)d^{(1/2)}_{-\nu',-\nu}(-\chi)d^{(1/2)}_{-\nu,-\lambda_3'}(\beta_1)=e^{i\pi\lambda_3'}\sum_{\nu\nu'}d^{(1/2)}_{\lambda_3''\nu'}(\beta_2)d^{(1/2)}_{\nu'\nu}(-\chi)d^{(1/2)}_{\nu\lambda_3'}(\beta_1)=e^{i\pi\lambda_3'}d^{(1/2)}_{\lambda_3''\lambda_3'}(\beta_1+\beta_2-\chi)$$
$$=e^{i\pi\lambda_3'}d(\beta_1+\beta_2-\chi), \tag{5.39}$$

where symmetry properties of the d functions have been used, and for compactness the $\lambda_3''\lambda_3'$ indices are suppressed in the final result. By similar arguments, the remaining three sums give

$$\sum_{\nu\nu'}4\nu'\nu\mathcal{I}=e^{i\pi\lambda_3'}d(\beta_1+\beta_2+\chi),$$

$$\sum_{\nu\nu'}2\nu'\mathcal{I}=e^{i\pi s_3}d(-\beta_1+\beta_2+\chi),$$

$$\sum_{\nu\nu'}2\nu\mathcal{I}=e^{i\pi s_3}d(-\beta_1+\beta_2-\chi). \tag{5.40}$$

Using these identities, the matrix element for particle 3 finally becomes

$$\mathcal{U}^{(3)}_{\lambda_3''\rho_3,\lambda_3'\rho_2}=e^{i\pi\lambda_3'}\mathcal{X}^{\rho_3\rho_2}_{\lambda_3''\lambda_3'}, \tag{5.41}$$

where the matrix $\mathcal{X}^{\rho'\rho}$ is

$$\mathcal{X}^{\rho'\rho}_{\lambda_3''\lambda_3'}=\begin{pmatrix} -[D_1A+D_5B] & -(-1)^{1/2-\lambda_3'}\left[\dfrac{E_{\tilde p''}}{m}D_5-(-1)^{\lambda_3''-\lambda_3'}\dfrac{\tilde p''}{m}D_1\right] \\ (-1)^{1/2-\lambda_3'}\left[\dfrac{E_{\tilde p'}}{m}D_5+\dfrac{\tilde p'}{m}D_1\right] & D_1 \end{pmatrix}, \tag{5.42}$$

with the notation

$$D_1 = d^{(1/2)}_{\lambda_3''\lambda_3'}(\beta_1+\beta_2-\chi)c'c - d^{(1/2)}_{\lambda_3''\lambda_3'}(\beta_1+\beta_2+\chi)s's,$$

$$D_5 = d^{(1/2)}_{\lambda_3''\lambda_3'}(-\beta_1+\beta_2+\chi)s'c - d^{(1/2)}_{\lambda_3''\lambda_3'}(-\beta_1+\beta_2-\chi)c's,$$

$$A = A_{++},$$

$$B = 2\lambda_3' B_{++}. \tag{5.43}$$

Combining the results (5.23), (5.26), and (5.41) gives the following expression for the reduced matrix element:

$$\langle\langle \tilde{p}'', \lambda_1(\lambda_2''\lambda_3'')\rho_3 | \tilde{p}', \lambda_2'(\lambda_1'\lambda_3')\rho_2\rangle\rangle = -e^{i\pi\lambda_3'} d^{(1/2)}_{\lambda_1\lambda_1'}(\beta_1) d^{(1/2)}_{\lambda_2''\lambda_2'}(-\beta_2) \mathcal{X}^{\rho_3\rho_2}_{\lambda_3''\lambda_3'}. \tag{5.44}$$

D. Symmetries of the permutation operator

The reduced matrix element (5.44) satisfies a symmetry condition which can be obtained from the following property of scalar products:

$$\langle\langle \tilde{p}'', \lambda_1(\lambda_2''\lambda_3'')\rho_3 | \tilde{p}', \lambda_2'(\lambda_1'\lambda_3')\rho_2\rangle\rangle = \langle\langle \tilde{p}', \lambda_2'(\lambda_1'\lambda_3')\rho_2 | \tilde{p}'', \lambda_1(\lambda_2''\lambda_3'')\rho_3\rangle\rangle^*. \tag{5.45}$$

Because the permutation operator is Hermitian and the initial and final states are composed of identical nucleons, this equation tells us that the matrix elements (5.9) must be identical under the substitution $q \leftrightarrow q'$ (which also implies $\tilde{p}'' \leftrightarrow \tilde{p}'$, $\tilde{\theta}'' \leftrightarrow \tilde{\theta}'$, and $\beta_1 \leftrightarrow \beta_2$) and

$$\begin{aligned} j &\leftrightarrow j', \quad \lambda_1 \leftrightarrow \lambda_2', \\ m &\leftrightarrow m', \quad \lambda_2'' \leftrightarrow \lambda_1', \\ \rho_3 &\leftrightarrow \rho_2, \quad \lambda_3'' \leftrightarrow \lambda_3'. \end{aligned} \tag{5.46}$$

Examination of the matrix elements shows that this implies

$$(-1)^{m-\lambda_1+\lambda_3'} d^{(J)}_{m-\lambda_1, m'-\lambda_2'}(\chi) d^{(1/2)}_{\lambda_1\lambda_1'}(\beta_1) d^{(1/2)}_{\lambda_2''\lambda_2'}(-\beta_2) \mathcal{X}^{\rho_3\rho_2}_{\lambda_3''\lambda_3'} = (-1)^{m'-\lambda_2'+\lambda_3''} d^{(J)}_{m'-\lambda_2', m-\lambda_1}(\chi) d^{(1/2)}_{\lambda_2'\lambda_2''}(\beta_2) d^{(1/2)}_{\lambda_1'\lambda_1}(-\beta_1) \mathcal{X}^{\rho_2\rho_3}_{\lambda_3'\lambda_3''}, \tag{5.47}$$

which reduces to the condition

$$(-1)^{\lambda_3'-\lambda_3''} \mathcal{X}^{\rho_3\rho_2}_{\lambda_3''\lambda_3'} = \mathcal{X}^{\rho_2\rho_3}_{\lambda_3'\lambda_3''}. \tag{5.48}$$

However, the transformation (5.46) gives $A \leftrightarrow A$, $B \leftrightarrow -(-1)^{\lambda_3''-\lambda_3'}B$, $D_1 \leftrightarrow (-1)^{\lambda_3''-\lambda_3'}D_1$, and $D_5 \leftrightarrow -D_5$, showing that the result (5.42) satisfies the symmetry condition (5.48).

Another symmetry of the matrix elements of the permutation operator follows from the fact that $\mathcal{P}_{12}$ commutes with the parity operator $\mathcal{P}$, which leads to the identity

$$\mathcal{P}_{12} = \mathcal{P}_{12}\mathcal{P}^2 = \mathcal{P}\mathcal{P}_{12}\mathcal{P}. \tag{5.49}$$

The action of the parity operator on the states defined in Eq. (4.13) can be worked out, giving

$$\mathcal{P}|qJM, \tilde{p}jm, \lambda_1(\lambda_2\lambda_3)\rho\rangle = (-1)^{j-1}(-1)^{J-j-1/2}\rho|qJM, \tilde{p}j-m, -\lambda_1(-\lambda_2-\lambda_3)\rho\rangle. \tag{5.50}$$

Hence the matrix elements of $\mathcal{P}_{12}$ must satisfy the identity

$$\begin{aligned} &\langle q'JM', \tilde{p}'j'm', \lambda_1'(\lambda_2'\lambda_3')\rho' | \mathcal{P}_{12} | qJM, \tilde{p}jm, \lambda_1(\lambda_2\lambda_3)\rho\rangle \\ &= \langle q'JM', \tilde{p}'j'm', \lambda_1'(\lambda_2'\lambda_3')\rho' | \mathcal{P}\mathcal{P}_{12}\mathcal{P} | qJM, \tilde{p}jm, \lambda_1(\lambda_2\lambda_3)\rho\rangle \\ &= (-1)^{2J-1}\rho\rho'\langle q'JM', \tilde{p}'\ j'-m', -\lambda_1'(-\lambda_2'-\lambda_3')\rho' | \mathcal{P}_{12} | qJM, \tilde{p}j-m, -\lambda_1(-\lambda_2-\lambda_3)\rho\rangle. \end{aligned} \tag{5.51}$$

For the triton, where $J=1/2$, this means that under the substitutions

$$\lambda_1' \leftrightarrow -\lambda_1', \quad \lambda_1 \leftrightarrow -\lambda_1,$$
$$\lambda_2' \leftrightarrow -\lambda_2', \quad \lambda_2 \leftrightarrow -\lambda_2,$$
$$\lambda_3' \leftrightarrow -\lambda_3', \quad \lambda_3 \leftrightarrow -\lambda_3,$$
$$m' \leftrightarrow -m', \quad m \leftrightarrow -m, \tag{5.52}$$

we should recover the same matrix element multiplied by a factor of $\rho\rho'$.

