Parametrization of nonaxial deformations in rotational nuclei

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A parametrization is proposed for hexadecapole tensors, which observes symmetries coming from the O_h group of transformations of the frame of reference defined through the quadrupole deformation and/or the rotation axes. A possible dependence of the hexadecapole deformation parameters on the quadrupole deformation is discussed. The inclusion of octupole deformation is considered. [S0556-2813(97)00307-5]

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I. INTRODUCTION

The quadrupole deformation is, indeed, dominant in atomic nuclei, but it is well known that deformations of higher multipolarities are also essential for a satisfactory description of nuclear properties. Although axially symmetric shapes prevail in nuclei, nonaxial quadrupole deformation γ has been considered for a long time. Occasions arise for taking into account also higher-multipolarity nonaxial deformations which are discussed much less so far. The second most important multipolarity in nuclear shape is the hexadecapole deformation, although the existence of hexadecapole vibrations is questionable. An interest in this multipolarity still increased after the observation of the $\Delta I=4$ staggering of superdeformed bands in some nuclei [1–3]. In the case of hexadecapole deformation, nonaxial shapes possessing three mutually perpendicular symmetry planes have only been discussed as yet [4–8]. A parametrization of such special hexadecapole shapes has been proposed in Ref. [7]. It is still not used in its general, three-parameter form (however, cf. [9]). In practical calculations a nonaxiality of the hexadecapole deformation is usually made dependent on the quadrupole triaxiality angle γ just to reduce the number of deformation parameters [10,7,8]. Some special forms of this dependence are used hitherto (cf. [4,6,11–13]). Another nuclear multipole mode of great importance is the octupole deformation. The octupole vibrations are observed for a long time (cf. [14] for a review). The octupole degrees of freedom appeared to be still more interesting when a significant role of octupole correlations in various nuclear phenomena, like fission, shape dependence of nuclear masses, superdeformation at high-spin states, etc., became recognized (see [15] for a comprehensive review). Axially symmetric, pearlike nuclear shapes have mainly been considered as yet (cf., e.g., [16]). Study of arbitrary nonaxial octupole deformation is exceptional [17]. Usually, some special octupole shapes are investigated by taking one or a few (but not all) spherical components of the octupole tensor (e.g., [18–21]). A general parametrization of the octupole deformation superimposed on the triaxial quadrupole shape was already proposed in [22].

The aim of the present paper is threefold. First, it is to propose a parametrization which is a generalization of that of [7] for an arbitrary hexadecapole deformation. Second, it is to discuss the most general dependence of hexadecapole shapes on the quadrupole deformation. These problems are presented in Secs. II and III, respectively. Third, it is to include the octupole deformation, treating it on the same footing with the hexadecapole deformation which is mainly considered here. This is discussed in Sec. IV. A conclusion from the present study is drawn in Sec. V.

The hexadecapole and octupole deformations are addressed here for rotational nuclei for two reasons. It is assumed throughout the paper that the shape always contains a substantial quadrupole component defining a system of axes up to their senses. Also, a possible external definition of one or all three axes as the rotation axes is considered. This means that the hexadecapole or the octupole component of the nuclear surface is always analyzed within a frame of axes fixed in advance and thus not only its shape, but also its orientation with respect to these axes is of physical relevance.

First of all we recapitulate briefly the well-known description of the quadrupole deformation. When it does not vanish the standard formula for the radius of the nuclear surface,

$$R(\Omega)=R_0\left(1+\sum_{\lambda,\mu}\alpha_{\lambda\mu}Y^{*}_{\lambda\mu}(\Omega)\right), \tag{1.1}$$

can be expressed in terms of spherical angles θ and ϕ describing the orientation Ω with respect to the principal axes $x,\ y,\ z$ of quadrupole tensor $\alpha_{2\mu}$ in the following form:

$$R(\theta,\phi)=R_0\Bigg[1+a_{20}Y_{20}(\theta,\phi)+a_{22}Y^{(+)}_{22}(\theta,\phi)+\sum_{\lambda\neq 2}\Bigg(a_{\lambda0}Y_{\lambda0}(\theta,\phi)+\sum_{\mu>0}[a_{\lambda\mu}Y^{(+)}_{\lambda\mu}(\theta,\phi)+b_{\lambda\mu}Y^{(-)}_{\lambda\mu}(\theta,\phi)]\Bigg)\Bigg], \tag{1.2}$$

where

TABLE I. Transformations of the coordinate system generating the O_h group.

Transformation	Symbol	Index i	Coordinates in the transformed system $x^{(i)}$	$y^{(i)}$	$z^{(i)}$
Identity	$\mathcal{E}$		x	y	z
Inversion	$\mathcal{I}$	0	$-x$	$-y$	$-z$
Bohr's	$\mathcal{R}_1$	1	x	$-y$	$-z$
rotat-	$\mathcal{R}_2$	2	y	$-x$	z
ions	$\mathcal{R}_3$	3	y	z	x

$$Y^{(+)}_{\lambda\mu}=\frac{1}{\sqrt{2}}[Y_{\lambda\mu}+(-1)^{\mu}Y_{\lambda-\mu}]$$

and

$$Y^{(-)}_{\lambda\mu}=\frac{1}{i\sqrt{2}}[Y_{\lambda\mu}-(-1)^{\mu}Y_{\lambda-\mu}] \tag{1.3}$$

are the real and imaginary parts of spherical harmonics $Y_{\lambda\mu}$ for $\mu\neq 0$ multiplied by $\sqrt{2}$ and, hence, $a_{\lambda\mu}$ and $b_{\lambda\mu}$ are, up to a coefficient, the real and imaginary parts of the intrinsic components of tensors $\alpha_{\lambda\mu}$. The quadrupole deformation is parametrized by means of two Bohr parameters [23] $\beta_2\geqslant 0$ and $-\pi\leqslant\gamma_2\leqslant\pi$ in the following way:

