Finite-size corrections in the spatial contribution to the density of states in nuclear multifragmentation

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The density of states in statistical approaches to nuclear multifragmentation includes a contribution from permutations of fragment positions in the physical space (the volume V_F) occupied by the fragments. For N pointlike noninteracting particles (perfect gas) this factor is simply proportional to V_F^N. We propose a correction due to the finite size of fragments which is based on an evaluation of the two-body overlap probability. The correction exhibits an approximate exponential dependence on the multiplicity N.
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I. INTRODUCTION

Prompt nuclear multifragmentation is a perhaps idealized image of a decay mechanism for highly excited atomic nuclei in which a set of fragments is instantaneously produced as a partition of the parent nucleus mass and charge [1,2]. Many authors have proposed statistical models of this process [3–6]. One of the most important problems for statistical models is to calculate the density of states (proportional to the probability of observation) associated with any given partition. A particular difficulty in this calculation concerns corrections to the spatial contribution to the density of phase-space states caused by the finite size of the fragments. An investigation of one such correction has been carried out by Randrup, Robinson, and Sneppen [7].

For a small number of fragments one approach to this problem is via Monte Carlo simulations. The methodology of such simulations provides a good introduction to the problem of finite size corrections. The usual procedure is to generate a set of trial events for any given partition. Suppose we stipulate that fragments should be constrained to lie with their mass centers inside some ''freeze-out'' volume V_F. In a particular trial, each fragment is placed at a uniformly random position inside the freeze-out volume, animated with some initial (thermal and/or collective) velocity and subjected to the Coulomb repulsion due to the other fragments. One rejects trials in which any two fragments overlap either at the initial stage or at any time in the subsequent evolution. The phase-space density (and thus the probability for observing the partition under consideration) is then modified by the survival probability which is estimated as the proportion of nonrejected trials. The modification may be rather important (several orders of magnitude) and the Monte Carlo method may even become impractical due to lack of statistics. It is therefore desirable to make an analytical estimation of the survival probability.

This work is concerned with the contribution to the survival probability due to finite size effects in the initial configuration for fragments constrained to lie with their mass centers inside the freeze-out volume (Sec. II). Our calculations of this contribution are compared with Monte Carlo simulations. The approach is semianalytic insofar as it is based on estimates of two-body overlap probabilities. Higher-order corrections are not rigorously treated but are approximated by an ''occupied volume'' correction. Nevertheless we have succeeded in producing a calculational scheme which predicts finite size corrections in agreement with the Monte Carlo estimates to within typically 20% for values in the range $1–10^{-4}$.

To within a factor of 5–10 the survival probability turns out to depend exponentially on the partition multiplicity. We show, in Sec. III, that this observation can be straightforwardly understood in terms of the two-body overlap probabilities. In this section we sacrifice the precision of the preceding analysis in order to emphasize the physical origin of the multiplicity dependence. In Sec. III B we also depart briefly from the situation analyzed in depth in the main body of the paper to consider configurations in which the fragments are entirely contained in the freeze-out volume [1]. In Sec. IV we present a summary of our work together with a brief discussion of related topics.

II. SPATIAL OVERLAPS OF FRAGMENT PAIRS INSIDE THE FREEZE-OUT VOLUME

A. Introductory remarks

We suppose that the position and momentum contributions to the density of states can be estimated separately (see, for example, Ref. [4]). This is trivially true if the system energy is independent of the spatial configuration of the fragments. It is also true in the canonical ensemble for a gas of interacting fragments whose energy can be expressed as a sum of kinetic energies which are independent of the spatial configuration and potential energies which depend on this configuration [7].

We shall be concerned with the contribution to the density of phase-space states due to permutations of fragment positions. We suppose the fragments to be spherical and to be confined such that their mass centers are randomly disposed within a spherical ''freeze-out'' volume with radius R_F. This corresponds to the ideal-gas picture, i.e., to the case of noninteracting fragments (in the limit of pointlike fragments). Extension to other geometries may be carried out with more or less difficulty but the physical picture which we analyze herein is thought to be a good starting point for the analysis of the multifragmentation mechanism. This choice coincides with that of Randrup and Koonin [5]. The main alternative (see Sec. III B) is to define the freeze-out volume as a spherical region which entirely englobes the fragments [1]. For pointlike fragments the two choices are obviously equivalent. For nonoverlapping finite sized fragments, according to the authors of Ref. [5] the two choices are roughly equivalent (to within an appropriate readjustment of the freeze-out radius).

