Coulomb-Sturmian separable expansion approach: Three-body Faddeev calculations for Coulomb-like interactions

Z. Papp
Institute of Nuclear Research of the Hungarian Academy of Sciences, Bem tér 18/c, P.O. Box 51, H–4001 Debrecen, Hungary

W. Plessas
Institute for Theoretical Physics, University of Graz, Universitätsplatz 5, A-8010 Graz, Austria
(Received 1 February 1996)

We demonstrate the feasibility and efficiency of the Coulomb-Sturmian separable expansion method for generating accurate solutions of the Faddeev equations. Results obtained with this method are reported for several benchmark cases of bosonic and fermionic three-body systems. Correct bound-state results in agreement with the ones established in the literature are achieved for short-range interactions. We outline the formalism for the treatment of three-body Coulomb systems and present a bound-state calculation for a three-boson system interacting via Coulomb plus short-range forces. The corresponding result is in good agreement with the answer from a recent stochastic-variational-method calculation. [S0556-2813(96)00207-5]

PACS number(s): 21.45.+v, 03.65.Ge, 02.30.Rz, 02.60.Nm

I. INTRODUCTION

Separable expansion schemes have always been extremely useful in solving few-body problems. For example, in three-body systems, at any time starting from the 1960s, important new results have been achieved by solving the Faddeev equations with some sort of separable representation of the two-body subsystems (for a review see, e.g., Ref. [1]). In more recent times, above all the separable expansion method proposed by Ernst, Shakin, and Thaler [2] (the so-called EST method) has proven very useful. Indeed, the first three-nucleon scattering results with such realistic meson-exchange N-N interactions as the Paris potential were achieved with this method [3], and they were later confirmed by a direct solution of the Faddeev equations [4].

Even nowadays, when three-body Faddeev equations can directly be solved on supercomputers (for a recent review of the state of the art, see Ref. [5]), separable expansion schemes have their relevance. Not only does an accurate separable representation of the input dynamics allow one to save much computer time in arriving at standard results, separable expansion methods may also help a lot in obtaining solutions to hitherto unsolved problems. In this respect we may mention the solution of the three-nucleon scattering problem with realistic N-N interactions and Coulomb forces at any energy; so far only limited solutions below or above breakup threshold have been obtained (see, e.g., Refs. [6] and [7]).

In this paper we deal with a separable expansion method that is well adapted to treating few-body problems including long-range forces. Its essence lies in the expansion of the potential operator v^s of the short-range part of any interaction with the use of Coulomb-Sturmian (CS) functions [8]. If the full potential also contains a Coulomb interaction v^C, this is kept in the Green's operator. Thereby all difficulties associated with a (separable) expansion of the Coulomb potential are avoided, while at the same time correct asymptotic properties of all quantities are guaranteed. Still the advantages of the separable expansion can be exploited in solving the two- and three-body integral equations.

The CS separable expansion method has been extensively tested before in two-body problems. Not only were bound and resonant states with a variety of short-range plus Coulomb potentials [8] investigated but scattering solutions were also obtained [9]. In these works also convergence studies were performed and subsequently extended to the multichannel Coulomb problem [11]. Computer codes for the CS separable expansion of any local or nonlocal two-body interaction in the presence of Coulomb-like potentials were published in Ref. [10]. In Sec. II below we shall recall some of the most important formulas for the two-body problem.

The principal advantage of the CS separable expansion is the fact that the matrix elements of the Coulomb Green's operator can be calculated analytically in the two-body system. Besides the separable representation of the short-range part of the full interaction, this turns out to be an essential requirement for an efficient and accurate solution of the three-body system. For the latter we may thus follow the integral-equation approach and thereby guarantee the implementation of the appropriate (Coulomb-like) asymptotics. In Sec. III below we shall demonstrate how the matrix elements of the three-body Coulomb Green's operator can be calculated in a reliable way.

We prove the efficiency of our method through the solution of the Faddeev equations for three-body bound states interacting via various short-range forces and in a case with additional Coulomb interaction among all three particles. We adhere to problems for which benchmark results from other methods have been obtained already. It is found that in all cases excellent agreement is achieved. The method therefore appears promising as an efficient tool for solving three-body systems, as it can be adapted to more general cases (including three-body forces) and extended to scattering under the presence of long-range interactions.

II. COULOMB-STURMIAN SEPARABLE EXPANSION

We give a short account of the formalism of the CS separable expansion in the two-body system with short-range

plus Coulomb interactions. Here we specify our notation and provide the formulas that are needed later in the solution of the three-body problem.

