# Calculation of shell model energies for states in $^{210}$Bi

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A total of 34 experimental levels of $^{210}$Bi have been fitted with a generalized intermediate coupling model involving a *n-p* interaction with a Gaussian radial dependence with and without coupling to an octupole phonon. $\chi^2$ values are somewhat lower with the octupole parameter. Comparison of $\chi^2$ values with those obtained by fitting the same 34 experimental levels using the delta force radial dependence indicate the Gaussian radial dependence is 8–10 times lower. The *n-p* interaction parameters are then used to calculate other $^{210}$Bi levels. [S0556-2813(97)02801-X]

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## I. INTRODUCTION

With just one proton and one neutron beyond the strongly double closed shell $^{208}$Pb, $^{210}$Bi is an ideal nucleus for testing the shell model calculations on odd-odd spherical nuclei. Furthermore, a considerable amount of experimental data on $^{210}$Bi has accumulated [1–12]. The most ambitious attempt at analysis [12] presents assignments for 13 configurations (some of which are not completely assigned) and a total of 83 states.

A number of theoretical calculations of $^{210}$Bi [13–21] have been attempted. The best agreement between experiment and theory for $^{210}$Bi results from the more phenomenological calculation of Kim and Rasmussen [15,16]. Their calculation utilized the tensor force and a very careful fitting of various force parameters to the $\pi h_{9/2}\otimes\nu g_{9/2}$ ground state configuration which they have shown to be very sensitive to the final results.

The calculation of Herling and Kuo [19] used a larger shell model space than Kim and Rasmussen, and pair reaction matrix elements deduced from the Hamada-Johnston [22] potential using the model of Kuo and Brown [23]. Thus the calculations of Herling and Kuo represent a more sophisticated and exacting test of theory since they allow no variation in the nucleon-nucleon interaction parameters from the free space values. However, it is clear that these more sophisticated theories do not do as well [12] in the prediction of experimentally observed states. This is particularly true of the $1^-$ and $0^-$ ground and first excited states which Herling and Kuo [19], and even the most recent rendition of Herling and Kuo by Warburton and Brown [21], reverse.

In this paper, then, we revert to the more phenomenological treatments. However, we attempt to consider carefully both the tensor and spin-orbit forces and octupole collectivity which has been shown to be important in nuclei just beyond $^{208}$Pb. The model we use in calculating the levels in $^{210}$Bi is the generalized intermediate coupling model (GICM). A description of this model follows.

## II. THEORETICAL DESCRIPTION OF ODD-ODD SPHERICAL NUCLEI IN THE GICM

In the GICM, odd-odd nuclei are assumed to consist of a vibrating even-even core and two outer nucleons (odd proton and odd neutron). Thus, the model Hamiltonian can be written in the form

$$H_{\text{odd-odd}}=H_{\text{core}}+H_n+H_p+H_{n\text{-core}}+H_{p\text{-core}}+H_{np}\,, \quad (1)$$

where $H_{\text{core}}$ is the Hamiltonian of the even-even core, $H_n$, $H_p$ Hamiltonians of odd nucleons, $H_{p\text{-core}}$, $H_{n\text{-core}}$ Hamiltonians of the interaction of odd nucleon and even-even core, and $H_{np}$ the Hamiltonian of the residual interaction between odd neutron and odd proton.

### A. Hamiltonian of the even-even core, $H_{\text{core}}$

Core vibrations are described phenomenologically with help of creation and annihilation phonon operators, $b^\dagger_{\lambda\mu}$ and $b_{\lambda\mu}$, respectively. In the harmonic approximation we can write [24]

$$H_{\text{core}}=\sum_\lambda \hbar\omega_\lambda \sum_{\mu=-\lambda}^{\lambda} (b^\dagger_{\lambda\mu}b_{\lambda\mu}+1/2), \quad (2)$$

where $\hbar\omega_\lambda$ is the energy of the phonon with multipolarity $\lambda$.

In our calculations of $^{210}$Bi, only one octupole phonon core vibration at 2615 keV (experimental energy of the lowest $3^-$ state in $^{208}$Pb) is considered.

### B. Hamiltonian of the odd neutron and odd proton, $H_n$ and $H_p$

For a description of one-quasiparticle states (or one-particle states in the case of $^{210}$Bi), the model of independent quasiparticles (particles) is used [25]:

$$H_n=H_{\text{av}}(n)+H_{\text{pair}}(n), \quad H_p=H_{\text{av}}(p)+H_{\text{pair}}(p). \quad (3)$$

 

$H_{\rm av}(n)$ and $H_{\rm av}(p)$, respectively, represent standard one-particle phenomenological Hamiltonian of the shell model for spherical nuclei [25]:

$$H_{\rm av}=-\frac{\hbar^2}{2m}\Delta+\frac{1}{2}m\omega_0^2r^2-V_{ls}(r)\boldsymbol{l}\cdot\boldsymbol{s}+D(l^2-\langle l^2\rangle), \quad (4)$$

where $m$ is the mass of nucleon, $\hbar\omega_0=41A^{1/3}$ MeV is the frequency of oscillations, $V_{ls}(r)$ is the radial part of the spin-orbit interaction, the term $D(l^2-\langle l^2\rangle)$ is the correction for heavy nuclei.

The Hamiltonian of the pairing interaction, $H_{\rm pair}$, is written in the form

$$H_{\rm pair}=-\frac{G}{4}\sum_{jm}\sum_{j'm'}(-1)^{j-m}(-1)^{-j'+m'}\times a^{\dagger}_{jm}a^{\dagger}_{j-m}a_{j'-m'}a_{j'm'}, \quad (5)$$

where $G$ is the interaction constant, and $a^{\dagger}_{jm}$ and $a_{jm}$ are particle creation and annihilation operators.

In our calculations of $^{210}$Bi, the pairing interaction is omitted since we have a doubly magic core $^{208}$Pb and only two outer nucleons. Therefore, neutron-particle and proton-particle states are taken from odd neighbors, $^{209}$Pb and $^{209}$Bi, respectively.

