# QCD sum rules for the isospin-breaking axial correlator with correct chiral behavior

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We revisit the QCD sum-rule treatment of the isospin-breaking correlator $\langle 0|T[A^3_\mu(x)A^8_\nu(0)]|0\rangle$, in light of the recent claim that a previous treatment produced results incompatible with known chiral constraints. The source of the error in the previous analysis is identified, and a corrected version of the sum-rule treatment obtained. It is then shown that, using input from chiral perturbation theory, one may use the resulting sum rule to extract information on the leading chiral behavior of isospin-breaking parameters associated with the coupling of excited pseudoscalar resonances to the axial currents. A rather accurate extraction is possible for the case of the $\eta'$. Demanding stability of the sum-rule analysis also allows us to improve the upper bound on the fourth-order low-energy constant, $L_7$. [S0556-2813(97)03409-2]

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## I. INTRODUCTION

One of the most attractive features of chiral perturbation theory (ChPT) [1] is that it provides a framework for constructing effective hadronic Lagrangians in the most general possible way that implements both the symmetries of QCD and the symmetry breaking pattern of the approximate chiral symmetries of QCD. As such, it fully incorporates the consequences of QCD in the low-energy regime. The price to be paid for using only symmetry arguments is that every term in the effective Lagrangian, $\mathcal{L}_{\rm eff}$, allowed by these arguments will appear, multiplied by an undetermined constant (referred to as a low-energy constant, or LEC). These LEC's could, in principle, be computed from QCD, but must be treated as parameters to be determined phenomenologically if one does not go beyond ChPT.

Although such effective Lagrangians are necessarily nonrenormalizable, Weinberg's counting argument [2] shows that only a finite number of terms in $\mathcal{L}_{\rm eff}$ contribute if one expands to fixed ''chiral'' order, that is in powers of the external momenta (generically denoted as $p$) and current quark masses [where $m_q$ counts as order $O(p^2)$]. As a result, in the chiral expansion of any low-energy observable, the general form of the dependence on external momenta and light quark masses, to a given order, can be computed straightforwardly from the form of the relevant terms in $\mathcal{L}_{\rm eff}$.

Since this formal dependence is a rigorous consequence of the symmetries and approximate symmetries of QCD, it follows that ChPT can be used to place constraints on treatments of the same observable using other methods. Indeed, if one makes a chiral expansion of the results obtained by any other method and finds that terms present in ChPT to a given order are missing, then one knows unambiguously that either the method itself, or some truncation employed in it, is incompatible with QCD. This is true regardless of the rapidity of convergence of the chiral series in question: all terms required by the symmetries of QCD must be present if the method is to correctly incorporate the consequences of QCD.

An example of the use of such constraints is provided by the analysis of the nucleon mass using QCD sum rules. Standard treatments were shown to produce an expression for $m_N$ in terms of condensates that implies the presence of certain chiral logarithms in $m_N$, although such contributions are known from ChPT to be absent [3,4]. The source of this problem was found to be a failure to treat properly the contribution of the $\pi N$ continuum to the sum rule in the original analyses [4]; including the leading contributions from such states restores the correct chiral behavior of $m_N$.

A more severe problem of the same type has been pointed out in the case of the isospin-breaking axial correlator

$$\Pi^{38}_{\mu\nu}(q)=i\int d^4x\ \ e^{iq\cdot x}\langle 0|T(A^3_\mu(x)A^8_\nu(0))|0\rangle$$
$$\equiv\Pi_1(q^2)q_\mu q_\nu-\Pi_2(q^2)q^2g_{\mu\nu}, \qquad (1)$$

where $A^3_\mu$, $A^8_\nu$ are the neutral members of the octet of axial currents $A^a_\mu=\overline{q}\gamma_\mu\gamma_5(\lambda^a/2)q$. This correlator was first analyzed using QCD sum rules in Ref. [5] (CHM). As shown in Ref. [6], however, if one writes $\Pi_1(q^2)$, which contains the $\pi^0$ and $\eta$ pole contributions, in the form

$$\Pi_1(q^2)=\left(\frac{g_\eta}{q^2-m^2_\eta}-\frac{g_\pi}{q^2-m^2_\pi}\right)+\cdots$$
$$=\left(\frac{q^2(g_\eta-g_\pi)+(g_\pi m^2_\eta-g_\eta m^2_\pi)}{(q^2-m^2_\eta)(q^2-m^2_\pi)}\right)+\cdots, \qquad (2)$$

then the expression for $g_\eta-g_\pi$ (given by the slope of the numerator with respect to $q^2$) obtained from the sum-rule analysis is lacking both the leading analytic and leading nonanalytic terms from its chiral expansion in terms of the light quark masses. (The demonstration of this is reviewed briefly below in Sec. II.)

 

In this paper we revisit the sum-rule analysis of the axial correlator above, and identify the source of this problem. We then obtain a corrected version of the relevant sum rule and show how it can be used to extract information on isospin-breaking couplings of the higher pseudoscalar resonances.

The paper is organized as follows. In Sec. II, we briefly review the sum-rule and ChPT analyses of the correlator. In Sec. III, we identify the problem with the previous sum-rule treatment and work out the corrected version of the relevant sum rules. In Sec. IV, we show how one can use information from ChPT as input into the sum rule. We also clarify the physical content of the corrected sum rule, extracting in the process information on the isospin-breaking couplings of the higher pseudoscalar resonances to the axial currents. We conclude in Sec. V with a brief summary.