$$a_{20}=\beta_2\cos\gamma_2, \quad a_{22}=\beta_2\sin\gamma_2. \tag{1.4}$$

The principal axes are defined up to the group O_h of 48 transformations changing their names and arrows. All these transformations are superpositions of the four following ones: $\mathcal{I}$, the inversion; $\mathcal{R}_1$, the rotation by angle π around axis x; $\mathcal{R}_2$, the rotation by $\pi/2$ around axis z; and $\mathcal{R}_3$, the circular transposition of axes $y\to x$, $z\to y$, $x\to z$. The three latter are well-known Bohr rotations [23,24]. The transformation rules for all four are listed in Table I. The transformations $\mathcal{R}_1$, $\mathcal{R}_2^2$, and $\mathcal{I}$ generate the subgroup $D_{2h}\subset O_h$ of the eight following transformations changing the arrows of axes: $\mathcal{E}$, the identity; $\mathcal{I}$, the inversion; $\mathcal{R}_x=\mathcal{R}_1$, $\mathcal{R}_y=\mathcal{R}_1\mathcal{R}_2^2$, $\mathcal{R}_z=\mathcal{R}_2^2$, the three rotations by π around axes x, y, z; and $\mathcal{S}_x=\mathcal{I}\mathcal{R}_x$, $\mathcal{S}_y=\mathcal{I}\mathcal{R}_y$, $\mathcal{S}_z=\mathcal{I}\mathcal{R}_z$, the three reflections with respect to the planes perpendicular to axes x, y, z, respectively. The entire group $O_h=D_{2h}\times S_3$ is the direct product of D_{2h} and S_3, the group of six permutations of axes: $\mathcal{P}_{xyz}=\mathcal{E}$, $\mathcal{P}_{yzx}=\mathcal{R}_3$, $\mathcal{P}_{zxy}=\mathcal{R}_3^2$, $\mathcal{P}_{yxz}=\mathcal{S}_y\mathcal{R}_2$, $\mathcal{P}_{zyx}=\mathcal{S}_y\mathcal{R}_2\mathcal{R}_3$, and $\mathcal{P}_{xzy}=\mathcal{S}_y\mathcal{R}_2\mathcal{R}_3^2$. In particular, $\mathcal{S}_y\mathcal{P}_{yxz}=\mathcal{R}_2$.

The deformation β_2 is O_h invariant, while γ_2 is invariant only under D_{2h} and is transformed under $\mathcal{R}_2$ and $\mathcal{R}_3$ generating S_3 (modulo destinations of the arrows) as follows:

$$\gamma_2^{(2)}=-\gamma_2, \quad \gamma_2^{(3)}=\gamma_2-\frac{2\pi}{3}, \tag{1.5}$$

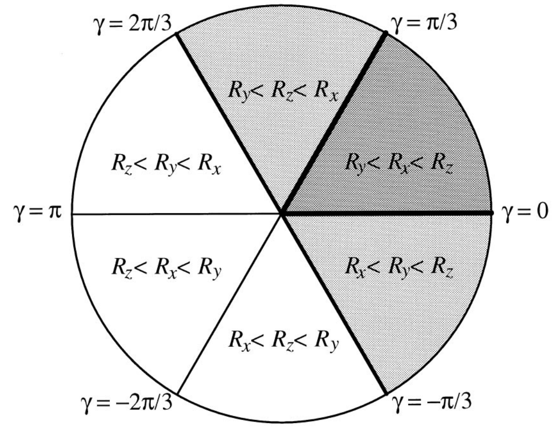

FIG. 1. Relations between the lengths of semiaxes R_x, R_y, R_z for different sectors of angle γ (γ_2 or γ_4) within the range $-\pi\leqslant\gamma\leqslant\pi$. All the border lines between different sectors correspond to the surfaces with two semiaxes equal to one another. In particular, the thicker line corresponds to $R_y=R_z$. The shaded sectors on the right-hand side of it ($-\pi/3<\gamma<2\pi/3$) correspond to $R_y<R_z$. Within the darker sector between the thickest lines ($0<\gamma<\pi/3$) one additionally has $R_y<R_x<R_z$.

where $\gamma_2^{(k)}$ is the angle γ_2 referring to the system of axes which is obtained by transformation $\mathcal{R}_k$ (the superscript $^{(0)}$ will be used in the case of transformation $\mathcal{I}$). The parameters β_2 and γ_2 determine in the following way a contribution coming from the quadrupole deformation to the lengths R_x, R_y, and R_z of the three semiaxes of the nuclear surface (1.2):

$$R_x=R\left(\theta=\frac{\pi}{2},\phi=0\right)=R_0\left[1+\sqrt{\frac{5}{4\pi}}\beta_2\cos\left(\gamma_2-\frac{2\pi}{3}\right)\right],$$

$$R_y=R\left(\theta=\frac{\pi}{2},\phi=\frac{\pi}{2}\right)=R_0\left[1+\sqrt{\frac{5}{4\pi}}\beta_2\cos\left(\gamma_2+\frac{2\pi}{3}\right)\right], \tag{1.6}$$

$$R_z=R(\theta=0,\phi)=R_0\left[1+\sqrt{\frac{5}{4\pi}}\beta_2\cos\gamma_2\right].$$

The names of the axes can be defined by saying that, for instance, $R_y\leqslant R_x\leqslant R_z$. According to Eq. (1.6), this definition corresponds to restriction of the range of values of γ_2 to $0\leqslant\gamma_2\leqslant\pi/3$. All the other values of γ_2 can be obtained by transformations $\mathcal{R}_2$ and $\mathcal{R}_3$ (Fig. 1). When, however, one of the axes, say x, is named *a priori*,[1] one has freedom only in defining axes y and z. The definition $R_y\leqslant R_z$ means the restriction of the range of γ_2's to $-\pi/3\leqslant\gamma_2\leqslant 2\pi/3$. Transformation $\mathcal{R}_3\mathcal{R}_2=\mathcal{S}_x\mathcal{P}_{xzy}$ exchanges the axes y and z and allows continuation of γ_2 onto another semiplane (Fig. 1). When the names of all three axes are fixed *a priori*, one does not have the $\mathcal{R}_2$ and $\mathcal{R}_3$ symmetry and the entire range

[1]One or all of the three axes can be determined not by the shape itself, but from outside, as is, for instance, in the cranking model in which the rotation axes are fixed in advance. Then, not only the shape of nuclear surface, but also its orientation with respect to the axes is relevant.

$-\pi \leqslant \gamma_2 \leqslant \pi$ should be considered. Since β_2 and γ_2 are insensitive to the D_{2h} transformations, the arrows of axes cannot be defined via the quadrupole deformation, which means that the quadrupole shapes are invariant under D_{2h}.

Within ''the Lund convention'' [25], which is often used, γ_2 is defined with the opposite sign (axes x and y exchanged) and all of the discussion should be changed accordingly.

II. THE MOST GENERAL HEXADECAPOLE DEFORMATION

Let us consider the nuclear surface of Eq. (1.2) with the hexadecapole deformation ($\lambda=4$) superimposed on top of the quadrupole shape. Then, not only different shapes of the quadrupole and hexadecapole components, but also their different relative orientations produce different, in general, total shapes of the surface. Only the D_{2h} symmetry of the quadrupole shape causes that we have equivalent total shapes for four different orientations of the hexadecapole component. This ambiguity can be eliminated through a definition of the arrows of axes or a restriction of the ranges of hexadecapole deformation parameters $a_{4\mu}$, $b_{4\mu}$.