### C. Hamiltonian of the interaction between the odd nucleon and vibrating core, $H_{n\text{-core}}+H_{p\text{-core}}$

We assume [24] that the interaction between the odd nucleon and vibrating core in the first approximation is proportional to $\alpha_{\lambda\mu}$,

$$H_{n\text{-core}} \text{ or } H_{p\text{-core}}=-\sum_{\lambda}k_{\lambda}(r)\sum_{\mu=-\lambda}^{\lambda}Y^{*}_{\lambda\mu}(r)\alpha_{\lambda\mu}, \quad (6)$$

where $Y_{\lambda\mu}(r)$ are spherical harmonics, and $k_{\lambda}(r)$ is a form factor, which depends on the nuclear mean field potential $U(\boldsymbol{r})=H_{\rm av}(\boldsymbol{r})$ [24]:

$$k_{\lambda}(r)=R_0\frac{\partial U}{\partial r}, \quad (7)$$

where $R_0$ is the nuclear radius. Equation (6) can be written in the form

$$H_{n\text{-core}}=\sum_{\lambda}(-1)^{\lambda+1}(2\lambda+1)^{1/2}k_{\lambda}(r)(Y_{\lambda}\alpha_{\lambda})_0. \quad (8)$$

The amplitudes $\alpha_{\lambda\mu}$ can be described by creation and annihilation phonon operators $b^{\dagger}_{\lambda\mu}$ and $b_{\overline{\lambda\mu}}$ [24]:

$$\alpha_{\lambda\mu}=i^{-\lambda}(\hbar^2/4C_{\lambda}D_{\lambda})^{1/4}(b^{\dagger}_{\lambda\mu}+b_{\overline{\lambda\mu}}), \quad (9)$$

where $b_{\overline{\lambda\mu}}=(-1)^{\lambda+\mu}b_{\lambda-\mu}$, and $C_{\lambda}$ and $D_{\lambda}$ are parameters of the Bohr liquid drop model, which we use for the description of the vibrating even-even core.

Instead of $k_{\lambda}$ in $H_{n\text{-core}}$ and $H_{p\text{-core}}$, $\xi_{\lambda}$ is used:

$$\xi_{\lambda}=\langle k_{\lambda}\rangle\left(\frac{2\lambda+1}{2\pi\hbar C_{\lambda}}\right)^{1/4}, \quad (10)$$

where $\langle k_{\lambda}\rangle$ is the mean value of the integral $\int_0^{\infty}k_{\lambda}(r)R_{n'l'j'}(r)R_{nlj}(r)r^2dr$ averaged over the whole set of the single-particle $|nljm\rangle$ states, $\xi_{\lambda}$ being the only free parameters of $H_{n\text{-core}}$ and $H_{p\text{-core}}$ fitted to experimental data. In our calculations, we assume the same values of $\xi_{\lambda}$ for $H_{n\text{-core}}$ and $H_{p\text{-core}}$.

### D. Hamiltonian of the interaction of the odd neutron and proton, $H_{np}$

The potential of the interaction of free nucleons cannot be used in our case, because a part of it has already been included in the mean field. Therefore one must use an effective phenomenological nucleon-nucleon potential. It is usually selected in a simplified form and fulfilling some expected symmetries (e.g., rotational invariance, time-reversal invariance, invariance under space reflection, Galileian invariance, and hermicity). The general nucleon-nucleon interaction can be composed of the central part $V_c$, noncentral tensor part $V_t$ and spin-orbit part $V_{ls}$ [26,27]:

$$H_{np}=V_c+V_t+V_{ls}. \quad (11)$$

For the central part $V_c$ we use [26]

$$V_c=V_c(r)(u_0+u_1\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n+u_2P_M+u_3P_M\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n), \quad (12)$$

where $r=|\boldsymbol{r}_n-\boldsymbol{r}_p|$, $\boldsymbol{\sigma}_i$ are Pauli spin matrices ($i=n,p$, respectively) and $P_M$ is the space exchange operator.

The tensor and spin-orbit parts of the $n$-$p$ interaction which act only on triplet spin states we use in the form [26]

$$V_t=V_t(r)(u_t+u_{tm}P_M)S_{12}, \quad (13)$$

where

$$S_{12}=\frac{1}{r^2}(\boldsymbol{\sigma}_p\cdot\boldsymbol{r})(\boldsymbol{\sigma}_n\cdot\boldsymbol{r})-\frac{1}{3}(\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n), \quad (14)$$

$$V_{ls}=V_{ls}(r)(u_e^s+u_o^sP_M)\boldsymbol{l}\cdot\boldsymbol{s}, \quad (15)$$

where $\boldsymbol{l}$ is the orbital angular momentum of the relative motion of proton and neutron and $\boldsymbol{s}$ is the total spin of both nucleons. $V_c(r)$, $V_t(r)$, and $V_{ls}(r)$ in Eqs. (12), (13), and (15) depend on the distance between the two nucleons. We use for them the Gaussian shape

$$V(r)=\exp(-r^2/r_0^2), \quad (16)$$

where $r_0=1.4$ fm is used in our calculations. This implies a long-range $n$-$p$ interaction. The eight (nine) free parameters $u_0$, $u_1$, $u_2$, $u_3$, $u_t$, $u_{tm}$, $u_e^s$, $u_o^s$ (and $\xi_3$ if one interacting octupole phonon is included) are fitted to the $^{210}$Bi experimental data.

For comparison, we use also the delta function for $V(r)$:

$$V(r)=\delta(r). \quad (17)$$

We have then only two (three) free parameters $u_0$, $u_1$ (and $\xi_3$ if one interacting octupole phonon is included) to be fitted to the $^{210}$Bi experimental data.

TABLE I. Parameters calculated from fitting 34 states in $^{210}$Bi with Gaussian shape *n-p* interaction. See text for more details.

| Parameter | No phonons (MeV) | One octupole phonon (MeV) |
|---|---|---|
| $u_0$ | $-40.4\pm2.5$ | $-40.8\pm0.8$ |
| $u_1$ | $-2.7\pm1.1$ | $-2.6\pm0.9$ |
| $u_2$ | $-32\pm6$ | $-31\pm5$ |
| $u_3$ | $-0.5\pm2.1$ | $-0.5\pm2.0$ |
| $u_t$ | $-73\pm6$ | $-73\pm5$ |
| $u_{tm}$ | $-108\pm16$ | $-108\pm14$ |
| $u_e^s$ | $-11\pm5$ | $-11\pm5$ |
| $u_o^s$ | $35\pm13$ | $35\pm12$ |
| $\xi_3$ | – | $0.37\pm0.11$ |
| $\chi^2$ | 40.9 | 36.6 |

### III. DATA ANALYSIS OF EXPERIMENTAL $^{210}$Bi LEVELS

Using the formalism described in Sec. II above, we have analyzed the experimental data for $^{210}$Bi. For this purpose we used the harmonic oscillator wave functions for seven neutron particle states from $^{209}$Pb, and four proton particle states from $^{209}$Bi. The neutron states used for the model space were $1g_{9/2}$ (0 keV), $0i_{11/2}$ (779 keV), $0j_{15/2}$ (1423 keV), $2d_{5/2}$ (1567 keV), $3s_{1/2}$ (2032 keV), $1g_{7/2}$ (2491 keV), and $2d_{3/2}$ (2491 keV), while the proton states were $0h_{9/2}$ (0 keV), $1f_{7/2}$ (896 keV), $0i_{13/2}$ (1609 keV), and $1f_{5/2}$ (2826 keV).

