Excluded volume hadron gas model for particle number ratios in $A+A$ collisions

Granddon D. Yen,[1] Mark I. Gorenstein,[2,3] Walter Greiner,[4] and Shin Nan Yang[3]

[1]*Institute of Physics, Academia Sinica, Taipei 11529, Taiwan*
[2]*Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine*
[3]*Department of Physics, National Taiwan University, Taipei 10617, Taiwan*
[4]*Institute for Theoretical Physics, Goethe University, Frankfurt, Germany*

We recapitulate a thermodynamically consistent excluded volume hadron gas model and examine its differences with other "thermal models" used in the literature. Preliminary experimental data for particle number ratios in the collisions of Au+Au at the BNL AGS ($11A$ GeV/c) and Pb+Pb at the CERN SPS ($160A$ GeV/c) are analyzed. For equal values of the hadron hard-core parameters the excluded volume model gives essentially the ideal gas predictions for the particle number ratios, which is similar to other thermal models. We observe, however, the systematic excess of experimental pion abundances compared to the ideal gas results. This effect can be explained in our model by a smaller pion hard-core volume compared to those of other hadrons. The absolute values for particle number and energy densities at the chemical freezeout are predicted with a simultaneous fit to all these AGS and SPS particle number ratios. [S0556-2813(97)05510-6]

PACS number(s): 25.75.$-$q, 24.10.Pa

I. INTRODUCTION

Preliminary data for nucleus-nucleus $(A+A)$ collisions with truly heavy beams have recently become available: Au+Au at $11A$ GeV/c at the BNL AGS and Pb+Pb at $160A$ GeV/c at the CERN SPS [1]. A systematic analysis of these data could yield clues to whether a short-lived phase with quark and gluon constituents, the quark-gluon plasma, exists during the hot and dense stage of these reactions.

For the studies of matter properties in $A+A$ collisions, it is of vital importance to determine whether local thermodynamical equilibrium in the system is reached. Assuming such a local thermodynamical equilibrium at the final (freeze-out) stage of the process, one can calculate the particle number ratios without detailed knowledge of the very complicated dynamical evolution history of the system. We remind that the *chemical* freeze-out which determines the hadron number ratios does not necessarily coincide with the *thermal* freeze-out which defines hadron momentum spectra.

The aim of the present paper is to analyze the preliminary data for particle number ratios of Au+Au (AGS) and Pb+Pb (SPS) collisions within the framework of the thermodynamical equilibrium hadron gas model. The name "thermal model" has often been used in the literature for this type of calculations. We stress, however, that those "thermal models" used in the literature to calculate particle number ratios are, in fact, very different. Therefore, our first step is to explain our model and clarify its difference from other versions of "thermal model."

It seems natural to start with the ideal hadron gas at the freeze-out stage. All known particles and resonances should be included in this gas and the resonance decay modes to the observed particles must be taken into account as well. However, such an ideal gas model becomes inadequate in high-energy $A+A$ collisions. The chemical freeze-out parameters, temperature T and baryonic chemical potential μ_b, obtained from fitting the particle number ratios at AGS and SPS energies lead to artificially large values of total particle number

densities which are much higher than any reasonable (or "intuitively expected") ones at the freeze-out. The total particle number density at the chemical freeze-out within the ideal gas approach is $n_{\text{tot}} \approx 4n_0$ for the AGS and $n_{\text{tot}} \approx 8n_0$ for the SPS, where $n_o \cong 0.16$ fm^{-3} is the normal nuclear density. These numbers also exceed any experimental estimate obtained from the particle multiplicities and from the measurement of the volume of the system at the freeze-out with pion-interferometry method. To suppress undesirable large values of particle number densities, the van der Waals (VDW) excluded volume procedure is used. We will follow in the present paper the thermodynamically consistent excluded volume model of Refs. [2, 3] which will be recapitulated in the next section. Some other "thermal models" Refs. [4–8] also include VDW "corrections," but in some *ad hoc* and inconsistent ways. It is often believed that the specific details of these VDW corrections are of minor importance. This is, however, not the case. The VDW repulsion does not essentially alter the particle number ratios, but it does always give a very strong suppression effect on the values of particle number densities themselves. As we shall see below the total values of particle number densities are suppressed by a factor of 8 and 14 for AGS and SPS chemical freeze-out states, respectively. Therefore, the influence of the VDW excluded volume procedure on particle densities, as well as on all other thermodynamical functions of the hadron gas, is very strong, and hence the correct form of the excluded volume model formulation should be employed.

II. van der WAALS REPULSION IN THE GRAND CANONICAL FORMULATION

For nonrelativistic statistical mechanics, the use of a grand canonical ensemble is usually just a matter of convenience. However, in the hadron gas considered below it is unavoidable. In a relativistic theory one can not fix the number of particles, the number of pions in a hadron gas system, for example. The average number of particles only makes sense in an equilibrium system, like in the case of a photon gas. This average number is not conserved, but increases

with increasing temperature of the system. This takes place at all temperatures, even at very low ones when the thermal motion of each individual pion can be treated nonrelativistically. The canonical ensemble with a fixed number of pions has no physical meaning. This statement is also valid, of course, for other hadrons. Particle chemical potentials, in general, regulate not particle numbers but the values of conserved charges. Strongly interacting matter has three conserved charges, viz., baryonic number B, strangeness S, and electric charge Q (strangeness is conserved as we neglect ''slow'' weak interactions). Therefore, the canonical ensemble can only be defined with fixed values of B, S, and Q, not the fixed numbers of pions, nucleons and other hadrons.