## II. PREVIOUS CHPT AND SUM-RULE TREATMENTS

We provide here only a very brief review, which will serve also to fix notation. For more details the reader is referred to Refs. [5] and [6] for the sum-rule and ChPT treatments, respectively.

We first review the sum-rule treatment.[1] As usual, the aim is to write dispersion relations for $\Pi_1(q^2)$ and $\Pi_2(q^2)$ which relate integrals over the relevant physical spectral functions to the behaviors at large spacelike $q^2$, where the operator product expansion (OPE) becomes valid. One then Borel transforms the resulting dispersion relations in order to exponentially suppress the higher-energy portions of the spectral integral on the phenomenological side and simultaneously factorially suppress the contributions of higher dimension operators on the OPE side. The scalar correlators $\Pi_1(q^2)$ and $\Pi_2(q^2)$ in Eq. (1) have been chosen in such a way that, from the asymptotic behavior of $\Pi^{38}_{\mu\nu}(q)$ in QCD, it is known that the relevant spectral integrals converge without subtraction. Note that the definition of $\Pi_2$ employed here agrees with that used in Ref. [5], but differs from that in Ref. [6] by a factor of $-q^2$.

On the phenomenological side, the axial-vector resonances contribute to both $\Pi_1(q^2)$ and $\Pi_2(q^2)$. In the narrow-width approximation, their contributions to the complete spectral function are written

$$\frac{1}{\pi}(\mathrm{Im}\ \Pi^{38}_{\mu\nu}(q))_A=\sum_A g^{(A)}[-g_{\mu\nu}+q_\mu q_\nu/M_A^2]\delta(q^2-M_A^2). \tag{3}$$

The pseudoscalar resonances, in contrast, contribute only to $\Pi_1$. Following the convention of earlier works, we write these contributions as

$$\frac{1}{\pi}(\mathrm{Im}\ \Pi_1(q^2))_P=g_\pi\delta(q^2-m_\pi^2)-g_\eta\delta(q^2-m_\eta^2) +g_{\eta'}\delta(q^2-m_{\eta'}^2)+g_{\pi'}\delta(q^2-m_{\pi'}^2) +\cdots. \tag{4}$$

(The minus sign in front of $g_\eta$ is conventional and related to the fact that, so defined, $g_\eta=g_\pi$ at leading order in the chiral expansion.)

On the OPE side of the sum rule, the expressions for the scalar correlators have been worked out by CHM up to and including operators of dimension 6, and to order $O(m_q,\alpha_s)$. Neglecting electromagnetic effects, the results have the form (with $Q^2=-q^2$)

$$\Pi_1(q^2)=\frac{1}{4\sqrt{3}}\left[C_0\ln Q^2+\frac{C_1}{Q^2}+\frac{C_2}{Q^4}+\frac{C_3}{Q^6}\right],$$
$$\Pi_2(q^2)=\frac{1}{4\sqrt{3}}\left[C_0\ln Q^2+\frac{C_1}{Q^2}-\frac{C_2}{Q^4}+\frac{C_3}{Q^6}\right], \tag{5}$$

where $C_0$ and $C_2$ vanish at the level of the truncations noted above, and

$$C_2=2[m_u\langle\bar{u}u\rangle-m_d\langle\bar{d}d\rangle],$$
$$C_3=\frac{352}{81}\pi\alpha_s[\langle\bar{u}u\rangle^2-\langle\bar{d}d\rangle]^2. \tag{6}$$

If one were to include higher-order terms in Eq. (5), $C_0$ would receive contributions at order $O(\alpha_{\rm EM},\alpha_s,m_q^2)$ and $C_1$ at $O(m_q^2)$. An argument analogous to that of Shifman, Vainshtein, and Zakharov [8] for the corresponding isospin-conserving correlator $\Pi^{33}_{\mu\nu}(q)$ shows that the higher-dimension operators not included in these expressions are also all explicitly of order $O(m_q^2)$. The form of the dimension 6 coefficient $C_3$ in Eq. (6) has been obtained assuming vacuum saturation.

As can be seen from the Lorentz structure of Eq. (3), it is possible to remove the contributions of the axial-vector mesons by considering the combination

$$\Pi_P(q^2)\equiv\Pi_1(q^2)-\Pi_2(q^2). \tag{7}$$

CHM, motivated by this observation, write a dispersion relation for $\Pi_P(q^2)$ in the form

$$\Pi_P(q^2)=\int\frac{1}{\pi}\frac{\mathrm{Im}\ \Pi_P(s)}{s-q^2}ds. \tag{8}$$

When Borel transformed, this relation gives CHM's sum rule,

$$2C_2\left(\frac{1}{M^4}\right)=\frac{4\sqrt{3}}{M^2}[g_\pi e^{-m_\pi^2/M^2}-g_\eta e^{-m_\eta^2/M^2}]+\cdots, \tag{9}$$

where $M$ is the Borel mass parameter and the dots refer to the contributions of higher pseudoscalar resonances. CHM then neglect higher resonance contributions and use this sum rule, together with its derivative with respect to the Borel mass, $M$, to solve for $g_\eta$ and $g_\pi$. This procedure leads to their result

[1]See, for example, Refs. [7–10] for details of the general method of QCD sum rules.

$$(g_\eta - g_\pi)_{\rm CHM} = \left(\frac{C_2}{2\sqrt{3}M^2}\right) \times \left[\frac{e^{m_\eta^2/M^2}(M^2+m_\pi^2) - e^{m_\pi^2/M^2}(M^2+m_\eta^2)}{m_\eta^2 - m_\pi^2}\right] \tag{10}$$

for the slope of the numerator on the RHS of Eq. (2).