Here, the most general hexadecapole deformation (i.e., arbitrary values of parameters $a_{4\mu}$, $b_{4\mu}$ which determine both the shape and the orientation of the hexadecapole component of the nuclear surface) is to be parametrized in terms of variables obeying simple transformation rules under O_h and, in particular, allow us to define the arrows of axes in a simple way. To do this one should first establish the transformation rules under O_h for $a_{4\mu}$, $b_{4\mu}$. To make the problem more transparent we rearrange, after [26], the hexadecapole terms on the right-hand side of Eq. (1.2) according to irreducible representations of O_h in the following way:

$$a_{40}Y_{40}(\theta,\phi)+\sum_{\mu>0}[a_{4\mu}Y_{4\mu}^{(+)}(\theta,\phi)+b_{4\mu}Y_{4\mu}^{(-)}(\theta,\phi)]=a_4A_4(\theta,\phi)+\sum_{\mu=0,2}e_{4\mu}E_{4\mu}(\theta,\phi)+\sum_{v=x,y,z}[f_{4v}F_{4v}(\theta,\phi)+g_{4v}G_{4v}(\theta,\phi)], \tag{2.1}$$

where

$$\begin{aligned}
A_4&=\sqrt{\frac{7}{12}}Y_{40}+\sqrt{\frac{5}{12}}Y_{44}^{(+)}=\frac{1}{4}\sqrt{\frac{21}{\pi}}\frac{1}{r^4}(x^4+y^4+z^4-3x^2y^2-3x^2z^2-3y^2z^2),\\
E_{40}&=\sqrt{\frac{5}{12}}Y_{40}-\sqrt{\frac{7}{12}}Y_{44}^{(+)}=\frac{1}{8}\sqrt{\frac{15}{\pi}}\frac{1}{r^4}[2z^4-x^4-y^4+6(2x^2y^2-x^2z^2-y^2z^2)],\\
E_{42}&=-Y_{42}^{(+)}=\frac{\sqrt{3}}{8}\sqrt{\frac{15}{\pi}}\frac{1}{r^4}[x^4-y^4-6z^2(x^2-y^2)],\\
F_{4x}&=\sqrt{\frac{7}{8}}Y_{41}^{(-)}+\frac{1}{\sqrt{8}}Y_{43}^{(-)}=\frac{3}{4}\sqrt{\frac{35}{\pi}}\frac{1}{r^4}yz(y^2-z^2),\\
F_{4y}&=-\sqrt{\frac{7}{8}}Y_{41}^{(+)}+\frac{1}{\sqrt{8}}Y_{43}^{(+)}=\frac{3}{4}\sqrt{\frac{35}{\pi}}\frac{1}{r^4}zx(z^2-x^2),\\
F_{4z}&=Y_{44}^{(-)}=\frac{3}{4}\sqrt{\frac{35}{\pi}}\frac{1}{r^4}xy(x^2-y^2),\\
G_{4x}&=\frac{1}{\sqrt{8}}Y_{41}^{(-)}-\sqrt{\frac{7}{8}}Y_{43}^{(-)}=\frac{3}{4}\sqrt{\frac{5}{\pi}}\frac{1}{r^4}yz(6x^2-y^2-z^2),\\
G_{4y}&=\frac{1}{\sqrt{8}}Y_{41}^{(+)}+\sqrt{\frac{7}{8}}Y_{43}^{(+)}=\frac{3}{4}\sqrt{\frac{5}{\pi}}\frac{1}{r^4}zx(6y^2-z^2-x^2),\\
G_{4z}&=Y_{42}^{(-)}=\frac{3}{4}\sqrt{\frac{5}{\pi}}\frac{1}{r^4}xy(6z^2-x^2-y^2)
\end{aligned} \tag{2.2}$$

are, apparently, bases for the one-dimensional, the two-dimensional, and the two three-dimensional irreducible representations of O_h, respectively (cf. [27]). The corresponding coefficients in front of functions $A_4,\ E_{4\mu}\quad(\mu=0,2),\ F_{4v},\ G_{4v}\quad(v=x,y,z)$ in Eq. (2.1),

$$
\begin{aligned}
a_4 &= \sqrt{\frac{7}{12}}a_{40}+\sqrt{\frac{5}{12}}a_{44},\\
e_{40} &= \sqrt{\frac{5}{12}}a_{40}-\sqrt{\frac{7}{12}}a_{44},\\
e_{42} &= -a_{42},\\
f_{4x} &= \sqrt{\frac{7}{8}}b_{41}+\frac{1}{\sqrt{8}}b_{43},\\
f_{4y} &= -\sqrt{\frac{7}{8}}a_{41}+\frac{1}{\sqrt{8}}a_{43},\\
f_{4z} &= b_{44},\\
g_{4x} &= \frac{1}{\sqrt{8}}b_{41}-\sqrt{\frac{7}{8}}b_{43},\\
g_{4y} &= \frac{1}{\sqrt{8}}a_{41}+\sqrt{\frac{7}{8}}a_{43},\\
g_{4z} &= b_{42}
\end{aligned}
\tag{2.3}
$$

should then constitute bases for the respective O_h irreducible representations in the space of hexadecapole deformation parameters. Hence, a_4 is O_h invariant, e_{40} and e_{42} (denoted as b_4 and c_4, respectively, in [7]) are transformed in the same way as $a_{20}\equiv e_{20}$ and $a_{22}\equiv e_{22}$ are, and $f_{4v},\ g_{4v}\quad(v=x,y,z)$ are transformed like the coordinates $x,\ y,\ z$ themselves with a possible additional change of sign, namely under $\mathcal{R}_1$ and $\mathcal{R}_3$: f_{4v}'s and g_{4v}'s are both transformed like the coordinates (cf. Table I), i.e.,

$$
\begin{aligned}
x^{(1)}&=x, \quad x^{(3)}=y,\\
y^{(1)}&=-y, \quad y^{(3)}=z,\\
z^{(1)}&=-z, \quad z^{(3)}=x,
\end{aligned}
\tag{2.4}
$$

under $\mathcal{R}_2$: f_{4v}'s are transformed like the coordinates, i.e.,

$$
\begin{aligned}
x^{(2)}&=y,\\
y^{(2)}&=-x,\\
z^{(2)}&=z,
\end{aligned}
\tag{2.5}
$$

whereas g_{4v}'s are transformed with the additional change of sign, under $\mathcal{I}$: f_{4v}'s and g_{4v}'s are invariants, i.e., unlike the coordinates, do not change their signs.