Let us start with the ideal gas of one particle species with temperature T, chemical potential μ and volume V. The pressure p^{id} is related to the grand canonical partition function $\mathcal{Z}^{id}$ and in thermodynamical limit $V \to \infty$, it has the following expression:

$$
\begin{aligned}
p^{id}(T,\mu) &\equiv T \lim_{V \to \infty} \frac{\ln \mathcal{Z}^{id}(T,\mu,V)}{V} \\
&= \frac{d}{6\pi^2} \int_0^\infty dk \frac{k^4}{(k^2+m^2)^{1/2}} f(k),
\end{aligned} \tag{1}
$$

with

$$
f(k) = \left[\exp\left(\frac{(k^2+m^2)^{1/2} - \mu}{T} \right) + \eta \right]^{-1} \tag{2}
$$

being the momentum distribution function. d is the number of particle internal degrees of freedom (degeneracy) and m is the mass. The value of η is -1 for bosons, $+1$ for fermions, and $\eta = 0$ gives the classical (Boltzmann) approximation. The ideal gas particle number density is given by

$$
n^{id}(T,\mu) \equiv \left(\frac{\partial p^{id}(T,\mu)}{\partial \mu} \right)_T = \frac{d}{2\pi^2} \int_0^\infty dk k^2 f(k). \tag{3}
$$

Furthermore, the entropy density is defined as $s^{id} = (\partial p^{id}/\partial T)_\mu$. The energy density can be found from another thermodynamical relation $\varepsilon^{id} = Ts^{id} - p^{id} + \mu n^{id}$ and is written as

$$
\varepsilon^{id}(T,\mu) = \frac{d}{2\pi^2} \int_0^\infty dk k^2 (k^2+m^2)^{1/2} f(k). \tag{4}
$$

For a fixed particle number N, the VDW excluded volume procedure means the substitution of the volume of the system V by $V - vN$, where v is the parameter corresponding to the proper volume of the particle. Note that this VDW procedure, interpreted in statistical mechanics as the gas of hard-sphere particles with radius r, requires that the volume parameter v equal to the ''hard-core particle volume,'' $\frac{4}{3}\pi r^3$, multiplied by a factor of 4 [9]. To introduce the excluded volume (á la van der Waals) in our grand canonical ensemble formulation, we start with the volume substitution of V by $V - vN$ in the canonical partition function Z for each fixed N separately. The grand canonical partition function for the ideal gas system

$$
\mathcal{Z}^{id}(T,\mu,V) = \sum_{N=0}^\infty \exp\left(\frac{\mu N}{T} \right) Z^{id}(T,N,V) \tag{5}
$$

then becomes

$$
\mathcal{Z}(T,\mu,V) = \sum_{N=0}^\infty \exp\left(\frac{\mu N}{T} \right) Z^{id}(T,N,V-vN) \theta(V-vN). \tag{6}
$$

There is a difficulty in the evaluation of the sum over N in Eq. (6) because of the N dependence of the ''available volume,'' $V - vN$. To overcome it, we perform a Laplace transform on Eq. (6) [2] and obtain

$$
\begin{aligned}
\hat{\mathcal{Z}}(T,\mu,x) &\equiv \int_0^\infty dV \exp(-xV) \mathcal{Z}(T,\mu,V) \\
&= \int_0^\infty d\hat{V} \exp(-x\hat{V}) \mathcal{Z}^{id}(T,\hat{\mu},\hat{V}),
\end{aligned} \tag{7}
$$

where $\hat{\mu} \equiv \mu - vTx$ and $\hat{V} \equiv V - vN$. The second equality in Eq. (7) can be easily understood as follows. We substitute $\mathcal{Z}(T,\mu,V)$ by the infinite sum of Eq. (6), change the integration variable V to $\hat{V}$ in each term of the series and then use Eq. (5) to sum up the series again.

From the definition of the pressure function

$$
p(T,\mu) \equiv T \lim_{V \to \infty} \frac{\ln \mathcal{Z}(T,\mu,V)}{V} \tag{8}
$$

one concludes that the grand canonical partition function of the system, in the thermodynamical limit, approaches

$$
\mathcal{Z}(T,\mu,V)|_{V \to \infty} \sim \exp\left[\frac{p(T,\mu)V}{T} \right]. \tag{9}
$$

From the first equality in Eq. (7) one sees that this exponentially increasing part of $\mathcal{Z}(T,\mu,V)$ generates an extreme right singularity in the function $\hat{\mathcal{Z}}(T,\mu,x)$ at some point x^*. For $x < p/T$ the integration over V for $\hat{\mathcal{Z}}(T,\mu,x)$ diverges at its upper limit. Therefore, the extreme right singularity of $\hat{\mathcal{Z}}(T,\mu,x)$ at $x^*(T,\mu)$ gives us the system pressure,

$$
p(T,\mu) = Tx^*(T,\mu). \tag{10}
$$

Note that this direct connection of the extreme right x singularity of $\hat{\mathcal{Z}}$ to the asymptotic behavior $V \to \infty$ of $\mathcal{Z}$ is a general mathematical property of the Laplace transform. Using the above equations we find

$$
x^*(T,\mu) = \lim_{\hat{V} \to \infty} \frac{\ln \mathcal{Z}^{id}(T,\hat{\mu},\hat{V})}{\hat{V}}, \quad \hat{\mu}_i \equiv \mu - vTx^*(T,\mu). \tag{11}
$$

With Eqs. (1) and (10) we find from Eq. (11) the transcendental equation for the pressure $p(T,\mu)$ of the gas with VDW repulsion in the grand canonical ensemble:

$$
p(T,\mu) = p^{id}(T,\tilde{\mu}), \quad \tilde{\mu} \equiv \mu - vp(T,\mu). \tag{12}
$$

At this point some comments are appropriate. First, we remind the reader that the singularity x^* of the $\hat{\mathcal{Z}}(T,\mu,x)$ function has nothing to do with phase transition singularities of the system. A singularity at $x^*(T,\mu)$ exists for the ideal gas as well — one can easily recover the ideal gas formulas by putting $v=0$ in the above equations. Another result which can puzzle a reader is the difference between the familiar form of the VDW repulsion,

$$p(V-vN)=NT, \qquad (13)$$

and our Eq. (12). According to Eq. (13) the ideal gas pressure, $p^{id}=NT/V$, increases due to the excluded volume repulsion as expected intuitively while Eqs. (1) and (12) clearly shows that the ideal gas pressure decreases with the inclusion of the excluded volume repulsion. To resolve this ''paradox,'' one should keep in mind the important difference between standard VDW gas treatment with Eq. (13) for fixed particle number N and our system with fixed chemical potential μ in the grand canonical ensemble. To clarify this point, let us calculate the particle number density:

$$n(T,\mu)\equiv\left(\frac{\partial p(T,\mu)}{\partial \mu}\right)_T=\frac{n^{id}(T,\hat{\mu})}{1+vn^{id}(T,\widetilde{\mu})} . \qquad (14)$$