We fitted 34 states from 5 multiplets in $^{210}$Bi. These included the 10 states each from the $\nu1g_{9/2}\otimes\pi0h_{9/2}$ and $\nu0i_{11/2}\otimes\pi0h_{9/2}$ configurations, the $9^-$ state from the $\nu0i_{11/2}\otimes\pi1f_{7/2}$ configuration, the 8 states from the $\nu1g_{9/2}\otimes\pi1f_{7/2}$ configuration, and the $3^+$–$6^+$ and $12^+$ states from the $\nu0j_{15/2}\otimes\pi0h_{9/2}$ configuration. The criteria for the choice of states fitted included the reliability of the experimental data and states below 2 MeV.

The calculated parameters which result from the fitting are given in Table I for the two cases with inclusion of no phonons and with the inclusion of one interacting octupole phonon. Plots of the experimental data vs the theoretical calculations for these two cases are presented in Figs. 1 and 2 with solid circles representing fitted energies.

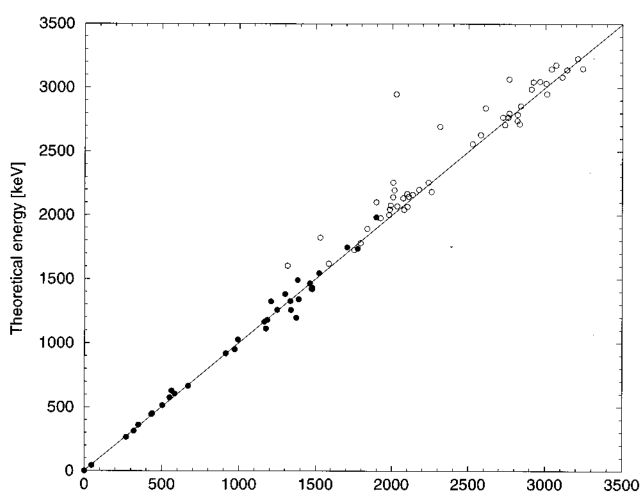


FIG. 1. Plot of the experimental data vs the theoretical calculations for the Gaussian force with inclusion of no phonons. The fitted levels are marked by solid circles, the unfitted levels by open circles.

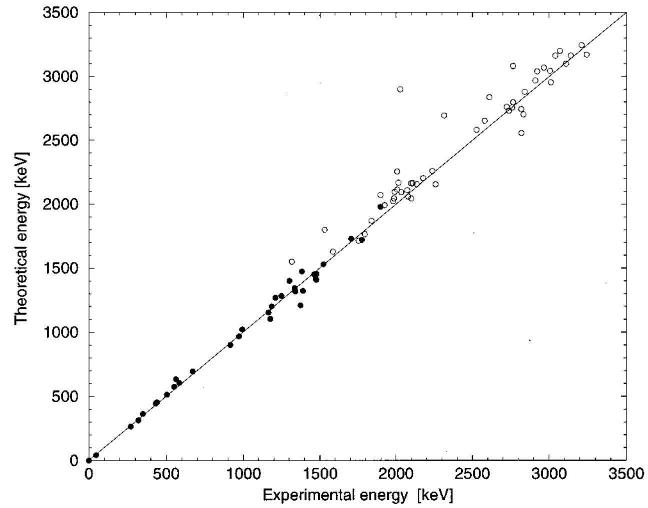


FIG. 2. Plot of the experimental data vs the theoretical calculations for the Gaussian force with the inclusion of one interacting octupole phonon. The fitted levels are marked by solid circles, the unfitted levels by open circles.

For comparison, the same data have been fitted with a delta force (without and with the inclusion of one interacting octupole phonon). The results are shown in Table II (the fitted parameters) and in Figs. 3 and 4 (plots of the experimental data vs the theoretical calculations).

It can be seen from the $\chi^2$ values in Tables I and II that the inclusion of one interacting octupole phonon slightly improves the fit for Gaussian shape neutron-proton interaction and slightly decreases the goodness of fit for the delta interaction. In the calculations of $\chi^2$ an error of 10 keV for all experimental energies is assumed.

### IV. NEUTRON-PROTON INTERACTION IN THE $^{210}$Bi GROUND STATE

The strength of the *n-p* interaction for the ground state, $E_0$, can be estimated from the experimental binding energies:

$$E_0(^{210}\mathrm{Bi})=B(^{209}\mathrm{Bi})+B(^{209}\mathrm{Pb})-B(^{210}\mathrm{Bi})-B(^{208}\mathrm{Pb}), \tag{18}$$

since $B(^{209}\mathrm{Bi})=B(^{208}\mathrm{Pb})-H_{\mathrm{int}}(^{208}\mathrm{Pb},p)$, $B(^{209}\mathrm{Pb})=B(^{208}\mathrm{Pb})-H_{\mathrm{int}}(^{208}\mathrm{Pb},n)$, and $B(^{210}\mathrm{Bi})=B(^{208}\mathrm{Pb})-H_{\mathrm{int}}(^{208}\mathrm{Pb},p)-H_{\mathrm{int}}(^{208}\mathrm{Pb},n)-H_{\mathrm{int}}(n,p)$.