To verify that the matrix elements of $\mathcal{P}_{12}$ satisfy this symmetry, return to the full expression given in Eq. (1.6), and use the identity

$$d^{(j)}_{\lambda,\lambda'}(\theta) = (-1)^{\lambda-\lambda'} d^{(j)}_{-\lambda,-\lambda'}(\theta) \tag{5.53}$$

to obtain the condition

$$\mathcal{N}^{\rho''\rho'}_{\lambda_3''\lambda_3'} = (-1)^{\lambda_3''-\lambda_3'} \rho''\rho' \mathcal{N}^{\rho''\rho'}_{-\lambda_3''-\lambda_3'}, \tag{5.54}$$

where $\mathcal{N}^{\rho''\rho'}_{\lambda_3''\lambda_3'}$ is defined in Eq. (6.2) below. Examination of this equation confirms that Eq. (5.54) is indeed satisfied.

In the next section we present our final results for the three-body equations.

VI. FINAL EQUATIONS

In this final section we collect the previous results together, and explain how it is that integration over the spectator momentum, q', is limited to a finite interval. Then we describe the changes in the equations which are required by isospin and the conservation of parity. Finally, we describe how the three-body channels are classified and counted.

A. Spectator equations in angular momentum space

Using Eqs. (5.42) and (4.29) gives the following compact result:

$$\sum_{\rho_2} \mathcal{X}^{\rho_3\rho_2}_{\lambda_3''\lambda_3'} O_{\rho_2\rho_1}(\tilde{p}', \lambda_3') = -\frac{E_{\tilde{p}'}}{m} \mathcal{N}^{\rho_3\rho_1}_{\lambda_3''\lambda_3'}, \tag{6.1}$$

where the matrix $\mathcal{N}^{\rho'\rho}$ is

$$\mathcal{N}^{\rho'\rho}_{\lambda_3''\lambda_3'} = \begin{pmatrix} \dfrac{E_{\tilde{p}''}}{m} D_1 - 4\lambda_3''\lambda_3' \dfrac{\tilde{p}''}{m} D_5 & -2\lambda_3'[D_1 B + D_5 A] \\ -2\lambda_3' D_5 & \dfrac{E_{\tilde{p}'}}{m} D_1 + \dfrac{\tilde{p}'}{m} D_5 \end{pmatrix}, \tag{6.2}$$

and A, B, D_1, and D_5 were defined in Eq. (5.43).

Combining Eqs. (5.44) and (6.1) and substituting into Eq. (5.16) gives

$$\langle J1(23)\rho|\Gamma^1\rangle = \sum_{j'm'} \sqrt{2j+1}\sqrt{2j'+1} \sum_{\substack{\lambda_2''\lambda_3''\rho'' \\ \lambda_1'\lambda_2'\lambda_3'\rho'}} \int_0^{q_{\rm crit}} q'^2 dq' \frac{m}{E_{q'}} \int_0^{\pi} d\chi \sin\chi d^{(J)}_{m-\lambda_1, m'-\lambda_2'}(\chi)$$
$$\times \langle j(23)\rho|M^1|j(2''3'')\rho''\rangle \frac{m}{E_{\tilde{p}''}} g^{\rho''}(q,\tilde{p}'')\, d^{(j)}_{m,\lambda_2''-\lambda_3''}(\tilde{\theta}'') d^{(j')}_{m',\lambda_1'-\lambda_3'}(\tilde{\theta}')(-1)^{m-\lambda_1+\lambda_3'}$$
$$\times d^{(1/2)}_{\lambda_1\lambda_1'}(\beta_1) d^{(1/2)}_{\lambda_2''\lambda_2'}(-\beta_2) \mathcal{N}^{\rho''\rho'}_{\lambda_3''\lambda_3'} \frac{m}{E_{\tilde{p}'}} \langle J'1'(2'3')\rho'|\Gamma^1\rangle. \tag{6.3}$$

This is identical to the final result given in Sec. I, Eqs. (1.3) and (1.6). It is a two-dimensional integral equation depending on the variables q' and χ, where the integration over the angle χ runs from 0 to π, independent of the value of q', and the integration over q' has been limited to the finite interval $[0,q_{\rm crit}]$, as discussed in the next subsection. The momenta $\tilde{p}'$ and $\tilde{p}''$, the angles $\tilde{\theta}'$ and $\tilde{\theta}''$, and the Wigner rotation angles β_1 and β_2 all depend on q, q', and χ, and are defined in Eqs. (5.17), (5.24) and (5.27). The matrix $\mathcal{N}$ is defined in Eq. (6.2).

B. Removal of the spacelike region

The physical reason for restricting the q' integration in Eq. (6.3) to the finite interval $0 \leq q' \leq q_{\rm crit}$ will be discussed now.

As given in Eq. (3.19), the invariant mass of the two-body subsystem decreases with increasing momentum of the spectator, q, and at the value

$$q = q_{\rm crit} = \frac{M_t^2 - m^2}{2M_t} \simeq \frac{4}{3}m \tag{6.4}$$

the mass of the two-body subsystem is zero. This means it is recoiling with the speed of light, and under such circumstances the relativistic effects are clearly enormous. Furthermore, as q increases beyond the critical value, we pass from a region where the two-body states are timelike into a region where they are spacelike. The two-body scattering calculations are carried out in the rest frame of the two-body system, which does not exist for spacelike states, and, more generally, it is unlikely that an effective theory designed to describe timelike scattering would be useful in the spacelike

region. Furthermore, since the spacelike two-body states appear only at rather high momentum (above 1200 MeV) where the amplitudes are expected to be very small anyway, it would be sensible to simply neglect the region $q \geqslant q_{\rm crit}$, and set the three-body amplitudes to zero in this region. As it turns out, the three-body amplitudes *go to zero automatically at the critical value of* q, permitting us to impose the condition that they are zero for $q \geqslant q_{\rm crit}$ *without making the three-body amplitudes discontinuous in* q.

To see that the Faddeev amplitudes $\langle J1(23)\rho|\Gamma^1\rangle \to 0$ as $q \to q_{\rm crit}$, note that the function $\tilde{p}_0(q,q',\chi)$ [defined in Eq. (5.15)] approaches infinity as $q \to q_{\rm crit}$ (as long as $q' > 0$, which is true over the entire region of the q' integration except at the boundary where the integrand is zero). Specifically,

$$\tilde{p}'' \underset{q \to q_{\rm crit}}{\to} \frac{q_{\rm crit}(E_{q'} + q'\cos\chi)}{W_q} = \frac{C}{W_q}, \tag{6.5}$$

and in this limit

$$\frac{m}{E_{\tilde{p}''}} g^+(\tilde{p}'')\mathcal{N}^{+\rho} \to \frac{m}{2\tilde{p}''^2}(\tilde{p}'' K_{+\rho}) \to (W_q)^1,$$

$$\frac{m}{E_{\tilde{p}''}} g^-(\tilde{p}'')\mathcal{N}^{-\rho} \to -\frac{1}{W_q}\left(\frac{W_q}{mC}\right) K_{-\rho} \to (W_q)^0, \tag{6.6}$$

where C and $K_{\pm\rho}$ are functions which are finite in the limit as $q \to q_{\rm crit}$. Note that the possible $1/W_q$ singularity from the negative energy part of the propagator is canceled by the $m/E_{\tilde{p}''}$ factor. Hence the Faddeev amplitudes go like

$$\langle J1(23)\rho|\Gamma^1\rangle \underset{W_q \to 0}{\to} c_{\rho+}(W_q)^{(n_{\rho+}+1)} + c_{\rho-}(W_q)^{n_{\rho-}}, \tag{6.7}$$

where $n_{\rho+}$ and $n_{\rho-}$ are powers with which the two-body amplitudes (4.50) fall with momentum as $\tilde{p}'' \to \infty$:

$$\langle j(23)\rho|M^1|j(2''3'')\rho_3\rangle \underset{\tilde{p}'' \to \infty}{\to} \left(\frac{1}{\tilde{p}''}\right)^{n_{\rho\rho_3}}. \tag{6.8}$$

We conclude that the Faddeev amplitudes not only go to zero as $q \to q_{\rm crit}$, but that they approach this limit smoothly.

C. Isospin

Since the main application of the three-body equations for spin 1/2 particles will be the three-nucleon system, we have to incorporate the isospin degree of freedom. This can be done separately from the other degrees of freedom, as described in this subsection. We will assume that isospin is conserved by the equations.