We now consider the problem of estimating, to within a global constant, the number of ways of distributing N spherical fragments in a spherical freeze-out volume V_F (with radius R_F) such that *no two fragments overlap*. Explicitly we will evaluate $\chi(R_F)$ such that the spatial contribution to the phase-space density is

$$\Gamma(R_F)=\chi(R_F){V_F}^N, \tag{1}$$

where χ is a quantity ($\chi\leqslant 1$) which formally expresses the suppression of spatial configurations which contain interpenetrating (overlapping) fragments (note that this definition differs from that used in Ref. [4]).

As stated in the introduction, one can solve this problem approximately (often quite accurately) by using Monte Carlo methods. This Monte Carlo technique consists in making a large number, N_{MC} of attempts to distribute the N fragments at random positions in the freeze-out volume. The number of retained events, N_R, is then that number of trials for which the distance between all pairs of fragments is greater or equal to the sum of their radii, i.e., for all i,j,

$$\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}\geqslant(R_i+R_j). \tag{2}$$

The spatial contribution to the phase-space density is then simply estimated as

$$\Gamma_{\mathrm{MC}}(V_F)={V_F}^N\frac{N_R}{N_{\mathrm{MC}}}, \tag{3}$$

so that the survival probability χ is estimated as N_R/N_{MC}. Clearly if N_R is small, the relative error in χ may be very large. A difficulty with this technique thus arises when the freeze-out volume V_F is sufficiently small so that the calculation time spent estimating χ becomes prohibitively long.

In Ref. [7], the authors discussed a virial approximation in which the survival probability χ is simply expressed as the product of two-body terms. We take basically the same starting point. However our estimation is based on an exact evaluation of the spatially averaged bare two-body term in a model in which all fragment mass centers are confined to lie within the freeze-out volume.

B. Two-body approximation

We consider a particular mass partition and we assign a label 1–N to each fragment. The probability that a configuration is retained is the probability that all distinct fragment pairs do not overlap. Thus the problem is solved if we can calculate the overlap probability for any given pair of fragments in the set. Let us first calculate this probability by ignoring the influence of fragments other than the specific pair considered.

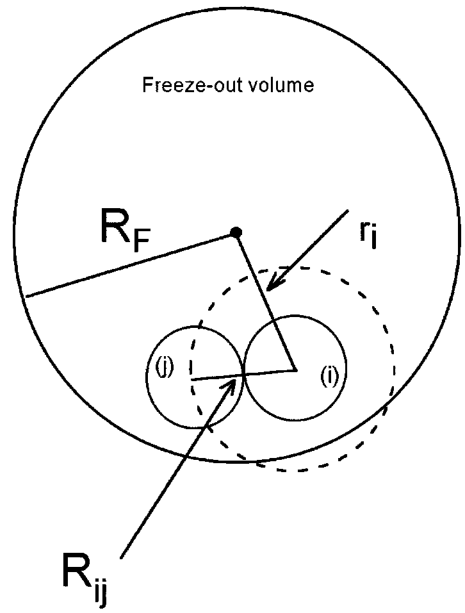

FIG. 1. A two-dimensional schematic picture of the two-body approximation. To avoid overlap, the fragment labeled j is excluded from the volume $V_{ij}(r_i)$ enclosed by the dotted line. In the example shown, a part of this volume lies outside the freeze-out volume (see text).

Suppose we position the center of the freeze-out volume at the origin of our coordinate system and place the fragment labeled i at the radial coordinate r_i. In order to avoid overlap with any other fragment j we need to exclude the center of mass of fragment j from the volume

$$V_{ij}=\frac{4\pi}{3}(R_i+R_j)^3. \tag{4}$$

The physical picture which corresponds to this situation is depicted in Fig. 1. If r_i is such that the volume V_{ij} is *entirely enclosed* within V_F then we see immediately that the probability that fragment j will overlap with fragment i is

$$\Phi_{ij}(r_i)=\frac{V_{ij}}{V_F}=\frac{(R_i+R_j)^3}{{R_F}^3}. \tag{5}$$

As we increase the value of r_i, it will eventually be the case that the volume V_{ij} will be only partially contained in the freeze-out volume. The overlap probability due to the fragment j is then simply given by the part of V_{ij} *which lies inside the freeze-out volume* divided by the freeze-out volume itself. Thus, provided that $V_{ij}\leqslant V_F$, then for all values of r_i

$$\Phi_{ij}(r_i)\leqslant 1. \tag{6}$$

This observation leads us to generalize Eq. (5) by writing

$$\Phi_{ij}(r_i)=\frac{V_{ij}^{(o)}(r_i)}{V_F}, \tag{7}$$

where $V_{ij}^{(o)}(r_i)$ is the part of the volume V_{ij} which overlaps the freeze-out volume. This definition is appropriate because even in cases where $V_{ij}>V_F$, the overlap requirement entails $V_{ij}^{(o)}=V_F$ and thus guarantees an upper bound of unity for $\Phi_{ij}(r_i)$ and also for the average value of this ratio taken over the freeze-out volume. Both of these quantities can therefore be interpreted as probabilities in the commonly accepted sense.