TABLE II. Parameters calculated from fitting 34 states in $^{210}$Bi with delta force. See text for more details.

| Parameter | No phonons (MeV) | One octupole phonon (MeV) |
|---|---|---|
| $u_0$ | $-1.16\pm0.22$ | $-1.17\pm0.21$ |
| $u_1$ | $-0.34\pm0.10$ | $-0.34\pm0.10$ |
| $\xi_3$ | – | $0.51+0.17-1.18$ |
| $\chi^2$ | 333.6 | 334.4 |

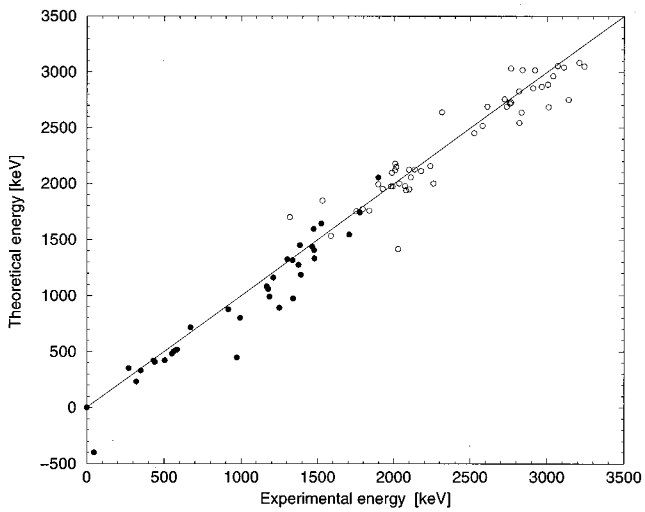


FIG. 3. Plot of the experimental data vs the theoretical calculations for the delta force with inclusion of no phonons. The fitted levels are marked by solid circles, the unfitted levels by open circles.

Using the 1993 Wapstra evaluation, $E_0(^{210}\mathrm{Bi})_{\mathrm{expt}} = (-0.67 \pm 0.01)$ MeV. Our model gives $E_0(^{210}\mathrm{Bi})_{\mathrm{theor}} = (-0.704 \pm 0.017)$ MeV (with inclusion of no phonons) and $E_0(^{210}\mathrm{Bi})_{\mathrm{theor}} = (-0.729 \pm 0.017)$ MeV (with the inclusion of one interacting octupole phonon). The error corresponds to a 10% change in the $\chi^2$ value, very similar to that of other parameters (see Table I).

For the delta force we get $E_0(^{210}\mathrm{Bi})_{\mathrm{theor}} = (-0.562 \pm 0.017)$ MeV with inclusion of no phonons and $E_0(^{210}\mathrm{Bi})_{\mathrm{theor}} = (-0.611 \pm 0.016)$ MeV with the inclusion of one interacting octupole phonon.

## V. CALCULATION OF UNFITTED LEVELS IN $^{210}$Bi

We have also calculated the unfitted experimental levels in $^{210}$Bi. They are higher in energy, less reliable experimentally, and in the region where core excitations are increasingly important. Therefore we must expect that they will be much more poorly reproduced. It is, however, to be hoped that severe discrepancies can point to experimental misassignments [12].

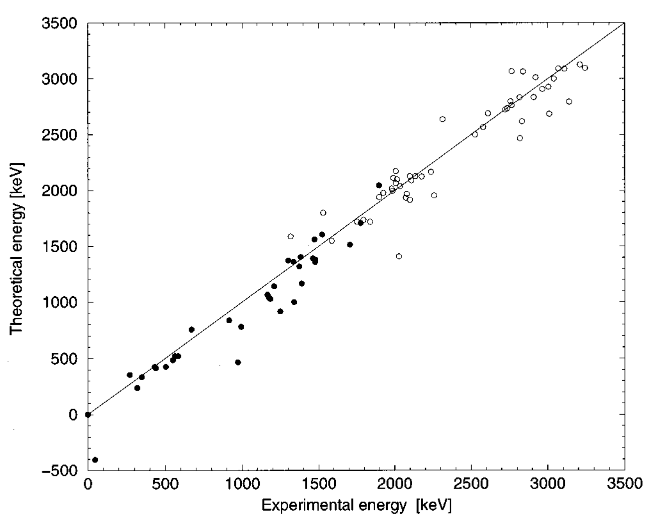


FIG. 4. Plot of the experimental data vs the theoretical calculations for the delta force with the inclusion of one interacting octupole phonon. The fitted levels are marked by solid circles, the unfitted levels by open circles.

### A. Suggestion for changes in the configurations of some states

For the $9^+$ level at 2072 keV experimentally interpreted as a member of the $\nu 0 j_{15/2} \otimes \pi 0 h_{9/2}$ multiplet, the model prefers a level at 2133 keV (or 2108 keV with the inclusion of one interacting octupole phonon) with the main component of about 65% $\nu 1 g_{9/2} \otimes \pi 0 i_{13/2}$ and 30% of $\nu 0 j_{15/2} \otimes \pi 0 h_{9/2}$. The new assignment of this level as a member of the $\nu 1 g_{9/2} \otimes \pi 0 i_{13/2}$ multiplet is supported by the experimentally observed transition from this level to the $8^-$ state at 916 keV ($\nu 1 g_{9/2} \otimes \pi 1 f_{7/2}$).

The $11^+$ state at 2833 keV tentatively assigned to the $\nu 0 i_{11/2} \otimes \pi 0 i_{13/2}$ multiplet could be a member of the $\nu 0 j_{15/2} \otimes \pi 1 f_{7/2}$ multiplet as our calculations suggest. This is supported also by the observed transition from this level to the $12^+$ state at 1473 keV ($\nu 0 j_{15/2} \otimes \pi 0 h_{9/2}$).

The experimental energy of the $1^+$ member of the $\pi 0 i_{13/2} \otimes \nu 0 i_{11/2}$ multiplet (2027 keV) does not agree with the theoretical prediction (theoretical value about 0.9 MeV higher with the Gaussian force and about 0.6 MeV lower with the delta force), which suggests that its experimental assignment may be in error.

States of the multiplets $\nu 2 d_{5/2} \otimes \pi 1 f_{7/2}$, $\nu 1 g_{7/2} \otimes \pi 0 h_{9/2}$, and $\nu 2 d_{3/2} \otimes \pi 0 h_{9/2}$ are highly mixed. Some of them, specifically $2^-$, $3^-$, and $5^-$, are also highly mixed with the multiplet $\nu 1 g_{9/2} \otimes \pi 1 f_{5/2}$, which has not been experimentally assigned and no results for this configuration have been shown in any of the previous calculations. For the $6^-$ levels observed at 2840 keV and 3141 keV and experimentally assigned to the mixed $\nu 1 g_{9/2} \otimes \pi 0 h_{9/2}$ and $\nu 2 d_{3/2} \otimes \pi 0 h_{9/2}$ configurations, the model prefers to reverse the configurations previously assigned.