To lay the foundation we return to the discussion in Sec. II. The exchange operators $\mathcal{P}_{ij}$ are a product of a part which acts *only* in isospin space, and a part which acts on all other coordinates, denoted by $\tilde{\mathcal{P}}_{ij}$. If the ij pair has isospin T_{ij}, the action of $\mathcal{P}_{ij}$ on the isospin part of the wave function will be denoted simply by its eigenvalue $(-1)^{T_{ij}-1}$. The phase ζ which occurs in Eq. (2.29) is -1 for fermions and is therefore a product of the phase $(-1)^{T_{ij}-1}$ from the exchange of the isospin variables, and the phase u, resulting from the operation of $\tilde{\mathcal{P}}_{ij}$. Hence

$$\zeta = -1 = u(-1)^{T_{ij}-1}. \tag{6.9}$$

Even though ζ is always -1 for fermions, there are in general two possible values of u corresponding to the two possible isospin channels, and Eq. (2.29) generalizes to

$$\tilde{\mathcal{P}}_{32} M^1_{22} = u M^1_{32} = M^1_{33}\tilde{\mathcal{P}}_{32},$$

$$\tilde{\mathcal{P}}_{23} M^1_{33} = u M^1_{23} = M^1_{22}\tilde{\mathcal{P}}_{23}. \tag{6.10}$$

The vectors $|\Gamma^1_2\rangle$ are also vectors in isospin space. Taking matrix elements of Eq. (2.38), and inserting $1 = \Sigma_T |T\rangle\langle T|$, gives

$$\langle T|\Gamma^1_2\rangle = 2\sum_{T'} \langle T|\mathcal{P}_{12}|T'\rangle M^{1T}_{22} G^1_2 \tilde{\mathcal{P}}_{12}\langle T'|\Gamma^1_2\rangle, \tag{6.11}$$

where $|T\rangle$ are the isospin wave functions discussed below, and $\langle T|\mathcal{P}_{12}|T'\rangle$ is the matrix element of the permutation operator in isospin space. The calculation of this matrix element is familiar from the nonrelativistic theory, but for completeness we will briefly present it here.

In more detail, the states in isospin space will be denoted

$$|T\rangle = |[(t_2 t_3)T t_1]\mathcal{T}\mathcal{T}_z\rangle, \tag{6.12}$$

where t_i is the isospin of particle i, T is the isospin of the pair, and $\mathcal{T}$ and $\mathcal{T}_z$ are the total three-body isospin and its projection. As the notation suggests, t_2 and t_3 are first coupled to T, and then T and t_1 are coupled to $\mathcal{T}$. These states form a complete, orthonormal basis

$$\langle[(t_2 t_3)T t_1]\mathcal{T}\mathcal{T}_z|[(t_2' t_3')T' t_1']\mathcal{T}'\mathcal{T}_z'\rangle = \delta_{t_1 t_1'}\delta_{t_2 t_2'}\delta_{t_3 t_3'}\delta_{TT'}\delta_{\mathcal{T}\mathcal{T}'}\delta_{\mathcal{T}_z\mathcal{T}_z'},$$

$$\sum_{t_1 t_2 t_3 T \mathcal{T}\mathcal{T}_z} |[(t_2 t_3)T t_1]\mathcal{T}\mathcal{T}_z\rangle\langle[(t_2 t_3)T t_1]\mathcal{T}\mathcal{T}_z| = \mathbf{1}. \tag{6.13}$$

The effect of $\mathcal{P}_{12}$ is to interchange particles 1 and 2:

$$\mathcal{P}_{12}|[(t_2 t_3)T' t_1]\mathcal{T}\mathcal{T}_z\rangle = |[(t_1 t_3)T' t_2]\mathcal{T}\mathcal{T}_z\rangle. \tag{6.14}$$

The matrix element of $\mathcal{P}_{12}$ in isospin space reduces therefore to a simple recoupling coefficient,

$$\begin{aligned}\langle T|\mathcal{P}_{12}|T'\rangle &= \langle[(t_2 t_3)T t_1]\mathcal{T}\mathcal{T}_z|\mathcal{P}_{12}|[(t_2 t_3)T' t_1]\mathcal{T}\mathcal{T}_z\rangle \\ &= \langle[(t_2 t_3)T t_1]\mathcal{T}\mathcal{T}_z|((t_1 t_3)T' t_2)\mathcal{T}\mathcal{T}_z\rangle \\ &= -\sqrt{2T+1}\sqrt{2T'+1}\begin{Bmatrix} t_2 & t_3 & T \\ t_1 & \mathcal{T} & T' \end{Bmatrix}.\end{aligned} \tag{6.15}$$

In the next subsection we complete the reduction of Eq. (6.11) by inserting a complete set of good parity eigenstates.

D. Parity eigenstates

The two-body scattering amplitudes which drive the three-body equations are separated into channels which are eigenstates of $\mathcal{P}^1$ (the parity operator on the 23 subspace) and isospin. Isospin was just discussed in the previous subsection, and need not be revisited again until the next subsection below where we explain how isospin (or exchange symmetry) plays a role in the description and counting of the channels. The role of the conservation of parity, which has not yet been taken into account, will be discussed in this subsection.

The helicity states $|J1(23)\rho\rangle$ are neither eigenstates of the full parity operator, $\mathcal{P}$, nor of the two-body parity operator, $\mathcal{P}^1$. Since the two-body scattering amplitudes which emerge from the two-body calculations are eigenstates of $\mathcal{P}^1$, the three-body equations must be reexpressed in terms of these states. This is not difficult because the eigenstates are merely linear combinations of the states we have already obtained.

First we return to the two-body helicity states $|jm(\lambda_2\lambda_3)\rho\rangle$, where the relative momentum, $\tilde{p}$ is suppressed because it will play no role in the discussion which follows. If we apply the operator $\mathcal{P}^1$ (referred to simply as $\mathcal{P}$ in Ref. [28]) to this state we get

$$\mathcal{P}^1|jm(\lambda_2\lambda_3)\rho\rangle=\rho\epsilon|jm(-\lambda_2-\lambda_3)\rho\rangle, \tag{6.16}$$

where $\epsilon=(-1)^{j-1}$. It is easy to see that the state

$$|j^r(m\lambda)\rho\rangle\equiv\frac{1}{\sqrt{2}}(1+r\mathcal{P}^1)|jm(\lambda_2\lambda_3)\rho\rangle=\frac{1}{\sqrt{2}}\{|jm(\lambda_2\lambda_3)\rho\rangle + r\rho\epsilon|jm(-\lambda_2-\lambda_3)\rho\rangle\}, \tag{6.17}$$

with $\lambda\equiv\lambda_2-\lambda_3$, is a normalized eigenstate of $\mathcal{P}^1$,

$$\mathcal{P}^1|j^r(m\lambda)\rho\rangle=r|j^r(m\lambda)\rho\rangle. \tag{6.18}$$

If we replace the individual particle helicities λ_2 and λ_3 on the RHS of Eq. (6.17) by $-\lambda_2$ and $-\lambda_3$, we obtain the same state, apart from a phase factor. We should therefore include in our new basis (6.17) only states that are not related to each other by changing the sign of both helicities. We choose the convention $\lambda_2=+\frac{1}{2}$ and label the states by the difference λ. With this convention λ can be 0 or 1, and the parity r can be $+$ and $-$, and we have again four independent states, just as before when each of the individual helicities were allowed to be $\pm\frac{1}{2}$. The selection rule

$$|\lambda_2-\lambda_3|\leqslant j \tag{6.19}$$

excludes $\lambda=1$ for states with $j=0$.

