Correlations of the deformation variables β and γ in even-even Hf, W, Os, Pt, and Hg nuclei

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(Received 23 August 1996)

In the framework of the triaxial rotor model of Davydov and Filippov, deformation parameters β and γ are extracted from both level energies and $E2$ transition rates in even-even Hf-Hg nuclei. Three results emerge: the two sets of β and γ values—energy-based and transition-rate-based—are in good agreement, with only a few exceptions, thus giving confidence in the extracted values; both β and γ follow smooth trajectories against N_pN_n; and the β and γ values themselves are correlated, pointing to the possibility of a simpler description of structural evolution. [S0556-2813(97)03201-9]

PACS number(s): 21.60.Ev, 21.10.Re, 23.20.−g, 27.70.+q

I. INTRODUCTION

The deformation parameters, β and γ, of the collective model [1] are basic descriptors of the nuclear equilibrium shape and structure. While values for these variables have been discussed for many nuclei [2–6], a systematic study in particular regions can nevertheless be revealing. We present here such a study, for the Hf-Hg nuclei where β and γ both vary strongly. We will show correlations of these variables with each other and with external parameters that may give clues to a simpler description of nuclear structure. In doing this, we will use the Davydov and Filippov model [7]. Even though this model embodies a nuclear shape with rigid triaxiality and these nuclei are known [8] to be γ-soft, the expectation or rms values of β and γ extracted should be valid. Differences between rigid and γ-soft models [7,9,10] mostly show up only in observables which are not used here (such as γ-band energy staggering) [8].

II. THE RIGID TRIAXIAL ROTOR MODEL (RTRM)

For a nucleus with quadrupole deformation, one can write the nuclear radius as $R=R_0[1+\Sigma_\mu \alpha_{2\mu}Y_{2\mu}(\Theta,\Phi)]$ where R_0 is the radius of the spherical nucleus with the same volume and the $Y_{2\mu}$ are spherical harmonics of order 2 [1]. The five expansion coefficients $\alpha_{2\mu}$ can be expressed as $\alpha_{21}=\alpha_{2-1}=0$, $\alpha_0=\beta\cos\gamma$ and $\alpha_{22}=\alpha_{2-2}=(1/\sqrt{2})\beta\sin\gamma$. The nuclear shape is then determined only in terms of β and γ where β represents the extent of quadrupole deformation and γ gives the degree of axial asymmetry. The relation between these deformation parameters and the nuclear radius can be evaluated by the change in radius $\delta R_k=R_k-R_0=\sqrt{5/(4\pi)}R_0\beta\cos(\gamma-2k\pi/3)$ with $k=1$, 2, 3. These equations show that it is sufficient to use only $\beta\geqslant 0$ and $0°\leqslant\gamma\leqslant 60°$ in order to describe the nuclear shape, because for every set of parameters outside this range, it is possible to find parameters inside this range which describe the same shape of the nucleus, with only the orientation in the coordinate system different. In Table I we have summarized the possible shapes of a nucleus with the corresponding values of β and γ.

The RTRM considers the nucleus as a rigid rotor with rigid triaxial asymmetry as specified by β and γ. The Davydov and Filippov [7] Hamiltonian can be written as

$$H=\frac{\hbar^2}{2}\sum \frac{I_i^2}{\Theta_i}, \tag{1}$$

where I_i are the projections of the angular momentum on the intrinsic axes. The moments of inertia are given by the hydrodynamical formula

$$\Theta_i=\frac{4}{3}\Theta_0\sin^2\left(\gamma-\frac{2\pi}{3}i\right). \tag{2}$$

Since Θ_0 depends on β, Θ_i involves both deformation parameters. From Eqs. (1) and (2) Davydov and Filippov obtain expressions for energies and $E2$ transition probabilities. The energies of the $2^+_{1,2}$ states are given by

$$E_{2^+_{1,2}}=\left(\frac{6\hbar^2}{2\Theta_0}\right)\frac{9-(-1)^{\sigma_{1,2}}\sqrt{81-72\sin^2(3\gamma)}}{4\sin^2(3\gamma)}, \tag{3}$$

where $\sigma_{1,2}=0,1$. The reduced $E2$ transition probabilities from the $2^+_{1,2}$ states to the ground state can be expressed as

$$B(E2;2^+_{1,2}\to 0^+_1)=\frac{1}{2}\left(\frac{e^2Q_0^2}{16\pi}\right)\left(1+(-1)^{\sigma_{1,2}}\frac{3-2\sin^2(3\gamma)}{\sqrt{9-8\sin^2(3\gamma)}}\right), \tag{4}$$

where $Q_0=3ZR^2\beta/\sqrt{5\pi}$, and the value of $B(E2;2^+_2\to 2^+_1)$ is given by

TABLE I. Possible nuclear shapes as a function of the deformation parameters β and γ.

