Thomas-Fermi theory of the breathing mode and nuclear incompressibility

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A Thomas-Fermi theory with a linear scaling assumption is proposed for the breathing mode of nuclear collective motion. It leads to a general result $K_A = \langle K(\rho,\delta)\rangle + K_{\rm GD} - 2E_C/A$ which states that the incompressibility K_A of a finite nucleus A mainly equals the nuclear matter incompressibility $K(\rho,\delta)$ averaged over the nucleon density distribution $\rho(\mathbf{r})$ of nucleus A, added to a term $K_{\rm GD}$ contributed from the gradients of nucleon densities, with twice the Coulomb energy per nucleon E_C/A subtracted. The nuclear matter equation of state given by the Thomas-Fermi statistical model with a Seyler-Blanchard-type interaction is employed to calculate the nuclear matter incompressibility $K(\rho,\delta)$ and a localized approximation of the Seyler-Blanchard-type interaction, which is shown to be similar to the Skyrme-type interaction, is developed to calculate the value of $K_{\rm GD}$. $K_{\rm GD}$ and $-2E_C/A$ contribute about 20–10 % and 1–5 %, respectively, to the nuclear incompressibility K_A, from the light to the heavy nuclei. The shell and the even-odd effects are discussed by a scaling model which shows that these effects can be neglected for medium and heavy nuclei. The anharmonic effect is shown to be significant only for light nuclei. The leptodermous expansion of K_A is obtained and the contribution from the curvature term proportional to $A^{-2/3}$ is discussed. The calculated isoscalar giant monopole resonance energy E_M for a variety of nuclei are shown to be in agreement with experimental measurements. [S0556-2813(97)01406-4]

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I. INTRODUCTION

The isoscalar giant monopole resonance of nuclei, well known as the breathing mode of nuclear collective motion, has been studied extensively in the context of various models, since it is expected to be able to provide very important information for the incompressibility K of nuclear matter, especially for the incompressibility K_0 of standard nuclear matter. By nuclear matter we mean the uncharged nucleon system distributed uniformly in the space, and by standard nuclear matter we mean the ground-state nuclear matter with equal neutron and proton numbers [1]. Theoretically, there are two separate problems. The first problem is how to relate the measured isoscalar giant monopole resonance energy E_M to the incompressibility K_A of a finite nucleus. This is a dynamic problem, and its solution requires a model on the breathing mode of nuclear collective motion. It is believed that the model based on the scaling assumption is an appropriate candidate [2]. The second problem is how to relate the incompressibility K_A of a finite nucleus to the incompressibility K of nuclear matter. This is a static problem in the adiabatic approximation, its solution depends on the model of nuclei and the model of the breathing mode, and the fundamental argument here is that the matter in a finite nucleus is different from the nuclear matter due to the electric charge and the finite size of nuclei. Therefore, the information about the incompressibility K of nuclear matter extracted from the isoscalar giant monopole resonance of nuclei is model dependent and even interaction dependent, since both the model of nuclei and the model of the breathing mode depend on the choice of nuclear interaction in general.

Serious attempts in order to minimize the model and the interaction dependence of the K extracted from the measured E_M have been made. One of these attempts is to consider the nuclear matter as a limit of the matter in finite nuclei. In this case, one is justified in expanding the incompressibility K_A of finite nuclei in a series of power of $A^{-1/3}$ [2], similar to the leptodermous expansion used in the droplet model of nuclei [3], i.e.,

$$K_A = K_0 - k_S A^{-1/3} + k_{cS} A^{-2/3} + [k_{sB} + k_{sS} A^{-1/3}]\left(\frac{N-Z}{A}\right)^2 - \kappa_C \frac{E_C}{A}, \tag{1}$$

where K_0, k_S, k_{cS}, k_{sB}, k_{sS}, and κ_C are adjustable parameters fitted to measured E_M, $A=N+Z$, N and Z are the neutron number and the proton number, respectively, and E_C is the Coulomb energy. The parameter K_0 in the expansion (1) can be identified as the incompressibility of standard nuclear matter in the scaling model [4,5], its value $K_0 = 220 \pm 20$ MeV obtained in this way [2] was accepted for a long time.

However, a reanalysis of the experimental data of E_M by a least-squares fit of the above semiempirical expansion gives K_0 ranging from 200 to 350 MeV [6,7]. It is shown that these adjustable parameters in the expansion (1) are not completely free and the physics involved in these parameters is important in this analysis [8]. This is emphasized also by the fact that the value of K_0 obtained from different nuclear measurements and astrophysical observations are spread over a large range from 180 to 800 MeV [9]. Therefore, theoret-

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ical attempts in order to clarify the model and the interaction dependence of the relationship between K_A and K are required [10].

In this regard, a general relationship between K_A and K_0 has been proposed recently [11]:

$$K_A=\frac{E_N}{-a_1A}K_0-\left(7-30\frac{a_1}{K_0}\right)\frac{E_C}{A}, \tag{2}$$

where a_1 is the binding energy coefficient and E_N the nuclear energy of the nucleus. This result depends only on the nuclear equation of state in the sense that only the general functional form of the equation of state given by the nuclear Thomas-Fermi (TF) model with a generalized Seyler-Blanchard interaction [12] has been used. In addition, it is interesting to note that Eq. (2) is also a nonperturbative formula, as no expansion in power of $A^{-1/3}$ has been used.

Therefore, a general relationship between K_A and K (not only K_0) is to be worked out, which is expected to depend mainly on the nuclear equation of state, within the TF model of nuclei and the scaling model of the breathing mode. In addition, an approximate expansion similar to Eq. (1) is expected to be derived from this general relation. This expectation is supported also by the numerical result based on microscopically constrained Hartree-Fock and random-phase approximation (RPA) sum rule calculations. As a matter of fact, independent of the chosen interactions, $K_A \approx 0.64K_0-3.5$ for ^{208}Pb and a similar relationship for ^{116}Sn are obtained [8]. The reason why we use the TF model of nuclei is that the incompressibility is essentially a macroscopic concept and the TF model is appropriate to describe the macroscopic properties of nuclei.

The purpose of the present paper is to derive this general relationship between the incompressibility K_A of finite nuclei and the incompressibility K of nuclear matter, by using the nuclear TF model and the linear scaling assumption, and to show that the expansion (1) holds approximately, independent of the choice of interaction. Section II describes the main model assumption and the general TF model treatment, where the TF nuclear energy is separated into two parts, one of which depends only on the local nucleon densities and the other on the gradients of densities. The result given in this section is interaction independent, since only the general formulation of the TF model and the linear scaling assumption are used. The specific interaction and model of nuclei are needed only when practical calculation is performed. In this context, the localized approximation of the generalized Seyler-Blanchard interaction is given in Sec. III. In applying the above theory to calculate the isoscalar giant monopole resonance energy E_M of nuclei, the constrained TF model calculation is given in Sec. IV. Section V discusses the leptodermous expansion of the incompressibility K_A of finite nuclei which gives a theoretical justification for the expansion (1). In Sec. VI a short discussion and summary are presented. The theoretical foundation of the scaling assumption based on a quantum mechanical model is discussed in Appendix A, the specific expressions of the quantities appeared in the numerical calculation of Secs. III and IV are collected in Appendix B, while some formulas for integrals involving Fermi function which appeared in the leptodermous expansion are given in Appendix C.

II. MODEL AND THEORY

The fact that there is only one isoscalar giant monopole resonance energy E_M for each nucleus experimentally provides a strong evidence that this resonance is a kind of simple collective harmonic oscillation with only one degree of freedom. This collective motion of nuclei corresponds to the breathing mode of radial oscillation. The most intuitive choice of the variable for the breathing mode is the radius of the nuclei [8,11]. But there are at least three characteristic radii for a spherically symmetric leptodermous distribution of nucleons: the central radius C, the equivalent sharp radius R, and the equivalent root mean square radius Q [13]. In order to avoid an obvious dependence of the theory on the specific choice of the radius, the scaling factor η in the radial linear scaling transformation

$$\mathbf{r}\rightarrow\mathbf{r}_s=\eta\mathbf{r} \tag{3}$$

is chosen as the dimensionless collective coordinate for the breathing mode in the present work.

In a TF model of nuclei, one of the basic quantities is the nucleon density $\rho(\mathbf{r})$ which can be normalized to the number of nucleons A. Under the scaling transformation (3), the scaled density $\rho_s(\mathbf{r})$ is given by

$$\rho(\mathbf{r})\rightarrow\rho_s(\mathbf{r})=\eta^3\rho(\eta\mathbf{r}). \tag{4}$$

Similar relations hold for the neutron density $\rho_n(\mathbf{r})$ and the proton density $\rho_p(\mathbf{r})$.

The kinetic energy of the breathing mode oscillation can be written as

$$E_k=\frac{1}{2}m_N\int d^3r\rho_s(\mathbf{r})[\mathbf{v}(\mathbf{r})]^2=\frac{1}{2}m_NA\langle r^2\rangle_s\left(\frac{\dot{\eta}}{\eta}\right)^2, \tag{5}$$

where m_N is the nucleon mass,

$$\langle r^2\rangle_s=\frac{1}{A}\int d^3r\rho_s(\mathbf{r})r^2=\frac{1}{\eta^2}\frac{1}{A}\int d^3r\rho(\mathbf{r})r^2=\frac{1}{\eta^2}\langle r^2\rangle, \tag{6}$$

and $\mathbf{v}(\mathbf{r})=-\mathbf{r}\dot{\eta}/\eta$ is the velocity field of the breathing mode defined by the equation of continuity $\partial\rho_s/\partial t+\nabla\cdot(\rho_s\mathbf{v})=0$. By introducing [14] $s=(\eta-1)/\eta$, the kinetic energy can be rewritten as

$$E_k=\frac{1}{2}m_NA\langle r^2\rangle(\dot{s})^2. \tag{7}$$

The ground-state energy E_A of nucleus (A,Z) can be written in the TF model as

$$E_A=E_N+E_C+E_{\rm res}, \tag{8}$$

where E_N is the nuclear energy, E_C the Coulomb energy, and $E_{\rm res}$ the residual energy which includes mainly the shell correction and the even-odd energy [15]. Furthermore, the nuclear energy E_N can be separated generally into a term $E_{\rm LD}$ depending on the local densities and a term $E_{\rm GD}$ depending on the gradients of densities,

$$E_N=E_{\rm LD}+E_{\rm GD}. \tag{9}$$

Specifically, the local density dependent term E_{LD} can be written generally as

$$E_{LD}=\int d^3r\rho(\mathbf{r})e(\rho,\delta), \tag{10}$$

where $e(\rho,\delta)$, the energy per nucleon, is the nuclear matter equation of state, and the relative neutron excess δ of nuclear matter is

$$\delta=\frac{\rho_n-\rho_p}{\rho}. \tag{11}$$

The Coulomb energy E_C can be divided into Coulomb direct energy E_{Coul} and Coulomb exchange energy E_{ex} [16], i.e., $E_C=E_{Coul}+E_{ex}$, where

$$E_{Coul}=\frac{1}{2}\int\frac{d^3rd^3r' e^2\rho_p(\mathbf{r})\rho_p(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},$$

$$E_{ex}=-e^2\frac{3}{4}\left(\frac{3}{\pi}\right)^{1/3}\int d^3r\rho_p^{4/3}(\mathbf{r}). \tag{12}$$

The potential energy of the breathing mode can be assumed, except by a constant, to be the scaling transformed energy $E_A(\eta)$ of nuclei, in an adiabatic approximation,

$$E_A(\eta)=E_N(\eta)+E_C(\eta)+E_{res}(\eta), \tag{13}$$

where $E_N(\eta)=E_{LD}(\eta)+E_{GD}(\eta)$,

$$E_{LD}(\eta)=\int d^3r\rho_s(\mathbf{r})e(\rho_s,\delta_s)=\int d^3r\rho(\mathbf{r})e(\eta^3\rho,\delta), \tag{14}$$

$$E_C(\eta)=\frac{1}{2}\int\frac{d^3rd^3r' e^2\rho_p^s(\mathbf{r})\rho_p^s(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} -e^2\frac{3}{4}\left(\frac{3}{\pi}\right)^{1/3}\int d^3r\rho_p^{s4/3}(\mathbf{r})=\eta E_C, \tag{15}$$

and

$$E_{res}(\eta)=E_{shell}(\eta)+E_{even\text{-}odd}(\eta), \tag{16}$$

where $E_{shell}(\eta)$ and $E_{even\text{-}odd}(\eta)$ are the scaled shell correction and even-odd energy, respectively.