For the construction of three-body parity eigenstates we can proceed in precisely the same way, treating the two-body subsystem as one elementary particle with spin j, helicity m, and (now well-defined) intrinsic parity r. The parity operator which acts in the three-body space will be denoted $\mathcal{P}$, and should be distinguished from $\mathcal{P}^1$. To carry out this construction, we first introduce the three-body states

$$|Jj^r\lambda_1(m\lambda)\rho\rangle=\frac{1}{\sqrt{2}}\{|J,jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle + r\rho\epsilon|J,jm;\lambda_1(-\lambda_2-\lambda_3)\rho\rangle\} =\frac{1}{\sqrt{2}}(1+r\mathcal{P}^1)|J,jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle, \tag{6.20}$$

where $|J,jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle=|J1(23)\rho\rangle$ are the same three-body states introduced in Sec. IV, Eq. (4.34), but with some of the notation restored for clarity. The parity operation $\mathcal{P}$ on these states yields

$$\mathcal{P}|Jj^r\lambda_1(m\lambda)\rho\rangle=\eta_1 r(-1)^{J-j-s_1}|Jj^r-\lambda_1(-m\lambda)\rho\rangle, \tag{6.21}$$

and the three-body eigenstates of parity, with eigenvalue $\Pi=\pm$ are therefore

$$|J^\Pi j^r(m\lambda)\rho\rangle\equiv\frac{1}{\sqrt{2}}\{|Jj^r\lambda_1(m\lambda)\rho\rangle + \Pi\,\eta_1 r(-1)^{J-j-s_1}|Jj^r-\lambda_1(-m\lambda)\rho\rangle\} =\frac{1}{\sqrt{2}}(1+\Pi\mathcal{P})|Jj^r\lambda_1(m\lambda)\rho\rangle. \tag{6.22}$$

In this case we adopt the convention $\lambda_1=+\frac{1}{2}$ and let m vary, subject to the condition that

$$|m-\lambda_1|=\left|m-\frac{1}{2}\right|\leqslant J. \tag{6.23}$$

Nucleon one is always in a positive-energy state, and therefore $\eta_1=1$ and $s_1=\frac{1}{2}$. The triton is characterized by $J^\Pi=\frac{1}{2}^+$. Three-body states with these quantum numbers are

$$|\tfrac{1}{2}^+,j^r(m\lambda)\rho\rangle=\frac{1}{\sqrt{2}}\{|\tfrac{1}{2}\,j^r\,\tfrac{1}{2}(m\lambda)\rho\rangle - r\epsilon|\tfrac{1}{2}j^r-\tfrac{1}{2}(-m\lambda)\rho\rangle\}. \tag{6.24}$$

Expanding these three-body parity eigenstates in terms of the original three-body helicity states (4.34) gives

$$|\tfrac{1}{2}^{+}j^r(m\lambda)\rho\rangle=\tfrac{1}{2}\{|\tfrac{1}{2},jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle+r\rho\epsilon|\tfrac{1}{2},jm;\lambda_1(-\lambda_2-\lambda_3)\rho\rangle-r\epsilon|\tfrac{1}{2},j-m;-\lambda_1(\lambda_2\lambda_3)\rho\rangle$$
$$-\rho|\tfrac{1}{2},j-m;-\lambda_1(-\lambda_2-\lambda_3)\rho\rangle\}=\tfrac{1}{2}(1+\mathcal{P})(1+r\mathcal{P}^1)|\tfrac{1}{2},jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle. \tag{6.25}$$

We now return to Eq. (6.11) and take the remaining spin-momentum matrix elements of the operators using the basis of good parity states, Eq. (6.25). For simplicity, we represent the states by $|\frac{1}{2}^{+}j^r(m\lambda)\rho\rangle=|j^r(m\lambda)\rho\rangle$, and the direct product states by $|T\rangle\otimes|j^r(m\lambda)\rho\rangle=|Tj^r(m\lambda)\rho\rangle$. Then, in our abbreviated notation,

$$\langle Tj^r(m\lambda)\rho|\Gamma_T^1\rangle=\sum_{\substack{j'r'\\m'T'}}\sum_{\substack{\lambda''\rho''\\\lambda'\rho'}}\int_0^{q_{\rm crit}}q'^2dq'\frac{m}{E_{q'}}\int_0^{\pi}d\chi\sin\chi\langle Tj^r(m\lambda)\rho|M^{1T}|Tj^r(m\lambda'')\rho''\rangle$$
$$\times\frac{m}{E_{\tilde{p}''}}g^{\rho''}(q,\tilde{p}'')\overline{\mathcal{P}}_{12}^{\rho''\rho'}[Tj^r(m\lambda'')\rho'',T'j'^{r'}(m'\lambda')\rho']\frac{m}{E_{\tilde{p}'}}\langle T'j'^{r'}(m'\lambda')\rho'|\Gamma_T^1\rangle, \tag{6.26}$$

where $\langle Tj^r(m\lambda)\rho|M^{1T}|Tj^r(m\lambda'')\rho''\rangle=M_{\rho\rho''}^{\lambda\lambda''}(Tj^r)$ is the two-body scattering amplitude for the jth partial wave with parity r and isospin T, and

$$\overline{\mathcal{P}}_{12}^{\rho''\rho'}[Tj^r(m\lambda'')\rho'',T'j'^{r'}(m'\lambda')\rho']=\langle T|\mathcal{P}_{12}|T'\rangle\times\langle j^r(m\lambda'')\rho''|\widetilde{\mathcal{P}}_{12}|j'^{r'}(m'\lambda')\rho'\rangle=\langle T|\mathcal{P}_{12}|T'\rangle\times\langle\langle\overline{\mathcal{P}_{12}}\rangle\rangle. \tag{6.27}$$

Equation (6.26) is our final result. It expresses the three-body equations in terms of the physical states with definite parity and isospin, driven by two-body amplitudes $M_{\rho\rho''}^{\lambda\lambda''}(Tj^r)$ which have been previously calculated as described in Ref. [28].

The new matrix element $\langle\langle\overline{\mathcal{P}_{12}}\rangle\rangle$ is readily obtained from the original matrix elements Eq. (5.5), which are, in the notation of this section,

$$\langle\langle\mathcal{P}_{12}\rangle\rangle=\langle\tfrac{1}{2},j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|\mathcal{P}_{12}|\tfrac{1}{2},jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle. \tag{6.28}$$

From definition (6.25) we have

$$\langle\langle\overline{\mathcal{P}_{12}}\rangle\rangle=\tfrac{1}{4}\langle\tfrac{1}{2},j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|(1+r'\mathcal{P}^1)(1+\mathcal{P})\mathcal{P}_{12}(1+\mathcal{P})(1+r\mathcal{P}^1)|\tfrac{1}{2},jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle$$
$$=\tfrac{1}{2}\langle\tfrac{1}{2},j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|(1+r'\mathcal{P}^1)\mathcal{P}_{12}(1+\mathcal{P})(1+r\mathcal{P}^1)|\tfrac{1}{2},jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle, \tag{6.29}$$

where we used the fact that $\mathcal{P}$ commutes with $\mathcal{P}_{12}$. Using Eqs. (6.16), (6.21), and (5.51), we obtain

$$\langle\langle\overline{\mathcal{P}_{12}}\rangle\rangle=\tfrac{1}{2}\{\langle\tfrac{1}{2},j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|\mathcal{P}_{12}|\tfrac{1}{2},jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle+r'\rho'\epsilon'\langle\tfrac{1}{2},j'm';\lambda_1'(-\lambda_2'-\lambda_3')\rho'|\mathcal{P}_{12}|\tfrac{1}{2},jm;\lambda_1(\lambda_2\lambda_3)\rho\rangle$$
$$+r\rho\epsilon\langle\tfrac{1}{2},j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|\mathcal{P}_{12}|\tfrac{1}{2},jm;\lambda_1(-\lambda_2-\lambda_3)\rho\rangle+r'r\rho'\rho\epsilon'\epsilon\langle\tfrac{1}{2},j'm';\lambda_1'(-\lambda_2'-\lambda_3')\rho'|$$
$$\times\mathcal{P}_{12}|\tfrac{1}{2},jm;\lambda_1(-\lambda_2-\lambda_3)\rho\rangle-\rho\langle\tfrac{1}{2},j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|\mathcal{P}_{12}|\tfrac{1}{2},j-m;-\lambda_1(-\lambda_2-\lambda_3)\rho\rangle$$
$$-r'\rho'\rho\epsilon'\langle\tfrac{1}{2},j'm';\lambda_1'(-\lambda_2'-\lambda_3')\rho'|\mathcal{P}_{12}|\tfrac{1}{2},j-m;-\lambda_1(-\lambda_2-\lambda_3)\rho\rangle-r\epsilon\langle\tfrac{1}{2},j'm';\lambda_1'(\lambda_2'\lambda_3')\rho'|$$
$$\times\mathcal{P}_{12}|\tfrac{1}{2},j-m;-\lambda_1(\lambda_2\lambda_3)\rho\rangle-r'r\rho'\epsilon'\epsilon\langle\tfrac{1}{3},j'm';\lambda_1'(-\lambda_2'-\lambda_3')\rho'|\mathcal{P}_{12}|\tfrac{1}{2},j-m;-\lambda_1(\lambda_2\lambda_3)\rho\rangle\}. \tag{6.30}$$

This is the correct form of the permutation operator to be used with the physical, good parity states.

E. Three-body channels

We conclude this paper by counting and classifying the channels which contribute to the final three-body equations.