Spherical:	$\beta=0$
Prolate:	$\beta>0,\ \gamma=0°$
Oblate:	$\beta>0,\ \gamma=60°$
Triaxial:	$\beta>0,\ 0°<\gamma<60°$

TABLE II. γ and β deformation parameters for even-even Hf, W, Os, Pt, and Hg nuclei obtained from indicated values of $R_b = B(E2;2_2^+ \to 2_1^+)/B(E2;2_2^+ \to 0_1^+)$, $B(E2;2_1^+ \to 0_1^+)$, $R_e = E_{2_2^+}/E_{2_1^+}$ and the energies of the 2_1^+ states. The energy is given in parenthesis, if the experimental data for spin and parity of the corresponding state are uncertain. The data are from Refs. [13–15].

Nucleus	$E(2_1^+)$ [keV]	$E(2_2^+)$ [keV]	$B(E2,2_1^+ \to 0_1^+)$ [e^2b^2]	R_e	R_b	β_e	β_b	γ_e [deg]	γ_b [deg]
^{164}Hf	211.1	(815.9)		3.86	7.8(11)	0.22		19.7	21.8(7)
^{166}Hf	158.5	(810.1)	0.692(36)	5.11	2.59(22)	0.25	0.255(8)	17.2	14.5(10)
^{168}Hf	124.0	(875.4)	0.858(40)	7.06	3.47(57)	0.27	0.284(8)	14.8	17.2(16)
^{170}Hf	100.8	(961.3)	1.000(22)	9.54	28(11)	0.29	0.300(2)	12.8	25.7(12)
^{172}Hf	95.2	952.4	0.878(60)	10.0	3.85(21)	0.29	0.283(11)	12.5	18.0(5)
^{174}Hf	91.0	900.2	0.960(58)	9.89	2.24(36)	0.30	0.290(11)	12.6	12.9(27)
^{176}Hf	88.4	1226.6	1.054(20)	13.9	2.69(41)	0.29	0.303(5)	10.7	14.9(19)
^{178}Hf	93.2	1174.6	0.964(12)	12.6	1.13(10)	0.28		11.2	
^{180}Hf	93.3	1183.4	0.930(16)	12.7	58(14)	0.28	0.276(2)	11.2	27.0(5)
^{182}Hf	97.8	(818.4)		8.37	1.89(51)	0.27		13.6	$<$13.7
^{178}W	106.1	1082.8		10.2		0.27		12.4	
^{180}W	103.6	(1117.3)	0.838(46)	10.8	1.94(10)	0.27	0.256(9)	12.1	10.8(10)
^{182}W	100.1	1221.4	0.83(2)	12.2	1.98(1)	0.27	0.254(4)	11.4	11.1(1)
^{184}W	111.2	903.3	0.746(14)	8.12	1.89(3)	0.25	0.238(3)	13.8	10.4(4)
^{186}W	122.6	737.9	0.688(12)	6.02	2.26(21)	0.24	0.229(4)	16.0	13.0(14)
^{174}Os	158.6	846.4		5.34	1.23(25)	0.23		16.9	
^{176}Os	135.0	863.6		6.40	3.7(6)	0.25		15.5	17.7(15)