A parametrization for $a_4,\ e_{40},\ e_{42}$ has been proposed in [7]. This is generalized here for an arbitrary hexadecapole deformation by adding the six new deformation parameters—$\varepsilon_4,\ \vartheta_4,\ \varphi_4,\ \xi_4,\ \eta_4$, and ζ_4—to the three old ones—$\beta_4,\ \delta_4$, and γ_4 in the following way:

$$
\begin{aligned}
a_4&=\beta_4\cos\varepsilon_4\cos\delta_4,\\
e_{40}&=\beta_4\cos\varepsilon_4\sin\delta_4\cos\gamma_4,\\
e_{42}&=\beta_4\cos\varepsilon_4\sin\delta_4\sin\gamma_4,\\
f_{4x}&=\beta_4\sin\varepsilon_4\sin\vartheta_4\cos\varphi_4\cos\xi_4,\\
f_{4y}&=\beta_4\sin\varepsilon_4\sin\vartheta_4\sin\varphi_4\cos\eta_4,\\
f_{4z}&=\beta_4\sin\varepsilon_4\cos\vartheta_4\cos\zeta_4,\\
g_{4x}&=\beta_4\sin\varepsilon_4\sin\vartheta_4\cos\varphi_4\sin\xi_4,\\
g_{4y}&=\beta_4\sin\varepsilon_4\sin\vartheta_4\sin\varphi_4\sin\eta_4,\\
g_{4z}&=\beta_4\sin\varepsilon_4\cos\vartheta_4\sin\zeta_4,
\end{aligned}
\tag{2.6}
$$

where all the parameters are within the following ranges:

$$
\begin{aligned}
&\beta_4\geqslant 0,\quad 0\leqslant\varepsilon_4\leqslant\frac{\pi}{2},\quad 0\leqslant\delta_4\leqslant\pi,\\
&-\pi\leqslant\gamma_4\leqslant\pi,\quad 0\leqslant\vartheta_4\leqslant\frac{\pi}{2},\quad 0\leqslant\varphi_4\leqslant\frac{\pi}{2},\\
&-\pi\leqslant\xi_4\leqslant\pi,\quad -\pi\leqslant\eta_4\leqslant\pi,\quad -\pi\leqslant\zeta_4\leqslant\pi.
\end{aligned}
\tag{2.7}
$$

The parameter β_4 is, as usual, a measure of the total hexadecapole deformation; ε_4 is a measure of deviation from shapes with the three, fixed in advance (for instance, by the quadrupole deformation), mutually perpendicular symmetry planes $xy,\ yz$, and zx (the D_{2h} symmetry); and δ_4 is a measure, discussed further below, of concavity or convexity of the shape in different quadrants. All three parameters are O_h invariants. The transformation rules for the six remaining parameters are the following:

$$
\begin{aligned}
&\gamma_4^{(1)}=\gamma_4,\quad \vartheta_4^{(1)}=\vartheta_4,\quad \varphi_4^{(1)}=\varphi_4,\\
&\xi_4^{(1)}=\xi_4,\quad \eta_4^{(1)}=\eta_4\pm\pi,\quad \zeta_4^{(1)}=\zeta_4\pm\pi,\\
&\gamma_4^{(2)}=-\gamma_4,\quad \vartheta_4^{(2)}=\vartheta_4,\quad \varphi_4^{(2)}=\frac{\pi}{2}-\varphi_4,\\
&\xi_4^{(2)}=-\eta_4,\quad \eta_4^{(2)}=\pm\pi-\xi_4,\quad \zeta_4^{(2)}=-\zeta_4,\\
&\gamma_4^{(3)}=\gamma_4-\frac{2\pi}{3},\quad \vartheta_4^{(3)}=\arccos(\sin\vartheta_4\cos\varphi_4),\\
&\varphi_4^{(3)}=\arctan\left(\frac{\cot\vartheta_4}{\sin\varphi_4}\right),\\
&\xi_4^{(3)}=\eta_4,\quad \eta_4^{(3)}=\zeta_4,\quad \zeta_4^{(3)}=\xi_4.
\end{aligned}
\tag{2.8}
$$

Because of the positive parity of the hexadecapole tensor all nine parameters are, of course, $\mathcal{I}$ invariants.

The angle γ_4 has the same transformation properties as γ_2 [cf. Eqs. (1.5)] and plays a similar role. Namely, for given β_4, ε_4, and δ_4, it determines, in an analogy to Eq. (1.6), the contribution to the lengths of semiaxes coming from the hexadecapole deformation:

$$R_x = R_0\left\{1+\sqrt{\frac{9}{4\pi}}\beta_4\cos\varepsilon_4\left[\sqrt{\frac{7}{12}}\cos\delta_4 + \sqrt{\frac{5}{12}}\sin\,\delta_4\cos\left(\gamma_4-\frac{2\pi}{3}\right)\right]\right\},$$

$$R_y = R_0\left\{1+\sqrt{\frac{9}{4\pi}}\beta_4\cos\varepsilon_4\left[\sqrt{\frac{7}{12}}\cos\delta_4 + \sqrt{\frac{5}{12}}\sin\delta_4\cos\left(\gamma_4+\frac{2\pi}{3}\right)\right]\right\}, \tag{2.9}$$

$$R_z = R_0\left\{1+\sqrt{\frac{9}{4\pi}}\beta_4\cos\varepsilon_4\left[\sqrt{\frac{7}{12}}\cos\delta_4 + \sqrt{\frac{5}{12}}\sin\delta_4\cos\gamma_4\right]\right\}.$$

Relations between the lengths of semiaxes are shown again in Fig. 1. The discussion at the end of the previous section on possible definitions of names of the axes and the variability ranges of γ_4 can be repeated without changes.