Let us use the symbol R_{ij} to denote the ''interaction radius'' R_i+R_j. We can state that Eq. (5) is valid as long as

$$r_i+R_{ij}\leqslant R_F. \tag{8}$$

For larger radii r_i (Fig. 1) the common volume is given by

$$V_{ij}^{(o)}(r_i)=\frac{2\pi}{3}(R_{ij}{}^3+R_F{}^3)+\frac{\pi r_i{}^3}{12}-\frac{\pi r_i}{2}(R_{ij}{}^2+R_F{}^2)-\frac{\pi}{4r_i}[(R_{ij}{}^2-R_F{}^2)^2]. \tag{9}$$

The average number of overlaps of fragment i at the radial position r_i is simply

$$\langle n_i(r_i)\rangle=\Sigma_{j\neq i}^{N}\frac{V_{ij}^{(o)}(r_i)}{V_F}, \tag{10}$$

and the probability that fragment i does not overlap with any other fragment is

$$p_i(r_i)=\Pi_{j\neq i}^{N}[1-\Phi_{ij}(r_i)]. \tag{11}$$

We now calculate the spatially averaged value of the overlap volume for the pair of fragments i and j which we denote as $\langle V_{ij}^{(o)}\rangle$. Here, it is useful to be explicit about the limiting value of r_i such that the sphere with radius R_{ij} is entirely enclosed within the freeze-out volume or, *alternately*, that the freeze-out volume is entirely enclosed in the sphere with radius R_{ij}. Thus we define $R_{\rm LIM}=|R_F-R_{ij}|$ and V_l as $\min(V_{ij},V_F)$. Then the required average value is expressed as

$$\langle V_{ij}^{(o)}(R_F)\rangle=\frac{V_l\int_0^{R_{\rm LIM}}r_i{}^2dr_i+\int_{R_{\rm LIM}}^{R_F}V_{ij}^{(o)}r_i{}^2dr_i}{\int_0^{R_F}r_i{}^2dr_i}. \tag{12}$$

We should state that this quantity is unchanged by interchanging the indices i and j. We can now obtain the global survival probability χ which is the probability that each and every fragment is not overlapped by one or more fragments. The construction of this quantity is not unique. We have adopted a method which is consistent with our two-body approximation, i.e., we take the product of all terms each of which corresponds to the *spatially averaged* nonoverlap probability of a distinct pair of fragments. Thus

$$\chi(R_F)=\Pi_{i=1}^{N-1}\Pi_{j>1}^{N}(1-\langle\Phi_{ij}\rangle). \tag{13}$$

Equation (13) can also be expressed by writing the logarithm of $\chi(R_F)$ as a sum over distinct pairs:

$$\ln[\chi(R_F)]=\Sigma_{i=1}^{N-1}\Sigma_{j>i}^{N}\ln(1-\langle\Phi_{ij}\rangle). \tag{14}$$

This last expression is interesting because, in the limit in which, for all pairs i,j, $\langle\Phi_{ij}\rangle\ll1$, we obtain using Eqs. (7) and (14)

$$\chi(R_F)=e\frac{-\Sigma\langle V_{ij}^{(o)}\rangle}{V_F}\sim1-\frac{\Sigma\langle V_{ij}^{(o)}\rangle}{V_F}, \tag{15}$$

in which the summation sign is to be understood as a shorthand expression of the double summation in Eq. (14). Equation (15), which is essentially the virial approximation used in Ref. [7], states that for large enough freeze-out volumes a first approximation to the survival probability can be found by simply dividing the sum of the average overlap volumes of the interaction spheres (with V_F) by the freeze-out volume itself.

C. Corrections for the presence of other fragments

In order to give a precise account of the corrections for the presence of other fragments we need a slightly more explicit notation. We will therefore denote the radially averaged overlap volume of the interaction sphere with radius $R_{ij}=R_i+R_j$ with a second sphere of radius R_K as $\langle V_{ij}^{(o)}(R_{ij},R_K)\rangle_R$ where the average is taken over the radial domain (for the center of the interaction sphere) from $r_i=0$ to $r_i=R$. Thus, for example, Eq. (12) would be represented as $\langle V_{ij}^{(o)}(R_{ij},R_F)\rangle_{R_F}$.

