Klein-Gordon equation in a coupled channels description of elastic and inelastic scattering

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We show that, in contrast to the case of elastic scattering of spinless particles from composite targets, Klein-Gordon dynamics is not a natural choice for the description of their inelastic scattering in the standard coupled channels method. We propose one way to overcome the difficulty, showing that Klein-Gordon dynamics may be used in practical calculations within the coupled channels framework provided that the optical potential is modified in a straightforward manner. [S0556-2813(97)04709-2]

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I. INTRODUCTION

Elastic scattering of spinless projectiles from composite targets (for example, pion elastic scattering from nuclei [1]) is well known to be capable of being described using the Klein-Gordon equation, with the optical potential given as the proper self-energy in a field theoretical framework [2]. However, for the case of inelastic scattering, where the projectile is able to induce transitions among various excited states of the system, Klein-Gordon dynamics is not as natural a description, as we shall show below. This fact is relevant for coupled channels models, where one envisions unifying elastic and inelastic scattering by formulating them in a common coupled channels framework. Models with spinless projectiles have been considered in this spirit for elastic and inelastic scattering of pions (e.g., [3–5]) and kaons [6] from nuclei.

We will show, in Sec. II, the origin of the differences between elastic and inelastic scattering that arises from the point of view of Klein-Gordon dynamics. It is also shown there that errors occur for inelastic scattering, but that they are confined to multistep components of the inelastic scattering wave function and therefore are expected to be of little consequence for most practical situations. We then show in Sec. III that the inelastic portion of the calculation can be remedied, when needed, by implementing a straightforward and presumably rapidly convergent expansion procedure. The general question of whether a more fundamental formulation of multiple scattering from a composite target with a relativistic propagator remains an open question worthy of additional investigation.

II. ANALYSIS OF TWO-STEP INELASTIC SCATTERING

Let us illustrate the origin of the problem by considering the inelastic reaction $^{14}\mathrm{C}_{\mathrm{g.s.}}(\pi,\pi')^{14}\mathrm{C}_F$, scattering from the ground state (g.s.) of ^{14}C to its excited final state F. In a coupled channels framework, the reaction would be allowed to proceed in a sequence of steps. Consider the two-step contribution in which the earlier inelastic scattering (occurring, say, at time t) is $^{14}\mathrm{C}_{\mathrm{g.s.}}(\pi,\pi'')^{14}\mathrm{C}_I$ (scattering to the excited intermediate state I of ^{14}C), and the latter (at time t') is $^{14}\mathrm{C}_I(\pi'',\pi')^{14}\mathrm{C}_F$. Note that since the language describing this reaction involves specifying a sequence of events in time, it can be represented by the time-ordered diagram in Fig. 1(a).

This stands in contrast to the situation described by the coupled channels equations using the Klein-Gordon propagator for the pion,

$$iS_k(t-t')=\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega(t-t')}d\omega}{\omega^2-k^2-m_\pi^2+i\eta} \tag{1}$$

$$=\frac{i}{2\pi}\frac{1}{2\omega_k}\int_{-\infty}^{\infty}\frac{e^{-i\omega(t-t')}d\omega}{\omega-\omega_k+i\eta}-\frac{i}{2\pi}\frac{1}{2\omega_k}\int_{-\infty}^{\infty}\frac{e^{-i\omega(t-t')}d\omega}{\omega+\omega_k-i\eta}, \tag{2}$$

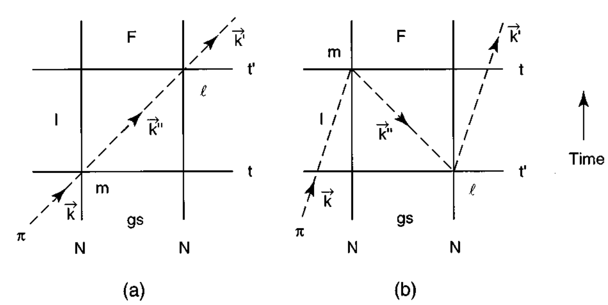

FIG. 1. Two time orderings describing pion propagation (a) forward and (b) backward in time for inelastic scattering. Only the struck nucleons N are shown.

with $\omega_k=(k^2+m_\pi^2)^{1/2}$. Here *two* time orderings are allowed, which can be seen by evaluating Eq. (1) by contour methods [7], in which case the first term of Eq. (2) is seen to act only when $t>t'$ and the second when $t<t'$. Of these, only the forward-in-time piece [the first in Eq. (2)] is relevant to the sequence of inelastic scatterings described above and in Fig. 1(a).