To have a direct comparison with Eq. (13) we now make the Boltzmann approximation, i.e., with $\eta=0$ in Eq. (2). We shall see later that at the AGS and SPS chemical freeze-out, the Boltzmann approximation leads to a very good agreement (only a few percent deviations) for all thermodynamical functions with those calculated with exact Bose and Fermi distributions. In Boltzmann approximation, the distribution function $f(k)$, as well as all thermodynamical functions of the ideal gas, possesses a momentum-independent factor $\exp(\mu/T)$. One can easily find that

$$p(T,\mu)=\exp\left(-\frac{vp(T,\mu)}{T}\right)p^{id}(T,\mu),$$
$$p^{id}(T,\mu)=Tn^{id}(T,\mu). \qquad (15)$$

After some simple algebra, Eqs. (14) and (15) give

$$p(T,\mu)[1-vn(T,\mu)]=n(T,\mu)T. \qquad (16)$$

The form of Eq. (16) amazingly coincides with that of Eq. (13), and therefore, we have no contradiction with the standard VDW gas physics. The crucial difference is that we have in Eq. (16) a $n(T,\mu)$ function instead of a fixed number N (or a fixed particle density N/V) as is the case in Eq. (13). In the grand canonical ensemble with fixed temperature and chemical potential, the VDW repulsion leads to a strong suppression of the particle number density. It is just this suppression on the number of particles which leads to the decrease of the ideal gas pressure after the VDW repulsion begins to come into effect. Equations (16) and (13) have similar form, but their physical meaning are very different. Equation (13) calculates the pressure as the function of N, V, T while Eq. (16) gives particle number density $n(T,\mu)$, but only after Eq. (15) for $p(T,\mu)$ function is solved.

With $p(T,\mu)$ as the solution of Eq. (12) the particle number density is given by Eq. (14). The entropy and energy densities are

$$s(T,\mu)\equiv\left(\frac{\partial p(T,\mu)}{\partial T}\right)_\mu=\frac{s^{id}(T,\widetilde{\mu})}{1+vn^{id}(T,\widetilde{\mu})} , \qquad (17)$$

$$\varepsilon(T,\mu)\equiv Ts-p+\mu n=\frac{\varepsilon^{id}(T,\widetilde{\mu})}{1+vn^{id}(T,\widetilde{\mu})} . \qquad (18)$$

Equations (14), (17), and (18) reveal two suppression effects on the particle number, entropy and energy densities because of the VDW repulsion, namely

(1) The modification of the chemical potential $\mu\to\widetilde{\mu}$ as given in Eq. (12). In Boltzmann approximation it just leads to an additional factor $\exp(-vp/T)$.

(2) A suppression factor $[1+vn^{id}(T,\widetilde{\mu})]^{-1}<1$.

For a ideal gas system of several particle species $i=1,\ldots,h$, the thermodynamical functions are additive and equal to the sums of their partial values for different particle species:

$$p^{id}(T,\mu_1,\ldots,\mu_h)=\sum_{i=1}^{h} p_i^{id}(T,\mu_i), \qquad (19)$$

and similar expressions for s^{id}, n^{id}, and ε^{id}. Note that the index i includes all the information about the ith particle, m_i, d_i, η_i, μ_i, etc.

The extension of the excluded volume procedure for several particle species is straightforward. The grand canonical partition function of the ideal gas equals to the product of $\mathcal{Z}_i^{id}(T,\mu_i,V)$ for each particle species ''i.'' The excluded volume grand canonical partition function for several particle species, $i=1,\ldots,h$, with proper volumes $v_1,\ldots,v_h$ can then be written as

$$\mathcal{Z}(T,\mu_1,\ldots,\mu_h,V)=\sum_{N_1=0}^{\infty}\cdots\sum_{N_h=0}^{\infty}\prod_{i=1}^{h} \exp\left(\frac{\mu_i N_i}{T}\right)$$
$$\times Z_i^{id}(T,N_i,\hat{V})\theta(\hat{V}) \qquad (20)$$

with available volume $\hat{V}\equiv V-\sum_{i=1}^{h}v_i N_i$. The Laplace transform of Eq. (20) gives

$$p(T,\mu_1,\ldots,\mu_h)\equiv T\lim_{V\to\infty}\frac{\ln\mathcal{Z}(T,\mu_1,\ldots,\mu_h,V)}{V}$$

$$=p^{id}(T,\widetilde{\mu}_1,\ldots,\widetilde{\mu}_h)=\sum_{i=1}^{h} p_i^{id}(T,\widetilde{\mu}_i),$$
$$\qquad (21)$$

$$\widetilde{\mu}_i\equiv\mu_i-v_i p(T,\mu_1,\ldots,\mu_h), \quad i=1,\ldots,h. \qquad (22)$$

Particle number density for the ith species can be calculated from Eqs. (21) and (22) and is found to be

$$n_i(T,\mu_1,\ldots,\mu_h) \equiv \left(\frac{\partial p}{\partial \mu_i}\right)_{T,\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_h}$$

$$= \frac{n_i^{id}(T,\widetilde{\mu}_i)}{1+\Sigma_{j=1}^h v_j n_j^{id}(T,\widetilde{\mu}_i)}. \tag{23}$$

The total particle number density is the sum of the partial values n_i from $i=1$ to $i=h$. Note that particle number density n_i depends on all proper volume parameters $v_1,\ldots,v_h$. The partial pressures $p_i=p_i^{id}(T,\widetilde{\mu}_i)$ are reduced to the ideal gas ones $p_i^{id}(T,\mu_i)$ in the limit $v_i \to 0$. This, however, does not take place for n_i because the particle number density n_i with $v_i=0$ still feels the presence of other particles with $v_j \neq 0$ due to the suppression factor $[1+\Sigma_{j=1}^h v_j n_i^{id}(T,\mu_i)]^{-1}$ in Eq. (23).