In the preceding section this volume was calculated by ignoring the influence of other fragments. We have attempted to take account of this influence by recalculating Eqs. (12) and (13) as the average overlap of V_{ij} with an *effective* volume which is simply the original freeze-out volume reduced by the sum of the average volume overlaps of all the other fragments with V_F. Explicitly, we replace $\langle V_{ij}^{(o)}(R_{ij},R_F)\rangle_{R_F}$ of Eq. (12) by $\langle V_{ij}^{(o)}(R_{ij},R_{F'})\rangle_{R_F}$ where

$$R_{F'}(i,j)=\left[\frac{3(V_F-\langle\Delta V_{ij}\rangle)}{4\pi}\right]^{1/3}. \tag{16}$$

The physical interpretation is that fragments other than the pair i,j under study are considered to form an outer spherical shell which is removed when constructing the volume overlap of the interaction sphere with the freeze-out sphere. The volume *enclosed* by this shell is considered to be empty so that the exact two-body term is valid in this region. In Eq. (16) the mean value $\langle\Delta V_{ij}\rangle$ of the excluded volume is therefore the average overlap of all other fragments (not i or j) with a volume equal to the freeze-out volume reduced by the average volume occupied by free fragments i and j $[\langle V_i^{(o)}(R_i,R_F)\rangle_{R_F}+\langle V_j^{(o)}(R_j,R_F)\rangle_{R_F}]$, i.e.,

$$\langle\Delta V_{ij}\rangle=\Sigma_{k\neq i,j}^{N}\langle V_{k,f}^{(o)}(R_k,R_f)\rangle_{R_F}, \tag{17}$$

where $\langle V_{k,f}^{(o)}(R_k,R_f)\rangle_{R_F}$ is the spatially averaged overlap volume of fragment k, with a sphere of radius

$$R_f(i,j)=\left[\frac{V_F-\langle V_i^{(o)}(R_i,R_F)\rangle_{R_F}-\langle V_j^{(o)}(R_j,R_F)\rangle_{R_F}}{4\pi/3}\right]^{1/3}. \tag{18}$$

We can now reconstruct the corresponding overlap probability $\langle \Phi_{ij} \rangle$ [see Eq. (13)] as

$$\langle \Phi_{ij} \rangle = \frac{\langle V_{ij}^{(o)}(R_{ij}, R_{F'}) \rangle_{R_F}}{V_F - \langle \Delta V_{ij} \rangle}. \tag{19}$$

We wish to draw attention to the fact that, according to our physical picture, the spatial average in (19) is still constructed over the radial domain from 0 to R_F. We further emphasize that the application of Eqs. (16)–(19) constitutes an *ad hoc* one-body correction. Other possible schemes may be considered and indeed we have studied a few alternatives. However, we have found this simply *ansatz* to be rather successful.

D. Comparison with Monte Carlo calculations

We begin by considering the survival probability for a particular fragment placed at a specified radial position, i.e., the quantities $p_i(r_i)$ of Eq. (11) calculated using the corrected theory discussed above. Figure 2 shows the calculated values for mass 8 fragments compared with Monte Carlo calculations (10^5 trials at each r_i value). We have ''embedded'' this fragment in three rather different partitions and have chosen three values of R_F which produce rather different spatial variations. In all cases the survival probability diminishes with decreasing r_i. Furthermore, in all three cases the theory is successful near the surface of the freeze-out volume and underpredicts the survival probability near the center (we should mention that the bare two-body approximation presented in Sec. II B is much more successful in reproducing the one-body profiles but overestimates the global survival probabilities).

The discrepancy cited above does not, however, strongly influence the calculated values of χ. The reason for this is not clear but must certainly originate in two-body (and possibly higher-order) spatial correlations. As can be seen in Fig. 3, where we show the calculations of χ as a function of R_F, the agreement between the corrected two-body theory and the corresponding Monte Carlo estimates is maintained over at least four orders of magnitude.

In Fig. 4, we show a global evaluation of our calculation by considering partitions of mass 12 in a freeze-out volume with radius $R_F = 4$ fm ($\rho_F/\rho_0 = 0.32$) and of mass 40 in a freeze-out sphere of radius $R_F = 8$ fm ($\rho_F/\rho_0 = 0.135$). Typically the prediction is accurate to within 20% for χ values down to the limit of accuracy in our Monte Carlo calculations (10^{-3}). However it will be noted that there is a slight tendency for the theory to overestimate the correction for small values of this latter quantity. This figure also emphasizes the importance of a correct evaluation of $\chi(R_F)$ since for a given nucleus and a given freeze-out volume the correction may extend over several orders of magnitude.