For a small vibration of the breathing mode, the scaling variable η oscillates around its stable value $\eta=1$ with a small amplitude $|s|\ll 1$, $E_A(\eta)$ can be written approximately as

$$E_A(\eta)=E_A+\frac{1}{2}AK_As^2+\frac{1}{6}AK_3s^3+\frac{1}{24}AK_4s^4, \tag{17}$$

where

$$K_A=\frac{1}{A}\left.\frac{d^2E_A(\eta)}{d\eta^2}\right|_{\eta=1} \tag{18}$$

is the incompressibility of the finite nucleus A [4], while

$$K_3=\frac{1}{A}\left[\frac{d^3E_A(\eta)}{d\eta^3}+6\frac{d^2E_A(\eta)}{d\eta^2}\right]_{\eta=1},$$

$$K_4=\frac{1}{A}\left[\frac{d^4E_A(\eta)}{d\eta^4}+12\frac{d^3E_A(\eta)}{d\eta^3}+36\frac{d^2E_A(\eta)}{d\eta^2}\right]_{\eta=1}. \tag{19}$$

In writing Eq. (17), the stability condition of the breathing mode vibration $[dE_A(\eta)/d\eta]_{\eta=1}=0$ has been applied. From Eqs. (14) and (15), this condition can be written explicitly as

$$\int d^3r\rho(\mathbf{r})3\rho(\mathbf{r})\frac{\partial e}{\partial\rho}+\left.\frac{dE_{GD}(\eta)}{d\eta}\right|_{\eta=1}+E_C+\left.\frac{dE_{res}(\eta)}{d\eta}\right|_{\eta=1} =0. \tag{20}$$

In an harmonic approximation, Eq. (17) is reduced to

$$E_A(\eta)\approx E_A+\frac{1}{2}AK_As^2, \tag{21}$$

the isoscalar giant monopole resonance energy E_M is given as the harmonic oscillator energy $E_M\approx\hbar\omega$, and from Eqs. (7) and (21),

$$\hbar\omega=\sqrt{\frac{\hbar^2K_A}{m_N\langle r^2\rangle}}. \tag{22}$$

From Eq. (13), the incompressibility K_A can be divided into three parts K_N, K_C, and K_{res},

$$K_A=K_N+K_C+K_{res}, \tag{23}$$

where

$$K_N=\frac{1}{A}\left.\frac{d^2E_N(\eta)}{d\eta^2}\right|_{\eta=1},\quad K_C=\frac{1}{A}\left.\frac{d^2E_C(\eta)}{d\eta^2}\right|_{\eta=1},$$

$$K_{res}=\frac{1}{A}\frac{d^2E_{res}(\eta)}{d\eta^2}_{\eta=1}. \tag{24}$$

The main contribution to the incompressibility K_A comes from the nuclear energy E_N. From Eq. (14), the following expression can be derived,

$$K_N=\frac{1}{A}\int d^3r\rho(\mathbf{r})\left[K(\rho,\delta)+6\rho(\mathbf{r})\frac{\partial e}{\partial\rho}\right]+\frac{1}{A}\left.\frac{d^2E_{GD}(\eta)}{d\eta^2}\right|_{\eta=1}, \tag{25}$$

where

$$K(\rho,\delta)=9\rho^2\frac{\partial^2e}{\partial\rho^2} \tag{26}$$

is the incompressibility of nuclear matter. Applying the stability condition (20), the following expression of K_N results:

$$K_N=\langle K(\rho,\delta)\rangle+K_{GD}-2\frac{E_C}{A}-2\left.\frac{dE_{res}(\eta)}{A\ d\eta}\right|_{\eta=1}, \tag{27}$$

where

$$\langle K(\rho,\delta)\rangle = \frac{1}{A}\int d^3r\rho(\mathbf{r})K(\rho,\delta) \tag{28}$$

is the nuclear matter incompressibility averaged over the nucleon distribution $\rho(\mathbf{r})$ of nuclei, and

$$K_{\rm GD} = \frac{1}{A}\left[\frac{d^2E_{\rm GD}(\eta)}{d\eta^2} - 2\frac{dE_{\rm GD}(\eta)}{d\eta}\right]_{\eta=1} \tag{29}$$

is the contribution from the gradients of densities.

Due to the scaling property (15) of the Coulomb energy, there is no contribution from the Coulomb energy directly to the incompressibility of finite nuclei, i.e., $K_C=0$, and due to the scaling assumption (16) of the residual energy, the contribution from the residual energy can be divided into two distinct parts, i.e., $K_{\rm res}=K_{\rm shell}+K_{\rm even\text{-}odd}$, where

$$K_{\rm shell} = \frac{1}{A}\frac{d^2E_{\rm shell}(\eta)}{d\eta^2}\bigg|_{\eta=1},$$

$$K_{\rm even\text{-}odd} = \frac{1}{A}\frac{d^2E_{\rm even\text{-}odd}(\eta)}{d\eta^2}\bigg|_{\eta=1}. \tag{30}$$

In a quantum mechanical model discussed in Appendix A, the following scaling properties are obtained:

$$E_{\rm shell}(\eta) = \eta^2 E_{\rm shell}, \quad E_{\rm even\text{-}odd}(\eta) = \eta^2 E_{\rm even\text{-}odd}, \tag{31}$$

so we have

$$E_{\rm res}(\eta) = \eta^2 E_{\rm res}, \quad K_{\rm res} = 2\frac{E_{\rm res}}{A}. \tag{32}$$

Substituting Eqs. (27), (32), and $K_C=0$ into Eq. (23), the following expression for K_A results:

$$K_A = \langle K(\rho,\delta)\rangle + K_{\rm GD} - 2\frac{E_C+E_{\rm res}}{A}. \tag{33}$$

Numerically, the residual energy term $-2E_{\rm res}/A$ can be neglected within an error of less than 1%, so we have finally

$$K_A = \langle K(\rho,\delta)\rangle + K_{\rm GD} - 2\frac{E_C}{A}. \tag{34}$$

The formula (33) or (34), similar to the Myers-Swiatecki's formula (2), is our main result. This formula is general, as only the general expression of the TF model energy together with the linear scaling assumption are employed. In the practical application of this formula, the specific nuclear matter equation of state $e(\rho,\delta)$, the nucleon distribution $\rho(\mathbf{r})$ of nuclei, the relative neutron excess δ of nuclear matter in the nuclei, the energy $E_{\rm GD}$ depending on the gradients of nucleon densities, and the specific expression of Coulomb energy E_C are needed, and all of them are model and interaction dependent.

In order to calculate the anharmonic effect given by the terms beyond the second order in Eq. (17), the following perturbation result is obtained, by neglecting higher order terms in the second order correction:

$$E_M = \hbar\omega + \frac{1}{8}\frac{K_4}{A\langle r^2\rangle^2}\frac{\hbar^4}{m_N^2}\frac{1}{(\hbar\omega)^2} - \frac{5}{24}\frac{K_3^2}{A\langle r^2\rangle^3}\frac{\hbar^6}{m_N^3}\frac{1}{(\hbar\omega)^4}. \tag{35}$$

It can be seen easily that, as the mean square of radius $\langle r^2\rangle$ is proportional to $A^{2/3}$ while K_3 and K_4 as well as $\hbar\omega$ have no serious changes with the nucleon number A, the anharmonic effect is significant only for light nuclei.

III. LOCALIZED APPROXIMATION OF SEYLER-BLANCHARD-TYPE INTERACTION

In the TF or the extended TF theory, the nuclear energy can be written generally as

$$E_N = \int d^3r h[\rho_n(\mathbf{r}),\rho_p(\mathbf{r})], \tag{36}$$

where the Hamiltonian density $h[\rho_n(\mathbf{r}),\rho_p(\mathbf{r})]$ is a functional of nucleon densities $\rho_n(\mathbf{r})$ and $\rho_p(\mathbf{r})$. For the extended TF theory or the TF theory with Skyrme-type interaction [17], $h[\rho_n(\mathbf{r}),\rho_p(\mathbf{r})]$ depends also on the gradients of $\rho_n(\mathbf{r})$ and $\rho_p(\mathbf{r})$, while for the TF theory with the Seyler-Blanchard-type interaction $h[\rho_n(\mathbf{r}),\rho_p(\mathbf{r})]$ is expressed as a space integral as

$$h[\rho_n(\mathbf{r}),\rho_p(\mathbf{r})] = \int d^3r' Y(\mathbf{r},\mathbf{r}')F(\mathbf{r},\mathbf{r}'), \tag{37}$$

where $F(\mathbf{r},\mathbf{r}')$ is the functional of $\rho_n(\mathbf{r})$ and $\rho_p(\mathbf{r})$ given at points $\mathbf{r}$ and $\mathbf{r}'$, and $Y(\mathbf{r},\mathbf{r}')$ is the Yukawa potential with the force range a:

$$Y(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi a^3}\frac{e^{-|\mathbf{r}-\mathbf{r}'|/a}}{|\mathbf{r}-\mathbf{r}'|/a}. \tag{38}$$

For finite nuclei, the nucleon distribution spreads over a finite region whose radius is much larger than a and the integral of Eq. (37) can be calculated approximately, for the spherically symmetric distribution of nucleons, by using the Taylor expansion of $F(r,r')$ at the point $r'=r$. It gives

$$\int d^3r' Y(\mathbf{r},\mathbf{r}')F(r,r') = \sum_{n=0}^{\infty}\frac{a^n}{n!}I_n(r/a)F^{(n)}(r), \tag{39}$$

where

$$F^{(n)}(r) = \frac{d^nF(r,r')}{dr'^n}\bigg|_{r'=r}, \tag{40}$$

$$I_n(x) = \frac{1}{x}[P_n(x) - Q_n(x)e^{-x}], \tag{41}$$

$$P_n(x) = \sum_{k=0}^{n}\binom{n}{k}(-x)^k\tilde{P}_m(x),$$

$$Q_n(x) = \sum_{k=0}^{n}\binom{n}{k}(-x)^k m!\frac{1}{2}[1+(-1)^m], \tag{42}$$

$$\widetilde{P}_m(x)=x^m+m(m-1)x^{m-2}+\cdots+m!x^{(1-\cos m\pi)/2},$$
$$m=n-k+1. \quad (43)$$

As examples which will be used in the present calculation, we have

$$I_0(x)=1, \quad I_1(x)=\frac{2}{x}(1-e^{-x}), \quad I_2(x)=2(1+2e^{-x}). \quad (44)$$

Substituting Eqs. (37) and (39) into Eq. (36), the nuclear energy E_N can be expressed as an expansion where the zero order term corresponds to the local density-dependent energy,

$$E_{\rm LD}=\int d^3rF(r,r)=\int d^3r\rho(r)e(\rho,\delta), \quad (45)$$

while the higher order terms contribute to the energy depending on the gradients of densities,

$$E_{\rm GD}=\sum_{n=1}^{\infty} E_{{\rm GD}n}=E_{\rm GD1}+E_{\rm GD2}+\cdots, \quad (46)$$

where

$$E_{{\rm GD}n}=\frac{a^n}{n!}\int d^3rI_n(r/a)F^{(n)}(r), \quad n=1,2,\cdots. \quad (47)$$

For finite nuclei, since the nucleon distribution presents a well-established uniform central region and a falloff surface region and the $F^{(n)}(r)$ is a function of r with sharp peaks located around the central radius C, the above expansion converges as a series of power of a/C. In this case, keeping the first few terms is enough for most purposes. We call this the localized approximation of the Seyler-Blanchard-type interaction. In this sense, the local density approximation is the special case of the localized approximation when keeping only the leading term $E_{\rm LD}$ in the expansion. In the present work we keep the terms up to $n=2$.