The calculated results compared with the modified experimental results for unfitted $^{210}$Bi levels are shown in Figs. 1, 2, 3, and 4 and marked by open circles.

The calculated and experimental energies are summarized in Table III. The theoretical assignment to a $n$-$p$ multiplet implies that it is the maximum component of the wave function.

The calculated standard deviations for the used $n$-$p$ interactions for the unfitted levels (with the $1^+$ state at 2027 keV excluded) are 124 keV for the Gaussian force without phonons, 127 keV for the Gaussian force with one interacting octupole phonon, 157 keV for the delta force without phonons and 152 keV for the delta force with one interacting octupole phonon.

## VI. CONCLUSIONS

The $n$-$p$ interaction parameters for $^{210}$Bi show a remarkable internal consistency in fitting the 34 levels in $^{210}$Bi shown in Figs. 1 and 2. The agreement between experiment and theory increases with the number of parameters used in the fitting theory. However, the factor of 8–10 in the $\chi^2$ values between the Gaussian and the delta force fits is clearly

TABLE III. Comparison of the energies of the experimentally identified states in $^{210}$Bi multiplets [12], $E_{\mathrm{expt}}$, to the model results with the Gaussian shape $n$-$p$ interaction without phonons, $E_A$, and with one interacting octupole phonon, $E_B$, with the delta force without phonons, $E_C$, and with one interacting octupole phonon, $E_D$, to the first results of Kim and Rasmussen [15] $E_{\mathrm{KR}}$ and to the corrected results of Kim and Rasmussen [16] $E_{\mathrm{KRcor}}$ and Warburton and Brown [21] $E_{\mathrm{W}}$. All energies are in keV. The last two configurations in this table have not previously been calculated and the last six are unknown experimentally.

| Major configuration | $I^\pi$ | $E_{\mathrm{expt}}$ | $E_A$ | $E_B$ | $E_C$ | $E_D$ | $E_{\mathrm{W}}$ | $E_{\mathrm{KR}}$ | $E_{\mathrm{KRcor}}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\nu 1g_{9/2}\otimes \pi 0h_{9/2}$ | $0^-$ | 46.5 | 42 | 40 | -402 | -406 | -84 | 22 | 44 |
| | $1^-$ | 0 | 0 | 0 | 0 | 0 | -23 | 0 | 0 |
| | $2^-$ | 320 | 311 | 312 | 233 | 236 | 272 | 283 | 304 |
| | $3^-$ | 348 | 360 | 362 | 330 | 332 | 347 | 343 | 358 |
| | $4^-$ | 503 | 511 | 512 | 422 | 425 | 497 | 459 | 481 |
| | $5^-$ | 439 | 450 | 452 | 408 | 412 | 454 | 392 | 411 |
| | $6^-$ | 550 | 575 | 575 | 481 | 484 | 579 | 510 | 532 |
| | $7^-$ | 433 | 442 | 443 | 420 | 424 | 453 | 376 | 397 |
| | $8^-$ | 583 | 603 | 603 | 517 | 521 | 615 | 532 | 554 |
| | $9^-$ | 271 | 263 | 263 | 352 | 354 | 291 | 284 | 308 |
| $\nu 0i_{11/2}\otimes \pi 0h_{9/2}$ | $1^-$ | 563 | 626 | 632 | 501 | 520 | 653 | 670 | 708 |
| | $2^-$ | 972 | 948 | 965 | 447 | 465 | 972 | 1128 | 1150 |
| | $3^-$ | 1374 | 1196 | 1208 | 1275 | 1319 | 1154 | 1167 | 1185 |
| | $4^-$ | 1248 | 1257 | 1280 | 891 | 916 | 1360 | 1282 | 1300 |
| | $5^-$ | 1336 | 1326 | 1340 | 1315 | 1360 | 1281 | 1249 | 1269 |
| | $6^-$ | 1339 | 1257 | 1316 | 976 | 998 | 1437 | 1240 | 1259 |
| | $7^-$ | 1301 | 1383 | 1398 | 1326 | 1372 | 1324 | 1302 | 1322 |
| | $8^-$ | 1184 | 1178 | 1200 | 992 | 1027 | 1357 | 939 | 1287 |
| | $9^-$ | 1478 | 1432 | 1453 | 1333 | 1379 | 1311 | 1347 | 1366 |
| | $10^-$ | 670 | 663 | 692 | 715 | 757 | 864 | 806 | |
| $\nu 1g_{9/2}\otimes \pi 1f_{7/2}$ | $1^-$ | 1165 | 1163 | 1151 | 1083 | 1065 | 1216 | 1123 | 1165 |
| | $2^-$ | 1175 | 1111 | 1101 | 1060 | 1039 | 1230 | 1295 | 1317 |
| | $3^-$ | 1476 | 1420 | 1407 | 1405 | 1358 | 1399 | 1390 | 1417 |
| | $4^-$ | 1390 | 1341 | 1320 | 1187 | 1165 | 1413 | 1370 | 1392 |
| | $5^-$ | 1463 | 1465 | 1449 | 1438 | 1392 | 1438 | 1412 | 1438 |
| | $6^-$ | 1208 | 1323 | 1266 | 1161 | 1139 | 1345 | 1355 | 1378 |
| | $7^-$ | 1383 | 1490 | 1472 | 1448 | 1403 | 1448 | 1448 | 1474 |
| | $8^-$ | 916 | 917 | 899 | 877 | 837 | 986 | 1265 | 962 |
| $\nu 0j_{15/2}\otimes \pi 0h_{9/2}$ | $3^+$ | 994 | 1026 | 1020 | 801 | 780 | 1094 | 1090 | |
| | $4^+$ | 1523 | 1544 | 1528 | 1642 | 1604 | 1570 | 1677 | |
| | $5^+$ | 1706 | 1747 | 1728 | 1545 | 1513 | 1756 | 1734 | |
| | $6^+$ | 1776 | 1736 | 1720 | 1741 | 1706 | 1761 | 1725 | |
| | $7^+$ | 1837 | 1893 | 1869 | 1758 | 1718 | 1911 | 1846 | |
| | $8^+$ | 1794 | 1780 | 1764 | 1770 | 1735 | 1720 | 1727 | |
| | $9^+$ | | 1909 | 1880 | 1825 | 1768 | 1947 | 1884 | |
| | $10^+$ | 1753 | 1726 | 1712 | 1751 | 1718 | 1725 | 1684 | |
| | $11^+$ | 2100 | 2066 | 2042 | 1950 | 1915 | 2031 | 1905 | |
| | $12^+$ | 1473 | 1425 | 1414 | 1593 | 1560 | 1431 | 1474 | |
| $\nu 2d_{5/2}\otimes \pi 0h_{9/2}$ | $2^-$ | 1925 | 1977 | 1990 | 1954 | 1977 | 1908 | 1609 | 1634 |
| | $3^-$ | 1990 | 2076 | 2094 | 1974 | 2110 | 2043 | 1984 | 2009 |
| | $4^-$ | 2079 | 2042 | 2058 | 1940 | 1969 | | 1984 | 2003 |
| | $5^-$ | 2034 | 2070 | 2092 | 2001 | 2036 | | 2015 | 2034 |
| | $6^-$ | 2108 | 2145 | 2162 | 2057 | 2090 | | 2077 | 2098 |
| | $7^-$ | 1980 | 2001 | 2022 | 1976 | 2015 | 2003 | 1996 | 2019 |
| $\nu 1g_{9/2}\otimes \pi 0i_{13/2}$ | $2^+$ | 1531 | 1825 | 1799 | 1847 | 1800 | 1823 | | |
| | $3^+$ | 1897 | 2101 | 2070 | 1995 | 1940 | 2096 | | |
| | $4^+$ | 2006 | 2141 | 2111 | 2120 | 2064 | 2149 | | |
| | $5^+$ | | 2175 | 2145 | 1993 | 1942 | 2178 | | |
| | $6^+$ | 2015 | 2195 | 2166 | 2152 | 2098 | 2203 | | |
| | $7^+$ | 2259 | 2182 | 2154 | 2001 | 1953 | 2169 | | |
| | $8^+$ | | 2225 | 2194 | 2161 | 2107 | 2225 | | |