A. Basis functions

The short-range potential operator v^s in some angular momentum state l will be expanded on the basis of Coulomb-Sturmian functions

$$\langle r|nl\rangle=\left[\frac{n!}{(n+2l+1)!}\right]^{1/2}(2br)^{l+1}e^{-br}L_n^{2l+1}(2br),$$
$$(n=0,1,2,\ldots), \tag{1}$$

which are the solutions of the Sturm-Liouville problem of the hydrogenic system [12]. Here, L_n^{2l+1} represent the Laguerre polynomials and b relates to the energy in the Sturm-Liouville equation. We take b as a fixed parameter, thus working with energy-independent CS functions. They form a complete set

$$\mathbf{1}=\lim_{N\to\infty}\sum_{n=0}^{N}|\widetilde{nl}\rangle\langle nl|=\lim_{N\to\infty}\mathbf{1}_N, \tag{2}$$

where

$$\langle r|\widetilde{nl}\rangle=\frac{1}{r}\langle r|nl\rangle. \tag{3}$$

With the Nth order unit operator $\mathbf{1}_N$ in Eq. (2) we can now expand the short-range potential operator in the form

$$v_l^s=\lim_{N\to\infty}\mathbf{1}_N v_l^s\mathbf{1}_N=\lim_{N\to\infty}\sum_{n,n'=0}^{N}|\widetilde{nl}\rangle\langle nl|v_l^s|n'l\rangle\langle\widetilde{n'l}|. \tag{4}$$

If N remains finite, we end up with a rank-N separable approximation. As a consequence the two-body problem can then be solved by algebraic methods [1].

B. Short-range plus Coulomb interactions

Let us now assume a two-potential case of short-range plus Coulomb-like interactions,

$$v=v^s+v^C, \tag{5}$$

and consider the homogeneous Lippmann-Schwinger equation for the bound state $|\psi_l\rangle$ in some partial wave l,

$$|\psi_l\rangle=g_l^C(E)v_l^s|\psi_l\rangle. \tag{6}$$

Here $g_l^C(E)$ is the two-body Coulomb Green's operator

$$g_l^C(E)=(E-h_l^0-v^C)^{-1} \tag{7}$$

with the free Hamiltonian denoted by h_l^0. Using the expansion (4) in Eq. (6) one arrives at a linear system of homogeneous quations for the wave-function coefficients $\underline{A}_{ln}=\langle\widetilde{nl}|\psi_l\rangle$:

$$\{[\underline{g}_l^C(E)]^{-1}-\underline{v}_l^s\}\underline{A}_l=0. \tag{8}$$

It has a unique solution if and only if

$$\det\{[\underline{g}_l^C(E)]^{-1}-\underline{v}_l^s\}=0. \tag{9}$$

The matrices involved are made up from the elements

$$\underline{g}_{lnn'}^C(E)=\langle\widetilde{nl}|g_l^C(E)|\widetilde{n'l}\rangle \tag{10}$$

and

$$\underline{v}_{lnn'}^s=\langle nl|v_l^s|n'l\rangle. \tag{11}$$

While the latter matrix elements may be evaluated (numerically) for any given short-range potential either in configuration or in momentum space, the matrix elements of the Coulomb Green's operator between CS states can be calculated analytically [8]; the corresponding computer code is available from Ref. [10]. This fact then also allows one to calculate the matrix elements of the full Green's operator in the whole complex plane,

$$\underline{g}_l(E)=\{[\underline{g}_l^C(E)]^{-1}-\underline{v}_l^s\}^{-1}, \tag{12}$$

which will be needed later on in the solution of the three-body problem with charged particles.

After solving Eq. (8) for the coefficients $\underline{A}_{ln}$ the bound state $|\psi_l\rangle$ can be expressed as

$$|\psi_l\rangle=\sum_{n=0}^{N}\underline{B}_{ln}g_l^C(E)|\widetilde{nl}\rangle, \tag{13}$$

where the new coefficients result from the matrix multiplication $\underline{B}_l=\underline{v}_l^s\underline{A}_l$. We note that expression (13) is distinct from the usual expansion of the state $|\psi_l\rangle$ with certain test functions. The explicit occurrence of the Coulomb Green's operator always ensures the correct asymptotic behavior [9]. This is an immediate consequence of the fact that only the short-range potential (but not the wave function) is expanded.