III. MULTIPLICITY DEPENDENCE

A. Mass centers confined in freeze-out volume

It is obvious from Fig. 4(a) that the corrections fall into groups which are quickly found to correspond to different

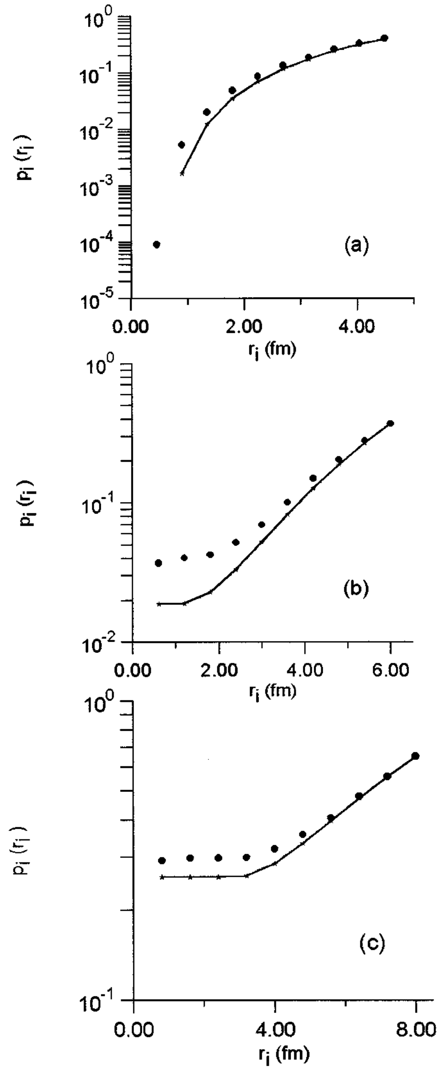

FIG. 2. Radial profiles of one-body survival probabilities for a single fragment in three typical partitions (a) one fragment of mass 8 in a partition containing three mass 8 fragments ($R_F = 4.5$ fm). (b) a fragment of mass 8 in the partition formed by masses 8,7,6,5,4,3,2,1 ($R_F = 6$ fm) (c) a fragment of mass 8 in the partition formed by 6 mass 8 fragments ($R_F = 8$ fm). All fragment radii are given by $R_k = 1.2 A_k^{1/3}$. The Monte Carlo calculations are shown as filled circles and the results of calculations made using the corrected two-body theory (Sec. II C) as continuous lines. All calculations were made with fragment mass centers enclosed in the freeze-out volume.

multiplicities. This is illustrated in Fig. 5 in which, for the two cases depicted in Fig. 4 we have plotted the theoretical correction (on a logarithmic scale) against the multiplicity (N). The variation is roughly exponential but falls off rather more slowly for small multiplicities. The exponential falloff can be qualitatively understood using the bare two-body term. We make use of the fact that for partitions within a given multiplicity class the correction exhibits relatively little variation so that we may reasonably represent the situation by partitioning the parent mass A into N equal masses. Let us denote the radius of each fragment as R_0 and use Eq. (14) to write