The specific form of the functional $F(r,r')$ depends on the type of interaction. As an example, we will use in the present work the generalized Seyler-Blanchard interaction which is introduced by Myers and Swiatecki [12] and thus will be referred to hereafter as the Myers-Swiatecki interaction.

The Myers-Swiatecki effective nuclear interaction $V(\mathbf{r},\mathbf{r}')$ is written as

$$V(\mathbf{r},\mathbf{r}')=\frac{T_0}{\hat{\rho}_0}Y(\mathbf{r},\mathbf{r}')\left[-\alpha+\beta\left(\frac{p}{p_0}\right)^2-\gamma\frac{p_0}{p}+\sigma\left(\frac{\overline{\rho}}{\hat{\rho}_0}\right)^{2/3}\right], \quad (48)$$

where p is the relative momentum of two nucleons situated at $\mathbf{r}$ and $\mathbf{r}'$, respectively, $\overline{\rho}$ is the average density defined as $\overline{\rho^{2/3}}=[\rho^{2/3}(\mathbf{r})+\rho^{2/3}(\mathbf{r}')]/2$, and $\hat{\rho}_0=\rho_0/2$. The quantities T_0, p_0, and ρ_0 are the Fermi energy, the Fermi momentum, and the standard nuclear matter density. The dimensionless interaction strength parameters α, β, γ, σ may be different for interactions between like and unlike nucleons. The difference is described by a parameter ξ for the leading part of the interaction

$$\alpha_{l,u}=\frac{1}{2}(1\mp\xi)\ \alpha \quad (49)$$

and by ζ for the remaining parts of the interaction

$$\beta_{l,u}=\frac{1}{2}(1\mp\zeta)\ \beta, \quad \gamma_{l,u}=\frac{1}{2}(1\mp\zeta)\ \gamma,$$
$$\sigma_{l,u}=\frac{1}{2}(1\mp\zeta)\ \sigma, \quad (50)$$

where l,u refer to ''like'' and ''unlike,'' and are associated with the minus and plus signs, respectively.

The first two terms in the square bracket of Eq. (48), with $\xi=\zeta$, are the original Seyler-Blanchard attractions with momentum-dependent repulsion [18]. The third term is an additional attraction, while the fourth term is an additional repulsion.

Calculating the potential energy with the above interaction (48), together with the kinetic energy calculated by the TF model, the $F(r,r')$ can be obtained [12,15]. From this $F(r,r')$, the nuclear equation of state $e(\rho,\delta)$ can be obtained as

$$e(\rho,\delta)=T_0\left[B(\delta)\left(\frac{\rho}{\rho_0}\right)^{2/3}-C(\delta)\left(\frac{\rho}{\rho_0}\right)^{3/3}+D(\delta)\left(\frac{\rho}{\rho_0}\right)^{5/3}\right], \quad (51)$$

where

$$B(\delta)=\frac{3}{10}(1-\gamma_l)[(1+\delta)^{5/3}+(1-\delta)^{5/3}] -\frac{3}{20}\gamma_u\begin{cases}5(1+\delta)^{2/3}(1-\delta)-(1-\delta)^{5/3}, & \text{for } \delta\geqslant 0,\\ 5(1+\delta)(1-\delta)^{2/3}-(1+\delta)^{5/3}, & \text{for } \delta\leqslant 0,\end{cases} \quad (52)$$

$$C(\delta)=\frac{1}{2}\alpha(1-\xi\delta^2), \quad (53)$$

$$D(\delta)=\frac{3}{10}\{B_l[(1+\delta)^{8/3}+(1-\delta)^{8/3}] +B_u(1-\delta^2)[(1+\delta)^{2/3}+(1-\delta)^{2/3}]\}, \quad (54)$$

where

$$B_{l,u}=\frac{1}{2}(1\mp\zeta)B, \quad B=\beta+\frac{5}{6}\sigma. \quad (55)$$

The term in $\rho^{2/3}$ in the equation of state (51) is related to the Fermi gas kinetic energy with an extra contribution from the $1/p$ attraction, the term in ρ is related to the normal Yukawa attraction, and the term in $\rho^{5/3}$ is due to the momentum-dependent repulsion with an additional contribution from the $\overline{\rho^{2/3}}$ repulsion.

The parameters $\alpha,\ \beta,\ \gamma,\ \sigma,\ \xi,\ \zeta$, and a can be determined by fitting to the nuclear masses, the nuclear fission barriers as well as the nuclear optical potential, when the radius constant r_0 of standard nuclear matter and the Süssmann width b [13] are kept as two geometrical constraints, giving [15]

$$\alpha=1.94684,\quad \beta=0.15311,\quad \gamma=1.13672,$$

$$\sigma=1.05,\quad \xi=0.27976,\quad \zeta=0.55665,$$

$$a=0.59294\ \text{fm},\quad r_0=1.14\ \text{fm},\quad b=1.0\ \text{fm}. \tag{56}$$

A relation between parameters $\alpha,\ \beta,\ \gamma$, and σ can be obtained from the stability of standard nuclear matter based on the above equation of state as [12]

$$10B-5\alpha+4(1-\gamma)=0. \tag{57}$$

This means that only six of the seven parameters $\alpha,\ \beta,\ \gamma,\ \sigma,\ \xi,\ \zeta$, and a are free. Furthermore, the following relationship between the incompressibility K_0 of standard nuclear matter and the volume binding coefficient a_1 as well as the Fermi energy T_0 can be derived from the above equation of state [12]:

$$K_0=15\,a_1+\frac{9}{5}\,(1-\gamma)T_0. \tag{58}$$

As $a_1\approx 16\pm 0.5$ MeV is well established, and T_0 is about 36.4 ± 3.2 MeV when $r_0\approx 1.15\pm 0.05$ fm, it can be seen from the above relationship with $\gamma=0$ that K_0 is in the range of 306 ± 13 MeV for the original Seyler-Blanchard interaction. However, this range of K_0 is too narrow for the adjustment of parameters, so the original Seyler-Blanchard interaction is not able to give a value of K_0 as low as around 220 MeV.

The higher order term E_{GD} contributes to the nuclear energy as the correction due to the gradients of nucleon densities. For the interaction (48), the first two terms in E_{GD} can be derived as

$$F^{(1)}(r)=T_0\left[\epsilon_{1n}\frac{d\rho_n}{dr}+\epsilon_{1p}\frac{d\rho_p}{dr}\right], \tag{59}$$

$$F^{(2)}(r)=T_0\left[\epsilon_{1n}\frac{d^2\rho_n}{dr^2}+\epsilon_{1p}\frac{d^2\rho_p}{dr^2}+\frac{\epsilon_{2n}}{\hat{\rho}_0}\left(\frac{d\rho_n}{dr}\right)^2+\frac{\epsilon_{2p}}{\hat{\rho}_0}\left(\frac{d\rho_p}{dr}\right)^2\right], \tag{60}$$

where $\epsilon_{1n},\ \epsilon_{1p},\ \epsilon_{2n}$, and ϵ_{2p} are the functionals of nucleon densities $\rho_n(r)$ and $\rho_p(r)$ whose specific expressions are given in Appendix B.

E_{GD1} depends on the first order derivatives of nucleon densities, while E_{GD2} depends on the second order derivatives as well as on the square of the first order derivatives of nucleon densities. Therefore, the localized approximation of the Seyler-Blanchard-type interaction, including the local density approximation term E_{LD} together with the two correction terms E_{GD1} and E_{GD2}, is shown to be similar to the Skyrme-type interaction with the gradient-dependent terms. This is the reason why the Seyler-Blanchard-type interaction can reproduce the nuclear measurement data so well, even if it has no gradient-dependent term explicitly.

For simplifying the analytical derivation of the leptodermous expansion given in Sec. V, the simplified Myers-Swiatecki interaction will be used in what follows. In the simplified Myers-Swiatecki interaction, the Yukawa potential (38) is applied only to the leading part of the interaction (with the interaction parameter α), while the Dirac delta function $\delta(\mathbf{r}-\mathbf{r}')$ is employed for the remaining part of the interaction (with the interaction parameters $\beta,\ \gamma$, and σ) [19]. In this case, only the leading part of the interaction is the Seyler-Blanchard-type one, for which the above localized approximation is applied, while the remaining part is reduced to the usual local interaction, which contributes only to the local-density dependent term. Therefore, in the simplified interaction, the terms with the parameters $\beta,\ \gamma$, and σ disappear in $F^{(n)}(r)$; Eq. (59) keeps the same form but Eq. (60) is reduced to

$$F^{(2)}(r)=T_0\left[\epsilon_{1n}\frac{d^2\rho_n}{dr^2}+\epsilon_{1p}\frac{d^2\rho_p}{dr^2}\right] \tag{61}$$

and ϵ_{1n} as well as ϵ_{1p} are simplified to

$$\epsilon_{1n}=-\epsilon_{1n\alpha}=-\frac{1}{2}\left[\alpha_l\frac{\rho_n}{\hat{\rho}_0}+\alpha_u\frac{\rho_p}{\hat{\rho}_0}\right],$$

$$\epsilon_{1p}=-\epsilon_{1p\alpha}=-\frac{1}{2}\left[\alpha_l\frac{\rho_p}{\hat{\rho}_0}+\alpha_u\frac{\rho_n}{\hat{\rho}_0}\right]. \tag{62}$$

In the TF theory, the total binding energy E_{TF} is the sum of the nuclear energy E_N and the Coulomb energy E_C:

$$E_{\mathrm{TF}}=E_N+E_C \tag{63}$$

and the nucleon densities $\rho_n(r)$ and $\rho_p(r)$ are determined by the minimization of E_{TF} with respect to their variations $\delta\rho_n(r)$ and $\delta\rho_p(r)$. In the present work, the following two-parameter Fermi distribution is taken for $\rho_n(r)$ and $\rho_p(r)$:

$$\rho_q(r)=\frac{\rho_{qc}}{1+\exp[(r-C_q)/d]}, \tag{64}$$

where $q=n$ or p. The ρ_{qc} can be determined by normalizing $\rho_n(r)$ and $\rho_p(r)$ to N and Z, respectively. The surface diffuseness d can be related to the Süssmann width b as [13] $d=\sqrt{3}b/\pi$. The central radius C_n and C_p are left as the free parameters to be determined by minimizing E_{TF}. As the densities are constrained to be Fermi distributions, the present calculation is similar to the constrained extended TF model calculation given in Ref. [5], where the densities are constrained to be generalized Fermi distributions.