The forward-in-time contribution in Eq. (2) corresponds to familiar Schrödinger dynamics (although with relativistic energies) of classical multiple-scattering theory. The backward-in-time contribution is, by distinction, an additional feature allowed in a quantum field theoretical description. In such theories mesons may be created and annihilated, and the backward-in-time contribution appears by necessity when the Klein-Gordon propagator is used.

The proper way to bring inelastic scattering together with the forward- and backward-in-time propagation of the pion requires that the two time orderings in Eq. (2) be calculated separately. The backward-in-time contribution would correspond to the sequence of events shown in Fig. 1(b), where the intermediate state has three pions simultaneously present, a contribution mandated by the field theoretical treatment mentioned above. The values for the two pieces of Fig. 1 are as follows:

$$\text{Fig. 1(a):}\quad \frac{i}{2\omega_{k''}}\langle F|\sum_l f_l e^{i\vec{r}_l\cdot(\vec{k}'-\vec{k}'')}|I\rangle\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega-\omega_{k''}+i\eta}\langle I|\sum_m f_m e^{i\vec{r}_m\cdot(\vec{k}''-\vec{k})}|\text{g.s.}\rangle, \tag{3}$$

$$\text{Fig. 1(b):}\quad \frac{-i}{2\omega_{k''}}\langle F|\sum_m f_m e^{i\vec{r}_m\cdot(\vec{k}''-\vec{k})}|I\rangle\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+\omega_{k''}-i\eta}\langle I|\sum_l f_l e^{i\vec{r}_l\cdot(\vec{k}'-\vec{k}'')}|\text{g.s.}\rangle, \tag{4}$$

where we have taken for notational simplicity the transition to be described by an energy-independent pion-nucleon amplitude f_l and taken the nuclear states $|\text{g.s.}\rangle$, $|I\rangle$, and $|F\rangle$ to be degenerate. The point is that these two processes, when added together, do not in general collapse back to the expression used in the coupled channels theory for the double-scattering term through intermediate state I, namely,

$$\langle F|\sum_l f_l e^{i\vec{r}_l\cdot(\vec{k}'-\vec{k}'')}|I\rangle\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{e^{-i\omega(t'-t)}}{\omega^2-k''^2-m_\pi^2+i\eta}\langle I|\sum_m f_m e^{i\vec{r}_m\cdot(\vec{k}''-\vec{k})}|\text{g.s.}\rangle. \tag{5}$$

[To recover the familiar time-independent form (see, e.g., Ref. [3]), one must take the Fourier transform $\int e^{i\omega_0 t}dt$, where ω_0 is the incident pion energy.]

In order for the propagators in Eqs. (3) and (4) to combine into their form given in Eq. (5), one would need to have

$$\langle F|\sum_l f_l e^{i\vec{r}_l\cdot(\vec{k}'-\vec{k}'')}|I\rangle=\langle I|\sum_l f_l e^{i\vec{r}_l\cdot(\vec{k}'-\vec{k}'')}|\text{g.s.}\rangle \equiv U(\vec{k}'-\vec{k}''), \tag{6}$$

which, of course, implies

$$\langle I|\sum_m f_m e^{i\vec{r}_m\cdot(\vec{k}''-\vec{k})}|\text{g.s.}\rangle=\langle F|\sum_m f_m e^{i\vec{r}_m\cdot(\vec{k}''-\vec{k})}|I\rangle = U(\vec{k}''-\vec{k}). \tag{7}$$

In this case, using Eq. (2), we would find

$$\text{Fig. 1(a)}+\text{Fig. 1(b)} = iU(k'-k'')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t'-t)}}{\omega^2-k''^2-m_\pi^2+i\eta}U(k''-k). \tag{8}$$

Note, however, that for inelastic scattering the conditions of Eqs. (6) and (7) are not generally satisfied, and so one does not obtain the second Born term, Eq. (5), that occurs in the coupled channels theory. Equation (5) contains the forward-in-time piece, Eq. (3), properly, but the backward-in-time contribution, given correctly in Eq. (4), is not what it should be. Thus, as a matter of principle, the coupled channels Klein-Gordon theory picks up an error at second order, and it remains to be seen how serious this can be.