TABLE III. (*Continued*).

| Major configuration | $I^{\pi}$ | $E_{expt}$ | $E_A$ | $E_B$ | $E_C$ | $E_D$ | $E_W$ | $E_{KR}$ | $E_{KRcor}$ |
|---|---|---|---|---|---|---|---|---|---|
| | $9^+$ | 2072 | 2133 | 2108 | 1977 | 1934 | 2100 | | |
| | $10^+$ | | 2250 | 2215 | 2166 | 2109 | 2237 | | |
| | $11^+$ | 1316 | 1604 | 1550 | 1700 | 1586 | 1652 | | |
| $\nu 0i_{11/2}\otimes\pi 1f_{7/2}$ | $2^-$ | 1585 | 1619 | 1626 | 1531 | 1548 | 1612 | 1952 | 1975 |
| | $3^-$ | 1985 | 2045 | 2043 | 2099 | 1997 | 2023 | 2063 | 2088 |
| | $4^-$ | 2177 | 2198 | 2201 | 2113 | 2124 | | 2123 | 2148 |
| | $5^-$ | 2099 | 2165 | 2161 | 2124 | 2127 | | 2092 | 2120 |
| | $6^-$ | 2238 | 2255 | 2257 | 2158 | 2166 | | 2167 | 2191 |
| | $7^-$ | 2135 | 2159 | 2155 | 2128 | 2127 | | 2084 | 2110 |
| | $8^-$ | 2006 | 2255 | 2254 | 2179 | 2173 | | 2194 | 2217 |
| | $9^-$ | 1897 | 1982 | 1977 | 2054 | 2044 | 2050 | 1918 | 1944 |
| $\nu 3s_{1/2}\otimes\pi 0h_{9/2}$ | $4^-$ | 2525 | 2557 | 2581 | 2452 | 2498 | | 2463 | 2488 |
| | $5^-$ | 2579 | 2628 | 2651 | 2519 | 2565 | | 2564 | 2581 |
| $\nu 0j_{15/2}\otimes\pi 1f_{7/2}$ | $4^+$ | 2819 | 2787 | 2555 | 2544 | 2465 | | 2645 | |
| | $5^+$ | | 2775 | 2739 | 2712 | 2639 | | 2837 | |
| | $6^+$ | | 2910 | 2859 | 2861 | 2675 | | 2824 | |
| | $7^+$ | | 2826 | 2790 | 2725 | 2650 | | 2815 | |
| | $8^+$ | | 2949 | 2896 | 2869 | 2793 | | 2863 | |
| | $9^+$ | | 2812 | 2776 | 2710 | 2640 | | 2771 | |
| | $10^+$ | | 2975 | 2921 | 2874 | 2791 | | 2885 | |
| | $11^+$ | 2833 | 2714 | 2701 | 2638 | 2615 | | 2563 | |
| $\nu 0i_{11/2}\otimes\pi 0i_{13/2}$ | $1^+$ | 2027 | 2946 | 2899 | 1415 | 1409 | 1929 | | |
| | $2^+$ | | 2228 | 2221 | 2465 | 2440 | | | |
| | $3^+$ | 2765 | 2797 | 2796 | 2728 | 2761 | | | |
| | $4^+$ | | 2599 | 2757 | 2833 | 2781 | | | |
| | $5^+$ | | 2946 | 2918 | 2823 | 2781 | | | |
| | $6^+$ | 2759 | 2765 | 2755 | 2717 | 2794 | | | |
| | $7^+$ | 2910 | 2987 | 2967 | 2854 | 2831 | | | |
| | $8^+$ | 2724 | 2766 | 2760 | 2755 | 2722 | | | |
| | $9^+$ | | 3005 | 2991 | 2884 | 2869 | | | |
| | $10^+$ | | 2686 | 2683 | 2748 | 2723 | | | |
| | $11^+$ | | 3035 | 3024 | 2914 | 2902 | | | |
| | $12^+$ | | 2295 | 2298 | 2663 | 2645 | | | |
| $\nu 2d_{5/2}\otimes\pi 1f_{7/2}$ | $1^-$ | | 2640 | 2633 | 2669 | 2656 | | 2660 | |
| | $2^-$ | 3011 | 2949 | 2953 | 2684 | 2681 | | 2850 | |
| | $3^-$ | | 2934 | 2932 | 2989 | 2983 | | 2877 | |
| | $4^-$ | 2610 | 2839 | 2837 | 2690 | 2687 | | 2857 | |
| | $5^-$ | 2921 | 3042 | 3038 | 3015 | 3010 | | 2987 | |
| | $6^-$ | 2314 | 2693 | 2692 | 2639 | 2635 | | 2720 | |
| $\nu 1g_{7/2}\otimes\pi 0h_{9/2}$ | $1^-$ | 2818 | 2738 | 2742 | 2828 | 2830 | | 2770 | |
| | $2^-$ | 3005 | 3033 | 3043 | 2888 | 2926 | | 3038 | |
| | $3^-$ | 2765 | 3066 | 3080 | 3032 | 3066 | | 2948 | |
| | $4^-$ | 2966 | 3046 | 3068 | 2869 | 2907 | | 2984 | |
| | $5^-$ | 3109 | 3082 | 3100 | 3041 | 3088 | | 3011 | |
| | $6^-$ | 3141 | 3139 | 3162 | 2751 | 2792 | | 2829 | |
| | $7^-$ | 3244 | 3148 | 3170 | 3048 | 3095 | | 3043 | |
| | $8^-$ | 2737 | 2708 | 2730 | 2690 | 2732 | | 2774 | |
| $\nu 2d_{3/2}\otimes\pi 0h_{9/2}$ | $3^-$ | 3040 | 3145 | 3162 | 2962 | 3000 | | 3079 | |
| | $4^-$ | 3069 | 3177 | 3200 | 3054 | 3090 | | 3115 | |
| | $5^-$ | 3210 | 3226 | 3243 | 3083 | 3124 | | 3099 | |
| | $6^-$ | 2840 | 2855 | 2878 | 3019 | 3064 | | 3070 | |
| $\nu 3s_{1/2}\otimes\pi 1f_{7/2}$ | $3^-$ | | 3392 | 3385 | 3353 | 3347 | | 3394 | |
| | $4^-$ | | 3394 | 3387 | 3370 | 3346 | | 3372 | |
| $\nu 1g_{7/2}\otimes\pi 1f_{7/2}$ | $0^-$ | | 3440 | 3437 | 2667 | 2660 | | 3124 | |
| | $1^-$ | | 3426 | 3411 | 3482 | 3455 | | 3292 | |