In order to clarify the following discussion we restore some of the notation which we previously suppressed, and denote the two-body helicity states (6.17) with good parity, $|j^r(m\lambda)\rho\rangle$, by $|\tilde{p}_0 j^r(m\lambda)\rho\rangle$, where

$$\tilde{p}_0=E_{\tilde{p}_0}-\tfrac{1}{2}W_{23} \tag{6.31}$$

is the difference in the energies of the two particles in the two-body rest frame (with particle two on shell) and W_{23} the rest frame energy of the two-body system. Note that $\tilde{p}_0$ is in general not zero because particle three is off shell, and that we continue to suppress explicit reference to the magnitude of the three component of the relative momentum, $\tilde{p}$, because it will play no role in the discussion which follows. This state satisfies the relation

$$\mathcal{Q}_2|\tilde{p}_0 j^r(m\lambda)\rho\rangle=|\tilde{p}_0 j^r(m\lambda)\rho\rangle, \tag{6.32}$$

where $\mathcal{Q}_2$ is the projection operator introduced in Sec. II which places particle two on shell, and is equivalent to the identity when operating on states where particle two is already on shell. Note that the state with relative energy $-\tilde{p}_0$ has particle three on shell and hence

$$\mathcal{Q}_3|-\tilde{p}_0 j^r(m\lambda)\rho\rangle=|-\tilde{p}_0 j^r(m\lambda)\rho\rangle. \tag{6.33}$$

In terms of the states (6.17), the interchange of space and spin coordinates (everything but isospin) has the following effect:

$$\tilde{\mathcal{P}}_{23}|\tilde{p}_0 j^r(m\lambda)\rho\rangle=(r\rho)^\lambda \epsilon^{(1-\lambda)}|-\tilde{p}_0 j^r(m\lambda)\rho\rangle. \tag{6.34}$$

This equation follows from the definition (6.17) and the relation

$$\tilde{\mathcal{P}}_{23}|\tilde{p}_0 jm(\lambda_2\lambda_3)\rho\rangle=\epsilon|-\tilde{p}_0 jm(\lambda_3\lambda_2)\rho\rangle, \tag{6.35}$$

which (except for notational changes) is Eq. (2.97) of Ref. [28].

Using Eq. (6.34), we can extend the discussion of the previous subsection and introduce states with *both* good parity and good exchange symmetry. Introduce the states

$$|j^{ru}(m\lambda)\rho\rangle\equiv\frac{1}{\sqrt{2}}(1+u\tilde{\mathcal{P}}_{23})|\tilde{p}_0 j^r(m\lambda)\rho\rangle. \tag{6.36}$$

These are normalized eigenstates of both $\mathcal{P}^1$ and $\mathcal{P}_{23}$

$$\mathcal{P}^1|j^{ru}(m\lambda)\rho\rangle=r|j^{ru}(m\lambda)\rho\rangle,$$

$$\tilde{\mathcal{P}}_{23}|j^{ru}(m\lambda)\rho\rangle=u|j^{ru}(m\lambda)\rho\rangle. \tag{6.37}$$

Since $u=(-1)^T$, which can be written $T=(1-u)/2$, these states are also the correct spin-momentum states to use with isospin.

The counting and classifying of three-body states depends in part on the number and classification of the two-body scattering states. Since both $\mathcal{P}^1$ and $\tilde{\mathcal{P}}_{23}$ are conserved by the two-body equations, two-body states can be classified by different possible values of the quantum numbers r and u. There are four combinations:

$$\text{singlet} \quad r=-\epsilon, \quad u=\epsilon,$$

$$\text{triplet} \quad r=-\epsilon, \quad u=-\epsilon,$$

$$\text{coupled} \quad r=\epsilon, \quad u=\epsilon,$$

$$\text{virtual} \quad r=\epsilon, \quad u=-\epsilon.$$

The last set of states, referred to as *virtual states* in Ref. [28], do not contribute to physical two-body scattering. This is because, in the positive ρ-spin sector, their parity assignment would require that $j=\not{\pm}1$, which in turn requires a total spin $S=1$. These assignments are consistent with an exchange symmetry of $u=-\epsilon$ only if these states are *odd under change of sign of the relative energy variable* $\tilde{p}_0$, which ensures that they are zero on shell. However, because the two-body quantum numbers r and u are *not conserved* in three-body scattering, they can contribute to relativistic three-body scattering and to the three-body bound state. In the calculations completed thus far [4] we have neglected these states, but we expect them to give a small contribution of purely relativistic origin.

TABLE I. Possible quantum numbers for three-body states with $J^\Pi=\frac{1}{2}^+$ (for the triton). Virtual two-body states have been neglected.

j	λ	r	u	ρ	m	Number of states
0	0	$\pm$	$-$	$\pm$	0	4
$\geqslant 1$	0,1	$\pm$	ϵ	$\pm$	0,1	16
$\geqslant 1$	0,1	$-\epsilon$	$-\epsilon$	$\pm$	0,1	8

Neglecting the virtual states, and recalling the selection rule (6.19) leads to the following counting rules:

$$\begin{aligned} j=0: \quad & (\lambda=0)\times(\rho=\pm 1)\times(r=\pm 1)\times(u=-1) \\ & =4 \quad \text{states}, \\ j>0: \quad & (\lambda=0,1)\times(\rho=\pm 1)\times(r=\pm\epsilon)\times(u=\epsilon) \\ & +(\lambda=0,1)\times(\rho=\pm 1)\times(r=-\epsilon) \\ & \times(u=-\epsilon)=12 \quad \text{states}. \end{aligned}$$

The total number of two-body states with angular momenta $j\leqslant j_{\max}$ is therefore $n_2=4+12j_{\max}$.

In numerical calculations of the three-nucleon bound state it has become customary to truncate the partial wave series according to the maximal included total pair angular momentum j. Table I shows how many different three-body states exist for a given j. The pattern is simple: applying the selection rule (6.23) for each $j>0$ gives 24 possible states corresponding to 12 two-body states with either $m=0$ or 1, or $2\times 12=24$ states. For $j=0$ only $m=0$ is allowed, and hence there are only four different states. For each combination of quantum numbers there is one particular pair isospin consistent with the Pauli principle. Since we have used exchange symmetry to count the states, the inclusion of the isospin does not lead to any further increase in the number of channels. The total number of states up to a given maximal value $j_{\max}$ is therefore $n_3=4+24j_{\max}$.

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APPENDIX A: OPERATOR FORM OF THE TWO-BODY EQUATIONS

In this appendix we present the operator form of the two-body equations, and their subsequent reduction to partial waves. Using the notation of Sec. II B, the two-body equations for the scattering amplitude are

$$M = V - VG_2\mathcal{Q}_1 M, \tag{A1}$$

where V is the symmetrized kernel and M is the two-body scattering amplitude describing the scattering of particles 1 and 2. Note that particle 1 in on shell in the intermediate state, in agreement with the conventions of Ref. [28] (see Sec. II B in that reference). In the three-body language, the M in Eq. (A1) is M^3, because the ''spectator'' (if it were present) would be particle three. To obtain a closed set of equations for Eq. (A1), multiply by $\mathcal{Q}_1$, giving

$$[\mathcal{Q}_1 M] = [\mathcal{Q}_1 V] - [\mathcal{Q}_1 V G_2 \mathcal{Q}_1][\mathcal{Q}_1 M], \tag{A2}$$

which shows that particle one is on shell throughout the interaction.

The two-body partial wave equations can be obtained from Eq. (A2) by inserting a complete set of the two-body angular momentum states defined in Eq. (4.8) (with the 23 pair relabeled 12). The completeness and generalized orthogonality relations for the two-body states, implied by the work in Sec. II C, is

$$\langle j'(1'2')\rho'|j(12)\rho\rangle = \delta_{j'j}\delta_{m'm}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} O_{\rho'\rho}(\tilde{p},\lambda_2) 2E_{\tilde{p}} \frac{\delta(\tilde{p}-\tilde{p}')}{\tilde{p}^2}\delta^4(P_{12}'-P_{12}),$$

$$\mathbf{1} \equiv \mathcal{Q}_{\alpha'\alpha}\delta_{\beta'\beta} = \int \frac{\tilde{p}^2 d\tilde{p}}{2E_{\tilde{p}}}\frac{m^2}{E_{\tilde{p}}^2} d^4P \sum_{\substack{jm\rho'\rho \\ \lambda_1\lambda_2\lambda_3}} |j(12)\rho'\rangle O_{\rho'\rho}(\tilde{p},\lambda_2)\langle j(12)\rho|, \tag{A3}$$

where the shorthand notation defined in Eq. (4.45) has been used. Substituting the completeness relation into Eq. (A2) gives

$$\langle j(12)\rho|M|j(1'2')\rho'\rangle = \langle j(12)\rho|V|j(1'2')\rho'\rangle - \sum_{\substack{\rho_1\rho_2 \\ \lambda_1''\lambda_2''}} \int \frac{k^2 dk}{2E_k}\frac{m^2}{E_k^2}\langle j(12)\rho|VG|j(1''2'')\rho_1\rangle O_{\rho_1\rho_2}(k,\lambda_2'')$$
$$\times \langle j(1''2'')\rho_2|M|j(1'2')\rho'\rangle. \tag{A4}$$