Before we examine the error for inelastic scattering, note that the conditions of Eqs. (6) and (7) are valid for the case of elastic scattering (the charges of the initial, intermediate, and final pions are then the same), and one easily recognizes Eq. (8) to be structurally the same as the second Born term (in a time-dependent scheme) for the pion elastic scattering amplitude with optical potential $U(q)$. So for elastic scattering the use of the Klein-Gordon equation does not suffer from the error discussed above.

To assess the consequences of including the spurious backward-in-time contribution for inelastic scattering, we compare the two terms in Eqs. (3) and (4), assuming that the nuclear form factors are each Gaussians,

$$U(k^2)=Ne^{-k^2/a^2}, \tag{9}$$

with N a normalization factor and with

$$a=\frac{2}{b}\approx 1.19\ \text{fm}^{-1}, \tag{10}$$

where the harmonic oscillator parameter b for ^{14}C was taken to be 1.68 fm. We then integrate over k'' and examine the ratio. The results are that for a nucleus as small as ^{14}C, the backward-in-time part, Eq. (4), is 6–9 % of the dominant forward-in-time part over the energy range of incident pion kinetic energy 5–100 MeV. This correction is sufficiently small for most purposes that it may be neglected.

III. CORRECTING THE CALCULATION

Although the backward-in-time contribution is quite small, one might want to consider making a correction when the double-scattering term is a particularly important part of the amplitude. Such occurs, for example, in pion double-charge exchange, where the double-scattering term is the leading contribution to the multiple scattering. In these cases, one might consider adding a correction term to the coupled channels potential so that the backward-in-time contribution is calculated correctly to second order, or dropping the backward-in-time contribution all together so that the result would correspond more closely to Schrödinger dynamics.

In actual coupled channels calculations such as those of Refs. [3–6], the most attractive option may be to retain the Klein-Gordon propagator throughout but then to add a perturbative correction to the optical potential that assures a correct two-step result. This correction δ is then the difference between the correct result, Eqs. (3) and (4), and the approximate result, Eq. (5). Expanding the Klein-Gordon propagator of Eq. (5) using Eqs. (1) and (2), we see that the forward-in-time pieces cancel identically, so that there is again no pole contribution to the correction. Thus, the correction δ has the form

$$\delta=\frac{i}{2\omega_{k''}}\langle F|\sum_m v_m(r_m)e^{i\vec{r}_m\cdot(\vec{k}''-\vec{k})}|I\rangle\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+\omega_{k''}+i\eta}\langle I|\sum_l v_l(r_l)e^{i\vec{r}_l\cdot(\vec{k}'-\vec{k}'')}|\text{g.s.}\rangle$$
$$-\frac{i}{2\omega_{k''}}\langle F|\sum_l v_l(r_l)e^{i\vec{r}_l\cdot(\vec{k}'-\vec{k}'')}|I\rangle\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+\omega_{k''}+i\eta}\langle I|\sum_m v_m(r_m)e^{i\vec{r}_m\cdot(\vec{k}''-\vec{k})}|\text{g.s.}\rangle$$
$$=-\frac{i}{2\omega_{k''}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega+\omega_{k''}+i\eta}[U_{FI}(k'-k'')U_{I\text{g.s.}}(k''-k)-U_{I\text{g.s.}}(k'-k'')U_{FI}(k''-k)].$$

Of course, to calculate the full contribution of δ one must integrate over all intermediate pion momenta $\vec{k}''$.

It is easy to see that the size of δ above will be sensitive to the quantum numbers in the transition g.s.$\to F$. Consider, for example, the value of δ in the forward direction ($\vec{k}\simeq\vec{k'}$), where δ often attains its largest values. If we use the dominant spin-independent part of the pion-nucleon scattering amplitude in Eqs. (6) and (7), it is easy to see by exchanging $\vec{k}$ ($=\vec{k'}$) and $\vec{k''}$ in one of the two terms on the last line of the equation above for δ that the magnitudes of these terms are equal and that they combine with a minus (plus) sign when the transition g.s.$\to F$ is parity nonchanging (parity changing). The change in sign results from the fact that U, for the amplitude being considered, is proportional to a spherical harmonic $Y^m_{\Delta l}(\vec{k}-\vec{k''})$, where Δl is the difference in the orbital angular momentum of the two nuclear states involved in the matrix element U. For parity nonchanging (changing) transitions the product of the two spherical harmonics is the same (opposite) when $\vec{k}$ and $\vec{k''}$ are exchanged. For these reasons, we may expect important cancellations to occur between the terms for parity nonchanging g.s.$\to F$ transitions, but that in other cases the two terms generally add coherently, increasing the size of δ. In any case, the correction is small enough to be considered perturbatively.