The analysis of $\Pi^{38}_{\mu\nu}(q)$ at next-to-leading (1-loop) order in ChPT is straightforward, and follows standard methods. We employ throughout the notation of Gasser and Leutwyler [1]. The result for $\Pi_2(q^2)$, recast so as to correspond to the definition employed in this paper, is [6]

$$\Pi_2^{1\,\rm loop}(q^2) = -\frac{B_0(m_d - m_u)}{\sqrt{3}q^2}\left[\frac{3}{32\pi^2}(\ln(m_K^2/\mu^2)+1) - 8L_5^r(\mu^2)\right], \tag{11}$$

where $B_0$ is the usual second-order LEC, related to the quark condensate in the chiral limit, $\mu$ is the renormalization scale, and $L_5^r(\mu^2)$ is a renormalized fourth-order LEC. Note that $\Pi_2(q^2)$ results solely from contact terms (that is, terms in $\mathcal{L}_{\rm eff}$ that are quadratic in the external axial sources). To this order, $\Pi_1(q^2)$ is saturated by the $\pi^0$ and $\eta$ pole terms. From a similar analysis, one finds, for the coefficients $g_\pi$ and $g_\eta$ appearing in Eq. (4),

$$g_\pi = f_\pi^2 \epsilon_1 \quad \text{and} \quad g_\eta = f_\eta^2 \epsilon_2, \tag{12}$$

where $F_\pi, f_\eta$ are the physical $\pi,\eta$ decay constants and $\epsilon_1$, $\epsilon_2$ are isospin-breaking parameters defined by

$$\langle 0|A_\mu^8|\pi\rangle = \mathrm{i} f_\pi \epsilon_1 q_\mu$$
$$\langle 0|A_\mu^3|\eta\rangle = -\mathrm{i} f_\eta \epsilon_2 q_\mu . \tag{13}$$

The expressions for $f_\pi$, $f_\eta$, $\epsilon_1$, and $\epsilon_2$ valid to one-loop order can be found in Ref. [1].

The problem with the sum-rule treatment is exposed when one uses the known chiral expansions of the meson masses and quark condensates to rewrite the sum-rule result, Eq. (10), as

$$(g_\eta - g_\pi)_{\rm CHM} = \theta_0 F^2\left(\frac{8}{9}\frac{B_0^2(m_s-\hat{m})(m_s+2\hat{m})}{M^4} + \cdots\right), \tag{14}$$

to order $O(m_q^2)$. Here $F$ is a second-order LEC, equal to $f_\pi$ in the chiral limit, and $\theta_0$ is the leading-order $\pi^0$-$\eta$ mixing angle,

$$\theta_0 = \frac{\sqrt{3}}{4}\left(\frac{m_d - m_u}{m_s - \hat{m}}\right), \tag{15}$$

with $\hat{m} = (m_u + m_d)/2$. Comparing this expression with the corresponding one obtained from the one-loop ChPT results,

$$(g_\eta - g_\pi)_{\rm ChPT} = \theta_0 F^2\left(\frac{(m_\pi^2 - \overline{m_K^2})}{8\pi^2F^2}\ln(\overline{m_K^2}/\mu^2) - \frac{B_0(m_s-\hat{m})}{8\pi^2F^2} + \frac{32B_0(m_s-\hat{m})}{3F^2}L_5^r(\mu^2) + \cdots\right), \tag{16}$$

one sees that the sum-rule expression is lacking both the leading analytic and leading non-analytic terms in its chiral expansion [6], and hence is incorrect. Moreover, the numerical consequences of this are significant: the sum-rule value for the slope is more than an order of magnitude smaller than that given by ChPT.

## III. CORRECTED VERSION OF THE SUM-RULE ANALYSIS

The key to understanding the origin of the problem with the CHM sum-rule analysis lies in Eq. (8). This relation follows from general properties of analyticity and unitarity under two assumptions: (a) that the singularities of $\Pi_P(q^2)$ consist solely of those associated with physical intermediate states and (b) that $\Pi_P(q^2)$ converges sufficiently fast that no subtractions are required. The latter assumption is explicitly verified by the known asymptotic behavior of $\Pi_1(q^2)$ and $\Pi_2(q^2)$ in QCD. The former, however, is more subtle, since there can also be singularities of purely kinematic origin. In the case at hand, Eq. (11) shows explicitly that $\Pi_2(q^2)$ has a kinematic pole at $q^2=0$. As a consequence, the correct version of the dispersion relation Eq. (8) must include the contribution of this kinematic pole to the underlying contour integral. Another way of saying this is that it is $q^2\Pi_2(q^2)$ which satisfies a dispersion relation without kinematic pole terms. The dispersion relation for this function, however, requires one subtraction in order to converge. The resulting subtraction constant gives rise to the kinematic pole term of $\Pi_2(q^2)$. Its value is calculable in ChPT, and turns out to correspond precisely to the contact contributions given in Eq. (11).