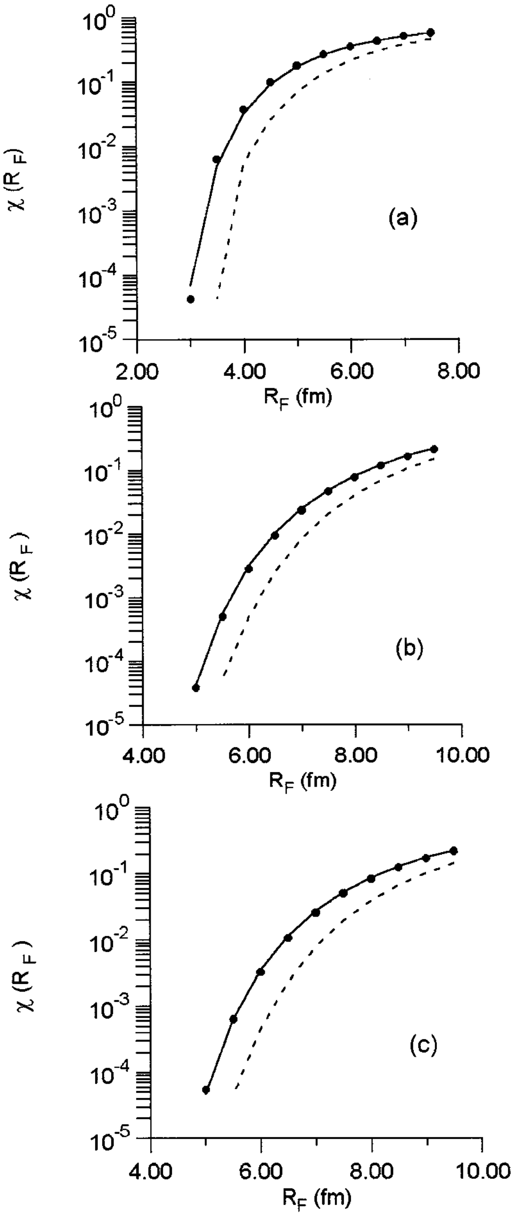

FIG. 3. Variation of calculated survival probabilities $\chi(R_F)$ (continuous lines) with freeze-out radius R_F for the three partitions considered in Fig. 2. The Monte Carlo calculations are shown, as before, as filled circles. For the largest value of R_F we used 10^5 trials. This number was linearly increased to 10^6 for the smallest value. The lower dashed lines show results obtained with Eq. (7) of Ref. [7].

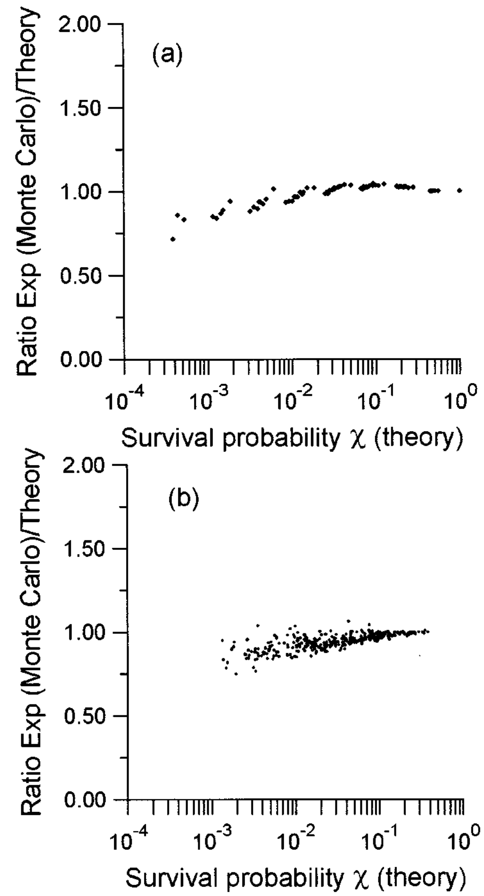

FIG. 4. Global evaluation of the proposed finite-size correction for configurations with mass centers enclosed in the freeze-out volume. The figure shows the mean of the ratio of the Monte Carlo result (exp) to the theoretical prediction (th) for: (a) partitions of mass 12 ($R_F=4$ fm) and (b) mass 40 ($R_F=8$ fm). For mass 12 all 77 partitions were included. For mass 40 the figure displays results for 1% of the total number (37 338) of partitions selected randomly. Monte Carlo calculations were run until 100 trials with no overlaps were obtained. The statistical accuracy is thus 10%. A limit of 10^6 trials was imposed on each mass 12 run so that χ values down to 10^{-4} could be estimated. This number was reduced to 10^5 for the mass 40 study.

$$\ln[\chi(R_F)]=\Sigma_{i=1}^{N-1}\Sigma_{j>i}^{N}\ln(1-\langle\Phi_{ij}\rangle)=\frac{N(N-1)}{2}\ln[1-\langle\Phi_{ij}\rangle]. \tag{20}$$

Neglecting edge effects we can approximate $\langle\phi_{ij}\rangle$ using Eq. (5), i.e.,

$$\langle\Phi_{ij}\rangle\approx\frac{(R_i+R_j)^3}{{R_F}^3}=\frac{8\rho_F}{N\rho_0}, \tag{21}$$

where ρ_F is the density at freeze-out and ρ_0 is the density of the fragments (or of the parent nucleus). If N is sufficiently large so that $\langle\Phi_{ij}\rangle<1$ we can expand the logarithm in Eq. (20) to first order to obtain

$$\ln[\chi(R_F)]\approx-\alpha(N-1), \tag{22}$$

which exhibits the required exponential behavior with $\alpha=4\rho_F/\rho_0$. It turns out that edge effects can be simply included because the average volume overlap of a sphere with radius R_s [in our case $R_s=2R_0=2r_0(A/N)^{1/3}$] and volume V_s, with a sphere of radius R_F ($R_s<R_F$) is quite closely approximated by

$$\frac{\langle V_s^{(o)}\rangle}{V_s}=1.0\text{–}0.53\frac{R_s}{R_F}. \tag{23}$$