In expressing the densities as the Fermi distribution (64), the Coulomb energy E_{Coul} and E_{ex} can be written, respectively, as [20]

$$E_{\mathrm{Coul}}\approx\frac{3}{5}\frac{Z^2e^2}{C_p}\left[1-\frac{7\pi^2}{6}\left(\frac{d}{C_p}\right)^2+18.031\left(\frac{d}{C_p}\right)^3\right],$$

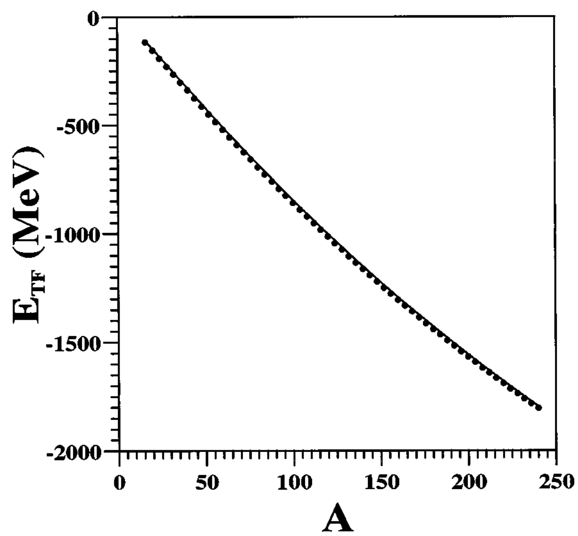

FIG. 1. The TF binding energy E_{TF} of the nuclei along the β-stability line. The solid curve is calculated by the present localized approximation and the full dots are taken from the exact numerical calculation given in Ref. [21]. The difference between them is within 1% for medium and heavy nuclei while less than 5% for light nuclei.

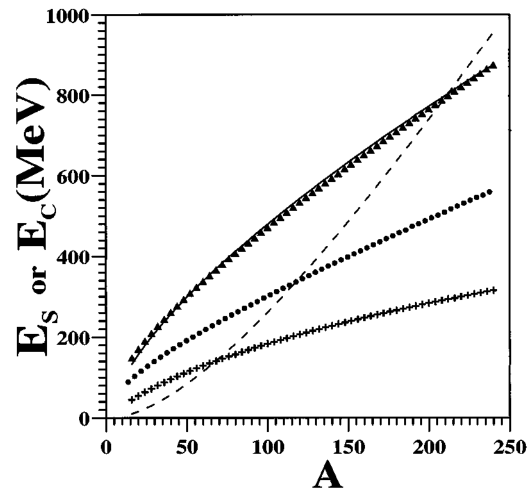

FIG. 2. The contributions to the TF surface energy E_S of finite nuclei along the β-stability line, calculated by the present localized approximation. The full dots are the energy E_{S0} of the local density approximation, the crosses are the density-gradient-dependent energy E_{GD}, the solid curve is the total TF surface energy E_S, and the triangles are the droplet model surface energy E_S (DLM) [15]. The Coulomb energy E_C is plotted also as the dashed curve for comparison. It can be seen that E_{GD} contributes about 34–38 % of the total surface energy.

$$E_{\mathrm{ex}} \approx -\frac{3}{4}\left(\frac{3}{2\pi}\right)^{2/3}\frac{e^2 Z^{4/3}}{C_p}\left[1-1.3355\frac{d}{C_p}\right]. \tag{65}$$

Figure 1 plots the calculated TF binding energy E_{TF} of nuclei along the β-stability line. The solid curve is calculated by the present localized approximation and the full dots are taken from the exact numerical calculation given in Ref. [21]. It can be seen that the present localized approximation is good enough, as the deviation from the exact TF calculation is within 1% for medium and heavy nuclei and less than 5% for light nuclei.

Since the inclusion of the gradient density dependent term E_{GD} represents an improvement to the local density approximation term E_{LD}, it is interesting to evaluate its contribution. In order to do that, we anticipate that the surface energy E_S, which will be defined in Sec. V, is just given by the sum of this new term E_{GD} and the term E_{S0} depending on the local density. In Fig. 2, we display E_S (solid curve), E_{S0} (full dots), E_{GD} (crosses), and the droplet model surface energy E_S (DLM) [15] (triangles). The Coulomb energy E_C is plotted also for comparison (dashed curve). All of these curves are for nuclei along the β-stability line. It can be seen that the energy E_{GD} depending on the gradients of densities contributes about 34–38 % of the total surface energy. This means that E_{GD} cannot be neglected at all in a realistic calculation.

IV. ISOSCALAR GIANT MONOPOLE RESONANCE ENERGY

In applying the general formula (33) to calculate the isoscalar giant monopole resonance energy E_M of nuclei, the constrained TF model calculation is given as an example in this section. The nuclear matter incompressibility $K(\rho,\delta)$ corresponding to the equation of state (51) is

$$K(\rho,\delta)=T_0\left[-2B(\delta)\left(\frac{\rho}{\rho_0}\right)^{2/3}+10D(\delta)\left(\frac{\rho}{\rho_0}\right)^{5/3}\right]. \tag{66}$$

Corresponding to the expansion (46), the second term in the general formula (33) can be written as

$$K_{\mathrm{GD}}=\sum_{n=1}^{\infty}K_{\mathrm{GD}n}=\sum_{n=1}^{\infty}\frac{1}{A}\left[\frac{d^2E_{\mathrm{GD}n}(\eta)}{d\eta^2}-2\frac{dE_{\mathrm{GD}n}(\eta)}{d\eta}\right]_{\eta=1}. \tag{67}$$

In the simplified interaction, substituting Eqs. (59) and (61) into Eq. (47), the first two terms of the above expansion can be obtained as

$$K_{\mathrm{GD1}}=\frac{aT_0}{A}\int d^3r\left\{-10[I_1(r/a)-f(r/a)]\times\left[\epsilon_{1n\alpha}\frac{d\rho_n}{dr}+\epsilon_{1p\alpha}\frac{d\rho_p}{dr}\right]\right\}, \tag{68}$$

$$K_{\mathrm{GD2}}=\frac{a^2T_0}{2A}\int d^3r\left\{-10\left[I_2(r/a)+2\frac{r}{a}f(r/a)\right]\times\left[\epsilon_{1n\alpha}\frac{d^2\rho_n}{dr^2}+\epsilon_{1p\alpha}\frac{d^2\rho_p}{dr^2}\right]\right\}, \tag{69}$$

where

$$f(x)=\frac{1}{5}(x+6)\,e^{-x}. \tag{70}$$

In the calculation of residual energy effects, the table given in Ref. [21] is used for the shell correction and the

TABLE I. The relevant quantities calculated for some nuclei, all in MeV except δ and ρ_C (in fm^{-3}).

	δ	ρ_C	K_A	$\langle K(\rho,\delta)\rangle$	K_{GD}	$2E_C/A$	K_{res}	$\hbar\omega$	E_M	$E_{M\ exp}$
^{16}O	0.0042	0.1385	101.191	72.808	28.446	1.369	1.306	22.196	19.559	
^{40}Ca	0.0176	0.1471	125.508	103.804	24.608	3.319	0.414	20.225	19.501	14.11
^{58}Ni	0.0371	0.1487	133.307	114.938	22.242	4.227	0.353	18.974	18.558	17.23
^{90}Zr	0.0825	0.1488	139.222	124.874	19.264	5.091	0.176	17.187	16.976	16.13
^{112}Sn	0.0866	0.1483	141.168	129.216	17.893	6.050	0.107	16.247	16.097	15.87
^{114}Sn	0.0952	0.1481	141.069	129.161	17.744	5.926	0.091	16.154	16.008	15.73
^{140}Ce	0.1277	0.1470	141.074	130.796	16.331	6.178	0.125	15.180	15.075	15.11
^{208}Pb	0.1649	0.1444	140.059	133.376	13.965	7.445	0.163	13.369	13.313	13.86
^{238}U	0.1791	0.1433	138.854	133.485	13.199	7.873	0.043	12.746	12.702	13.88

even-odd energy. The formulas for the anharmonic coefficients K_3 and K_4 are given in Appendix B.

The relevant quantities calculated for some nuclei are shown in Table I. First, it can be seen that the central density ρ_C is almost a constant near the standard nuclear matter density ρ_0, while the relative neutron excess δ is close to zero, i.e., $\rho_C \approx \rho_0$, $0<\delta<0.18$. Secondly, the average incompressibility $\langle K(\rho,\delta)\rangle$ increases slowly from about 73 to about 133 MeV, the gradient density dependent term K_{GD} decreases from about 28 to 13 MeV, the Coulomb energy term $-2E_C/A$ decreases from -1.4 to -7.9 MeV, while the residual energy term K_{res} decreases from 1.3 to 0.04 MeV, as the incompressibility K_A of the nuclei increase slowly from 101 to 140 MeV when the nucleon number increases from ^{16}O to ^{238}U. As the Coulomb energy contributes to K_A by less than 5%, while the residual energy contributes by less than 1%, the main contribution of K_A comes from the average incompressibility $\langle K(\rho,\delta)\rangle$ and the gradient density-dependent term K_{GD} which contributes with about 10–20 %. At last, it can be seen also that the anharmonic effect, as given by the difference between E_M and $\hbar\omega$, is about 6% for light nuclei while it is less than 0.5% for heavy nuclei. As the contribution $-2E_{res}/A$ of the residual energy is very small for most of the nuclei, this term can be safely neglected in the practical calculation.

The contributions to the incompressibility K_A of finite nuclei along the β-stability line, calculated by the present constrained TF minimization, are shown in Fig. 3. The full dots are the average incompressibility $\langle K(\rho,\delta)\rangle$, the crosses are the gradient density-dependent term K_{GD}, the triangles are the Coulomb energy term $-2E_C/A$, and the solid curve is the total incompressibility K_A.

Figure 4 shows the isoscalar giant monopole resonance energy E_M times $A^{1/3}$ calculated along the β-stability line in comparison with the measured data. The lower solid curve is calculated by the present calculation with the simplified Myers-Swiatecki interaction, the dashed curve by the approximate calculation which will be explained in the next section, the upper solid curve by Myers and Swiatecki's formula (2) with their parameters [11], the triangles by Nayak *et al.*'s leptodermous expansion of K_A with their parameters for the SkM* interaction [5], while the experimental data (full dots) are taken from the compilation of Shlomo and Youngblood [7] with the prescription [2,6,7,10] $E_M^2 = E_0^2 + 3(\Gamma/2.35)^2$, where E_0 is the measured centroid energy and Γ the experimental width. It can be seen from this figure that the agreement between the two calculated curves (the lower solid and the dashed curves) and the experimental data is satisfactory, since there is no parameter being adjusted in the present work.

In calculating E_M by Myers and Swiatecki's and Nayak *et al.*'s expression of K_A, the harmonic approximation (22) is applied, where $\langle r^2\rangle$ is the same as that used in our calculation. However, as the anharmonic effect is included in our E_M, the comparison between our E_M and that of Myers and Swiatecki as well as that of Nayak *et al.* is not fully in the same base.

The calculated energy E_M for some spherical nuclei is given in Table II in comparison with other calculations. MS is the present result using the simplified Myers-Swiatecki interaction, JJ the result of Jennings and Jackson by using the hydrodynamic model [4], GBMQ the TF model calculation given by Gleissl *et al.* [22] with the Skyrme force, and BMS the relativistic mean-field calculation by Boersma *et al.* [14].