The parameter δ_4 describes the shape of a D_{2h} symmetric ($\varepsilon_4=0$) surface for given β_4 and γ_4. The types of shapes can be classified by means of sections of the surface in the symmetry planes. For $\varepsilon_4=0$ the section in the xy plane is described by the following equation:

$$R\left(\theta=\frac{\pi}{2},\phi\right) = R_0\left\{1+\sqrt{\frac{9}{4\pi}}\beta_4\left[\frac{3}{8}\left(\sqrt{\frac{7}{12}}\cos\delta_4 + \sqrt{\frac{5}{12}}\sin\delta_4\sin\gamma_4\right) + \frac{\sqrt{5}}{4}\sin\delta_4\sin\gamma_4\cos 2\phi + \frac{\sqrt{35}}{8}\left(\sqrt{\frac{5}{12}}\cos\delta_4 - \sqrt{\frac{7}{12}}\sin\delta_4\cos\gamma_4\right)\cos 4\phi\right]\right\}. \tag{2.10}$$

To give the figures of Eq. (2.10) names we notice that, for small deformations β_4, the equation represents a superposition of an ellipse[2] and a tetratrochoid.[3] The radius $R(\theta=\pi/2,\phi)$ always has an extremum at the intersection points of the figure (2.10) with axes x and y ($\phi=0,\pm\pi/2,\pm\pi$). When there are no other extrema of the radius, the figure is to be named the section of elliptic type. When, however, the radius has four other maxima (minima), the figure is to be named the section of epitetratrochoidal (hypotetratrochoidal) type. All three types of sections are shown in Fig. 2. For a given γ_4, section (2.10) is hypotetratrochoidal when

$$0 \leqslant \delta_4 < \delta_h(\gamma_4), \tag{2.11}$$

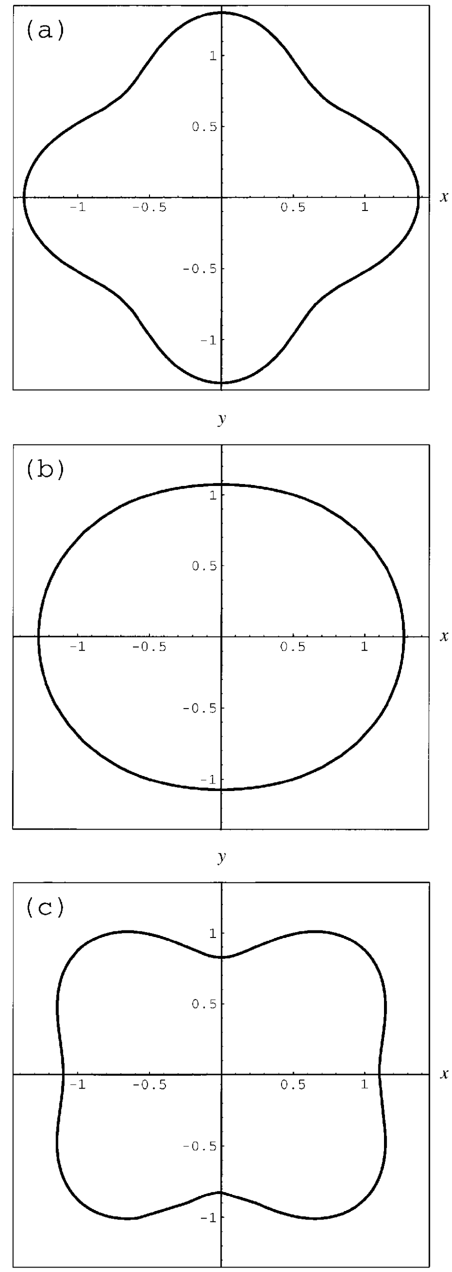

FIG. 2. Sections of the hexadecapole surfaces in plane xy for $\beta_4=0.6$, $\varepsilon_4=0$, $\gamma_4=\pi/6$. The units are in R_0. (a) Section of the hypotetratrochoidal type, $\delta_4=\pi/12<\delta_h(\pi/6)$, (b) section of the elliptic type, $\delta_4=3\pi/12$, $\delta_h(\pi/6)<3\pi/12<\delta_e(\pi/6)$, (c) section of the epitetratrochoidal type, $\delta_4=5\pi/12>\delta_e(\pi/6)$.

[2]The ellipse is also a hypotrochoid generated by a fixed interior point of a circle rolling inside a fixed circle twice as large.

[3]The prefix *tetra* is added here to indicate that for the trochoids in question, the ratio of radii of the fixed and rolling circles is 4:1.

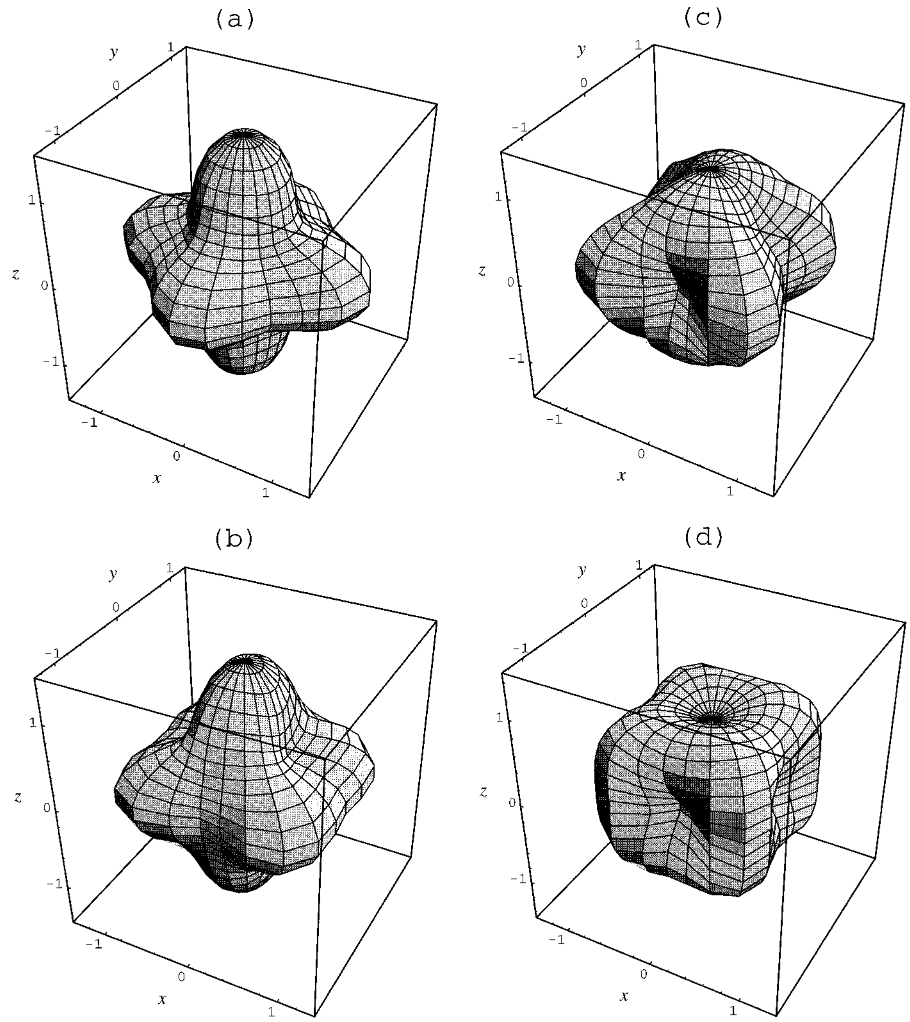

FIG. 3. Hexadecapole surfaces for $\beta_4 = 0.6$, $\varepsilon_4 = 0$, $\gamma_4 = \pi/6$ and (a) $\delta_4 = \pi/12$, (b) $\delta_4 = 5\pi/12$, (c) $\delta_4 = 8\pi/12$, (d) $\delta_4 = 11\pi/12$. Description of the shapes is in the text. The units are the same as in Fig. 2.