^{178}Os	132.2	864.3		6.54	1.77(20)	0.24		15.3	9.1(30)
^{180}Os	132.4	870.7		6.58	0.61(14)	0.24		15.3	
^{182}Os	126.9	890.5	0.762(66)	7.02	2.30(15)	0.24	0.238(12)	14.8	13.2(9)
^{184}Os	119.7	942.7	0.64(3)	7.88	2.19(8)	0.24	0.216(6)	14.0	12.6(6)
^{186}Os	137.2	767.5	0.582(20)	5.59	2.26(12)	0.23	0.205(5)	16.5	13.0(8)
^{188}Os	155.0	633.0	0.508(12)	4.08	3.28(10)	0.22	0.192(3)	19.2	16.7(3)
^{190}Os	186.7	558.0	0.46(2)	2.99	5.58(19)	0.21	0.182(5)	22.3	20.2(3)
^{192}Os	205.8	489.1	0.41(1)	2.38	7.6(3)	0.20	0.170(3)	25.2	21.7(3)
^{194}Os	(218.5)	(656.5)		3.00	4.0(6)	0.19		22.2	18.2(12)
^{180}Pt	153.3	(677.5)		4.42	10.0(18)	0.23		18.5	22.8(8)
^{182}Pt	154.8	667.1		4.31	$<$5.44	0.23		18.7	$<$20.1
^{184}Pt	163.0	648.8	0.79(3)	3.98	8.2(16)	0.22	0.237(5)	19.4	22.1(11)
^{186}Pt	191.6	607.2	0.596(22)	3.17	1.81(68)	0.21	0.200(6)	21.6	$<$14.2
^{188}Pt	265.6	605.7	0.52(9)	2.28	27(2)	0.18	0.187(18)	25.9	25.6(2)
^{190}Pt	295.8	597.6	0.35(4)	2.02	74(6)	0.17	0.15(1)	31.1	32.6(2)
^{192}Pt	316.5	612.5	0.382(12)	1.94	203(10)		0.155(3)		31.60(4)
^{194}Pt	328.5	622.0	0.332(12)	1.89	320(15)		0.144(3)		31.27(4)
^{196}Pt	355.7	688.7	0.28(1)	1.94	$1.4(4)\times 10^5$		0.131(3)		30.0
^{198}Pt	407.2	774.7	0.212(10)	1.90	980(270)		0.113(3)		30.7(2)
^{188}Hg	412.8	881.0		2.13	12.5(50)	0.15		32.8	36.4(10)
^{190}Hg	416.4	1099.9		2.64	13.9(56)	0.14		36.3	36.1(18)
^{192}Hg	422.8	1113.6		2.63	33(7)	0.14		36.2	33.9(6)
^{194}Hg	428.0	(1073.2)		2.51	11-56	0.14		35.6	34.9(20)
^{196}Hg	426.1	1036.2	0.23(1)	2.43	43(12)	0.13	0.117(4)	35.1	33.5(6)
^{198}Hg	411.8	1087.7	0.198(2)	2.64	29(4)	0.13	0.1084(9)	36.3	34.2(4)
^{200}Hg	367.9	1254.1	0.171(2)	3.41	9.61(95)	0.14	0.1014(8)	39.1	37.3(5)
^{202}Hg	439.6	959.7	0.122(2)	2.18	78(10)	0.13	0.0832(9)	33.3	32.6(2)