As our calculation is a constrained minimization, it would be interesting to make a comparison with the result based on an absolute minimization, where the nucleon density distribution is not restricted to assume a specific form but is ob-

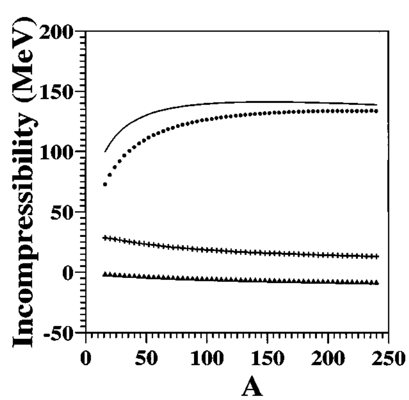

FIG. 3. The contributions to the incompressibility K_A of finite nuclei along the β-stability line, calculated by the present constrained TF minimization. The full dots are the average incompressibility $\langle K(\rho,\delta)\rangle$, the crosses are the gradient-density-dependent term K_{GD}, the triangles are the Coulomb energy term $-2E_C/A$, and the solid curve is the total incompressibility K_A. It can be seen that the main contribution of K_A comes from the average incompressibility $\langle K(\rho,\delta)\rangle$, the gradient-density-dependent term K_{GD} contributes with about 10–20 %, while the Coulomb energy term $-2E_C/A$ contributes by less than 5%.

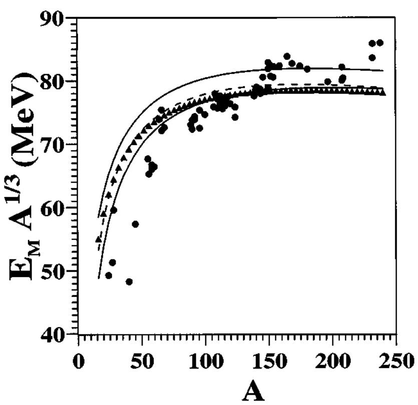

FIG. 4. Isoscalar giant monopole resonance energy E_M times $A^{1/3}$ versus A along the β-stability line. The lower solid curve is calculated by the present constrained TF minimization, the dashed curve by the approximate calculation based on the droplet-model-like formulas (94) and (95), the upper solid curve by Myers and Swiatecki's formula (2) with their parameters [11], the triangles by the Nayak *et al.*'s leptodermous expansion of K_A with their parameters for the SkM* interaction [5], while the full dots are the experimental data taken from the compilation of Shlomo and Youngblood with their prescription [7].

tained by a numerical procedure. However, this is out of the scope of the present work and we leave it as a future study.

V. LEPTODERMOUS EXPANSION

We will derive in this section Eq. (1) from our general formula (34), by its leptodermous expansion similar to that used in the droplet model of nuclei where the nuclear energy $E_N(A,Z)$ of the nucleus (A,Z) can be written as

$$E_N(A,Z)=E_B+E_S, \tag{71}$$

where

$$E_B=\int d^3r\rho(\mathbf{r})e(\rho_b,\delta_b)=e(\rho_b,\delta_b)\ A \tag{72}$$

is the bulk energy, and

$$E_S=E_{S0}+E_{\rm GD}=\int d^3r\rho(\mathbf{r})[e(\rho,\delta)-e(\rho_b,\delta_b)]+E_{\rm GD} \tag{73}$$

TABLE II. Isoscalar giant monopole resonance energy E_M (in MeV) for some spherical nuclei.

Nucleus	E_0	Γ	MS	JJ	GBMQ	BMS
^{16}O			19.56	22.4	18.6	18.0
^{40}Ca	14.11		19.50	18.2	18.6	19.0
^{58}Ni	17.06	3.28	18.56		18.9	
^{90}Zr	15.95	3.29	16.98	15.0	17.7	22.7
^{112}Sn	15.64	3.67	16.10		16.7	
^{114}Sn	15.51	3.52	16.01			22.5
^{140}Ce	14.95	3.00	15.08		15.6	
^{208}Pb	13.73	2.58	13.31	11.7	13.5	21.0

is the surface energy [23].

For a leptodermous distribution $\rho(\mathbf{r})$ of nucleons, the bulk density ρ_b equals approximately the central density ρ_C which is close to ρ_0. For nuclei not far from the β-stability line, the bulk neutron excess δ_b is small. Therefore, the equation of state $e(\rho_b,\delta_b)$ in Eq. (72) can be expanded at the point $(\rho_0,0)$ as

$$e(\rho_b,\delta_b)=-a_1+\frac{1}{2}K_0\epsilon^2+J\delta_b^2-L\epsilon\delta_b^2+\frac{1}{2}M\delta_b^4+\cdots, \tag{74}$$

where ϵ is the density variance

$$\epsilon=-\frac{\rho_b-\rho_0}{3\rho_0}, \tag{75}$$

while a_1, K_0, J, L, and M are the droplet model parameters

$$a_1=-e(\rho_0,0),\quad K_0=K(\rho_0,0),\quad J=\frac{1}{2}\left.\frac{\partial^2 e}{\partial\delta^2}\right|_0,$$

$$L=\frac{3}{2}\rho_0\left.\frac{\partial^3 e}{\partial\rho\partial\delta^2}\right|_0,\quad M=\frac{1}{12}\left.\frac{\partial^4 e}{\partial\delta^4}\right|_0. \tag{76}$$

The subscript 0 here stands for the standard nuclear matter $\rho=\rho_0$, $\delta=0$. As $(\rho_0,0)$ is the stable point of nuclear matter, the condition $\partial e/\partial\rho|_0=0$ has been applied in the expansion (74). As the nuclear force is symmetric in the proton and neutron, the equation of state $e(\rho,\delta)$ is an even function of δ, there is no odd order terms in δ_b.

For the surface energy E_S, we can expand the equation of state $e(\rho,\delta)$ at (ρ_b,δ_b):

$$E_S\approx\rho_b\left.\frac{\partial e}{\partial\rho}\right|_b\int d^3r\rho(\mathbf{r})\frac{\rho-\rho_b}{\rho_b}+\frac{1}{18}K(\rho_b,\delta)\int d^3r\rho(\mathbf{r})\times\left(\frac{\rho-\rho_b}{\rho_b}\right)^2+\frac{1}{2}\left.\frac{\partial^2 e}{\partial\delta^2}\right|_b\int d^3r\rho(\mathbf{r})[\delta(r)-\delta_b]^2+E_{\rm GD}, \tag{77}$$

where the subscript b stands for the bulk point (ρ_b,δ_b). Since the integrals in the above expansion as well as those involved in the expression of $E_{\rm GD}$ depend on the characteristic radius R of the nuclei and the relative neutron excess δ_b; E_S is generally a function of $A^{1/3}$ and δ_b. For a leptodermous distribution $\rho(\mathbf{r})$, these integrals will essentially be performed around the surface area, and the surface energy can be expressed as

$$E_S=4\pi R^2\left[\frac{1}{4\pi r_0^2}(a_2+F\epsilon+H\tau^2+2P\tau\delta_b-G\delta_b^2)\right]+8\pi R\left[\frac{a_3}{8\pi r_0}\right]+\cdots, \tag{78}$$

where a_2, a_3, F, H, P, and G are the droplet model parameters and $\tau=(C_n-C_p)/r_0$ is the reduced neutron skin thickness. R is defined as $4\pi R^3\rho_b/3=A$.

If the Fermi distribution (64) is assumed for the nucleon density $\rho_q(r)$, the above integrals can be performed analytically. Dropping higher order terms, the result is

$$\int d^3r\rho(\mathbf{r})\frac{\rho-\rho_b}{\rho_b}=-4\pi R^2 d\rho_b\left[1+\frac{r_0^2}{24d^2}\tau^2\right], \quad (79)$$

$$\int d^3r\rho(\mathbf{r})\left(\frac{\rho-\rho_b}{\rho_b}\right)^2=4\pi R^2 d\rho_b\left[\frac{1}{2}+\frac{r_0^2}{48d^2}\tau^2\right]+8\pi Rd^2\rho_b\left[\frac{1}{2}+\frac{r_0^2}{32d^2}\tau^2\right], \quad (80)$$

$$\int d^3r\rho(\mathbf{r})[\delta(r)-\delta_b]^2=8\pi Rd^2\ \rho_b\frac{1}{8}\frac{r_0^2}{d^2}\tau^2, \quad (81)$$

$$E_{\mathrm{GD1}}=\frac{2\pi Ra^2\rho_b^2T_0}{\rho_0}\alpha(1-\xi\delta_b^2), \quad (82)$$

$$E_{\mathrm{GD2}}=\frac{\pi R^2d\rho_b^2T_0}{3\rho_0}\frac{a^2}{d^2}\left[\alpha-\xi\alpha\delta_b^2-\frac{r_0^2}{20d^2}(1+\xi)\alpha\tau^2\right]-\frac{2\pi Ra^2\rho_b^2T_0}{\rho_0}\left[\alpha-\xi\alpha\delta_b^2+\frac{r_0}{6d}\xi\alpha\tau\delta_b\right]. \quad (83)$$

In performing the integrals of E_{GD1} and E_{GD2}, the approximation of $I_1(x)\approx 2/x$ and $I_2(x)\approx 2$ is employed as the integrals are mainly in the surface area where $\exp(-x)\ll 1$. Here we omit the details of the derivation of the above formulas, and give only some formulas used in this derivation in Appendix C.

Substituting the above result into the Eq. (77), the following formulas can be obtained:

$$a_2=\frac{1}{12}\frac{d}{r_0}K_0\left(1+3\alpha\frac{a^2}{d^2}\frac{T_0}{K_0}\right), \quad (84)$$

$$a_3=\frac{1}{6}\frac{d^2}{r_0^2}K_0, \quad (85)$$

$$F=\frac{3}{4}\frac{d}{r_0}K_0\left(1-\frac{1}{3}\kappa_{10}-2\alpha\frac{a^2}{d^2}\frac{T_0}{K_0}\right), \quad (86)$$

$$G=\frac{d}{r_0}K_0\left(\frac{L}{K_0}-\frac{1}{24}\kappa_{02}+\frac{1}{4}\xi\alpha\frac{a^2}{d^2}\frac{T_0}{K_0}\right), \quad (87)$$

$$H=\frac{1}{288}\frac{r_0}{d}K_0\left[1-\frac{18}{5}(1+\xi)\alpha\frac{a^2}{d^2}\frac{T_0}{K_0}\right], \quad (88)$$

$$P=0, \quad (89)$$

where

$$\kappa_{\mu\nu}=\frac{\rho_0^\mu}{K_0}\frac{\partial^{\mu+\nu}K}{\partial\rho^\mu\partial\delta^\nu}\bigg|_0, \quad \mu,\nu=0,1,2,\ldots. \quad (90)$$

In these formulas, the terms involving the Yukawa range a come from the gradient dependent part of energy E_{GD}, while the others come from the local density approximation. Especially, it is worthwhile to note that the surface energy coefficient a_2 involves the contributions both from the local density dependent energy E_{LD} as well as the gradient density-dependent energy E_{GD}. Numerically, each of them contributes about half a_2 if the parameters (56) are used. On the other hand, the curvature energy coefficient a_3 comes only from E_{LD} in the present model.