III. SOLUTION OF THE THREE-BODY BOUND-STATE PROBLEM

We now extend the CS basis to the three-body system and demonstrate the solution of the Faddeev equations for bound states of three particles with any short-range interactions and under the presence of Coulomb forces.

A. Short-range interactions

The integral equations for the three Faddeev components Ψ_α of the bound-state wave function Ψ read

$$|\Psi_\alpha\rangle=G_\alpha(E)[v_\alpha^s|\Psi_\beta\rangle+v_\alpha^s|\Psi_\gamma\rangle] \tag{14}$$

with α,β,γ a cyclic permutation. Here the channel Green's operators are defined by

$$G_\alpha(E)=(E-H^0-v_\alpha^s)^{-1}, \tag{15}$$

where H^0 is the free three-particle Hamiltonian and v_α^s the short-range interaction of the pair (β,γ). In the angular momentum representation (omitting the explicit spin and isospin

dependence from our notation) we define the CS basis for the expansion of the short-range interactions in the three-particle system as

$$|n\nu l\lambda\rangle_\alpha = |nl\rangle_\alpha \otimes |\nu\lambda\rangle_\alpha \quad (n,\nu = 0,1,2,\ldots), \tag{16}$$

with the CS states from Eq. (1). Here l and λ denote the angular momenta of the two-body pair (β,γ) and of the third particle α relative to the center of mass of this pair, respectively. In the three-particle Hilbert space we have (with angular momentum summation implicitly included)

$$\mathbf{1} = \lim_{N\to\infty} \sum_{n,\nu=0}^{N} |\widetilde{n\nu l\lambda}\rangle_{\alpha\alpha}\langle n\nu l\lambda| = \lim_{N\to\infty} \mathbf{1}_{N,\alpha} \tag{17}$$

where the configuration-space representation in terms of Jacobi coordinates ξ_α and η_α reads

$$\langle \xi_\alpha \eta_\alpha | \widetilde{n\nu l\lambda}\rangle_\alpha = \frac{1}{\xi_\alpha \eta_\alpha} \langle \xi_\alpha \eta_\alpha | n\nu l\lambda\rangle_\alpha . \tag{18}$$

After the CS expansion of the potentials v^s_α, v^s_β, and v^s_γ in the three-particle space, the Faddeev equations can be rewritten as

$$|\Psi_\alpha\rangle = G_\alpha(E)[\mathbf{1}_{N,\alpha} v^s_\alpha \mathbf{1}_{N,\beta} |\Psi_\beta\rangle + \mathbf{1}_{N,\alpha} v^s_\alpha \mathbf{1}_{N,\gamma} |\Psi_\gamma\rangle]. \tag{19}$$

By applying the CS states ${}_\alpha\langle\widetilde{n\nu l\lambda}|$ from the left, Eqs. (19) turn into a linear system of homogeneous equations for the coefficients of the Faddeev components $\underline{A}_{l_\alpha\lambda_\alpha n\nu} = {}_\alpha\langle\widetilde{n\nu l\lambda}|\Psi_\alpha\rangle$:

$$\{[\underline{G}(E)]^{-1} - \underline{v}\}\underline{A} = 0. \tag{20}$$

A unique solution thereof exists if and only if

$$\det\{[\underline{G}(E)]^{-1} - \underline{v}\} = 0. \tag{21}$$

The matrices $\underline{G}(E)$ and $\underline{v}$ have a block structure and the matrix elements are given by

$$\underline{v}_{l_\alpha\lambda_\alpha n\nu, l'_\beta\lambda'_\beta n'\nu'} = (1-\delta_{\alpha\beta})\,{}_\alpha\langle n\nu l\lambda | v^s_\alpha | n'\nu' l'\lambda'\rangle_\beta \tag{22}$$

and

$$\underline{G}_{l_\alpha\lambda_\alpha n\nu, l'_\alpha\lambda'_\alpha n'\nu'}(E) = \delta_{\alpha\beta}\,{}_\alpha\langle\widetilde{n\nu l\lambda}|G_\alpha(E)|\widetilde{n'\nu' l'\lambda'}\rangle_\alpha , \tag{23}$$

respectively. Notice that the matrix elements of the Green's operator are needed only within the same partition α whereas the matrix elements of the potentials occur only between different partitions α and β. The latter may again be evaluated numerically either in configuration or in momentum space. We have adopted the configuration-space version of the Balian-Brézin method [13].