$$B(E2;2_2^+ \to 2_1^+) = \frac{10}{7}\left(\frac{e^2 Q_0^2}{16\pi}\right)\frac{\sin^2(3\gamma)}{9-8\sin^2(3\gamma)}. \tag{5}$$

Equations (3), (4), and (5) were used to evaluate the β and γ deformation parameters of even-even Hf, W, Os, Pt, and Hg nuclei. γ may be determined in two ways. The ratio $R_e = E_{2_2^+}/E_{2_1^+}$ depends only on γ. We denote the parameter obtained from these energies by γ_e. An independent approach is based on the reduced $E2$ transition probabilities of the $2_{1,2}^+$ states. Gupta *et al.* [3] have used the experimental quantities $E_{2_{1,2}^+}$, $B(E2;2_1^+ \to 0_1^+)$ and $B(E2;2_2^+ \to 0_1^+)$ in order to determine β and γ. But in many cases the required values $B(E2;2_2^+ \to 0_1^+)$ are unknown. However, often the branching ratio $R_b = B(E2;2_2^+ \to 2_1^+)/B(E2;2_2^+ \to 0_1^+)$ is ob-

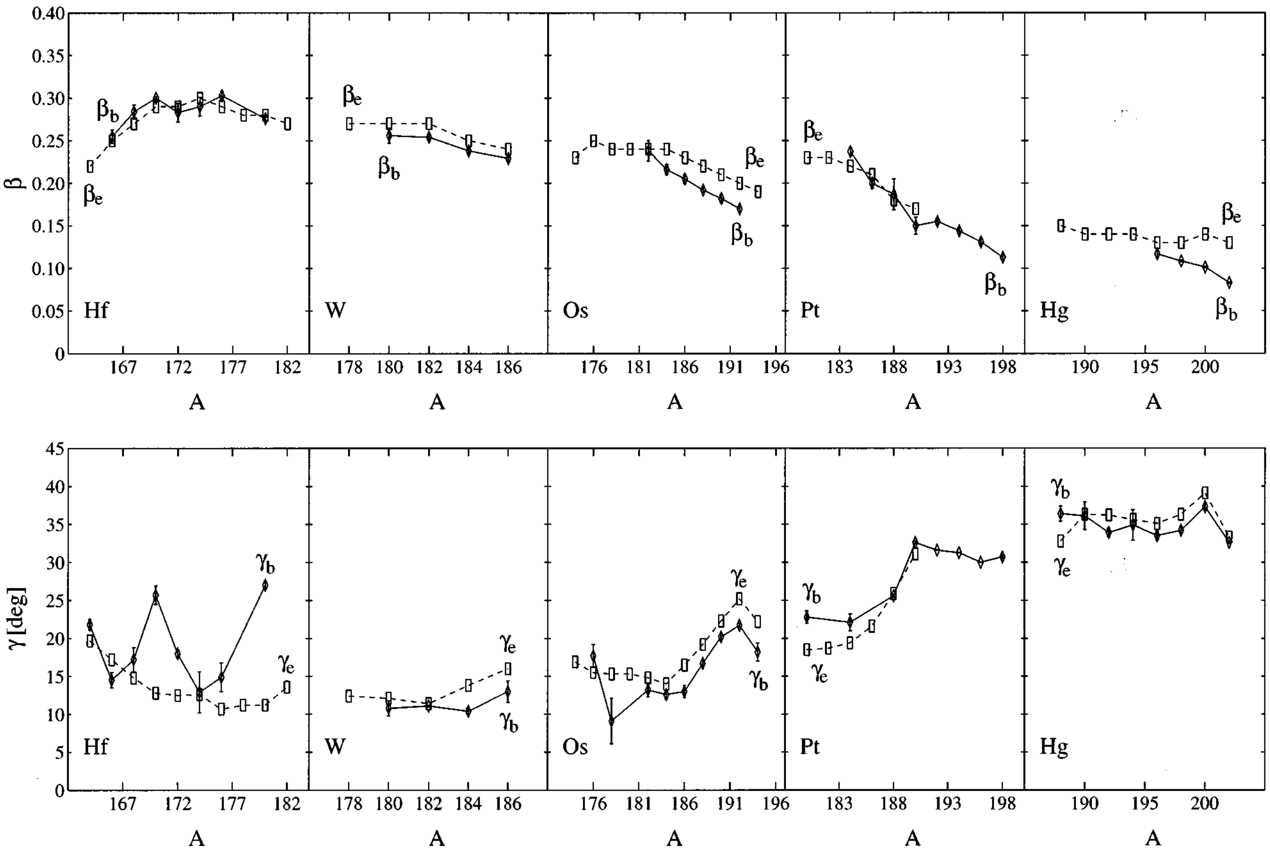

FIG. 1. Comparison of (β_e, γ_e) and (β_b, γ_b) values.

served which is only a function of γ and allows therefore the determination of this parameter. Asymmetries so determined are denoted γ_b. β is easily determined from Eq. (4), for the $2_1^+ \rightarrow 0_1^+$ transition using these γ_b values and $Q_0 = 3ZR^2\beta/\sqrt{5\pi}$. Similarly to the γ values we denote the β values obtained solely from the $B(E2)$ value by β_b. In a second way, especially where the $B(E2;2_1^+ \rightarrow 0_1^+)$ value is not known, we estimate β by using the approximate empirical Grodzins relation [11]

$$E_{2_1^+} \cdot B(E2;2_1^+ \rightarrow 0_1^+) = 2.5 \times 10^{-3} Z^2 A^{-1} \text{ [MeV } e^2\text{b}^2]. \tag{6}$$