Bearing this in mind, it is straightforward to write down the corrected dispersion relation for $\Pi_P(q^2)$,

$$\Pi_P(q^2) = -\frac{1}{q^2}\frac{B_0(m_d - m_u)}{\sqrt{3}}\left[\frac{3}{32\pi^2}(\ln(m_K^2/\mu^2)+1) - 8L_5^r(\mu^2)\right] + \int \frac{1}{\pi}\frac{\mathrm{Im}\,\Pi_P(s)}{s-q^2}ds, \tag{17}$$

where Im $\Pi_P(s)$ includes only the spectral strength associated with pseudoscalar states. The corresponding Borel-transformed sum rule is then

$$\frac{C_2}{2\sqrt{3}M^2}=\Bigg[\left(-\frac{B_0(m_d-m_u)}{\sqrt{3}}\right)\left(\frac{3}{32\pi^2}(\ln(m_K^2/\mu^2)+1)-8L_5^r\right)+g_\pi e^{-m_\pi^2/M^2}-g_\eta e^{-m_\eta^2/M^2}+g_{\eta'}e^{-m_{\eta'}^2/M^2}+g_{\pi'}e^{-m_{\pi'}^2/M^2}+\dots\Bigg]. \quad (18)$$

As one might expect, the inclusion of the kinematic-pole contribution cures the problem of the incorrect chiral behavior of $g_\eta - g_\pi$. To see this, consider the $O(M^0)$ terms of Eq. (18). Bearing in mind that $g_{\eta'}$, $g_{\pi'}$, . . . are all of order $O(m_q^2)$ [8], one has

$$0=\left(-\frac{B_0(m_d-m_u)}{\sqrt{3}}\right)\left(\frac{3}{32\pi^2}(\ln(m_K^2/\mu^2)+1)-8L_5^r(\mu^2)\right)+g_\pi-g_\eta+O(m_q^2), \quad (19)$$

where the first term on the RHS results from the kinematic pole in Eq. (17). Without this term, one gets $g_\eta - g_\pi = O(m_q^2)$, as found by CHM. In contrast, using the corrected sum rule, one finds that Eq. (19) is simply an alternate form of Eq. (16), as required.

To clarify the physical content of the remaining pieces of the sum rule, Eq. (18), it is useful to note the chiral order of various quantities appearing therein. In particular, the chiral expansions of $g_\pi$, $g_\eta$, $m_{\eta'}^2$, and $m_{\pi'}^2$ start at order $O(p^0)$, $m_\pi^2$, $m_\eta^2$, and $C_2$ at $O(p^2)$, and (as already noted above) $g_{\eta'}$, $g_{\pi'}$ at $O(m_q^2)=O(p^4)$. After the cancellation embodied in Eq. (19), the only $O(p^2)$ terms remaining in Eq. (18) are those in $C_2$ and $-g_\pi m_\pi^2 + g_\eta m_\eta^2$. Using the leading-order expressions $\langle \bar{u}u\rangle = \langle \bar{d}d\rangle = -B_0F^2$ and $g_\pi = g_\eta = \theta_0 F^2$, it is straightforward to show that the $O(p^2)$ terms on both sides of the sum rule also match properly. To this order, the matching is just an isospin-breaking version of the Gell-Mann–Oakes–Renner relation.

The information obtained in the previous paragraph is all that we can extract from Eq. (18) in its present form. This is because terms of $O(p^4)$ have not been included on the OPE side of the sum rule. If one wishes to use the sum rule to obtain information about anything beyond the leading and next-to-leading order behavior of $g_\pi$ and $g_\eta$, one must, therefore, restore the $O(m_q^2)$ terms to the OPE. This is easily accomplished starting from the expression for the corresponding terms in the OPE of the analogous isospin-conserving correlator, as given in Ref. [11]. The result is

$$[\Pi_P(q^2)]_{\rm OPE}=-\frac{1}{\sqrt{3}Q^2}\left[\frac{3(m_d^2-m_u^2)}{8\pi^2}\ln\left(\frac{Q^2}{\mu^2}\right)+\frac{m_d\langle\bar{d}d\rangle-m_u\langle\bar{u}u\rangle}{Q^2}+\frac{(m_d^2-m_u^2)}{\pi Q^4}\langle\alpha_s G^2\rangle\right], \quad (20)$$

where we have kept terms only up to dimension 4 and written down the coefficient functions only to leading order in $\alpha_s$. Substituting the expression $[\Pi_P(q^2)]_{\rm OPE}$ into the LHS of Eq. (17) and Borel transforming, we obtain an improved version of the corrected CHM sum rule, Eq. (18). To facilitate subsequent analysis, it is convenient to multiply both expressions for the correlator by $Q^2$ before Borel transforming (thereby eliminating the contribution of the kinematic pole). We also follow standard practice and introduce a continuum threshold parameter, $s_0$, representing the point beyond which the hadronic spectral function is modelled by its perturbative QCD counterpart. The contribution corresponding to the integral over that portion of the phenomenological spectral function can then be moved to the OPE side of the sum rule. The result of these manipulations is the sum rule

$$g_\pi m_\pi^2 e^{-m_\pi^2/M^2}-g_\eta m_\eta^2 e^{-m_\eta^2/M^2}+\sum_{P\neq\pi,\eta} g_P m_P^2 e^{-m_P^2/M^2}=\frac{1}{\sqrt{3}}\left[\frac{3(m_d^2-m_u^2)}{8\pi^2}M^2(e^{-s_0/M^2}-1)+(m_d\langle\bar{d}d\rangle-m_u\langle\bar{u}u\rangle)-\frac{(m_d^2-m_u^2)}{\pi M^2}\langle\alpha_s G^2\rangle\right], \quad (21)$$

where the sum on the LHS now runs over pseudoscalar resonances with squared masses less than $s_0$.