Moreover, the total entropy and energy densities of the VDW hadron gas are given as

$$s(T,\mu_1,\ldots,\mu_h) \equiv \left(\frac{\partial p}{\partial T}\right)_{\mu_1,\ldots,\mu_h} = \frac{\Sigma_{i=1}^h s_i^{id}(T,\widetilde{\mu}_i)}{1+\Sigma_{j=1}^h v_j n_j^{id}(T,\widetilde{\mu}_j)}, \tag{24}$$

$$\varepsilon(T,\mu_1,\ldots,\mu_h) \equiv Ts-p+\sum_{i=1}^h \mu_i n_i$$

$$= \frac{\Sigma_{i=1}^h \varepsilon_i^{id}(T,\widetilde{\mu}_i)}{1+\Sigma_{j=1}^h v_j n_j^{id}(T,\widetilde{\mu}_j)}. \tag{25}$$

III. COMPARISON WITH OTHER "THERMAL MODELS"

We recapitulate in the last section a thermodynamically consistent formulation of the VDW repulsion in the grand canonical ensemble. The problem appeared to be not trivial and many thermal model formulations of the VDW "corrections" used in the literature do not meet the requirement of self-consistency. The essential difference between the model that we use and those of Refs. [4–7] is the modification of the hadron chemical potentials according to Eq. (22). The VDW "correction" of the ideal gas formulas by the suppression factor $[1+\Sigma_{j=1}^h v_j n_j^{id}(T,\mu_j)]^{-1}$ was postulated in Refs. [4–7] for all thermodynamical functions including the pressure. These "corrected" functions, however, *do not* satisfy fundamental thermodynamical relations. Besides, the consistent formulation allows the dependence on the excluded volumes v_i's for particle number ratios, while in the formulation of Refs. [4–7], particle ratios are always the same as in the ideal gas for any choices of v_i values because the suppression factor in the denominator is identical for each ith particle and therefore cancel each other out.

In Ref. [8], the VDW "correction" was postulated in the form of

$$p_i(T,\mu_i) = \frac{p_i^{id}(T,\mu_i)}{1+\Sigma_{j=1}^h v_j n_j^{id}(T,\mu_j)}. \tag{26}$$

The total pressure p is just the sum of above partial pressures over all particle species i. As we have shown in the previous section, the VDW formulation leads to the transcendental equations (21) and (22) for the pressure function. To com-

pare it with ansatz of Eq. (26), let us make again the Boltzmann approximation in Eq. (21). Using Eqs. (5) and (1) one finds from Eq. (21) that

$$p(T,\mu_1,\ldots,\mu_h) \cong \sum_{i=1}^h \exp\left(-\frac{v_i p}{T}\right) p_i^{id}(T,\mu_i). \tag{27}$$

Assuming $v_i p/T \ll 1$ so that $\exp(-v_i p/T) \approx (1+v_i p/T)^{-1}$, we use then the expansion

$$p(T,\mu_1,\ldots,\mu_h) \cong p^{id}(T,\mu_1,\ldots,\mu_h)+O(v_i)$$

and the relation

$$p^{id}(T,\mu_1,\ldots,\mu_h) = \sum_{i=1}^h p_i^{id}(T,\mu_i) \cong T\sum_{i=1}^h n_i^{id}(T,\mu_i) \tag{28}$$

[the second equality in Eq. (28) follows because of the Boltzmann approximation] to find finally

$$p_i(T,\mu_i) \approx \frac{p_i^{id}(T,\mu_i)}{1+v_i\Sigma_{j=1}^h n_j^{id}(T,\mu_j)}. \tag{29}$$

Equation (29) is still rather different from the prescription of Eq. (26) used in Ref. [8]. In Eq. (29) $p_i=p_i^{id}$ when $v_i=0$. This is generally true as can be clearly seen in the last equality of Eq. (21). However, this does not take place in Eq. (26) if some other $v_j \neq 0$. We conclude therefore that the "excluded volume correction" in Eq. (26) [8] has nothing to do with VDW excluded volume procedure even in the limit when all v_i's are small, i.e., in the first order expansion over v_i.

IV. PARTICLE NUMBER RATIOS AT AGS AND SPS ENERGIES

As mentioned before, particle chemical potentials μ_i regulate the values of conserved charges. For simplicity we neglect the effects of nonzero electrical chemical potential which were considered in Ref. [10]. Electrical chemical potential is responsible, for example, for $\pi^+ \pi^-$ asymmetry when colliding ions are heavy and therefore have isotopic asymmetry, i.e., their number of neutrons is larger than number of protons. These interesting effects are, however, not large numerically at AGS and especially at SPS energies.

The chemical potential of the ith particle can be written as

$$\mu_i = b_i \mu_b + s_i \mu_s, \tag{30}$$

in terms of baryonic chemical potential μ_b and strange chemical potential μ_s, where b_i and s_i are the corresponding baryonic number and strangeness of the ith particle. The hadronic gas state is defined by two independent thermodynamical parameters, T and μ_b. The strange chemical potential $\mu_s(T,\mu_b)$ is determined from the requirement of zero strangeness

$$n_S(T,\mu_b,\mu_s) \equiv \sum_{i=1}^h s_i n_i(T,\mu_i) = 0. \tag{31}$$

In addition to T and μ_b, the thermodynamical functions also depend on the excluded volume parameters v_i because of Eq. (22).

Baryon and meson resonances and their subsequent decays to observed hadrons are of great importance for the measured particle number ratios at AGS and SPS energies. All known resonance states with mass up to 2 GeV are included in our calculations. Preliminary experimental data for particle number ratios in the collisions of Au+Au at the BNL AGS (11A GeV/c) and Pb+Pb at the CERN SPS (160A GeV/c) are analyzed. We use the compilation of the experimental data which were presented by Stachel at QM'96 (see Ref. [11] and references therein).

The enhancement of strange particle production in $A+A$ collisions has attracted special attention as a possible signal of the quark-gluon plasma formation [12]. However, the ''thermal models'' of Refs. [4–7] which, in fact, gives the same results as the ideal hadron gas model have been successful in reproducing many of the ratios with strange hadrons measured at the AGS and SPS without any additional parameter to control strangeness abundance. Instead, they have a problem with pion abundances—for the SPS data it has been observed [4–6] that a simple ideal gas model is unable, within a single set of freeze-out parameters, to reproduce simultaneously the strange particles yields and antibaryon to baryon ratios together with ratios where pions are involved. Specifically, experimental pion to nucleon ratio and ratios of pions to other hadrons are larger than the ideal gas predictions. As pion multiplicity can be related to the entropy of the system, the ideal hadron gas model appears unable to account for the large entropy per baryon of the freeze-out system. The deficiency of pions in the ideal hadron gas calculations has become the main problem in the theoretical ''thermal model'' in the interpretation of the SPS particle production data. Several mechanisms [5,6,13] have been proposed to remedy this problem, but no satisfactory answer has yet been found.