By substituting the result for an axially symmetric nucleus $B(E2;2_1^+ \rightarrow 0_1^+) = e^2 Q_0^2/16\pi = 9e^2 Z^2 R^4 \beta^2/80\pi^2$ (in units of $e^2\text{b}^2$) we can relate β and $E_{2_1^+}$. We obtain

$$\beta_G^2 \cong \frac{1224}{E_{2_1^+} A^{7/3}}, \tag{7}$$

where $E_{2_1^+}$ is in MeV. Since, in practice $\gamma \neq 0°$, this result needs to be corrected by the factor multiplying $6\hbar^2/2\Theta_0$ in Eq. (3) for $E_{2_1^+}$, giving

$$\beta = \beta_G \left(\frac{9 - \sqrt{81 - 72\sin^2(3\gamma)}}{4\sin^2(3\gamma)} \right)^{1/2}. \tag{8}$$

We label the β values obtained from $E_{2_1^+}$ in this way by β_e.

The method to calculate γ from the energy ratio $R_e = E_{2_2^+}/E_{2_1^+}$ fails for nuclei where the ratio is lower than two, because this is the lowest possible value in the RTRM. However, with the method based on the transition probabilities, we could avoid this difficulty and determined β and γ for many nuclei even if $R_e < 2$.

For any ratios R_e or R_b, there exist simultaneously two solutions for the γ deformation parameter (if $\gamma \neq 30°$), one less than 30° for a more prolate shape and the other larger than 30° for a more oblate shape. These solutions cannot be distinguished without supplementary information. Here we use guidance from Möller *et al.* [12] who have calculated ground-state electric quadrupole moments for a nucleus with a sharp surface. They found that the quadrupole moments for all mercury ($^{188-202}$Hg) isotopes investigated as well as for $^{190-198}$Pt have negative signs which suggests an oblate shape for these nuclei. The signs for $^{164-182}$Hf, $^{178-186}$W, $^{174-194}$Os, and $^{180-188}$Pt are positive which suggests a prolate shape. We therefore use the calculated quadrupole moments to give guidance if the nucleus is more oblate or prolate and to distinguish the two solutions in the RTRM. Therefore, we give in Table II, for nuclei with negative calculated quadrupole moments, the solution where γ is larger than 30° and, for nuclei with positive calculated quadrupole moment, the solution where γ is less than 30°.

III. DISCUSSION

The deformation parameters (β_e, γ_e) and (β_b, γ_b) extracted above from energies and from γ-band transition rates

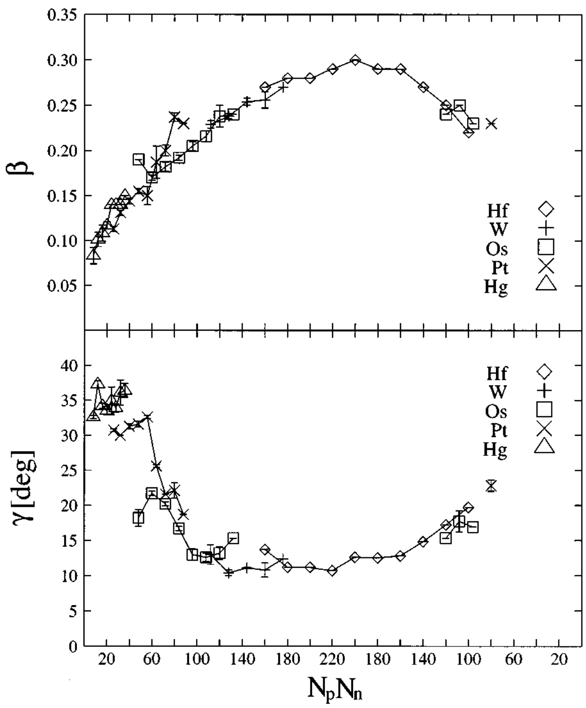

FIG. 2. β and γ values against N_pN_n. Where values based on energies and transition rates differ substantially, those values giving the smoother systematics are used. If values based on transition rates are unknown or uncertain, values based on energies are given. The points on the left side up to $N_pN_n = 220$ represent nuclei where the number of neutrons is nearer to the magic shell closure at $N=126$. The points on the right side represent nuclei where the number of neutrons is nearer to the magic shell closure at $N=82$.

for Hf-Hg are compared in Fig. 1. Generally, there is excellent accord. Only for a couple of isolated points in Hf and Os are the γ_b and γ_e values substantially different. In each of these cases the γ_b values seems erratic. This may be due in some cases to unknown or incorrect $M1$ components for the $2_2^+ \rightarrow 2_1^+$ transitions.