The chiral expansion of the sum rule, Eq. (21), contains terms of order $O(p^2)$ and higher, together with the usual chiral logs, which start at order $O(p^4 \ln p)$. The $O(p^2)$ terms are the same as those in Eq. (18) and so it is easy to see the sum rule is consistent to this order. Since only the light-quark condensate and the quantities $g_\pi$, $g_\eta$, $m_\pi^2$, and $m_\eta^2$ contain leading chiral logs, these contributions must also cancel in Eq. (21) (as verified below). Finally, the expansion of $g_\pi m_\pi^2 e^{-m_\pi^2/M^2} - g_\eta m_\eta^2 e^{-m_\eta^2/M^2}$ to order $O(p^4)$ can be found from known one-loop expansions, and that for $m_d\langle\bar{d}d\rangle - m_u\langle\bar{u}u\rangle$ can be obtained from a straightforward one-loop calculation. With these results, we may employ this sum rule to obtain a relation describing the leading chiral behavior $[O(m_q^2)=O(p^4)]$ of the isospin-breaking parameters $g_{\eta'}, g_{\pi'}, \dots$ for the heavy pseudoscalar mesons. No further information can be extracted from Eq. (21) without two-loop ChPT calculations as input.

To verify the cancellation of the chiral logs, and to obtain the promised sum rule for the leading chiral behavior of $g_{\eta'}$, $g_{\pi'}$, . . . , we expand the $\pi$, $\eta$ and condensate terms in Eq. (21) to order $O(p^4)$. To do so for the $\pi$ and $\eta$ terms appearing on the LHS requires only the one-loop expressions for $f_\pi$, $f_\eta$, $\epsilon_1$, and $\epsilon_2$ given by Gasser and Leutwyler [1]. The results are

$$g_\pi m_\pi^2 - g_\eta m_\eta^2 = -\theta_0 F^2 \Bigg[ \frac{4}{3} B_0 (m_s - \hat{m}) - B_0 \left( \frac{9(m_s - 2\hat{m})\ell_\pi + 6 m_s \ell_K + (m_s + 2\hat{m})\ell_\eta}{72\pi^2 F^2} \right) + \frac{128 B_0^2 (m_s - \hat{m})}{3F^2} [(m_s + 2\hat{m}) L_6^r(\mu^2) - 2(m_s - \hat{m}) L_7^r + 2\hat{m} L_8^r(\mu^2)] + \frac{B_0^2 \hat{m}^2}{6\pi^2 F^2} \ln(m_K^2/\mu^2) - \frac{B_0^2 \hat{m}(m_s - \hat{m})}{12\pi^2 F^2} \Bigg] \tag{22}$$

and

$$\frac{1}{M^2}(-g_\pi m_\pi^4 + g_\eta m_\eta^4) = \frac{16\theta_0 F^2}{9M^2} B_0^2 (m_s - \hat{m})^2, \tag{23}$$

where $\ell_P = m_P^2 \ln(m_P^2/\mu^2)$ and all other notation is as in Gasser and Leutwyler [1]. For the condensate contributions on the RHS, we require the expressions for $\langle \bar{d} d \rangle$ and $\langle \bar{u} u \rangle$ valid to order $O(m_d - m_u)$. These are easily obtained, and can be written

$$\langle \bar{u} u \rangle = \langle \bar{u} u \rangle_I + \delta,$$
$$\langle \bar{d} d \rangle = \langle \bar{u} u \rangle_I - \delta, \tag{24}$$

where $\langle \bar{u} u \rangle_I$ is the one-loop expression for the condensate in the isospin-symmetric limit, also to be found in Ref. [1], and

$$\delta = (m_d - m_u) \left[ \frac{\ell_\eta - \ell_\pi}{64\pi^2 F^2 (m_s - \hat{m})} + \frac{B_0 (1 + \ln(m_K^2/\mu^2))}{32\pi^2 F^2} - \frac{4B_0}{F^2} (2L_8^r(\mu^2) + H_2^r(\mu^2)) \right]. \tag{25}$$

From Eqs. (24) and (25) it follows that, to order $O(p^4)$,

$$\frac{[m_d \langle \bar{d} d \rangle - m_u \langle \bar{u} u \rangle]}{\sqrt{3}} = -\theta_0 F^2 \Bigg[ \frac{4}{3} B_0 (m_s - \hat{m}) - \frac{B_0^2 \hat{m}(m_s - \hat{m})}{12\pi^2 F^2} + \frac{B_0^2 \hat{m}^2}{6\pi^2 F^2} \ln\left( \frac{m_K^2}{\mu^2} \right) - B_0 \left( \frac{9(m_s - 2\hat{m})\ell_\pi + 6 m_s \ell_K + (m_s + 2\hat{m})\ell_\eta}{72\pi^2 F^2} \right) + \frac{64 B_0^2 (m_s - \hat{m})}{3F^2} [2(m_s + 2\hat{m}) L_6^r(\mu^2) + 2\hat{m} L_8^r(\mu^2) + \hat{m} H_2^r(\mu^2)] \Bigg]. \tag{26}$$