elliptic (still hypotrochoidal) when

$$\delta_h(\gamma_4) \leqslant \delta_4 \leqslant \delta_e(\gamma_4), \tag{2.12}$$

epitetratrochoidal when

$$\delta_e(\gamma_4) < \delta_4 \leqslant \pi, \tag{2.13}$$

where

$$\delta_h(\gamma_4) = \arccos \frac{7\cos\gamma_4 + \sqrt{3}|\sin\gamma_4|}{\sqrt{35 + (7\cos\gamma_4 + \sqrt{3}|\sin\gamma_4|)^2}},$$

$$\delta_e(\gamma_4) = \arccos \frac{7\cos\gamma_4 - \sqrt{3}|\sin\gamma_4|}{\sqrt{35 + (7\cos\gamma_4 - \sqrt{3}|\sin\gamma_4|)^2}}. \tag{2.14}$$

The angles δ_h and δ_e are rising functions of γ_4. Therefore, the D_{2h} symmetric hexadecapole surfaces are of the four following types for greater and greater δ_4.

(a) The surface is convex at all the six intersection points with the symmetry axes (all the sections are hypotrochoidal).

(b) The surface is convex on the longest symmetry axes and has saddle points on the two other symmetry axes (the section in the symmetry plane perpendicular to the longest axis becomes epitetratrochoidal).

(c) The surface is concave on the shortest symmetry axis and has saddle points on the two other symmetry axes (only the section in the symmetry plane perpendicular to the shortest axis remains hypotrochoidal).

(d) The surface is concave on all symmetry axes (all the sections are epitetratrochoidal).

All four types of shapes are shown in Fig. 3.

For $\varepsilon_4 \neq 0$ the ''vectors'' f_{4v} and $g_{4v}, v = x,y,z$, begin contributing to the hexadecapole surface (this becomes the only contribution when $\varepsilon_4 = \pi/2$). Since $\mathcal{R}_v f_{4v} = \mathcal{S}_v f_{4v} = f_{4v}$ and $\mathcal{R}_u f_{4v} = \mathcal{S}_u f_{4v} = -f_{4v}$ for $u \neq v$ (and the same for g_{4v}), the D_{2h} symmetry is broken and planes xy, yz, and zx quit, in general, being the symmetry planes. The contribution of f_{4v} and/or g_{4v} to Eq. (2.1) for a given v conserves the symmetry with respect to the plane perpendicular to axis v only. Hence, the parameters ϑ_4 and φ_4 determine with respect to which plane and to what extent the symmetry is broken. For given ϑ_4 and φ_4 the angles ξ_4, η_4, and ζ_4 parametrize shapes of the sections in planes yz, zx, and xy, respectively. This is seen when looking at the following equation of section in plane xy for $\varepsilon_4 = \pi/2$:

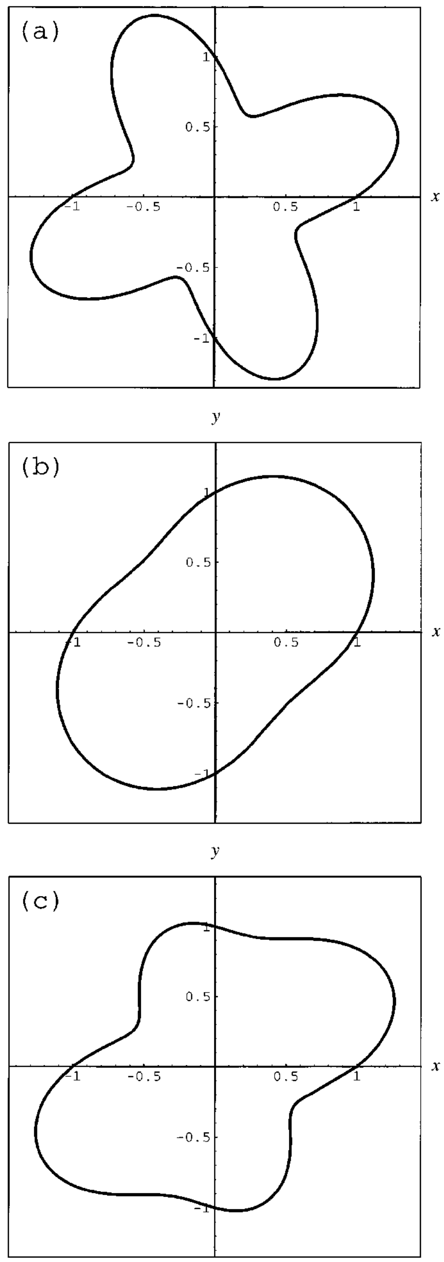

FIG. 4. Sections of the hexadecapole surfaces in plane xy for $\beta_4=0.6$, $\varepsilon_4=\pi/2$, $\vartheta_4=0$. The units are the same as in Fig. 2. (a) $\zeta_4=0$, section of the tetratrochoidal type with the symmetry axes turned by $\pi/8$ with respect to the coordinate axes x and y, (b) $\zeta_4=-\pi/2$, section of the elliptic type (large deformation β_4 causes a deviation from the elliptic shape) with the symmetry axes turned by $\pi/4$ with respect to the coordinate axes x and y; (c) $\zeta_4=-\pi/3$, section being a superposition of types (a) and (b) with no symmetry axes.

$$R\left(\theta=\frac{\pi}{2},\phi\right)=R_0\left\{1+\frac{3}{2}\sqrt{\frac{5}{2\pi}}\beta_4\cos\vartheta_4\right.$$
$$\left.\times\left[\sqrt{\frac{7}{8}}\cos\zeta_4\cos2\phi-\frac{1}{\sqrt{8}}\sin\zeta_4\right]\sin2\phi\right\}. \tag{2.15}$$

For $\zeta_4=0,\ \pm\pi/2,\ \pm\pi$ it represents figures still with two perpendicular symmetry axes, but turned with respect to the coordinate axes. For other ζ_4's the figures have no symmetry axes. This is shown in Fig. 4.