Our model procedure is as follows. For given values of T and μ_b, we solve the system of equations, Eqs. (21), (22), and (31), to find $p(T,\mu_b)$ and $\mu_s(T,\mu_b)$. The values of T and μ_b are then determined from the ''best fit'' to particle number ratios not involving pions. These best fits to the Au+Au (AGS) and Pb+Pb (SPS) preliminary data are shown in Fig. 1. The chemical freeze-out parameters are found to be $T \cong 140$ MeV, $\mu_b \cong 590$ MeV for Au+Au AGS, and $T \cong 185$ MeV, $\mu_b \cong 270$ MeV for Pb+Pb SPS collisions.

All particle number densities are calculated from Eq. (23) for all known stable particles and resonances with mass up to 2.0 GeV. The total production yield of hadron i is then proportional to the sum of its thermal density and all possible resonance decay contributions to that hadron i:

$$n_i^{tot} = n_i + n_i^{dec} = n_i + \sum_{j \neq i} n_j \alpha(j,i), \qquad (32)$$

where $\alpha(j,i)$ is the probability (branching ratio) for resonance j to strongly decay into hadron i.

In our calculations we will examine the dependence of hadron ratios on particle volume parameters v_i's. Results shown in Fig. 1 correspond to the same proper volume parameter $v_i = v$ for all hadrons. In this case particle number
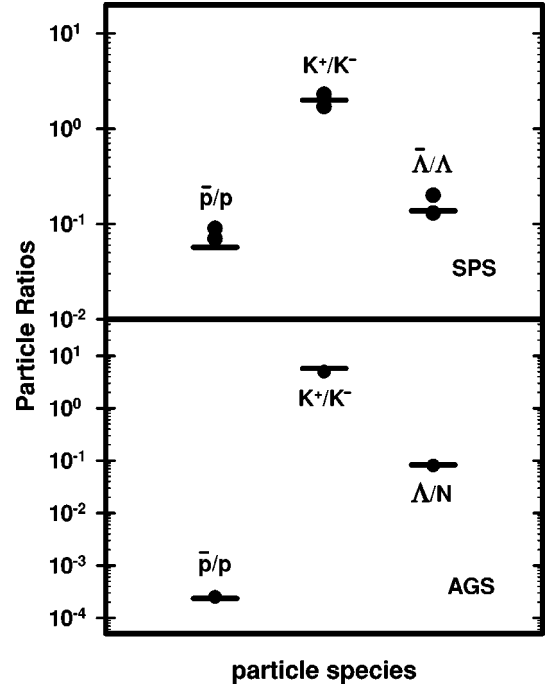

FIG. 1. Points are the preliminary experimental data for the particle number ratios (see Ref. [11] and references therein) for Au+Au AGS and Pb+Pb SPS collisions (in the lower and upper part of the figure, respectively). The short horizontal lines are the model fit with $T \cong 140$ MeV, $\mu_b \cong 590$ MeV (AGS), and $T \cong 185$ MeV, $\mu_b \cong 270$ MeV (SPS).

ratios are almost independent on the value of v and are the same as in the ideal gas and as in Refs. [4–7]. It occurs because of the relation

$$n_i^{id}(T, \widetilde{\mu}_i) \cong \exp\left(-\frac{v p(T, \mu_b)}{T} \right) n_i^{id}(T, \mu_i). \qquad (33)$$

Note that the above equality becomes exact in the Boltzmann approximation. In this case a common VDW ''denominator,'' $1 + \sum_{j=1}^{h} v_j n_j^{id}(T, \widetilde{\mu}_j)$, and a common ''numerator,'' $\exp(-vp/T)$, are canceled and Eq. (23) leads to

$$\frac{n_i(T, \mu_b)}{n_j(T, \mu_b)} \cong \frac{n_i^{id}(T, \mu_i)}{n_j^{id}(T, \mu_j)}. \qquad (34)$$

The value of the parameter v is, however, still crucial for the absolute values of particle number densities as well as for all other thermodynamical functions of the hadron gas. At the same fixed T and μ_b, all thermodynamical functions of the hadron gas are smaller than in the ideal hadron gas and strongly decrease with increasing v.

We use quantum statistics in our calculations, but some of our qualitative arguments depend on the validity of the Boltzmann approximation. We have checked for all particle number ratios that the Boltzmann approximation in the ideal hadron gas (i.e., $v_i = 0$) gives an accurate estimate to the corresponding quantum statistics values: $\sim 1-3$% for both AGS and SPS chemical freeze-out parameters. In our consideration with $v_i > 0$, each chemical potential is shifted by $-v_i p$ and the Boltzmann approximation always becomes even much better.

In the case when not all of the v_i's are equal, the hadron volume parameters v_i do influence the particle number ratios through the modification of Eq. (22) in the particle chemical potentials, required by thermodynamical self-consistency. The effect is quite evident that hadrons which take up less space (i.e., smaller values of v_i), and hence smaller influence on the excluded volume, have the advantage. The particle number ratios of those small hadrons to larger ones increase in comparison with the ideal gas results.