To obtain a sum rule for the leading chiral behavior of the higher pseudoscalar resonances, we make use of Eqs. (22) and (23) to replace the leading terms of the $\pi$ and $\eta$ contributions in Eq. (21). The terms of higher order in $m_\pi$ and $m_\eta$ may be neglected since they are at least of order $O(p^6)$ in the chiral expansion, and they are numerically small for the Borel masses of interest. Finally, inserting the chiral expansion of the quark condensates, Eq. (26), into this sum rule, we get

$$\sum_{P \neq \pi, \eta} g_P m_P^2 e^{-m_P^2/M^2} = \frac{\sqrt{3} M^2}{8\pi^2} (e^{-s_0/M^2} - 1)(m_d^2 - m_u^2) + \frac{(m_d^2 - m_u^2)}{8\sqrt{3}\pi M^2} \langle \alpha_s G^2 \rangle - \frac{64 B_0^2 (m_s - \hat{m}) \theta_0}{3} [4(m_s - \hat{m}) L_7^r - 2\hat{m} L_8^r(\mu^2) + \hat{m} H_2^r(\mu^2)] - \frac{16}{9M^2} \theta_0 F^2 B_0^2 (m_s - \hat{m})^2. \tag{27}$$

Note that all of the chiral logarithms have cancelled, leaving only terms that start at order $O(p^4)$.

It is worth noting that the term involving the chiral LEC's makes a numerically significant contribution to the sum rule and is dominated by $L_7^r$. Moreover, phenomenological treatments that use resonance exchanges to generate the LEC's in the effective Lagrangian of ChPT [20,21] show that $L_7^r$ receives contributions only from flavor-singlet pseudoscalar states. Hence it already follows, without any more detailed analysis, that the sum rule implies the existence of significant isospin-breaking $\eta'$ coupling. Before presenting the results of our analysis for the isospin-breaking parameters, $g_P$, it is also worth stressing a number of features of the sum rule, Eq. (27), which imply that, once $L_7^r$ is fixed, these results for at least the $\eta'$ coupling should be quite reliable.

The renormalized LEC's $L_7^r, L_8^r$ have been determined phenomenologically and are reasonably well-known (see, for example, Refs. [12,13] for recently updated values). The remaining LEC, $H_2^r$, can be related, for example, to the isospin-breaking condensate ratio $\gamma \equiv [\langle \bar{d}d\rangle - \langle \bar{u}u\rangle]/\langle \bar{u}u\rangle$. This ratio has been estimated in a number of sum-rule analyses [14–16,10,17,18]. Using any of the values of $\gamma$ obtained in these treatments to estimate $H_2^r$, the resulting values are such that the LEC combination in Eq. (27) is dominated by the $L_7^r$ term. In particular, the uncertainty in the LEC combination associated with the sum of the errors on the phenomenological determinations of $L_8^r$ and $H_2^r$ (where the latter error is taken to correspond to the entire range of values cited above) is an order of magnitude smaller than that associated with the error on the existing phenomenological determination of $L_7^r$. We may, therefore, ignore the effect of the uncertainties in the values of $L_8^r$ and $H_2^r$. This feature of the analysis results from the fact that the coefficients of $L_8^r$ and $H_2^r$ are suppressed by a factor of $(m_s - \hat{m})/\hat{m} \sim 23$ [19] relative to that of $L_7^r$. The uncertainty in the ratio $m_s/\hat{m}$ which enters this suppression is, of course, also completely negligible. Note that we do not require an explicit input value for $m_s$ since, to the order considered in the chiral expansion, we may take

$$B_0^2 (m_s - \hat{m})^2 = (m_K^2 - m_\pi^2)^2. \tag{28}$$

On the phenomenological side of the sum rule, we expect contributions from all of the higher pseudoscalar resonances, $\eta'(958), \eta(1295), \pi'(1300), \eta(1440), \pi'(1800), \ldots$. The $\pi'(1300)$ is relatively broad ($\Gamma = 325$ MeV [22]) and spans the region between the $\eta(1295)$ and the $\eta(1440)$. Therefore, without keeping terms of yet higher dimension in the OPE, we have too little information in the sum rule to both adequately parametrize the spectral function in the region between $\sim 1300$ and $\sim 1450$ MeV and at the same time to use the sum rule to extract the values of all such parameters. Hence we concentrate on the extraction of $g_{\eta'}$, parametrizing the $\eta(1295)$, $\pi'(1300)$, $\eta(1440)$ region in terms of a single effective contribution of zero width located at around 1375 MeV. By varying the position of this contribution between 1300 and 1450 MeV, we have verified that the extracted value of $g_{\eta'}$ is not sensitive to this approximation, varying by $\sim \pm 6\%$ over this range. This is a factor of 6 smaller than the variation induced by the uncertainty in the input value of $L_7^r$, which we discuss in more detail below.