TABLE III. (*Continued*).

| Major configuration | $I^\pi$ | $E_{\rm expt}$ | $E_A$ | $E_B$ | $E_C$ | $E_D$ | $E_{\rm W}$ | $E_{\rm KR}$ | $E_{\rm KRcor}$ |
|---|---|---|---|---|---|---|---|---|---|
| | $2^-$ | | 3833 | 3829 | 3763 | 3760 | | 3628 | |
| | $3^-$ | | 3826 | 3821 | 3768 | 3764 | | 3629 | |
| | $4^-$ | | 3938 | 3934 | 3833 | 3830 | | 3814 | |
| | $5^-$ | | 3834 | 3829 | 3798 | 3795 | | 3650 | |
| | $6^-$ | | 3982 | 3979 | 3877 | 3873 | | 3862 | |
| | $7^-$ | | 3499 | 3637 | 3744 | 3740 | | 3536 | |
| $\nu 2d_{3/2}\otimes \pi 1f_{7/2}$ | $2^-$ | | 3772 | 3770 | 3647 | 3640 | | 3688 | |
| | $3^-$ | | 3907 | 3905 | 3849 | 3847 | | 3841 | |
| | $4^-$ | | 4013 | 4010 | 3924 | 3920 | | 3921 | |
| | $5^-$ | | 3736 | 3734 | 3773 | 3769 | | 3750 | |
| $\nu 0j_{15/2}\otimes \pi 0i_{13/2}$ | $1^-$ | | 3284 | 3238 | 3024 | 2973 | | | |
| | $2^-$ | | 3263 | 3208 | 3252 | 3178 | | | |
| | $3^-$ | | 3472 | 3426 | 3531 | 3436 | | | |
| | $4^-$ | | 3510 | 3453 | 3298 | 3250 | | | |
| | $5^-$ | | 3542 | 3493 | 3565 | 3476 | | | |
| | $6^-$ | | 3572 | 3516 | 3369 | 3234 | | | |
| | $7^-$ | | 3667 | 3478 | 3576 | 3485 | | | |
| | $8^-$ | | 3577 | 3525 | 3356 | 3268 | | | |
| | $9^-$ | | 3637 | 3585 | 3581 | 3490 | | | |
| | $10^-$ | | 3540 | 3491 | 3326 | 3241 | | | |
| | $11^-$ | | 3653 | 3601 | 3585 | 3494 | | | |
| | $12^-$ | | 3437 | 3389 | 3263 | 3177 | | | |
| | $13^-$ | | 3675 | 3620 | 3588 | 3498 | | | |
| | $14^-$ | | 2898 | 2853 | 2966 | 2871 | 3137 | | |
| $\nu 1g_{9/2}\otimes \pi 1f_{5/2}$ | $2^-$ | | 2547 | 2543 | 2543 | 2518 | | | |
| | $3^-$ | | 2993 | 3004 | 3090 | 3112 | | | |
| | $4^-$ | | 3290 | 3289 | 3140 | 3128 | | | |
| | $5^-$ | | 3184 | 3210 | 3138 | 3149 | | | |
| | $6^-$ | | 3381 | 3382 | 3276 | 3327 | | | |
| | $7^-$ | | 2971 | 2972 | 3005 | 3009 | | | |
| $\nu 2d_{5/2}\otimes \pi 0i_{13/2}$ | $4^+$ | | 3627 | 3617 | 3586 | 3568 | | | |
| | $5^+$ | | 3718 | 3708 | 3618 | 3601 | | | |
| | $6^+$ | | 3770 | 3761 | 3695 | 3684 | | | |
| | $7^+$ | | 3699 | 3691 | 3590 | 3574 | | | |
| | $8^+$ | | 3818 | 3809 | 3724 | 3714 | | | |
| | $9^+$ | | 3597 | 3588 | 3531 | 3509 | | | |

significant. The addition of the octupole parameter $\xi_3$ increases the consistency of the Gaussian fit (comparison of Figs. 1 and 2 and Table I) but interestingly not for the delta force (Figs. 3 and 4 and Table II).