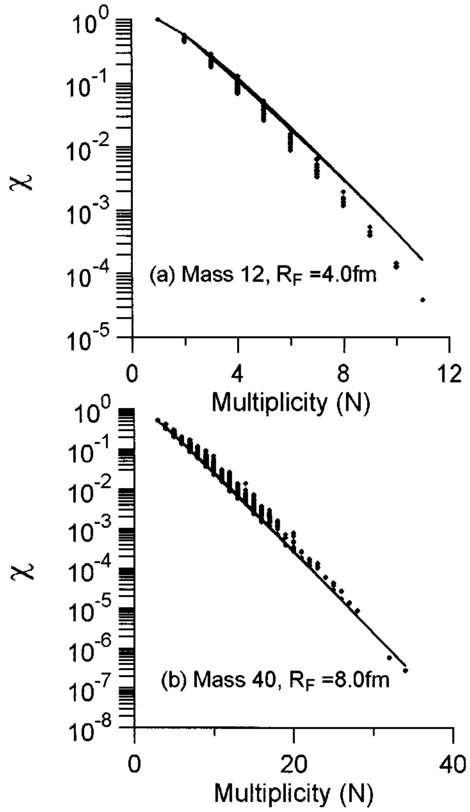

FIG. 5. Theoretical calculations (filled circles) of the survival probabilities (χ) for the systems considered in the previous figure displayed as a function of the multiplicity. The continuous lines are estimates using the simplified theory described in Sec. III A.

We may therefore simply correct for this effect by *redefining* the slope parameter α of Eq. (22) as

$$\alpha=\frac{4\rho_F}{\rho_0}\left[1.0-0.53\frac{2R_0}{R_F}\right]. \tag{24}$$

Clearly this redefined value depends weakly on the multiplicity (via R_0). In fact we have included the calculation of Eq. (24) in Fig. 5. Despite the approximations the observed slope is reasonably accounted for especially for the smaller freeze-out density.

B. Fragments completely contained within freeze-out volume

We have also carried out simulations in which fragments were *completely contained* in the freeze-out volume. For equal mass fragments one quickly sees that this strategy is exactly equivalent to that of the preceding subsection provided we modify the freeze-out radius by the common radius of the fragments. Thus the expected slope of the exponential falloff for large multiplicities would, in this case be

$$\alpha'=\frac{4\rho_{F'}}{\rho_0}\left[1.0-0.53\frac{2R_0}{R_{F'}}\right], \tag{25}$$

where $R_{F'}=R_F-R_0$ and $\rho_{F'}=4\pi R_{F'^3}/3A$ is the corresponding density. One may note that Eq. (25) exhibits a modified multiplicity dependence because both $R_{F'}$ and $\rho_{F'}$ depend explicitly on the multiplicity. To investigate this point we have carried out a number of Monte Carlo calculations made by enclosing fragments entirely within the freeze-out volume. In Fig. 6 we show plots of ln(χ) versus the multiplicity N for a parent mass of 40 and various values of the freeze-out volume. One observes a rather uniform exponential falloff except for very small freeze-out volumes. We have also extracted the slope parameters for various masses as a function of the ratio of the freeze-out volume to the volume of the parent (V_F/V_p), which are compared with the simple two-body prediction of Eq. (25) in Fig. 7. The figure demonstrates firstly that, as expected, the slope parameters depend essentially on V_F/V_p and also that the two-body prediction is rather accurate for values of $R_F/R_p>2$.

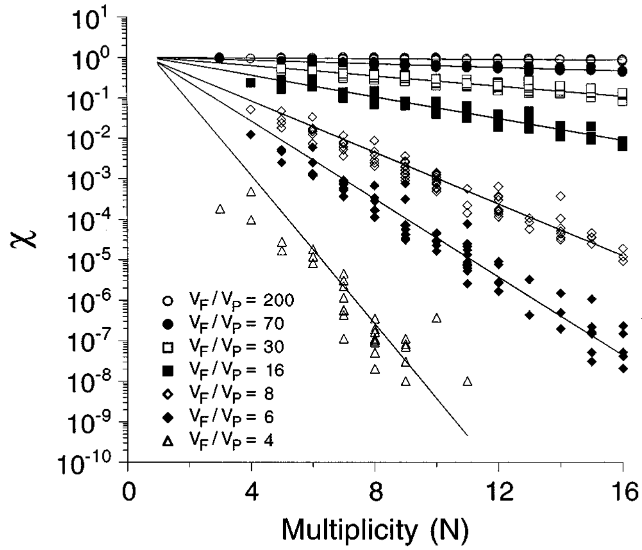

FIG. 6. Plots of χ versus the multiplicity obtained from Monte Carlo simulations in which the fragments were entirely enclosed in the freeze-out volume. All plots are for mass 40. The various curves are labeled by the ratio $V_F/V_p=\rho_0/\rho_F$ (see Sec. III B).