Having shown that similar sets of deformation variables result from both energy and transition rate observables, we can have some confidence in the values obtained. We therefore now inspect the systematics of these values in this region and note two significant conclusions.

Figure 2 shows β and γ values plotted against the valence nucleon product N_pN_n [16]. This quantity is known to correlate extremely well with observables reflecting the equilibrium shape and structure of the nucleus, such as $E(2_1^+)$,$E(4_1^+)$,$R_{4/2} \equiv E(4_1^+)/E(2_1^+)$, $B(E2;2_1^+ \rightarrow 0_1^+)$ values, nuclear radii, and the like [17]. Since β is closely related to the above $B(E2)$ values (indeed it was extracted from it), it is not surprising that β correlates with N_pN_n. However, the smooth behavior of γ with N_pN_n is striking. To our knowledge, this has been shown in only one other region, the γ-soft Xe-Ba nuclei near $A=130$ [18]. However, in that region the γ values are limited to $20° - 30°$ whereas here they span nearly the full range observed in actual nuclei, namely from $\gamma \geqslant 10°$. This result suggests that a systematic study over all nuclei would be worthwhile.

The second correlation is already apparent in comparing the two parts of Fig. 2 and is made explicit in Fig. 3 which

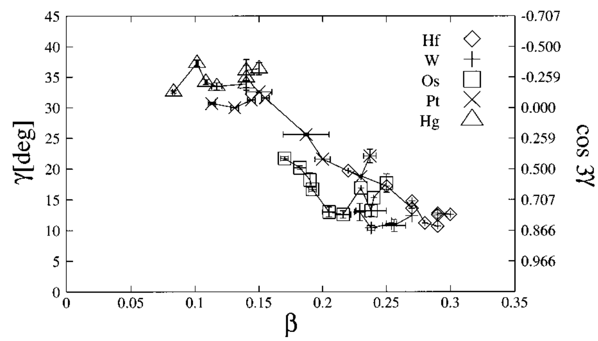

FIG. 3. Correlation of γ (and $\cos 3\gamma$) with β.

shows the γ values (as well as $\cos 3\gamma$) plotted against β. The correlation is excellent. A similar result has been noted by Andrejtscheff *et al.* [5] using the method of shape invariants [19,20]. That method is, in principle, model independent. However, it requires more-difficult-to-obtain data, including quadrupole moments. That limits the number of nuclei (and the accuracy of the γ values) where it can be applied. Nevertheless, over a broad range of nuclei from $A \sim 90-190$, Ref. [5] showed an approximate correlation of β and γ. Our results, shown in Fig. 3, represent a more detailed look at a specific region where we have used the more readily observable data required in Eqs. (3)–(5) to obtain results for over 40 nuclei. The same striking correlation emerges.

If β and γ are correlated, this suggests the possibility of obtaining a one-parameter description of nuclear shapes by relating one to the other. For example, Fig. 3 suggests that γ is approximately linear in β for the $A \sim 160-200$ region.

IV. CONCLUSIONS

We have investigated the systematic behavior of the β and γ shape variables in the Hf-Hg region. Three conclusions emerge:

(i) Sets of β and γ values extracted independently from energies and transition rates are generally very consistent (with only a few exceptions), thus giving confidence in the values obtained.

(ii) γ (and β) values were shown to be excellently correlated with N_pN_n. This extends (both to a new mass region and to a larger range of γ values) a conclusion recently noted in the $A=130$ nuclei [18].

(iii) γ and β are themselves closely correlated. This result and that of the global but less detailed survey of Ref. [5], suggest that these variables may be generally correlated. If this is so, one could write β and γ in terms of a single variable and obtain a simpler one-parameter description of nuclear shapes. At the least, our results point to the interest in further study of this issue. Finally, we note that our results do not distinguish whether the origin of finite γ values lies in rigid triaxiality or if it stems from γ softness.

ACKNOWLEDGMENTS

We are grateful to W. Andrejtscheff, N. V. Zamfir, and N. Pietralla for useful discussions. This work was supported in part by the BMBF under Contract No. 06 OK 668 and by the U.S. DOE under Contract No. DE–FG02-91ER40609.

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