Since, according to Eq. (2.8), transformations $\mathcal{R}_x$, $\mathcal{R}_y$, and $\mathcal{R}_z$ shift the respective pairs of ξ_4, η_4, ζ_4 by $\pm\pi$, the range of one pair, say ξ_4 and η_4, can be restricted, for instance, to positive values only:

$$0\leqslant\xi_4<\pi,\quad 0\leqslant\eta_4<\pi \tag{2.16}$$

and this way the arrows of axes x and y are defined. Then the arrow of axis z still remains undefined due to the inversion invariance of the quadrupole-hexadecapole shapes.

As there is one to one correspondence between $a_{4\mu}$ and $b_{4\mu}$, the components of the hexadecapole tensor, and the nine deformation parameters defined by Eqs. (2.6), the latter, when taking the values within the ranges (2.7), describe all possible hexadecapole surfaces and thus all possible quadrupole-hexadecapole shapes for given β_2 and γ_2. Inequalities (2.16) eliminate three out of the four equivalent shapes.

Using parametrization (2.6) it is difficult to distinguish the hexadecapole surfaces not having symmetry planes from those still having them, but turned with respect to the xy, yz, and zx planes. This question is connected with the complicated problem of the definition of an intrinsic frame linked to the hexadecapole surface (cf. Refs. [26,28] for the same problem in the case of an octupole surface).

III. HEXADECAPOLE SHAPES PARAMETRIZED IN TERMS OF THE QUADRUPOLE DEFORMATION PARAMETERS

The most general quadrupole-hexadecapole surface of Eq. (1.2) is, as has been seen in the previous sections, parametrized in terms of 11 parameters. The D_{2h}-symmetric $(\varepsilon_4=0)$ shapes have been hitherto considered. Still, these need five deformation parameters. Most of the nuclei exhibit a quadrupole deformation and higher multipoles only give corrections to the quadrupole shape. This is why we try and we may use as few higher-multipole deformation parameters as possible making some of them dependent on others.[4]

Some of the hexadecapole shapes can, in fact, be parametrized in terms of the quadrupole deformation parameters. Here, all such possible shapes are to be discussed. This is a long-standing problem [10]. It consists of constructing hexadecapole tensors of a quadrupole tensor. All hexadecapole tensors being isotropic functions of the quadrupole tensor $\alpha_{2\mu}$ take the following form (cf. [29,30,24]):

$$\alpha_{4\mu}(\alpha_2)=\frac{\sqrt{70}}{6}\{\chi_2(\beta_2^2,\beta_2^3\cos3\gamma_2)[\alpha_2\times\alpha_2]_{4\mu}$$
$$+\chi_3(\beta_2^2,\beta_2^3\cos3\gamma_2)[\alpha_2\times\sigma_2]_{4\mu}$$
$$+\chi_4(\beta_2^2,\beta_2^3\cos3\gamma_2)[\sigma_2\times\sigma_2]_{4\mu}\}, \tag{3.1}$$

[4]For instance, we assume that higher-multipole deformations follow the quadrupole one just to minimize the energy.

where $\sigma_{2\mu}=-\sqrt{7/2}[\alpha_2\times\alpha_2]_{2\mu}$ and χ_2, χ_3, χ_4 are the arbitrary scalar functions of the two independent quadrupole invariants. Not assuming the analyticity of functions χ_i $(i=2,3,4)$ in the invariants β_2^2 and $\beta_2^3\cos 3\gamma_2$, Eq. (3.1) is rewritten as follows:

$$\alpha_{4\mu}=\frac{\sqrt{70}}{6}\Bigg\{h_2(\beta_2,\cos 3\gamma_2)\frac{1}{\beta_2^2}[\alpha_2\times\alpha_2]_{4\mu} + h_3(\beta_2,\cos 3\gamma_2)\frac{1}{\beta_2^3}[\alpha_2\times\sigma_2]_{4\mu} + h_4(\beta_2,\cos 3\gamma_2)\frac{1}{\beta_2^4}[\sigma_2\times\sigma_2]_{4\mu}\Bigg\}, \tag{3.2}$$

where h_2, h_3, h_4 are the arbitrary functions of β_2 and $\cos 3\gamma_2$. The components of the tensor of Eq. (3.2) defined in Eq. (2.3) read

$$a_4=\sqrt{\frac{7}{12}}(h_2+h_3\cos 3\gamma_2+h_4),$$

$$e_{40}=\sqrt{\frac{5}{12}}(h_2\cos 2\gamma_2+h_3\cos\gamma_2+h_4\cos 4\gamma_2),$$

$$e_{42}=\sqrt{\frac{5}{12}}(-h_2\sin 2\gamma_2+h_3\sin\gamma_2+h_4\sin 4\gamma_2),$$

$$f_{4v}=g_{4v}=0 \quad \text{for} \quad v=x,y,z. \tag{3.3}$$

Early parametrizations [4,6] of the nonaxial hexadecapole deformation as a function of γ_2 are not in the form of Eqs. (3.3) and thus do not have correct tensor properties. Apparently, only the D_{2h}-symmetrical hexadecapole shapes (with $\varepsilon_4=0$) can be parametrized by means of a quadrupole tensor and made dependent on the quadrupole deformation. This dependence has the following form

$$\beta_4=\Bigg[h_2^2+h_3^2\left(\frac{5}{12}+\frac{7}{12}\cos^2 3\gamma_2\right)+h_4^2+2(h_2+h_4)h_3\cos 3\gamma_2+2h_2h_4\times\left(\frac{7}{12}+\frac{5}{12}\cos 6\gamma_2\right)\Bigg]^{1/2},$$

$$\cos\delta_4=\sqrt{\frac{7}{12}}\frac{h_2+h_3\cos 3\gamma_2+h_4}{\beta_4},$$

$$\sin\gamma_4=\frac{-h_2\sin 2\gamma_2+h_3\sin\gamma_2+h_4\sin 4\gamma_2}{[h_2^2+h_3^2+h_4^2+2(h_2+h_4)h_3\cos 3\gamma_2+2h_2h_4\cos 6\gamma_2]^{1/2}},$$

$$\cos\gamma_4=\frac{h_2\cos 2\gamma_2+h_3\cos\gamma_2+h_4\cos 4\gamma_2}{[h_2^2+h_3^2+h_4^2+2(h_2+h_4)h_3\cos 3\gamma_2+2h_2h_4\cos 6\gamma_2]^{1/2}},$$

$$\varepsilon_4=0. \tag{3.4}$$

In practical calculations very simple forms of functions h_2, h_3, h_4 have been used. The three one-parameter parametrizations of the nonaxial hexadecapole deformation have been proposed in [8] by replacing one of the functions h_i $(i=2,3,4)$ with a constant and putting two others equal to zero. Usually, h_3 and h_4 have been put equal to zero [11–13]. It is not obvious that this is really the optimal parametrization. For instance, Magierski *et al.* [31] have shown that in a rotating system of particles which occupy one j shell and interact via the quadrupole-quadrupole and hexadecapole-hexadecapole forces, the hexadecapole deformation is, for a weak hexadecapole interaction, related to the quadrupole one in such a way that just $h_3\neq 0$.