The evaluation of the matrix element of VG is identical to the evaluation of M^1G_3 carried out in Eq. (4.48). Substituting this into Eq. (A4) gives

$$\langle j(12)\rho|M|j(1'2')\rho'\rangle = \langle j(12)\rho|V|j(1'2')\rho'\rangle - \sum_{\rho''\lambda_1''\lambda_2''} \int k^2 dk \langle j(12)\rho|V|j(1''2'')\rho''\rangle \frac{m^2}{E_k^2} g^{\rho''}(q,k)$$
$$\times \langle j(1''2'')\rho''|M|j(1'2')\rho'\rangle. \tag{A5}$$

Multiplying both sides of the equation by $(2\pi)^3 m^2/E_{\tilde{p}}E_{\tilde{p}'}$, and introducing the amplitudes

$$\frac{(2\pi)^3 m^2}{E_{\tilde{p}}E_{\tilde{p}'}}\langle j(12)\rho|M|j(1'2')\rho'\rangle = M^{\rho\rho' j}_{\lambda_1\lambda_1',\lambda_2\lambda_2'}(\tilde{p},\tilde{p}';P_{12}),$$

$$\frac{(2\pi)^3 m^2}{E_{\tilde{p}}E_{\tilde{p}'}}\langle j(12)\rho|V|j(1'2')\rho'\rangle = V^{\rho\rho' j}_{\lambda_1\lambda_1',\lambda_2\lambda_2'}(\tilde{p},\tilde{p}';P_{12}),$$

$$\frac{k^2}{(2\pi)^3} g^{\rho}(q,k) = g^{\rho}(k), \tag{A6}$$

gives the equation

$$M^{\rho\rho' j}_{\lambda_1\lambda_1',\lambda_2\lambda_2'}(\tilde{p},\tilde{p}';P_{12}) = V^{\rho\rho' j}_{\lambda_1\lambda_1',\lambda_2\lambda_2'}(\tilde{p},\tilde{p}';P_{12}) - \sum_{\rho''\lambda_1''\lambda_2''} \int dk V^{\rho\rho'' j}_{\lambda_1\lambda_1'',\lambda_2\lambda_2''}(\tilde{p},k;P_{12}) g^{\rho''}(k) M^{\rho''\rho' j}_{\lambda_1''\lambda_1',\lambda_2''\lambda_2'}(k,\tilde{p}';P_{12}). \tag{A7}$$

This is identical to Eq. (2.88) of Ref. [28], establishing the relationship between this paper and previous work on the two-body problem.

APPENDIX B: REDUCTION OF THE δ FUNCTIONS IN $\mathcal{P}_{12}$

In this appendix we derive Eq. (5.13) for the δ functions

$$2E_{k_1}\delta^{(3)}(k_1'^0-k_1)2E_{k_2}\delta^{(3)}(k_2'^0-k_2) \tag{B1}$$

which appear in the matrix element of the permutation operator, Eq. (5.11).

1. Radial δ functions

First find the vectors $k_1'^0$ and $k_2'^0$ shown in Fig. 5(a), and $k_1''^0$ and $k_2''^0$ shown in Fig. 5(b). The vectors $k_1'^0$ and $k_2''^0$ are

$$k_1'^0=\begin{pmatrix} E_{q'} \\ 0 \\ 0 \\ -q' \end{pmatrix}, \quad k_2''^0=\begin{pmatrix} E_q \\ 0 \\ 0 \\ -q \end{pmatrix}, \tag{B2}$$

where we anticipate that $\{k''\}\to\{k\}$ and hence use $|\mathbf{k}_2''|=q$ instead of $|\mathbf{k}_2''|=q''$. This agrees with the notation in Eq. (5.12). The vectors $k_2'^0$ and $k_1''^0$ can be found by boosting the vectors $\tilde{k}_2'^0$ and $\tilde{k}_1''^0$ defined in the two-body rest frame:

$$\tilde{k}_2'^0=\begin{pmatrix} E_{\tilde{p}'} \\ \tilde{p}'\sin\tilde{\theta}' \\ 0 \\ \tilde{p}'\cos\tilde{\theta}' \end{pmatrix}, \quad \tilde{k}_1''^0=\begin{pmatrix} E_{\tilde{p}''} \\ \tilde{p}''\sin\tilde{\theta}'' \\ 0 \\ \tilde{p}''\cos\tilde{\theta}'' \end{pmatrix}. \tag{B3}$$

The boost in the z direction for the configuration shown in Fig. 5(a) is

$$Z_{q'}=\begin{pmatrix} C' & 0 & 0 & S' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S' & 0 & 0 & C' \end{pmatrix}, \tag{B4}$$

where

$$C'=\frac{M_t-E_{q'}}{W_{q'}}=\frac{\sqrt{W_{q'}^2+q'^2}}{W_{q'}}, \quad S'=\frac{q'}{W_{q'}}. \tag{B5}$$

The boost for the configuration shown in Fig. 5(b) is obtained from Eq. (B4) by replacing q' by q. Hence

$$k_2'^0=Z_{q'}\tilde{k}_2'^0=\begin{pmatrix} C'E_{\tilde{p}'}+S'\tilde{p}'\cos\tilde{\theta}' \\ \tilde{p}'\sin\tilde{\theta}' \\ 0 \\ S'E_{\tilde{p}'}+C'\tilde{p}'\cos\tilde{\theta}' \end{pmatrix}, \tag{B6}$$

$$k_1''^0=Z_q\tilde{k}_1''^0=\begin{pmatrix} CE_{\tilde{p}''}+S\tilde{p}''\cos\tilde{\theta}'' \\ \tilde{p}''\sin\tilde{\theta}'' \\ 0 \\ SE_{\tilde{p}''}+C\tilde{p}''\cos\tilde{\theta}'' \end{pmatrix}.$$

Now the vectors which appear in the δ functions (B1) are $\mathbf{k}_1'^0$, $\mathbf{k}_2'^0$, $\mathbf{k}_1$, and $\mathbf{k}_2$. The second two of these are obtained by applying the rotation $R_{V'}$ to $\mathbf{k}_1''^0$ and $\mathbf{k}_2''^0$, as illustrated in Fig. 5(c). Since the rotation $R_{V'}$ does not change the length of the three-vectors, the condition $k_2'^0=k_2$ imposed by one of the radial δ functions becomes

$$\begin{aligned} q&=\{\tilde{p}'^2\sin^2\tilde{\theta}'+S'^2E_{\tilde{p}'}^2+C'^2\tilde{p}'^2\cos^2\tilde{\theta}' \\ &\quad +2C'S'\tilde{p}'E_{\tilde{p}'}\cos\tilde{\theta}'\}^{1/2} \\ &=\{[C'E_{\tilde{p}'}+S'\tilde{p}'\cos\tilde{\theta}']^2-m^2\}^{1/2}. \end{aligned} \tag{B7}$$

Requiring that $|\cos\tilde{\theta}'|<1$ gives the unique solution

$$\cos\tilde{\theta}'=\cos\tilde{\theta}_0'=\frac{E_q-C'E_{\tilde{p}'}}{S'\tilde{p}'}=\frac{W_{q'}E_q-(M_t-E_{q'})E_{\tilde{p}'}}{q'\tilde{p}'}. \tag{B8}$$

The radial δ function can therefore be rewritten

$$\delta(k_2'^0-k_2)=\delta(\cos\tilde{\theta}'-\cos\tilde{\theta}_0')\left|\frac{dk_2}{d\cos\tilde{\theta}'}\right|^{-1}_{\cos\tilde{\theta}'=\cos\tilde{\theta}_0'}.$$

The derivative is

$$\left|\frac{dk_2}{d\cos\tilde{\theta}'}\right|_{\cos\tilde{\theta}'=\cos\tilde{\theta}_0'}=\frac{S'E_q\tilde{p}'}{q},$$

giving finally

$$\begin{aligned} \frac{\delta(k_2'^0-k_2)}{k_2^2}&=\frac{1}{S'qE_q\tilde{p}'}\,\delta(\cos\tilde{\theta}'-\cos\tilde{\theta}_0') \\ &=\frac{W_{q'}}{qq'E_q\tilde{p}'}\,\delta(\cos\tilde{\theta}'-\cos\tilde{\theta}_0'), \end{aligned} \tag{B9}$$

where $\cos\tilde{\theta}_0'$ was defined in Eq. (B8).