We have to note also that, different than the usual droplet model result, F is not 0 but P is 0. This is so because in the present case the nucleon density is restricted to be the Fermi distribution (64), similar to the case of Nayak *et al.*'s work where the nucleon density is restricted to be the generalized Fermi distribution and the F term is not 0 [5]. This situation means that the Fermi distribution and even the generalized Fermi distribution are not very appropriate for expressing the nucleon distribution in a calculation with high accuracy. However, keeping this in mind, we can still expect to get some general ideas from the simplified analytical derivation by using these distributions.

Thus we have the leptodermous expansion of the total energy E_{TF} as

$$E_{\mathrm{TF}}(A,Z)=\left[-a_1+\frac{1}{2}K_0\epsilon^2+J\delta_b^2-L\epsilon\delta_b^2+\frac{1}{2}M\delta_b^4\right]A+[a_2+(2a_2+F)\epsilon+H\tau^2+2P\tau\delta_b-G\delta_b^2]A^{2/3}+a_3A^{1/3}+c_1\frac{Z^2}{A^{1/3}}\left[1-\epsilon+\frac{\tau}{2A^{1/3}}\right], \quad (91)$$

where the first term is the volume energy, the next two terms the surface energy, and the last term the Coulomb energy. In obtaining the above expression of Coulomb energy, the geometrical relation [23]

$$\tau=\frac{2}{3}(I-\delta_b)A^{1/3} \quad (92)$$

and the approximation

$$E_C\approx\frac{3}{5}\frac{Z^2e^2}{R_p} \quad (93)$$

are employed, where $I=(N-Z)/A$, $R_p=(3Z/4\pi\rho_{pb})^{1/3}$ is the equivalent sharp charge radius, and $\rho_{pb}=(1-\delta_b)\rho_b/2$.

Minimizing the total energy with respect to ϵ and δ_b, the following droplet-model-like relations can be obtained:

$$\epsilon=\frac{L\delta_b^2-(2a_2+F)A^{-1/3}+c_1Z^2A^{-4/3}}{K_0}, \quad (94)$$

$$\delta_b=\frac{I+(3/8)(c_1/H)Z^2A^{-5/3}}{1+(9/4)(J/H)A^{-1/3}-(9/4)(G/H)A^{-2/3}}. \quad (95)$$

As the above formulas can be used to calculate the nucleon densities $\rho_q(r)$ approximately, when the Fermi distribution (64) is assumed, we can also perform the calculation explained in the last section by using the $\rho_q(r)$ obtained in this way. The calculated $E_MA^{1/3}$ and K_A are shown in Figs. 4 and 5, respectively, by the dashed curves.

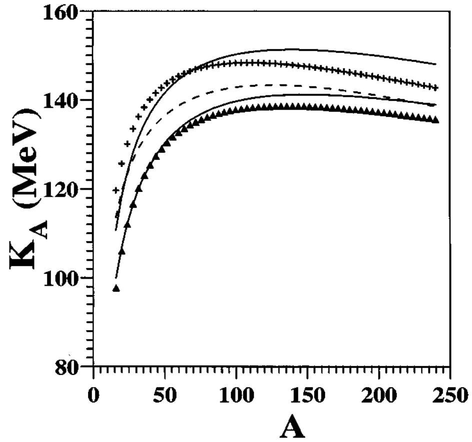

FIG. 5. The calculated incompressibility K_A of finite nuclei along the β-stability line by different methods. The lower solid curve is calculated by the present constrained TF minimization, the dashed curve by the approximate calculation based on the droplet-model-like formulas (94) and (95), the crosses by the present leptodermous expansion (110), the upper solid curve by Myers and Swiatecki's formula (2) with their parameters [11], and the triangles by Nayak *et al.*'s leptodermous expansion with their parameters for the SkM* interaction [5].

Let us now consider the leptodermous expansion of the incompressibility K_A. Similarly, the nuclear part K_N of the finite nuclei incompressibility can be written as

$$K_N=\frac{1}{A}\int d^3r\rho(\mathbf{r})K(\rho,\delta)+K_{\rm GD}=\frac{1}{A}\int d^3r\rho(\mathbf{r})[K(\rho_b,\delta_b)+K(\rho,\delta)-K(\rho_b,\delta_b)]+K_{\rm GD}=K_B+K_S, \tag{96}$$

where K_B and K_S are the bulk and surface incompressibility, respectively:

$$K_B=\frac{1}{A}\int d^3r\rho(\mathbf{r})K(\rho_b,\delta_b)=K(\rho_b,\delta_b), \tag{97}$$

$$K_S=\frac{1}{A}\int d^3r\rho(\mathbf{r})[K(\rho,\delta)-K(\rho_b,\delta_b)]+K_{\rm GD}=\langle K(\rho,\delta)\rangle-K(\rho_b,\delta_b)+K_{\rm GD}. \tag{98}$$

The bulk incompressibility K_B can be expanded also at the point $(\rho_0,0)$. As the bulk point (ρ_b,δ_b) is close to $(\rho_0,0)$, the variation of K_B is very small around K_0. Therefore, it is a good approximation to keep only the following five terms in the expansion:

$$K_B\approx K_0\left(1-3\kappa_{10}\epsilon+\frac{1}{2}\kappa_{02}\delta_b^2-\frac{3}{2}\kappa_{12}\epsilon\delta_b^2+\frac{1}{24}\kappa_{04}\delta_b^4\right). \tag{99}$$

Substituting the formula (94) and $\delta_b\approx I$ into the above equation, we have

$$K_B\approx K_0+3\kappa_{10}(2a_2+F)A^{-1/3}+\left(\frac{1}{2}\kappa_{02}K_0-3\kappa_{10}L\right)\left(\frac{N-Z}{A}\right)^2-3\kappa_{10}c_1Z^2A^{-4/3}+\frac{3}{2}\kappa_{12}(2a_2+F)\left(\frac{N-Z}{A}\right)^2A^{-1/3}+\left(\frac{1}{24}\kappa_{04}K_0-\frac{3}{2}\kappa_{12}L\right)\left(\frac{N-Z}{A}\right)^4, \tag{100}$$

where the last two terms come from the last two terms in the expansion (99) and will be shown to be negligibly small.

For the surface incompressibility K_S, we can expand the incompressibility $K(\rho,\delta)$ at (ρ_b,δ_b):

$$K_S=\rho_b\frac{\partial K}{\partial\rho}\bigg|_b\frac{1}{A}\int d^3r\rho(\mathbf{r})\frac{\rho-\rho_b}{\rho_b}+\frac{1}{2}\rho_b^2\frac{\partial^2K}{\partial\rho^2}\bigg|_b\frac{1}{A}\int d^3r\rho(\mathbf{r})\left(\frac{\rho-\rho_b}{\rho_b}\right)^2+\frac{1}{2}\frac{\partial^2K}{\partial\delta^2}\bigg|_b\int d^3r\rho(\mathbf{r})[\delta(r)-\delta_b]^2+K_{\rm GD}. \tag{101}$$

As the integrals are exactly the same as those in Eq. (77), we can use the results given in the Eqs. (79)–(81). Besides, similar to the approximation used in performing the integrals of $E_{\rm GD}$, the term involving $f(x)$ in $K_{\rm GD}$ [Eqs. (68) and (69)] can be neglected and the following relations are obtained:

$$K_{\rm GD1}\approx\frac{10}{A}E_{\rm GD1},\quad K_{\rm GD2}\approx\frac{10}{A}E_{\rm GD2},\quad K_{\rm GD}\approx\frac{10}{A}E_{\rm GD}. \tag{102}$$

Therefore, we can have

$$K_S=-k_S^0A^{-1/3}+k_{cS}A^{-2/3}+k_{sS}^0\left(\frac{N-Z}{A}\right)^2A^{-1/3}+\cdots, \tag{103}$$

where k_S^0, k_{cS}, and k_{sS}^0 can be expressed in the present model as

$$k_S^0=\frac{d}{r_0}K_0\left[3\kappa_{10}-\frac{3}{4}\kappa_{20}-\frac{5}{2}\alpha\frac{a^2}{d^2}\frac{T_0}{K_0}\right], \tag{104}$$

$$k_{cS}=-\frac{d}{r_0}K_0\left[3\kappa_{10}+6\kappa_{20}-\frac{9}{4}\kappa_{30}-15\alpha\frac{a^2}{d^2}\frac{T_0}{K_0}\right]\frac{2a_2+F}{K_0}+\frac{3}{2}\frac{d^2}{r_0^2}\kappa_{20}K_0, \tag{105}$$

$$k_{sS}^0=\frac{d}{r_0}K_0\left\{\left[\frac{3}{8}\kappa_{22}-\frac{3}{2}\kappa_{12}+\left(3\kappa_{10}+6\kappa_{20}-\frac{9}{4}\kappa_{30}\right)\frac{L}{K_0}\right]-\left[15\frac{L}{K_0}+\frac{5}{2}\xi\right]\alpha\frac{a^2}{d^2}\frac{T_0}{K_0}\right\}. \tag{106}$$

Thus, we have the following leptodermous expansion for the incompressibility K_A:

$$K_A \approx K_0 - [k_S^0 - 3\kappa_{10}(2a_2+F)]A^{-1/3} + k_{cS}A^{-2/3} + \left(\frac{1}{2}\kappa_{02}K_0 - 3\kappa_{10}L\right)\left(\frac{N-Z}{A}\right)^2 + \left[k_{sS}^0 + \frac{3}{2}\kappa_{12}(2a_2+F)\right]\left(\frac{N-Z}{A}\right)^2 A^{-1/3} - (2+3\kappa_{10})c_1 Z^2 A^{-4/3} + \left(\frac{1}{24}\kappa_{04}K_0 - \frac{3}{2}\kappa_{12}L\right) \times \left(\frac{N-Z}{A}\right)^4. \quad (107)$$

The expansion (107), except the term in I^4, is exactly the same as Eq. (1) formally. It is worthwhile to note that the coefficient $2+3\kappa_{10}$ of the Coulomb energy term $c_1Z^2A^{-4/3}$ and the coefficient $(1/2)\kappa_{02}K_0 - 3\kappa_{10}L$ of the asymmetry term $[(N-Z)/A]^2$ in the above equation are identical to Blaizot's formulas [24,25]. It is worthwhile also to note that, as the variation of the incompressibility K_A comes essentially from the surface incompressibility K_S, the contribution from the terms involving k_{cS} and k_{sS}^0 are as important as that from the term involving k_S^0 only, so the term proportional to $A^{-2/3}$ should be taken into account in the data fitting.

The coefficients $\kappa_{\mu\nu}$ can be easily calculated with the help of Eqs. (52), (54), and (66), based on the Myers-Swiatecki interaction, and the use of the corresponding parameters (56). In particular, our value $3\kappa_{10}=4.92$ is in the range $3\sim5$ given by Blaizot *et al.* [8] where the symbol S/K_{nm} is used. This stresses the point that the Seyler-Blanchard-like nonlocal interaction is similar to the Skyrme-type interaction in the description of nuclear matter equation of state.