Clearly, the procedure of correcting the amplitude could be extended to higher order in a systematic fashion. One may therefore conclude that although not as natural a framework for inelastic as for elastic scattering, Klein-Gordon dynamics may be used with the caveat that the expansion of the inelastic transition interaction contains the correction terms discussed here.

In other cases it may be more convenient to suppress the backward-in-time term, in which case the result would correspond to a calculation in relativistic Schrödinger dynamics. We will show next that dropping the backward-in-time term leads to a result very similar in form to the original Klein-Gordon theory, but with a modified form-factor cutoff in the off-shell amplitude f_l. To see this, suppose f_l is separable, $\langle k|f_l|k'\rangle=v(k)\lambda_l v(k')$. We may then write the forward-in-time piece of Fig. 1(a) in terms of the combination

$$v(k)\frac{1}{2\omega_k}\frac{1}{\omega-\omega_k+i\eta}v(k)$$
$$=v(k)\sqrt{\frac{\omega+\omega_k}{2\omega_k}}\frac{1}{k_0^2-k^2+i\eta}\sqrt{\frac{\omega+\omega_k}{2\omega_k}}v(k),\qquad(11)$$

with $k_0^2=\omega^2-m_\pi^2$. For many processes, the dominant effect of the projectile-nucleon form factor is given by its range, Λ, which is related to its falloff at small k. In this spirit, if we assume

$$v(k)\approx 1-\frac{k^2}{\Lambda^2},\qquad(12)$$

Eq. (12) suggests the introduction of a modified projectile-nucleon form factor $\tilde{v}(k)$,

$$\tilde{v}(k)=v(k)\sqrt{\frac{\omega+\omega_k}{2\omega_k}}$$
$$\approx\sqrt{\frac{m_\pi+\omega}{2m_\pi}}\left[1-\left(\frac{\omega}{m_\pi+\omega}\frac{1}{4m_\pi^2}+\frac{1}{\Lambda^2}\right)k^2\right]$$
$$\approx\sqrt{\frac{m_\pi+\omega}{2m_\pi}}\left(1-\frac{k^2}{\tilde{\Lambda}^2}\right), \tag{13}$$

from which we see that the effect of the suppression of the crossed diagram is to renormalize the form factor by

$$\sqrt{\frac{m_\pi+\omega}{2m_\pi}}, \tag{14}$$

and to increase its range so that

$$\tilde{\Lambda}^2=\left(\frac{\omega}{m_\pi+\omega}\frac{1}{4m_\pi^2}+\frac{1}{\Lambda^2}\right)^{-1}=\frac{4m_\pi^2\Lambda^2(m_\pi+\omega)}{\Lambda^2\omega+4m_\pi^2(m_\pi+\omega)}. \tag{15}$$

This type of correction would be particularly useful for the double-charge exchange theory of Refs. [8,9], where the quadratic form of the propagator leads to some particular simplifications in the formulation of the theory. In the theory of Ref. [8] the range Λ is quite large ($\sim 6\ \mathrm{fm}^{-1}$) for the dominant p wave, so that the effective $\tilde{\Lambda}$ is approximately

$$\tilde{\Lambda}\approx 2m_\pi\approx 1.42\ \ \mathrm{fm}^{-1}. \tag{16}$$

IV. CONCLUDING REMARKS

We have shown in this paper that although the Klein-Gordon equation is routinely used to describe the elastic scattering of pions and kaons, this equation is not the natural choice for describing their inelastic scattering in the standard coupled channels method. In particular, for inelastic scattering, a small error is made for two- (and more) step contributions to the scattering amplitude.

We do not want to rule out the possibility that a more elegant description of multiple scattering free of this error can be formulated. However, we have shown that a systematic proceedure for correcting the errors within the standard coupled channels method can be found, and we present an expression for this correction at the leading, two-step level. For most inelastic transitions, the two-step contribution to the cross section is relatively small, and the errors we discuss thus have little quantitative significance. However, for pion double-charge exchange the errors can be more important, since in this case the two-step term is the leading contribution to the cross section. We have shown in the discussion in connection with Eq. (5) that even in this case the standard coupled channels description is a reasonable approximation for transitions g.s.$\rightarrow F$ that do not change parity. For parity-changing double-charge exchange transitions, the correction we discuss may be particularly important and should be made explicitly.

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