IV. SUMMARY AND DISCUSSION

The objective of this work was to construct an improved finite size correction to the spatial permutation factor which appears in the perfect gas density of states and which is used in statistical models of nuclear multifragmentation. The work was mainly focused on the situation in which the fragment mass centers are constrained to lie within the freeze-out volume. Our proposed correction is semianalytic as it involves more or less intuitive choices in the detailed construction of the survival probabilities and in the estimation of the excluded volume. Nevertheless, it is based on the rigorous evaluation of the two-body overlap probability and has been shown to successfully reproduce results of Monte Carlo calculations for survival probabilities down to at least 10^{-4}.

The calculations reveal that the survival probability falls off approximately as an exponential in the partition multiplicity. We were able to understand this behavior using the two-body approximation. An obvious consequence is that one should be very careful to interpret correctly experimental measurements of the freeze-out volume especially for small values of this quantity.

At present there is no universally accepted manner for

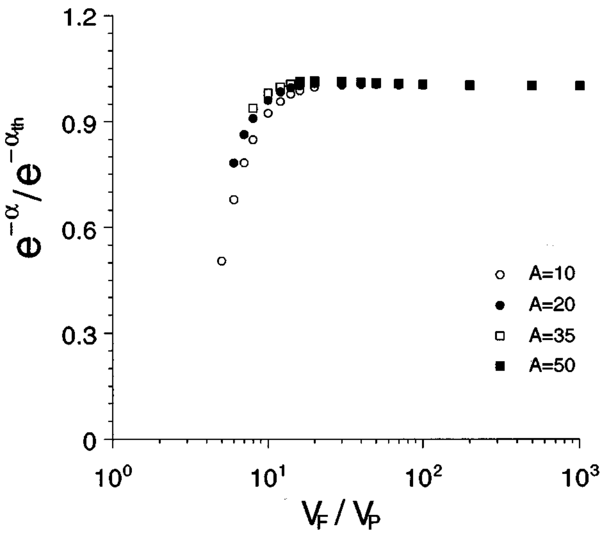

FIG. 7. Values of $e^{-\alpha}$ (obtained from Monte Carlo simulations in which the fragments were entirely enclosed in the freeze-out volume) compared with the two-body theory [Eq. (25)]. The parent masses (A) are indicated in the figure. The theoretical values were obtained as average values over multiplicities $\geqslant 2$. The figure demonstrates the accuracy of the simple two-body theory for $V_F/V_p > 8$.

defining the freeze-out volume. This is one of the typical difficulties of small systems. As mentioned in the introduction it would be possible to insist that all fragments are entirely contained within this spherical region (briefly explored in Sec. III B). This definition however would involve a modification of the basic V_F^N factor and thus a significant departure from the perfect gas formula. This is the main reason why we have concentrated our analysis on the technique of randomly distributing the centers of mass of the spherical fragments inside the freeze-out volume.

We have not attempted to determine additional suppression factors due to overlaps encountered in the dynamic evolution of the initial spatial configuration. This is, in principle, a complicated problem. In the limit where the motion of fragments is dominated by a radial collective expansion one would expect that initially nonoverlapping fragments will tend to increase their relative distances as for pure Coulomb expansion. Dynamically induced spatial overlaps would then not occur. In the opposing limit where random thermal motion dominates one would certainly expect interfragment collisions to play a non-negligible role. It is perhaps appropriate to mention here that a detailed discussion of reliable methods of assigning fixed values of collective dynamic variables to multifragmentation configurations (including modifications to the density of phase-space states) has been given by Randrup [8].

We would like to emphasize that our calculation assumes that all partitions are contained in the same *constant* freeze-out volume. This is an assumption which seems justified by recent experimental work [9]. It would also be possible to consider models in which the freeze-out volume depends explicitly on the partition. In this case a further modification of the phase-space density would be required. Finally we should perhaps mention that an adaptation of the present work should be applicable to the problem of a real gas, specifically to the calculation of the energy distribution of a confined set of interacting particles.

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