IV. INCLUSION OF THE OCTUPOLE DEFORMATION

When the terms with $\lambda=3$ are also included in Eq. (1.2), the results of [22] are applicable. Here these results are rewritten briefly in the spirit of [26] and the present paper. Also, the notation of the present paper is used.

The octupole tensor is resolved according to the one-dimensional and the two three-dimensional irreducible representations of O_h in the following way:

$$b_3=b_{32},$$

$$f_{3x}=\sqrt{\frac{3}{8}}a_{31}-\sqrt{\frac{5}{8}}a_{33},$$

$$f_{3y}=\sqrt{\frac{3}{8}}b_{31}+\sqrt{\frac{5}{8}}b_{33},$$

$$f_{3z}=a_{30}, \tag{4.1}$$

$$g_{3x}=\sqrt{\frac{5}{8}}a_{31}+\sqrt{\frac{3}{8}}a_{33},$$

$$g_{3y}=-\sqrt{\frac{5}{8}}b_{31}+\sqrt{\frac{3}{8}}b_{33},$$

$$g_{3z}=a_{32}.$$

The component b_3 is invariant under rotations $\mathcal{R}_1$ and $\mathcal{R}_3$, and changes its sign under rotation $\mathcal{R}_2$ and inversion $\mathcal{I}$. The ''vectors'' f_{3v} and g_{3v} $(v=x,y,z)$ are transformed under $\mathcal{R}_i$ like f_{4v} and g_{4v}, respectively, and change their signs under $\mathcal{I}$. All of them can be parametrized as follows:

$$b_3=\beta_3\sin\varepsilon_3,$$

$$f_{3x}=\beta_3\cos\varepsilon_3\sin\vartheta_3\cos\varphi_3\cos\xi_3,$$

$$f_{3y}=\beta_3\cos\varepsilon_3\sin\vartheta_3\sin\varphi_3\cos\eta_3,$$

$$f_{3z}=\beta_3\cos\varepsilon_3\cos\vartheta_3\cos\zeta_3, \tag{4.2}$$

$$g_{3x}=\beta_3\cos\varepsilon_3\sin\vartheta_3\cos\varphi_3\sin\xi_3,$$

$$g_{3y}=\beta_3\cos\varepsilon_3\sin\vartheta_3\sin\varphi_3\sin\eta_3,$$

$$g_{3z}=\beta_3\cos\varepsilon_3\cos\vartheta_3\sin\zeta_3.$$

This is, in fact, the same parametrization as that of [22] with the following correspondence of the parameters:

$$
\begin{aligned}
\beta_3 &\Leftrightarrow \beta, \\
\varepsilon_3 &\Leftrightarrow \delta_0, \\
\vartheta_3 &\Leftrightarrow \delta_1, \\
\varphi_3 &\Leftrightarrow \delta_2, \\
\xi_3 &\Leftrightarrow \gamma_2 + c, \\
\eta_3 &\Leftrightarrow \gamma_3 - c, \\
\zeta_3 &\Leftrightarrow \gamma_1,
\end{aligned} \tag{4.3}
$$

where $c = \arcsin\sqrt{5/8}$.

The transformation rules for β_3, ϑ_3, φ_3, ξ_3, η_3, ζ_3 under $\mathcal{R}_k$ are identical with those for β_4, ϑ_4, φ_4, ξ_4, η_4, ζ_4, respectively. The differences arise in the case of inversion, namely,

$$
\begin{aligned}
\xi_3^{(0)} &= \xi_3 \pm \pi, \\
\eta_3^{(0)} &= \eta_3 \pm \pi, \\
\zeta_3^{(0)} &= \zeta_3 \pm \pi,
\end{aligned} \tag{4.4}
$$

while the hexadecapole counterparts are all invariant under $\mathcal{I}$. The octupole counterparts of Eqs. (2.8) now being accompanied by Eqs. (4.4) allow for defining the arrows of all three axes by restricting the ranges of ξ_3, η_3, ζ_3 to positive values:[5]

$$
\begin{aligned}
0 &\leqslant \xi_3 < \pi, \\
0 &\leqslant \eta_3 < \pi, \\
0 &\leqslant \zeta_3 < \pi.
\end{aligned} \tag{4.5}
$$

When the arrows of axes x and y are already defined by (2.16) only the latter of inequalities (4.5) have to be fulfilled in order to define the arrow of axis z. The parameter ε_3 is hardly a counterpart of $\pi/2 - \varepsilon_4$ since a_4 is the O_h scalar while b_3 is not. The negative values for ε_3 are allowed and its range is $-\pi/2 \leqslant \varepsilon_3 \leqslant \pi/2$. It is, like b_3, invariant under $\mathcal{R}_1$ and $\mathcal{R}_3$ and changes its sign under $\mathcal{R}_2$ and $\mathcal{I}$.

[5] Inequalities (4.5) eliminate seven out of the eight possible quadrupole-octupole shapes equivalent due to D_{2h} symmetry.

V. CONCLUSION

A parametrization of the most general quadrupole-octupole-hexadecapole shapes are described in the frame defined by the principal axes of the quadrupole deformation tensor or the rotation axes. It is suggested from the present study that when we consider nonaxial shapes of higher multipolarities, we should deal with the deformations of types a, b, e, f, g corresponding to the irreducible representations of the O_h group (called A_1, A_2, E, F_1, F_2, respectively, according to Hamermesh's classification [27]) rather than particular spherical components of the deformation tensors. This is already well understood when dealing with the hexadecapole deformation although early calculations have usually been done in the space of β_2, γ_2, and a_{40} (cf., e.g., [32]). However, this is still not realized in the case of octupole deformation. This is perhaps because more than one deformation parameter should then be used to describe the shape.

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