To solve the problem with pion multiplicities in the VDW model we now introduce different hard-core radii: r_π for pions and r for all other hadrons ($r > r_\pi$). Such a possibility with $r_\pi = 0$ was considered in Ref. [14]. We remind the reader that the excluded volume parameters are $v_i = 4(4\pi/3)r_i^3$. Using Eqs. (32) and (34), we then find

$$
\begin{aligned}
\frac{n_\pi^{\text{tot}}}{n_i^{\text{tot}}} &\cong \frac{\exp(-v_\pi p/T)n_\pi^{id}(T,\mu_\pi=0)+\Sigma_{j\neq\pi}\exp(-vp/T)n_j^{id}(T,\mu_j)\alpha(j,\pi)}{\exp(-vp/T)n_i^{id}(T,\mu_i)+\Sigma_{j\neq i}\exp(-vp/T)n_j^{id}(T,\mu_j)\alpha(j,i)} \\[2mm]
&\equiv \frac{\exp(\mu_\pi^*/T)n_\pi^{id}(T,\mu_\pi=0)+\Sigma_{j\neq\pi}n_j^{id}(T,\mu_j)\alpha(j,\pi)}{n_i^{id}(T,\mu_i)+\Sigma_{j\neq i}n_j^{id}(T,\mu_j)\alpha(j,\pi)},
\end{aligned}
\tag{35}
$$

where

$$
\mu_\pi^* \equiv (v-v_\pi)p = \frac{16\pi}{3}(r^3-r_\pi^3)p(T,\mu_b;r,r_\pi). \tag{36}
$$

In Eq. (36) we add r and r_π in the arguments of the pressure function to remind the pressure dependence on particle hard-core volumes through Eqs. (21) and (22). Equation (35) shows that the ratio of pions to any hadron i is changed in comparison to the ideal gas calculations. From the second equality of Eq. (35) the increase of the thermal pion density appears to be due to an effective pion chemical potential μ_π^*. In the Boltzmann approximation, it leads just to the additional factor of $\exp(\mu_\pi^*/T)$ in the pion number density of the ideal gas. Note that all chemical potentials μ_i are transformed to $\widetilde{\mu}_i = \mu_i - v_i p$ and therefore $\widetilde{\mu}_\pi = -v_\pi p$ becomes negative because $\mu_\pi = 0$. If v_π is smaller than $v_i = v$, in the ratios n_π/n_i, μ_π^* looks like a positive pion chemical potential in the ideal gas formalism. From this explanation of the origin of μ_π^*, it is clear that there is no restriction on its possible values by the pion mass, in contrast to the ideal Bose gas where always $\mu \leqslant m$.

In Figs. 2 and 3 different pion to hadron ratios are shown for Au+Au AGS and Pb+Pb SPS collisions, respectively. Preliminary experimental values are designated by dotted lines, while our model results are represented by solid curves, as functions of μ_π^*. The value $\mu_\pi^* = 0$ corresponds to the ideal gas results. We have already demonstrated that particle number ratios for $v_i = 0$ (the ideal hadron gas) remain the same as those for $v_i = v = $ constant. For all ratios in Figs. 2 and 3 we find that the experimental values systematically exceed the ideal gas results ($\mu_\pi^* = 0$). To fit data, one needs $\mu_\pi^* > 0$ and from Eq. (36) it means $r > r_\pi$. Our T and μ_b values are already fixed both for AGS and SPS from ratios given in Fig. 1. Assuming r_π different from $r_i = r$ for other hadrons we obtain no changes in the VDW model values for the ratios shown in Fig. 1. As pions have no influence on those ratios, they remain the same as the ideal hadron gas results.

At fixed T and μ_b the value of μ_π^* is a complicated function of r_π and r. The ratios in Figs. 2 and 3 feel not specific values of (r_π, r) but μ_π^* value. From the preliminary experimental data we find $\mu_\pi^* \cong 100$ MeV for Au+Au AGS collisions and $\mu_\pi^* \cong 180$ MeV for Pb+Pb SPS ones. We stress that the problem with pion deficiency observed in Refs. [5, 6] for SPS energies looks similar to what we find in *both* AGS and SPS data analyzed above. This deficiency of pions for the preliminary data in Pb+Pb SPS collisions is not so drastic as in S+Pb SPS data, where the pion to nucleon ratio
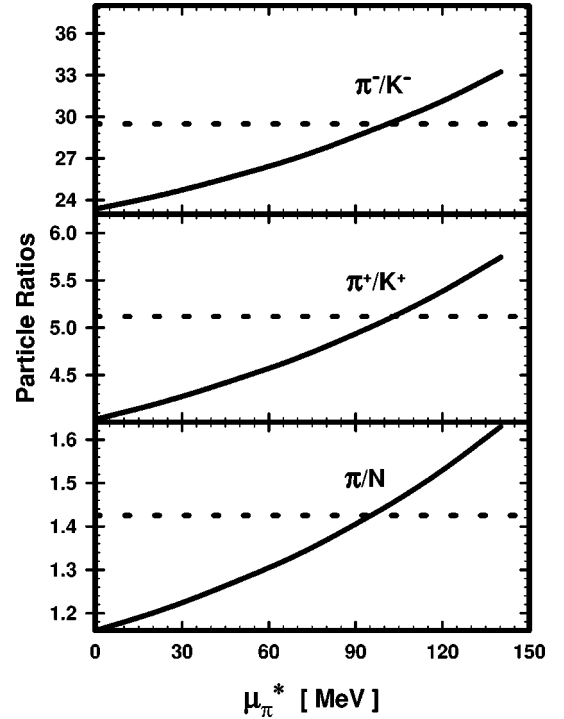