The nucleon-octupole phonon interaction especially changes the mixing of the 1339 keV and 1208 keV $6^-$ states with configurations $\nu 0i_{11/2}\otimes \pi 0h_{9/2}$ and $\nu 1g_{9/2}\otimes \pi 1f_{7/2}$ in favor of the main experimentally assigned components. The mixing of the $6^-$ states of the two multiplets is indicated by experimentally observed transitions from the $7^+$ level at 1837 keV ($\nu 1g_{9/2}\otimes \pi 0h_{9/2}$), the $7^+$ level at 2910 keV ($\nu 0i_{11/2}\otimes \pi 0h_{9/2}$), and the $7^-$ level at 1980 keV (relatively pure $\nu 2d_{5/2}\otimes \pi 0h_{9/2}$) to the $6^-$ state at 1208 keV with major configuration $\nu 1g_{9/2}\otimes \pi 1f_{7/2}$.

Although we do not see much improvement in using octupole degrees of freedom, we must be careful in drawing this conclusion about their role. The nucleon-octupole phonon interaction strength is fitted to the experimental levels below 2 MeV where this interaction does not play an important role. In the region around the $^{208}$Pb core octupole excitation at 2.6 MeV where the role could be amplified, no totally reliably assigned experimental data exist.

Using the parameters obtained by fitting certain of the levels in $^{210}$Bi to predict other levels in $^{210}$Bi (Figs. 1 and 2) indicates that most of these levels are properly assigned. However, certain levels do not agree with the theoretical prediction, which suggests that their experimental assignment may be in error. Nevertheless, in spite of mixing, tentative assignments and being above core particle-hole excitations in the 3 MeV region the overall agreement is quite good.

The comparison to the other model calculations [15,16,21] is not straightforward since they did not calculate as many states as in this paper. Therefore we refer the reader to Table III where our results are compared to the calculations of Kim and Rasmussen [15,16], Warburton and Brown

[21] and to experimental data [12]. We point out that especially for the low energy region our calculations give a better description of the experimental data. The enlarged model space enabled us to calculate the members of the $\nu 1g_{9/2}\otimes\pi 1f_{5/2}$ lying around 3 MeV and mixed to other multiplets in this region. The states of this multiplet have not been experimentally assigned and in the calculations of Ref. [15,16] are not incorporated in the model space.

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[1] J.R. Erskine, W.W. Buechner, and H.A. Enge, Phys. Rev. **128,** 720 (1961).
[2] H.T. Motz, E.T. Jurney, E.R. Shera, and R.K. Sheline, Phys. Rev. Lett. **26,** 854 (1971).
[3] P.D. Ellegaard, P.D. Barnes, R. Eisenstein, and T.R. Canada, Phys. Lett. **35B,** 149 (1971).
[4] J.J. Kolata and W.W. Daehnick, Phys. Rev. C **5,** 568 (1972).
[5] C.K. Cline, W.P. Alford, H.E. Gove, and R. Tickle, Nucl. Phys. **A186,** 273 (1972).
[6] T.R. Canada, R.A. Eisenstein, C. Ellegaard, P.D. Barnes, and J. Miller, Nucl. Phys. **A205,** 145 (1973).
[7] D. Proetel, F. Riess, E. Grosse, R. Ley, M.R. Maier, and P. von Brentano, Phys. Rev. C **7,** 2137 (1973).
[8] D.J. Donahue, O. Hausser, R.L. Hershberger, R. Lutter, and F. Riess, Phys. Rev. C **12,** 1547 (1975).
[9] W.W. Daehnick, M.J. Spisak, R.M. Del Vecchio, and W. Oelert, Phys. Rev. C **15,** 594 (1977).
[10] W.W. Daehnick, M.J. Spisak, and J.R. Comfort, Phys. Rev. C **23,** 1906 (1981).
[11] R.L. Ponting, R.K. Sheline, E.T. Jurney, L. Nkwambiaya, and B. Bombaka, Rev. Zair. Sci. Nucl. **7**, 1 (1986).
[12] R.K. Sheline, R.L. Ponting, A.K. Jain, J. Kvasil, B. bu Nianga, and L. Nkwambiaya, Czech. J. Phys. B **39,** 22 (1989).
[13] N. Newby and E.J. Konopinski, Phys. Rev. **115,** 434 (1959).
[14] I.Yu. Kharitonov, L.A. Sliv, and G.A. Sogomonova, Nucl. Phys. **28,** 210 (1961).
[15] Y.E. Kim and J.O. Rasmussen, Nucl. Phys. **47,** 184 (1963).
[16] Y.E. Kim and J.O. Rasmussen, Nucl. Phys. **61,** 173 (1965).
[17] T.A. Hughes, R. Snow, and W.T. Pinkston, Nucl. Phys. **82,** 129 (1966).
[18] J. Vary and J.N. Ginocchio, Nucl. Phys. **A166,** 479 (1971); J. Vary, Ph.D. thesis, Yale University, 1970 (unpublished).
[19] G.H. Herling and T.T.S. Kuo, Nucl. Phys. **A181,** 113 (1972); T.T.S. Kuo and G.H. Herling, Memorandum No. 2258, Naval Research Lab, Washington, D.C., 1971 (unpublished).
[20] C.W. Ma and W.W. True, Phys. Rev. C **8,** 2313 (1973).
[21] E.K. Warburton and B.A. Brown, Phys. Rev. C **43,** 602 (1991).
[22] T. Hamada and I.D. Johnston, Nucl. Phys. **34,** 382 (1962).
[23] T.T.S. Kuo and G.E. Brown, Nucl. Phys. **85,** 40 (1966); T.T.S. Kuo and G.E. Brown, Nucl. Phys. **A114,** 241 (1968).
[24] A. Bohr and B.R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. 2.
[25] V.G. Soloviev, *Theory of Complex Nuclei* (Pergamon Press, Oxford, 1976).
[26] J.P. Boisson, R. Piepenbring, and W. Ogle, Phys. Rep. **26,** 99 (1976).
[27] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin, 1980).