Finally, we have

$$K_B \approx 234.422 + 84.206A^{-1/3} - 392.62\left(\frac{N-Z}{A}\right)^2 - 3.7305Z^2A^{-4/3} - 40.08A^{-1/3}\left(\frac{N-Z}{A}\right)^2 + 124.67\left(\frac{N-Z}{A}\right)^4, \quad (108)$$

$$K_S \approx -365.725A^{-1/3} + 31.930A^{-2/3} + 373.21A^{-1/3}\left(\frac{N-Z}{A}\right)^2, \quad (109)$$

$$K_A \approx 234.422 - 281.518A^{-1/3} + 31.930A^{-2/3} - 392.62\left(\frac{N-Z}{A}\right)^2 + 333.13A^{-1/3}\left(\frac{N-Z}{A}\right)^2 - 5.2462Z^2A^{-4/3} + 124.67\left(\frac{N-Z}{A}\right)^4. \quad (110)$$

The last two terms in Eq. (108) come from the last two terms in the expansion (99). Numerically, it can be seen that the contributions of these two terms are negligible in comparing with the other terms. This means that at least the term in I^4 can be omitted in the final result [5].

Figure 5 gives the comparison between the incompressibility K_A obtained by different methods. The lower solid curve is calculated by the present constrained TF minimization, the dashed curve by the approximate calculation based on the droplet-model-like formulas (94) and (95), the crosses by the present leptodermous expansion (110), the upper solid curve by Myers and Swiatecki's formula (2) with their parameters [11], and the triangles by the Nayak *et al.*'s leptodermous expansion with their parameters for the SkM* interaction [5]. The calculations are along the β-stability line nuclei. The relatively big deviation of the present leptodermous expansion from the exact result is due to the defect of the Fermi distribution discussed in the second paragraph after Eq. (90).

VI. DISCUSSION AND SUMMARY

In the present work, the calculation is made under the spherical symmetry approximation only. However, it can be seen from Fig. 4 that there are ''bumps'' in the regions of deformed nuclei where the spherical symmetry distribution is obviously not a good approximation. Therefore, it would be interesting to work out a more realistic theory for deformed nuclei, to see if this kind of bump is due to the effect of deformation or not.

As the trend of the experimental data shown in Fig. 4 can be well reproduced by the present calculation without adjustment of any existing parameter, the nuclear TF model with the simplified Myers-Swiatecki interaction is shown to be able to give a consistent description for the isoscalar giant monopole resonance, besides the nuclear masses and the other properties of nuclei. This suggests that the measurement data of the isoscalar giant monopole resonance energy E_M should be included in the fitting of the model parameters when the calculation for deformed nuclei is included.

Experimentally, it is well known that the measurements of the nuclear masses and the fission barriers can provide information about the nuclear matter equation of state $e(\rho,\delta)$, such as that given by the droplet model parameters a_1, K_0, J, L, and M. Now the present work shows that the measurement of the energy E_M can provide information about the nuclear matter incompressibility $K(\rho,\delta)$, such as that given by the coefficients K_0 and $\kappa_{\mu\nu}$ in the leptodermous expansion of K_A. However, in order to extract this information of nuclear matter, high precision measurements for closed shell nuclei and new data for nuclei away from the β-stability line are needed. Even our purpose in the present work is not to extract this information of nuclear matter; we can say that the value of the standard nuclear matter incompressibility $K_0=234$ MeV, given by Myers and Swiatecki [15], is supported by the present calculation, and this value is close to that given by Blaizot *et al.* recently [8].

In summary, the main results and conclusions of the present work are as follows.

(1) The finite nucleus incompressibility K_A is essentially the nuclear matter incompressibility $K(\rho,\delta)$ averaged over its nucleon distribution $\rho(\mathbf{r})$, added to a term $K_{\rm GD}$ contributed from the gradients of nucleon densities, and with twice the Coulomb energy per nucleon E_C/A subtracted. $K_{\rm GD}$ and

$-2E_C/A$ contribute about 20–10 % and 1–5 %, respectively, to the nuclear incompressibility K_A, from the light to the heavy nuclei, in this order, whereas the shell and the even-odd energy corrections are negligibly small. It is interesting to note that the relation (34) is interaction independent, and interaction-independent information of nuclear matter incompressibility is expected to be extracted by using this relation.

(2) The leptodermous expansion (1) used in the data fitting is justified again and is shown to be an approximation to the exact result by the present TF theory with the linear scaling assumption. The approximation is good for heavy nuclei but not for light ones. It is shown also by the present work that the curvature term $k_{cS}A^{-2/3}$ cannot be neglected, as it is very important in the description of the surface incompressibility K_S. All of these results are also interaction independent. The reason why the curvature term has not been included in some data fitting should be carefully investigated further.

(3) The anharmonic effect of the breathing mode contributes to the isoscalar giant monopole resonance energy E_M in about 6% of light nuclei and less than 0.5% of heavy nuclei. However, as this effect depends on the anharmonic terms K_3 and K_4, and the latter depends on the choice of interaction as well as the values of the model parameters, the evaluation of the anharmonic effect is both model and interaction dependent.

(4) In developing a localized approximation, the Seyler-Blanchard-type interaction is shown to be similar to the Skyrme-type interaction, in the sense that the density-gradient-dependent terms appear in the Hamiltonian functional.

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APPENDIX A: A QUANTUM MECHANICAL MODEL FOR SCALING

The Schrödinger equation for a nucleon with mass m_N moving in the potential $V(\mathbf{r})$ is

$$-\frac{\hbar^2}{2m_N}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi_k(\mathbf{r})+V(\mathbf{r})\varphi_k(\mathbf{r})=\varepsilon_k\varphi_k(\mathbf{r}), \tag{A1}$$

where $\varphi_k(\mathbf{r})$ is the single-particle wave function and ε_k the single-particle energy of nucleon. If the potential $V(\mathbf{r})$ is changed into $V_s(\mathbf{r})$, the wave function $\varphi_k(\mathbf{r})$ and the energy ε_k will be changed into $\varphi_k^s(\mathbf{r})$ and ε_k^s, respectively, according to the changed equation

$$-\frac{\hbar^2}{2m_N}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi_k^s(\mathbf{r})+V_s(\mathbf{r})\varphi_k^s(\mathbf{r})=\varepsilon_k^s\varphi_k^s(\mathbf{r}). \tag{A2}$$

If the potential obeys a two-dimensional scaling

$$V(\mathbf{r})\rightarrow V_s(\mathbf{r})=\eta^2V(\eta\mathbf{r}), \tag{A3}$$

where η is the scaling parameter, Eq. (A2) can be rewritten as

$$-\frac{\hbar^2}{2m_N}\left(\frac{\partial^2}{\partial x_s^2}+\frac{\partial^2}{\partial y_s^2}+\frac{\partial^2}{\partial z_s^2}\right)\varphi_k^s(\mathbf{r})+V(\mathbf{r}_s)\varphi_k^s(\mathbf{r})$$

$$=\frac{1}{\eta^2}\varepsilon_k^s\varphi_k^s(\mathbf{r}), \tag{A4}$$

where $\mathbf{r}_s=\eta\mathbf{r}$. Comparing the above equation to the original one (A1), it can be seen that

$$\varphi_k^s(\mathbf{r})\propto\varphi_k(\eta\mathbf{r}) \tag{A5}$$

and

$$\varepsilon_k^s=\eta^2\varepsilon_k. \tag{A6}$$

Up to an arbitrary phase factor, the proportional constant in Eq. (A5) can be determined by the normalization of wave functions $\varphi_k(\mathbf{r})$ and $\varphi_k^s(\mathbf{r})$. It gives

$$\varphi_k^s(\mathbf{r})=\eta^{3/2}\varphi_k(\eta\mathbf{r}). \tag{A7}$$

Therefore, we have the three-dimensional scaling of density

$$\rho(\mathbf{r})\rightarrow\rho_s(\mathbf{r})=\eta^3\rho(\eta\mathbf{r}). \tag{A8}$$

The spherical harmonic potential $V(\mathbf{r})=\frac{1}{2}m_N\omega^2r^2$ is an example for this scaling transformation. If the potential parameter ω has a change as

$$\omega\rightarrow\omega_s=\eta^2\omega, \tag{A9}$$

it can be shown easily that the potential and the energy eigenvalues have the two-dimensional scaling property (A3) and (A6), i.e.,

$$V(\mathbf{r})\rightarrow V_s(\mathbf{r})=\frac{1}{2}m_N\omega_s^2r^2=\eta^2\frac{1}{2}m_N\omega^2(\eta r)^2=\eta^2V(\eta\mathbf{r}), \tag{A10}$$

$$\varepsilon_k\rightarrow\varepsilon_k^s=\hbar\omega_s\left(n_k+\frac{1}{2}\right)=\eta^2\hbar\omega\left(n_k+\frac{1}{2}\right)=\eta^2\varepsilon_k. \tag{A11}$$

The change of the potential parameter ω (A9) means a squeezing (for $\eta>1$) or an extension (for $\eta<1$) of the potential radially, so it results in a squeezing (for $\eta>1$) or an extension (for $\eta<1$) of the nuclear density radially, as shown by Eq. (A8). In this way, the scaling assumption assumed in the text is suitable to describe the breathing mode.

The scaling of the shell correction E_{shell} and the even-odd energy $E_{\text{even-odd}}$ can be determined by the scaling (A6) of the single-particle energy of nucleons. The shell correction E_{shell} can be defined, according to the Strutinsky method [26], as [1]

$$E_{\text{shell}}=E-\widetilde{E}, \tag{A12}$$

where $E=\Sigma_{i=1}^{A}\varepsilon_{ki}$, and $\widetilde{E}$ is the smooth part of the above summation E after some arithmetic smoothing procedure. As both E and $\widetilde{E}$ are linear in ε_k, the summations as well as their difference (A12) will have the same scaling as ε_k (A6), i.e.,

$$E_{\text{shell}}\rightarrow E_{\text{shell}}(\eta)=\eta^2 E_{\text{shell}}, \tag{A13}$$

where $E_{\text{shell}}(\eta)$ is the scaled shell correction.

The even-odd energy $E_{\text{even-odd}}$ is the double of the energy gap Δ which is determined according to BCS theory [27] by the following gap equation [1]:

$$\frac{2}{G}=\int_{\varepsilon_F-S}^{\varepsilon_F+S}\frac{1}{2}\frac{g(\varepsilon)d\varepsilon}{\sqrt{(\varepsilon-\varepsilon_F)^2+\Delta^2}}, \tag{A14}$$

where G is the interaction constant, $g(\varepsilon)$ the single-particle energy level density, ε_F the Fermi energy of energy level, and S is the energy interval around the Fermi energy which is relevant for the pair correlation of nucleons. In case the single-particle potential of the mean field has the two-dimensional scaling (A3), it is consistent to assume that the residual interaction between the nucleons also has the two-dimensional scaling

$$G\rightarrow G_s=\eta^2 G. \tag{A15}$$

In addition, as the single-particle energy ε_k has the two-dimensional scaling (A6), it can be seen easily that the Fermi energy ε_F and the energy interval S also have the two-dimensional scaling

$$\varepsilon_F\rightarrow\varepsilon_F^s=\eta^2\varepsilon_F,\quad S\rightarrow S_s=\eta^2 S, \tag{A16}$$

whereas the energy level density $g(\varepsilon)$ has the following inverse two-dimensional scaling:

$$g(\varepsilon)\rightarrow g_s(\varepsilon_s)=\frac{f}{\Delta\varepsilon_s}=\frac{f}{\eta^2\Delta\varepsilon}=\frac{1}{\eta^2}g(\varepsilon), \tag{A17}$$

where f is the degeneracy of the single-particle energy level. Therefore, it can be shown from Eq. (A14) that the energy gap has two-dimensional scaling

$$\Delta\rightarrow\Delta_s=\Delta(\eta)=\eta^2\Delta. \tag{A18}$$

Accordingly, the even-odd energy $E_{\text{even-odd}}$ also has the two-dimensional scaling

$$E^s_{\text{even-odd}}=E_{\text{even-odd}}(\eta)=\eta^2 E_{\text{even-odd}} \tag{A19}$$

where $E_{\text{even-odd}}(\eta)$ is the scaled even-odd energy. All of the results given above are based on the model assumption (A3).