FIG. 2. Pion to hadron ratios for Au+Au AGS collisions. The experimental data (see Ref. [11] and references therein) are shown by the dotted horizontal lines. The solid lines are the VDW hadron gas model results as the functions of μ_π^* (see the text for details). An agreement with data corresponds to $\mu_\pi^* \cong 100$ MeV.
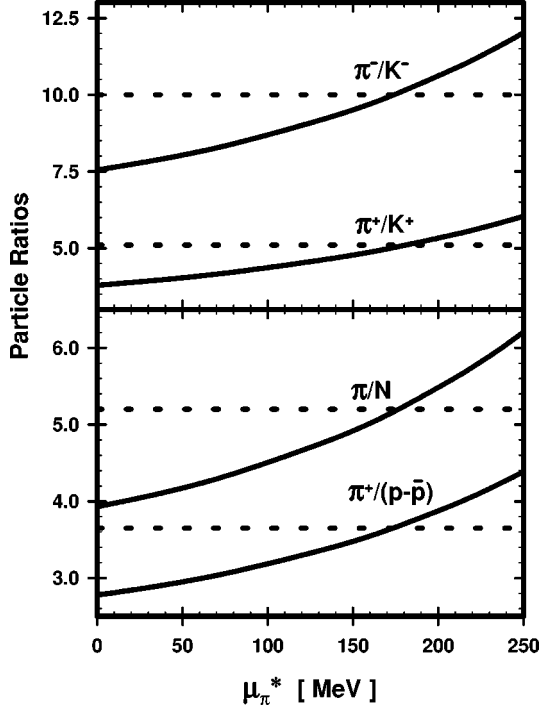
FIG. 3. Pion to hadron ratios for Pb+Pb SPS collisions. The experimental data (see Ref. [11] and references therein) are shown by the dotted horizontal lines. The solid lines are the VDW hadron gas model results as the functions of μ_π^* (see the text for details). An agreement with data corresponds to $\mu_\pi^* \cong 180$ MeV.
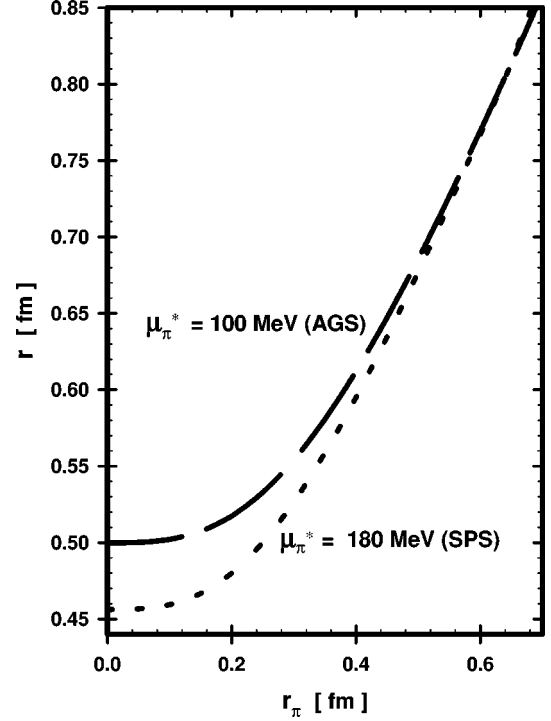

FIG. 4. The solutions of the equations $\mu_\pi^* =$ constant for AGS parameters, Eq. (37), (the dashed curve) and SPS ones, Eq. (38), (the dotted curve) shown in (r_π, r) plane. The intersection point is approximately at $(r_\pi \cong 0.62$ fm, $r \cong 0.8$ fm).

is close to 8 [15] and is approximately 2 times larger than the ideal gas result. To have an agreement with data in Figs. 2 and 3 we need only about 30% larger pion to other hadrons ratios than those in the ideal hadron gas. With $r_\pi < r$ (and therefore $\mu_\pi^* > 0$) the thermal pion number density would increase by a factor of $\exp(\mu_\pi^*/T)$. It is about 2 for Au+Au AGS and about 2.6 for Pb+Pb SPS collisions to have an agreement with data. The effect for the pion thermal density is therefore quite strong. However, it does not strongly alter the pion to hadron ratios, Eq. (35), because at both AGS and SPS energies the pion production is essentially dominated by the resonance decay contributions. Let us also remind the possibility of the chemical nonequilibrium effects for pions discussed in Ref. [16]. It would lead to the chemical potential $\mu_\pi > 0$ with its values being always smaller than the pion mass m_π. For $r_\pi < r$, we have obtained in the VDW model the effective pion chemical potential μ_π^* whose value is not restricted by the pion mass, and no chemical nonequilibrium effects are required.

V. PARTICLE HARD-CORE RADII

Our fits to the preliminary data of Au+Au (AGS) and Pb+Pb (SPS) on the particle number ratios with VDW hadron gas model are shown in Figs. 1–3. From fitting the data we have found the following model parameters:

$$\text{AGS:} \quad T \cong 140 \text{ MeV}, \quad \mu_b \cong 590 \text{ MeV}, \quad \mu_\pi^* \cong 100 \text{ MeV},$$
$$\tag{37}$$

$$\text{SPS:} \quad T \cong 185 \text{ MeV}, \quad \mu_b \cong 270 \text{ MeV}, \quad \mu_\pi^* \cong 180 \text{ MeV}.$$
$$\tag{38}$$

The set of all possible values of r_π and r which give the same value of μ_π^* in Eq. (36) for the AGS or SPS defines a curve in (r_π, r)-plane. The two curves corresponding to the AGS and SPS, respectively, are shown in Fig. 4, together with their intersection point ($r_\pi \cong 0.62$ fm, $r \cong 0.8$ fm). This intersection point is a solution for r_π and r in the VDW

TABLE I. Meson number density n_m, baryon number density n_b, total particle number density n_{tot}, energy density ε, and total pion number density n_π^{tot}, which includes thermal pions and contributions from decays, at the freeze-out for AGS Au+Au collision at $11A$ GeV/c.

(r_π, r) [fm]	n_m [fm^{-3}]	n_b [fm^{-3}]	n_{tot} [fm^{-3}]	ε [GeV/fm^3]	n_π^{tot} [fm^{-3}]
(0.00, 0.00)	0.200	0.402	0.603	0.722	0.400
(0.00, 0.50)	0.097	0.130	0.227	0.248	0.161
(0.20, 0.52)	0.089	0.120	0.209	0.228	0.149
(0.40, 0.61)	0.061	0.084	0.145	0.159	0.103
(0.62, 0.80)	0.032	0.043	0.075	0.083	0.054

TABLE II. Meson number density n_m, baryon number density n_b, total particle number density n_{tot}, energy density ε, and total pion number density n_π^{tot}, which includes thermal pions and contributions from decays, at the freeze-out for SPS Pb+Pb collision at $160A$ GeV/c.

(r_π, r) [fm]	n_m [fm^{-3}]	n_b [fm^{-3}]	n_{tot} [fm^{-3}]	ε [GeV/fm^3]	n_π^{tot} [fm^{-3}]
(0.00, 0.00)	0.771	0.404	1.251	1.585	1.213
(0.00, 0.46)	0.278	0.095	0.391	0.429	0.382
(0.20, 0.48)	0.244	0.084	0.344	0.378	0.336
(0.40, 0.59)	0.139	0.049	0.197	0.219	0.193
(0.62, 0.80)	0.064	0.022	0.090	0.100	0.088

hadron gas model to fit simultaneously the AGS and SPS data for all particle number ratios.