The effective strength parameter describing the $\eta(1295)$, $\pi'(1300)$, and $\eta(1440)$ region (which we denote by $g_{\pi'}$ in what follows) is, of course, much more sensitive to the assumed position of this strength. The corresponding uncertainty in the extraction of $g_{\pi'}$ is $\sim 15\%$, which is significant, although still much less than the $\sim 60\%$ associated with $L_7^r$. The stability of the determination of $g_{\eta'}$ is attributable, to a large extent, to the fact that the residual term proportional to $L_7^r$ provides the major contribution to the sum rule; as already noted above, $L_7^r$ is known to receive contributions only from flavor-singlet states, of which the $\eta'$ is nearest and hence should provide the dominant contribution. This feature of the sum rule is also responsible for the greater sensitivity of $g_{\pi'}$ to the input value chosen for the location of the effective strength describing the $\eta(1295)$, $\pi'(1300)$, and $\eta(1440)$ region: the combined effective contribution to the sum rule is small relative to the dominant $\eta'$ term and the extracted value can therefore depend sensitively on the assumed separation from the $\eta'$ peak.

Having employed information from ChPT to fix the low-lying $\pi$ and $\eta$ contributions to the original sum rule, and explicitly modeled the contributions up to 1.44 GeV, we note that there is now a significant gap to the next resonance contribution at 1.8 GeV. We therefore expect that Borel masses of order 1–1.5 GeV will suppress the contributions of higher resonance on the phenomenological side of the sum rule.

On the OPE side it turns out that the situation is also rather favorable. First, the gluon condensate term turns out to be numerically very small compared to the dominant $L_7^r$ contribution. Indeed, if we take for definiteness the value for this condensate advocated in Ref. [23] (which is similar to that employed, for example, in Ref. [24]),

$$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.03 \pm 0.015 \ \ \mathrm{GeV}^4, \tag{29}$$

which includes rather conservative errors, then we find that this uncertainty corresponds to $< 0.3\%$ variations in $g_{\eta'}$ and $g_{\pi'}$.

The perturbative contribution [the first term on the RHS of Eq. (27)] is similarly small. This is fortunate since recent analyses [25–27] suggest that conventional sum-rule determinations of the light current quark masses [24,28,29] may have overestimated these masses by as much as a factor of 2. For the central value of $L_7^r$, allowing $m_u + m_d$ to vary between the conventional value, 12 MeV [24] and 6 MeV produces a variation of only 2.5% in $g_{\eta'}$ and $g_{\pi'}$. Such an uncertainty is again much smaller than that arising from the errors on $L_7^r$, and hence can be neglected. The smallness of this perturbative contribution also implies that the analysis should be rather insensitive to the continuum threshold parameter, $s_0$. We expect that this should lie somewhere in the vicinity of the onset of the $\pi'(1800)$ resonance. In our analysis, we find, for example, that varying $s_0$ by $\pm 1$ GeV$^2$ about a central value $s_0 = 3$ GeV$^2$ produces variations of $< 1\%$ in $g_{\eta'}$ and $g_{\pi'}$.

From the above discussion, we see that the RHS of the sum rule in Eq. (27) is dominated by the terms that are directly calculable using ChPT. The first of these, involving the $O(p^4)$ LEC's, is the piece of the quark condensate term from the OPE that remains after cancellation against $\pi$ and $\eta$ contributions from the phenomenological side of the sum rule. The second consists of the remaining $O(m_q^2)$ $\pi$ and $\eta$ contributions from the phenomenological side. Numerically it is more than a factor of 2 smaller than the LEC term, for $M > 1$ GeV$^2$, and of the same sign. The major uncertainty in the values of these terms is that arising from the phenomenological determination of the (scale-independent) LEC [12,13],

$$L_7^r = (-0.4 \pm 0.15) \times 10^{-3}. \tag{30}$$

For completeness we list below the remaining input values (apart from well-determined meson masses):

$$m_u + m_d = 9 \ \text{MeV},$$
$$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.03 \ \text{GeV}^4,$$
$$s_0 = 3.0 \ \text{GeV},$$
$$L_8^r(m_\rho^2) = 0.9 \times 10^{-3},$$
$$H_2^r(m_\rho^2) = -7.5 \times 10^{-4},$$
$$m_{\pi'} = 1.375 \pm 0.075 \ \text{GeV},$$
$$m_s / \hat{m} = 24.4,$$
$$r = \frac{m_d - m_u}{m_d + m_u} = 0.3 \pm 0.05, \tag{31}$$

where by $m_{\pi'}$ we mean the location of the effective strength for the $\eta(1295)$, $\pi'(1300)$, $\eta(1440)$ region, as discussed above. In most cases we have not shown the corresponding uncertainties, since, as already noted, the variations in the results associated with them are small. Apart from $L_7^r$, the largest uncertainty is that associated with the choice $m_{\pi'}$, which parametrizes the strength lying above the $\eta'$.