APPENDIX B: SOME FORMULAS FOR E_{GD}, K_3, AND K_4

The following are the functionals ϵ_{1n} and ϵ_{1p} that appear in the E_{GD}:

$$\epsilon_{1n}=-\epsilon_{1n\gamma}-\epsilon_{1n\alpha}+\epsilon_{1n\beta}, \tag{B1}$$

$$\epsilon_{1p}=-\epsilon_{1p\gamma}-\epsilon_{1p\alpha}+\epsilon_{1p\beta}, \tag{B2}$$

where

$$\epsilon_{1n\gamma}=\frac{1}{2}\gamma_l\left(\frac{\rho_n}{\hat{\rho}_0}\right)^{2/3}+\frac{1}{4}\gamma_u\begin{cases}3\left(\frac{\rho_p}{\hat{\rho}_0}\right)^{2/3}-\left(\frac{\rho_n}{\hat{\rho}_0}\right)^{2/3}, & \text{for}\quad \rho_n<\rho_p\\ 2\frac{\rho_p}{\hat{\rho}_0}\left(\frac{\rho_n}{\hat{\rho}_0}\right)^{-1/3}, & \text{for}\quad \rho_n>\rho_p,\end{cases} \tag{B3}$$

$$\epsilon_{1n\alpha}=\frac{1}{2}\left[\alpha_l\frac{\rho_n}{\hat{\rho}_0}+\alpha_u\frac{\rho_p}{\hat{\rho}_0}\right], \tag{B4}$$

$$\epsilon_{1n\beta}=\frac{3}{5}\left[B_l\left(\frac{\rho_n}{\hat{\rho}_0}\right)^{5/3}+B_u\left(\frac{\rho_p}{\hat{\rho}_0}\right)^{5/3}\right]; \tag{B5}$$

the expressions of $\epsilon_{1p\gamma}$, $\epsilon_{1p\alpha}$, and $\epsilon_{1p\beta}$ can be obtained from Eqs. (B3), (B4), and (B5), respectively, by interchanging the subscript n and p.

The following is the functional ϵ_{2n} that appears in the E_{GD}:

$$\epsilon_{2n}=\frac{1}{6}\left(\frac{\rho_n}{\hat{\rho}_0}\right)^{-1/3}\left[\gamma_l+\gamma_u\begin{cases}1, & \text{for}\quad\rho_n<\rho_p\\ \frac{\rho_p}{\rho_n}, & \text{for}\quad\rho_n>\rho_p\end{cases}\right]. \tag{B6}$$

The expression of ϵ_{2p} can be obtained by interchanging the subscript n and p.

The following formulas are used for calculating the anharmonic coefficients K_3 and K_4:

$$\frac{1}{A}\frac{d^3E_A(\eta)}{d\eta^3}\bigg|_{\eta=1}=\frac{\rho_0T_0}{A}\int d^3r\left[-6C(\delta)\left(\frac{\rho}{\rho_0}\right)^{6/3}+60D(\delta)\times\left(\frac{\rho}{\rho_0}\right)^{8/3}\right]+\frac{1}{A}\frac{d^3E_{\text{GD}}(\eta)}{d\eta^3}\bigg|_{\eta=1}, \tag{B7}$$

$$\frac{1}{A}\frac{d^4E_A(\eta)}{d\eta^4}\bigg|_{\eta=1}=\frac{\rho_0T_0}{A}\int d^3r\left[120D(\delta)\left(\frac{\rho}{\rho_0}\right)^{8/3}\right]+\frac{1}{A}\frac{d^4E_{\text{GD}}(\eta)}{d\eta^4}\bigg|_{\eta=1}, \tag{B8}$$

where $[d^3E_{\text{GD}}(\eta)/d\eta^3]_{\eta=1}$ and $[d^4E_{\text{GD}}(\eta)/d\eta^4]_{\eta=1}$ can be obtained by the standard scaling calculation.

APPENDIX C: SOME FORMULAS FOR INTEGRALS INVOLVING FERMI FUNCTION

Let us consider the integral

$$G_\mu(c)=\int_0^\infty dx f^\mu(x,c)g(x),\quad \mu>0, \tag{C1}$$

where $f(x,c)$ is the Fermi function

$$f(x,c)=\frac{1}{1+e^{x-c}},\quad 0\leqslant x<\infty,\quad 0\leqslant c<\infty, \tag{C2}$$

which has the following symmetry properties:

$$f(-x,-c)=f(c,x),\quad f(x,c)+f(c,x)=1, \tag{C3}$$

$$\frac{\partial^n f}{\partial x^n}=(-1)^n\frac{\partial^n f}{\partial c^n}. \tag{C4}$$

The nth order derivative of f can be expressed as a $n+1$ order polynomial of f as

$$\frac{\partial f}{\partial x}=f^2-f,\quad \frac{\partial^2 f}{\partial x^2}=2f^3-3f^2+f, \tag{C5}$$

and so on. For a well behaved function $g(x)$, the following formulas can be obtained by using Eqs. (C4) and (C5):

$$G_{\mu+1}(c)=G_{\mu}(c)-\frac{1}{\mu}\frac{dG_{\mu}(c)}{dc}, \tag{C6}$$

$$\int_0^\infty dx\frac{\partial f^{\mu}(x,c)}{\partial x}f^{\nu}(x,c)g(x)=\mu[G_{\mu+\nu+1}(c)-G_{\mu+\nu}(c)], \tag{C7}$$

$$\int_0^\infty dx\frac{\partial^2 f^{\mu}(x,c)}{\partial x^2}f^{\nu}(x,c)g(x)=\mu(\mu+1)G_{\mu+\nu+2}(c)-\mu(2\mu+1)G_{\mu+\nu+1}(c)+\mu^2G_{\mu+\nu}(c). \tag{C8}$$

Applying Eq. (C6), the above Eqs. (C7) and (C8) can be rewritten as

$$\int_0^\infty dx\frac{\partial f^{\mu}(x,c)}{\partial x}f^{\nu}(x,c)g(x)=-\frac{\mu}{\mu+\nu}\frac{dG_{\mu+\nu}(c)}{dc}, \tag{C9}$$

$$\int_0^\infty dx\frac{\partial^2 f^{\mu}(x,c)}{\partial x^2}f^{\nu}(x,c)g(x)=-\frac{\mu(\mu+1)}{\mu+\nu+1}\frac{dG_{\mu+\nu+1}(c)}{dc}+\frac{\mu^2}{\mu+\nu}\frac{dG_{\mu+\nu}(c)}{dc}. \tag{C10}$$

Furthermore, if $g(0)=0$, $h(x)=dg(x)/dx$, and $\lim_{x\to\infty}f^{\mu}(x,c)g(x)=0$, it can be shown that

$$\frac{dG_{\mu}(c)}{dc}=H_{\mu}(c),\quad G_{\mu}(c)=G_{\mu}(0)+\int_0^c dc\ H_{\mu}(c), \tag{C11}$$

where

$$H_{\mu}(c)=\int_0^\infty dxf^{\mu}(x,c)h(x). \tag{C12}$$

Similarly, we also have

$$\int_0^\infty dx\frac{\partial f^{\mu}(x,c)}{\partial x}f^{\nu}(x,c)g(x)=-\frac{\mu}{\mu+\nu}H_{\mu+\nu}(c). \tag{C13}$$

Using Eq. (C7), we can obtain another recursion relation

$$G_{\mu+1}(c)-G_{\mu}(c)=-\frac{1}{\mu}H_{\mu}(c). \tag{C14}$$

In the present application, the following integral with $g(x)=x^{\nu}$ is needed:

$$I_{\mu\nu}(c)=\int_0^\infty dxf^{\mu}(x,c)x^{\nu},\quad \mu>0,\quad \nu>-1. \tag{C15}$$

According to Eqs. (C6), (C11), and (C14), the following recursion relations of $I_{\mu\nu}(c)$ can be obtained, respectively:

$$I_{\mu\nu}(c)=I_{\mu-1,\nu}(c)-\frac{1}{\mu-1}\frac{dI_{\mu-1,\nu}(c)}{dc}, \tag{C16}$$

$$\frac{dI_{\mu\nu}(c)}{dc}=\nu I_{\mu\nu-1}(c),\quad I_{\mu\nu}(c)=I_{\mu\nu}(0)+\nu\int_0^c dcI_{\mu\nu-1}(c), \tag{C17}$$

$$I_{\mu+1,\nu}(c)=I_{\mu\nu}(c)-\frac{\nu}{\mu}I_{\mu,\nu-1}(c). \tag{C18}$$

The general expression of $I_{\mu\nu}(c)$ for integer $\nu\geqslant 0$ can be found in Refs. [28–30]. We will give some simple relations here for both $\mu=m$ and $\nu=n$ being integers. In this case, the specific expression of $I_{mn}(c)$ can be obtained from the simplest $I_{10}(c)$ and the value of $I_{1n}(0)$:

$$I_{10}(c)=\int_0^\infty\frac{dx}{1+e^{x-c}}=c+\omega_{10}(c),$$

$$\omega_{10}(c)=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}e^{-kc}, \tag{C19}$$

$$I_{1n}(0)=\int_0^\infty\frac{x^n dx}{1+e^x}=\Gamma(n+1)\left(1-\frac{1}{2^n}\right)\zeta(n+1),\quad n>-1, \tag{C20}$$

where $\Gamma(p)$ is the gamma function, and $\zeta(p)$ is the Riemann function.

The expression of $I_{1n}(c)$ can be obtained from $I_{10}(c)$ and $I_{1n}(0)$ by Eq. (C17):

$$I_{1n}(c)=I_{1n}(0)+n\int_0^c dcI_{1n-1}(c)=P_{1n}(c)+\omega_{1n}(c), \tag{C21}$$

where

$$P_{1n}(c)=n\int_0^c dcP_{1n-1}(c)+P_{1n}(0) \tag{C22}$$

is a $n+1$ order polynominal of c and

$$\omega_{1n}(c) = -n\int_c^\infty dc\ \omega_{1n-1}(c)$$
$$= (-1)^{n-1} n! \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^{n+1}} e^{-kc}. \quad \text{(C23)}$$

The integral constant $P_{1n}(0)$ in Eq. (C22) can be calculated as

$$P_{1n}(0) = I_{1n}(0) - \omega_{1n}(0)$$
$$= [1-(-1)^n](1-2^{-n})\ n!\ \zeta(n+1). \quad \text{(C24)}$$

The expression of $I_{mn}(c)$ can be obtained from $I_{1n}(c)$ by Eq. (C16):

$$I_{mn}(c) = I_{m-1,n}(c) - \frac{1}{m-1}\frac{dI_{m-1,n}(c)}{dc} = P_{mn}(c) + \omega_{mn}(c), \quad \text{(C25)}$$

where

$$P_{mn}(c) = P_{m-1,n}(c) - \frac{1}{m-1}\frac{dP_{m-1,n}(c)}{dc} \quad \text{(C26)}$$

is a $n+1$ order polynominal of c and

$$\omega_{mn}(c) = \omega_{m-1,n}(c) - \frac{1}{m-1}\frac{d\omega_{m-1,n}(c)}{dc} \quad \text{(C27)}$$

is an infinite series in power of $\exp(-c)$.

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