In Tables I and II, we show the values of the total meson number density n_m, baryon number density n_b, total particle number density n_{tot} (antibaryons included), total energy density ε and total pion number density n_π^{tot}, which includes both thermal pions and contributions from resonance decays, at chemical freeze-out. The ideal hadron gas ($r_\pi = r = 0$) results are given in the first row, and those from our calculations with different (r_π, r) values along the curves in Fig. 4 are shown in the remaining four rows. We emphasize that the last rows in the tables give our results for the intersection point ($r_\pi \cong 0.62$ fm, $r \cong 0.8$ fm), and that these values are our predictions within the VDW hadron gas model for the chemical freeze-out at Au+Au AGS and Pb+Pb SPS collisions, respectively. We believe that these r_π, r values, along with the corresponding densities, give a reasonable physical solution for the chemical freeze-out state in $A+A$ collisions considered.

VI. SUMMARY

A self-consistent hadron gas model with the VDW excluded volume is considered and critically compared with other ''thermal models'' used in the literature. This approach is then adopted to analyze the preliminary data of Au+Au (AGS) and Pb+Pb (SPS). Within the VDW model, the obtained values of particle number and energy densities, and other thermodynamic functions at chemical freeze-out are very different from those obtained in the ideal hadron gas, as seen in the first and last rows in Tables I and II. Because of their strong effects on hadron thermodynamical functions, the VDW gas formulation should be properly treated.

The preliminary data of Au+Au (AGS) and Pb+Pb (SPS) for the particle number ratios can be fitted in the VDW hadron gas as shown in Figs. 1–3. The model parameters, given in Eqs. (37) and (38), lead to the enhancement of pions in pion to hadron ratios as compared to the ideal hadron gas model predictions. This enhancement in pions is regulated by μ_π^* (>0) and is explained in the VDW model by a smaller pion ''hard-core radius'' than those of all other hadrons.

The obtained parameters for Au+Au (AGS) and Pb+Pb (SPS) define two curves in the (r_π, r) plane, respectively.

These two curves shown in Fig. 4 intersect at the point ($r_\pi \cong 0.62$ fm, $r \cong 0.8$ fm). It is the solution in the VDW hadron gas model to fit simultaneously the AGS and SPS data for all particle number ratios. The absolute values of the particle number and energy densities in the VDW hadron gas for this solution are listed in the last rows of Tables I and II. We stress that these values are much smaller than those in the ideal hadron gas at the same T and μ_b shown in the first rows of Tables I and II. There is an experimental estimate for the freeze-out pion number density in Si+Pb central collisions at the AGS energy: $n_\pi^{\text{expt}} \cong 0.063$ fm^{-3} [17]. Our result $n_\pi^{\text{tot}} \cong 0.054$ fm^{-3} for Au+Au AGS collisions shown in the last row of the Table I is quite close to this experimental estimate.

The next step is naturally to fit hadron momentum spectra. It requires the inclusion of the longitudinal and transverse collective flow effects. The temperature $T \cong 185$ MeV looks a little too high for use in the fitting of the transverse momentum spectra in Pb+Pb (SPS) collisions. For example, the pion inverse slope parameter is near 190 MeV [18] and the freeze-out temperature $T \cong 185$ MeV seems to leave almost no room for the transverse collective motion effects. However, two facts should be taken into account. First, the temperature determined from particle number ratios is for the chemical freeze-out which could be higher than the thermal freeze-out temperature used in particle spectra calculations. Second, we remind the reader again that there are large resonance decay contributions to pion production. Even after the enhancement of thermal pions with $\mu_\pi^* \cong 180$ MeV, resonance decays contribute more than 60% to the final pions. These resonance decays are known to lead to a lower pion ''effective temperature'' (the inverse slope parameter) at the transverse pion mass less than 1 GeV.

[1] Proceedings of Quark Matter 96 [Nucl. Phys. **A610**, (1996)].

[2] D. H. Rischke, M. I. Gorenstein, H. Stöcker, and W. Greiner, Z. Phys. C **51**, 485 (1991).

[3] J. Cleymans, M. I. Gorenstein, J. Stalnacke, and E. Suhonen, Phys. Scr. **48**, 277 (1993).

[4] J. Cleymans and H. Satz, Z. Phys. C **57**, 135 (1993).

[5] J. Cleymans, K. Redlich, H. Satz, and E. Suhonen, Z. Phys. C **8**, 347 (1993).

[6] J. Cleymans, K. Redlich, H. Satz, and E. Suhonen, Nucl. Phys. **A566**, 391c (1994).

[7] P. Braun-Munziger, J. Stachel, J. P. Wessels, and N. Xu, Phys. Lett. B **344**, 43 (1995).

[8] P. Braun-Munziger, J. Stachel, J. P. Wessels, and N. Xu, Phys. Lett. B **365**, 1 (1995).

[9] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1975).

[10] M. I. Gorenstein and S. N. Yang, Phys. Rev. C **44**, 2875 (1991); M. I. Gorenstein, H. G. Miller, R. M. Quick, and S. N. Yang, *ibid.* **50**, 2232 (1994); M. I. Gorenstein and H. G. Miller, *ibid.* **55**, 2002 (1997).

[11] J. Stachel, Nucl. Phys. **A610**, 509c (1996).

[12] J. Rafelski, Phys. Rep. **88**, 331 (1982); P. Koch, B. Müller, and J. Rafelski, *ibid.* **142**, 167 (1986).

[13] J. Letessier, A. Tounsi, U. Heinz, J. Solfrank, and J. Rafelski, Phys. Rev. Lett. **70**, 3530 (1993); Phys. Rev. D **51**, 3408 (1995).

[14] R. A. Ritchie, M. I. Gorenstein, and H. G. Miller, Z. Phys. C (to be published).

[15] EMU05 Collaboration, A. Iyono *et al.*, Nucl. Phys. **A544**, 455c (1992); R. Holyński, Nucl. Phys. **A556**, 191c (1994).

[16] M. Kataja and P. V. Ruuskanen, Phys. Lett. B **243**, 181 (1990); P. Gerber, H. Leutwyler, and J. L. Goity, *ibid.* **246**, 513 (1990).

[17] E814 Collaboration, J. Barrette *et al.*, Phys. Lett. B **333**, 33 (1994).

[18] P. G. Jones, Nucl. Phys. **A610**, 188c (1996).