Also significant is the uncertainty associated with the isospin-breaking mass ratio, $r$ [19]. The quoted range covers a wide range of possibilities for the degree of breaking of Dashen's theorem [30] for the electromagnetic contribution to the kaon mass splitting. The recent results of Refs. [31–33] would appear to confirm a larger value for the breaking, as suggested by earlier analyses [34–36], and hence larger values of $r$ in the quoted range, with a somewhat smaller resulting error. Since the subject is not yet fully resolved (see Ref. [31] for a detailed list of recent work on the subject, including some work advocating smaller violations of Dashen's theorem [37]), we have refrained from attempting to make a revised estimate for the input central value and error on $r$. In any case, every term on the RHS of Eq. (27) contains one factor of $m_d - m_u$, so that this uncertainty enters only into the overall normalization of the final results. It does not, therefore, affect the stability analysis of the sum rule, and it can be removed by quoting results in the form $g_P / \theta_0 F^2$.

For a given set of values for the input parameters $L_7^r$ and $m_{\pi'}$, we look for values of $g_{\eta'}$ and $g_{\pi'}$ that bring the two sides of the sum rule into agreement over a range of Borel mass values. A convenient way to do this is to use the sum rule, Eq. (27), and its derivative with respect to $M$, at a fixed value of the Borel mass, as simultaneous linear equations for $g_{\eta'}$ and $g_{\pi'}$. If a region is found where the results of this procedure are independent of $M$, then this indicates the existence of a stability window where the two sides of the sum rule match. In Fig. 1 we show some typical results for a case where we obtain good stability, $L_7^r = -0.34 \times 10^{-3}$ and $m_{\pi'} = 1375$ MeV, with $g_{\eta'} = 2.88 \times 10^{-5}$ and $g_{\pi'} = -5.57 \times 10^{-6}$. The two curves are essentially indistinguishable, except at the very lower end of Borel masses displayed.

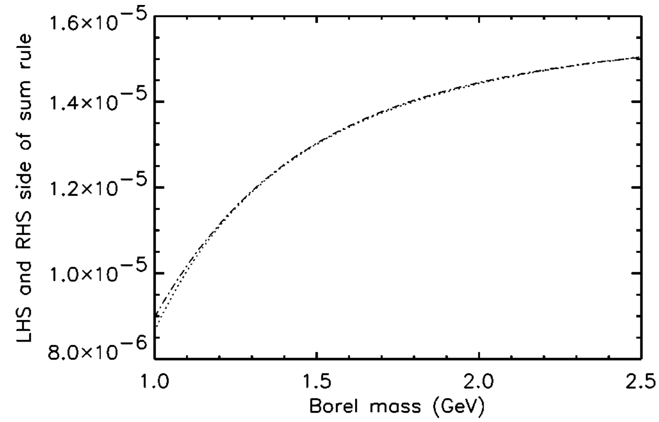


FIG. 1. The OPE versus the phenomenological side of the improved sum rule as a function of the Borel mass, $M$. The dotted line is the OPE side, the dash-dotted line the phenomenological side in units of $\text{GeV}^2$.

As $|L_7^r|$ is decreased, the stability window moves to larger values of $M$. In this region, the perturbatively modeled continuum becomes increasingly important in the spectral representation of the correlator and so the sum rule becomes unreliable for the determination of resonance properties. In contrast, as $|L_7^r|$ is increased the stability window moves to smaller values of $M$ and also becomes very much narrower. In fact, for values of $|L_7^r|$ that are larger than about $0.48 \times 10^{-3}$ we are unable to find a stable matching between the two sides of the sum rule. This occurs before the window reaches sufficiently small values of $M$ that the convergence of the OPE becomes questionable. We are thus able to use the sum rule to make a somewhat improved determination of the LEC $L_7^r$, reducing by about a factor of 2 the distance to the upper bound on its magnitude compared to the ChPT result, Eq. (30) [12,13].

For values of $L_7^r$ in the range $-0.25 \times 10^{-3}$ to $-0.48 \times 10^{-3}$, we obtain

$$g_{\eta'} / \theta_0 F^2 = 0.42 \pm 0.15,$$
$$g_{\pi'} / \theta_0 F^2 = -0.13 \pm 0.07. \tag{32}$$

The dependence on $r$ has been scaled out of these results, as discussed above, and so the dominant uncertainties quoted in Eqs. (32) are those associated with the range of values for $L_7^r$. Allowing for the uncertainty in $r$ taken from [19], our values for the isospin-breaking parameters are

$$g_{\eta'} = (3.6 \pm 1.9) \times 10^{-5} \ \text{GeV}^2,$$
$$g_{\pi'} = (-1.1 \pm 0.8) \times 10^{-5} \ \text{GeV}^2. \tag{33}$$

## IV. SUMMARY

In this paper, we have revisited the sum-rule treatment for the isospin-breaking axial correlator, correcting the error in a previous treatment which led to the incorrect chiral behavior of the slope parameter $g_\eta - g_\pi$. Including the kinematic pole omitted from the previous treatment restores the correct chiral behavior of the correlator. We have then used the explicit evaluation of the $\pi$ and $\eta$ contributions to the correlator at next-to-leading order in ChPT to obtain a rather well-behaved sum rule for the leading chiral behavior of the isospin-breaking parameters, $g_P$, of the higher pseudoscalar resonances. This sum rule has been analyzed and shown to provide a rather reliable estimate for $g_{\eta'}$, once one has fixed the chiral LEC, $L_7^r$. The requirement of the stability of this sum rule is shown, moreover, to provide a somewhat improved determination this LEC by reducing the upper bound on its magnitude.

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