The result for the other radial δ function follows immediately by interchanging primed and unprimed variables.

2. Angular δ functions

Evaluation of the angular δ functions requires explicit consideration of the rotation $R_{V'}=R_{\alpha,\chi',\beta}$ which rotates the vectors $k_1''^0$ and $k_2''^0$ into k_1 and k_2 as discussed in Sec. V A and illustrated in Fig. 5(c). First we consider the δ function

$$\delta^{(2)}(k_2'^0-k_2)=\delta(\cos\theta_2'^0-\cos\theta_2)\,\delta(\phi_2'^0-\phi_2), \tag{B10}$$

where (θ_2,ϕ_2) and $(\theta_2'^0,\phi_2'^0)$ are the polar and azimuthal angles of $\mathbf{k}_2$ and $\mathbf{k}_2'^0$, respectively. The three-vector $\mathbf{k}_2'^0$ was given in Eq. (B6), and $\mathbf{k}_2$ is

$$\mathbf{k}_2=R_{\alpha,\chi',\beta}\mathbf{k}_2''^0=R_{\alpha,\chi',\beta}\begin{pmatrix}0\\0\\-q\end{pmatrix}=-q\begin{pmatrix}\sin\chi'\cos\alpha\\\sin\chi'\sin\alpha\\\cos\chi'\end{pmatrix}. \tag{B11}$$

Since $k_{2y}=k_{2y}'^0=0$ and $k_{2x}=k_{2x}'^0\geqslant 0$, the azimuthal part of Eq. (B10) becomes

$$\delta(\phi_2'^0-\phi_2)=\delta(\pi-\alpha). \tag{B12}$$

The polar part of the δ function is

$$\delta(\cos\theta_2'^0-\cos\theta_2)=\delta\left(\frac{1}{q}[S'E_{\tilde{p}'}+C'\tilde{p}'\cos\tilde{\theta}']+\cos\chi'\right). \tag{B13}$$

If this δ function is used to eliminate the χ', integration, one is left with an integration over $\tilde{p}'$ with upper and lower limits depending on the two external momenta and on the second integration variable. This makes numerical solutions of the resulting equations awkward. Because χ' is the angle between the momenta $\mathbf{q}$ and $\mathbf{q}'$, and is therefore symmetric under interchange of initial and final states, it is more convenient to retain χ' as the independent variable and eliminate instead the integration over $\tilde{p}'$. The limits on the χ' integration turn out to be independent of the other variables, running over the expected range from 0 to π. Using Eq. (B8) to replace $\cos\tilde{\theta}'$, the δ function becomes

$$\delta(\cos\theta_2'^0-\cos\theta_2)=\delta\left(\frac{C'E_q-E_{\tilde{p}'}}{S'q}+\cos\chi\right), \tag{B14}$$

with the solution

$$E_{\tilde{p}'}=E_{\tilde{p}_0'}=C'E_q+S'q\cos\chi=\frac{(M_t-E_{q'})E_q+qq'\cos\chi}{W_{q'}}, \tag{B15}$$

and, since the $\mathbf{k}_2'^0=\mathbf{k}_2$ is now satisfied, we have replaced χ' by χ, as discussed in Sec. V A. It is easy to show that Eq. (B14) implies that $E_{\tilde{p}'}\geqslant m$ for all values of χ, so that the δ function (B14) places no additional restrictions on the χ integration. Hence

$$\delta(\cos\theta_2'^0-\cos\theta_2)=\delta(\tilde{p}'-\tilde{p}_0')\left|\frac{d\cos\theta_2'^0}{d\tilde{p}'}\right|^{-1}_{\tilde{p}'=\tilde{p}_0'}$$

$$=\frac{S'qE_{\tilde{p}_0'}}{\tilde{p}_0'}\delta(\tilde{p}'-\tilde{p}_0'). \tag{B16}$$

The final angular δ function is

$$\delta^{(2)}(k_1'^0-k_1)=\delta(\cos\theta_1'^0-\cos\theta_1)\delta(\phi_1'^0-\phi_1), \tag{B17}$$

where (θ_1,ϕ_1) and $(\theta_1'^0,\phi_1'^0)$ are the polar and azimuthal angles of $\mathbf{k}_1$ and $\mathbf{k}_1'^0$, respectively. The vector $\mathbf{k}_1'^0$ is given in Eq. (B2); the vector $\mathbf{k}_1$ is

$$\mathbf{k}_1=R_{\pi,\chi,\beta}\mathbf{k}_1''^0=R_{\pi,\chi,\beta}\begin{pmatrix}v_x\\0\\v_z\end{pmatrix}=R_{\pi,\chi,0}\begin{pmatrix}v_x\cos\beta\\v_x\sin\beta\\v_z\end{pmatrix}$$

$$=-\begin{pmatrix}v_x\cos\beta\cos\chi+v_z\sin\chi\\v_x\sin\beta\\v_x\cos\beta\sin\chi-v_z\cos\chi\end{pmatrix}. \tag{B18}$$

where

$$v_x=\tilde{p}''\sin\tilde{\theta}'',$$

$$v_z=SE_{\tilde{p}''}+C\tilde{p}''\cos\tilde{\theta}''=\frac{CE_{q'}-E_{\tilde{p}''}}{S}. \tag{B19}$$

Setting $\mathbf{k}_1=\mathbf{k}_1'^0$ gives three equations:

$$0=v_x\cos\beta\cos\chi+v_z\sin\chi=q'\sin\theta_1\cos\phi_1,$$

$$0=v_x\sin\beta=q'\sin\theta_1\sin\phi_1,$$

$$q=v_x\cos\beta\sin\chi-v_z\cos\chi=-q'\cos\theta_1. \tag{B20}$$

We will first use the left-hand set of Eqs. (B20) to obtain the values of β and $\tilde{p}''$ which are fixed by the δ functions (B17). Then we will use the right-hand set to find the Jacobian of the transformation from the variables $\cos\theta_2,\phi_2$ to the variables $\tilde{p}'',\beta$.

The angle β must be 0 or π. Allowing for either possibility, the first and third of Eqs. (B20) give

$$q'\cos\chi=-v_z. \tag{B21}$$

Substituting this result back into the third equation gives

$$q'\sin^2\chi=v_x\cos\beta\sin\chi=\tilde{p}''\sin\tilde{\theta}''\cos\beta\sin\chi. \tag{B22}$$

Since the sin of all angles under consideration is positive, this equation shows that $\cos\beta$ is also and that therefore $\beta=0$. Finally, from Eq. (B21) we obtain

$$\cos\chi+\frac{CE_{q'}-E_{\tilde{p}''}}{q'S}=0 \tag{B23}$$

which is the condition (B14), with q and q' interchanged, showing that $E_{\tilde{p}''}$ satisfies Eq. (B15) (with q and q' interchanged). We have shown that the second angular δ function is

$$\delta^{(2)}(k_1'^0-k_1)=\delta(\beta)\,\delta(\tilde{p}''-\tilde{p}_0)J, \tag{B24}$$

where $\tilde{p}_0$ is $\tilde{p}_0'$ with q and q' interchanged and J is the Jacobian of the transformation from the variables $\cos\theta_2,\phi_2$ to the variables $\tilde{p}'',\beta$.

We return to the right-hand set of Eqs. (B20) to calculate this Jacobian. Unlike the previous cases it is necessary to

calculate a full Jacobian because the variables are all coupled unless we go to the limit $\sin\theta_1=0$, which gives singular results. Postponing this limit until the end, we first eliminate v_z from Eqs. (B20) and obtain,

$$v_x\sin\beta=q'\sin\theta_1\sin\phi_1,$$

$$v_x\cos\beta=q'[\sin\theta_1\cos\phi_1\cos\chi-\cos\theta_1\sin\chi]. \quad \text{(B25)}$$

Differentiating both of these equations with respect to $\cos\theta_1$ and ϕ_1, computing the Jacobian, and *then* taking the limits $\theta_1=\pi$, $\phi_1=0$, and $\beta=0$ gives

$$\left|\frac{\partial v_x}{\partial\tilde{p}''}\right| J=\begin{vmatrix} \dfrac{\partial\beta}{\partial\phi_1} & \dfrac{\partial v_x}{\partial\phi_1} \\ \dfrac{\partial\beta}{\partial\cos\theta_1} & \dfrac{\partial v_x}{\partial\cos\theta_1}\end{vmatrix}=\frac{q'^2}{v_x}\cos\chi. \quad \text{(B26)}$$

The radial δ functions have fixed $\cos\tilde{\theta}''$ in terms of $\tilde{p}''\to\tilde{p}_0$ [Eq. (B8) with $q'\to q$ and $\tilde{p}'\to\tilde{p}''\to\tilde{p}_0$], and using this relation we find that

$$\frac{\partial v_x}{\partial\tilde{p}''}=-\frac{q'W_q\cos\chi}{qE_{\tilde{p}_0}\sin\tilde{\theta}''}, \quad \text{(B27)}$$

and hence the Jacobian is

$$J=\frac{qq'E_{\tilde{p}_0}}{\tilde{p}_0W_q}=\frac{Sq'E_{\tilde{p}_0}}{\tilde{p}_0}. \quad \text{(B28)}$$

Combining Eq. (B9), its companion, Eqs. (B16), (B24), and (B28), and anticipating the fact that the δ functions fix the rotation so that $\tilde{p}''\to\tilde{p}$ and $\tilde{\theta}''\to\tilde{\theta}$, we obtain our final result

$$2E_{k_1}\delta^{(3)}(k_1'^0-k_1)2E_{k_2}\delta^{(3)}(k_2'^0-k_2)=4E_{\tilde{p}_0}E_{\tilde{p}_0'}\delta(\alpha-\pi)\delta(\beta)\frac{\delta(\tilde{p}'-\tilde{p}_0')}{\tilde{p}_0'^2}\frac{\delta(\tilde{p}-\tilde{p}_0)}{\tilde{p}_0^2}\delta(\cos\tilde{\theta}'-\cos\tilde{\theta}_0')$$
$$\times\delta(\cos\tilde{\theta}-\cos\tilde{\theta}_0). \quad \text{(B29)}$$

APPENDIX C: WIGNER ROTATIONS

In this appendix the Wigner rotation angle for the standard boost (5.18) that occurs in the matrix elements of the permutation operator is derived.

Consider a spin-1/2 particle with mass m, helicity λ, and three-momentum $\tilde{\mathbf{p}}$ which lies in the xz plane (with the x component positive, by convention). Under the boost Z_q in the $+z$ direction the state transforms like

$$Z_q|\tilde{p},\lambda\rangle=R(Q)|p,\lambda\rangle=\sum_\nu|p,\nu\rangle d^{(1/2)}_{\nu,\lambda}(\beta), \quad \text{(C1)}$$

where Z_q will by used to denote *both* the boost (B4) in four-dimensional space-time, *and* its representation on the space of states. Hence $p=Z_q\tilde{p}$. In agreement with the notation used in Sec. IV, the magnitude $|\mathbf{p}|=q'$, and angle between $\mathbf{p}$ and the $+z$ axis is $\theta=\pi-\chi$. We will show that Q is a rotation about the y axis, and find the rotation angle, β, in terms of q, q', and χ.

As previously described, the helicity states are constructed from rest states by first boosting in the $+z$ direction, and then rotating through the proper angle. For the states $|p,\lambda\rangle$ and $|\tilde{p},\lambda\rangle$, this construction gives

$$|\tilde{p},\lambda\rangle=e^{-iJ_y\tilde{\theta}}L_{\tilde{p}}|\hat{m},\lambda\rangle,$$

$$|p,\lambda\rangle=e^{-iJ_y\theta}L_{q'}|\hat{m},\lambda\rangle, \quad \text{(C2)}$$

where L_k was defined in Eq. (5.19) and $\hat{m}=(m,\mathbf{0})$ is the four-momentum vector of a particle of mass m at rest. Hence the Wigner rotation operator is given by

$$R(Q)=(e^{-iJ_y\theta}L_{q'})^{-1}Z_qe^{-iJ_y\tilde{\theta}}L_{\tilde{p}}. \quad \text{(C3)}$$

Assuming that $R(Q)$ is a pure rotation about the y axis, the equation reduces to the following set of equations involving β:

$$e^{-iJ_y\theta}L_{q'}e^{-iJ_y\beta}=Z_qe^{-iJ_y\tilde{\theta}}L_{\tilde{p}}. \quad \text{(C4)}$$

We will solve these equations using the Dirac representation for the operators. In this representation the pure boosts in the z direction are

$$L_{q'}=e^{\alpha_z\eta_{q'}/2}=c'+s'\alpha_z, \quad \text{(C5)}$$

where $\tanh\eta_{q'}=q'/E_{q'}$, $c'=\cosh(\eta_{q'}/2)$, and $s'=\sinh(\eta_{q'}/2)$ [these relations were previously defined in Eq. (5.36)]. The boost Z_q has a structure similar to Eq. (C5) but with $\eta_{q'}\to n_q$ where

$$C=\frac{M_t-E_q}{W_q}=\cosh n_q, \quad S=\frac{q}{W_q}=\sinh n_q \quad \text{(C6)}$$

as in Eq. (B5), but with $q'\to q$. To simplify the notation in what follows, we denote the hyperbolic functions of $n_q/2$ by

$$c_n=\cosh(n_q/2), \quad s_n=\sinh(n_q/2). \quad \text{(C7)}$$

The pure rotations about the y axis are

$$e^{-iJ_y\theta}=e^{-i\theta\gamma^5\alpha_y/2}=\cos(\theta/2)-i\sin(\theta/2)\gamma^5\alpha_y. \quad \text{(C8)}$$

Hence Eq. (C4) becomes

$$[\cos(\theta/2)-i\gamma^5\alpha_y\sin(\theta/2)][c'+s'\alpha_z][\cos(\beta/2)-i\gamma^5\alpha_y\sin(\beta/2)]=[c_n+\alpha_z s_n][\cos(\tilde{\theta}/2)-i\gamma^5\alpha_y\sin(\tilde{\theta}/2)][c_p+\alpha_z s_p]. \tag{C9}$$

Using $\{\gamma^5\alpha_y,\alpha_z\}=0$, Eq. (C9) becomes

$$\{\cos[(\theta+\beta)/2]-i\gamma^5\alpha_y\sin[(\theta+\beta)/2]\}c'+\{\cos[(\theta-\beta)/2]-i\gamma^5\alpha_y\sin[(\theta-\beta)/2]\}s'\alpha_z$$
$$=\{c_nc_p+s_ns_p+\alpha_z(s_nc_p+c_ns_p)\}\cos(\tilde{\theta}/2)-i\{c_nc_p-s_ns_p+\alpha_z(s_nc_p-c_ns_p)\}\gamma^5\alpha_y\sin(\tilde{\theta}/2). \tag{C10}$$

Equating the coefficients of the independent operators on each side of this equation gives four coupled equations

$$c'\cos[(\theta+\beta)/2]=(c_nc_p+s_ns_p)\cos(\tilde{\theta}/2),$$
$$c'\sin[(\theta+\beta)/2]=(c_nc_p-s_ns_p)\sin(\tilde{\theta}/2),$$
$$s'\sin[(\theta-\beta)/2]=(c_ns_p-s_nc_p)\sin(\tilde{\theta}/2),$$
$$s'\cos[(\theta-\beta)/2]=(c_ns_p+s_nc_p)\cos(\tilde{\theta}/2). \tag{C11}$$

These are not all independent; squaring each of these and then adding the first two and subtracting the second two gives an identity (1=1). Hence only three are independent, and given the quantities q, q', and $\chi=\pi-\theta$, these three independent equations can be regarded as equations for the unknown quantities $\tilde{\theta}$, $\tilde{p}$, and β.

We are only interested in an equation for β. This is obtained by multiplying the first equation by the last equation, and adding it to the product of the second equation and the third. The result is

$$\cos\beta=\frac{C\sinh\eta_p+S\cosh\eta_p\cos\tilde{\theta}}{\sinh\eta'}. \tag{C12}$$

To eliminate $\cos\tilde{\theta}$, we find an equation for it by summing the squares of each of the equations. This gives

$$\cos\tilde{\theta}=\frac{\cosh\eta'-C\cosh\eta_p}{S\sinh\eta_p}=\frac{E_{q'}-CE_{\tilde{p}}}{S\tilde{p}}. \tag{C13}$$

Note that this is identical to Eq. (B8) (with the primed and unprimed variables exchanged), showing that the calculations are consistent. Substituting for $\cos\tilde{\theta}$ gives

$$\cos\beta=\frac{\cosh\eta_p\cosh\eta'-C}{\sinh\eta_p\sinh\eta'}=\frac{W_qE_{\tilde{p}}E_{q'}-m^2(M_t-E_q)}{\tilde{p}q'W_q}=\frac{q'(M_t-E_q)+qE_{q'}\cos\chi}{\sqrt{q'^2W_q^2+q^2E_{q'}^2+2qq'E_{q'}(M_t-E_q)\cos\chi+(qq'\cos\chi)^2}}. \tag{C14}$$

This is the formula for $\cos[\beta(q,q',\chi)]$.

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