For the calculation of the matrix elements of the Green's operator in Eq. (23) we proceed in the following way. We split the three-particle free Hamiltonian into

$$H^0 = h^0_{\xi_\alpha} + h^0_{\eta_\alpha}, \tag{24}$$

i.e., the free motions in the two Jacobi coordinates. Then we define the two-body Hamiltonian h_{ξ_α} as

$$h_{\xi_\alpha} = h^0_{\xi_\alpha} + v^s_\alpha . \tag{25}$$

Since the commutator of the Hamiltonians h_{ξ_α} and $h^0_{\eta_\alpha}$ vanishes,

$$[h_{\xi_\alpha}, h^0_{\eta_\alpha}] = 0, \tag{26}$$

we may apply the convolution theorem by Bianchi and Favella [14]

$$G_\alpha(E) = (E - h_{\xi_\alpha} - h^0_{\eta_\alpha})^{-1} = \frac{1}{2\pi i}\oint_C d\epsilon\,(E-\epsilon-h_{\xi_\alpha})^{-1} \times (\epsilon - h^0_{\eta_\alpha})^{-1}. \tag{27}$$

Here the contour C encircles the spectrum of $h^0_{\eta_\alpha}$ without penetrating into the spectrum of h_{ξ_α} (cf. Fig. 1). We note that in this integral the roles of $h^0_{\eta_\alpha}$ and h_{ξ_α} may also be interchanged.

After sandwiching the above Green's operator between the CS states, the integral in Eq. (27) appears in the form

$$\underline{G}_{l_\alpha\lambda_\alpha n\nu, l'_\alpha\lambda'_\alpha n'\nu'}(E) = \frac{1}{2\pi i}\oint_C d\epsilon\,{}_\alpha\langle\widetilde{nl}|(E-\epsilon-h_{\xi_\alpha})^{-1}|\widetilde{n'l'}\rangle_\alpha \times {}_\alpha\langle\widetilde{\nu\lambda}|(\epsilon-h^0_{\eta_\alpha})^{-1}|\widetilde{\nu'\lambda'}\rangle_\alpha , \tag{28}$$

where the separate matrix elements occurring in the integrand are known from the two-particle case of the previous section [cf. Eq. (12)].

After solving Eq. (20) for the coefficients $\underline{A}_{l_\alpha\lambda_\alpha n\nu}$ the Faddeev components can be expressed as

$$|\Psi_\alpha\rangle = \sum_{n,\nu=0}^{N} \underline{B}_{l_\alpha\lambda_\alpha n\nu} G_\alpha(E)|\widetilde{n}\nu l\lambda\rangle_\alpha , \tag{29}$$

where the new coefficients again result from the matrix multiplication $\underline{B} = \underline{v}\,\underline{A}$. As before we hint at the advantage that the representation (29) of $|\Psi_\alpha\rangle$ guarantees for the correct asymptotic behavior of the Faddeev component, due to the explicit occurrence of the Green's operator. This is of particular importance in the Coulomb case below.

B. Coulomb-like interactions

In this section we extend the formulation of the three-body problem to the case of long-range interactions. We as-

FIG. 1. Contour C for the integral in Eq. (27) in case of the three-body bound-state problem. To the right of the dotted vertical line lies the (continuous) spectrum of $h^0_{\eta_\alpha}$, to its left the (discrete and continuous) spectrum of h_{ξ_α}.

sume the subsystem interaction to be a sum of short-range plus (repulsive) Coulomb interactions

$$v^{\alpha}=v^{s}_{\alpha}+v^{C}_{\alpha}, \tag{30}$$

and adhere to the Faddeev equations in the form as modified by Noble [15]:

$$|\Psi_{\alpha}\rangle=G^{C}_{\alpha}(E)[v^{s}_{\alpha}|\Psi_{\beta}\rangle+v^{s}_{\alpha}|\Psi_{\gamma}\rangle] \tag{31}$$

with the Coulomb-like Green's operator

$$G^{C}_{\alpha}(E)=(E-H^{0}-v^{s}_{\alpha}-v^{C}_{\alpha}-v^{C}_{\beta}-v^{C}_{\gamma})^{-1}. \tag{32}$$

Herein all long-range interactions for all subsystems are collected, in complete analogy to the two-body case [cf. Eq. (6)].

As in the previous section the short-range potentials are expanded on the CS basis leading to an equation similar to (19). A unique solution thereof exists if and only if

$$\det\{[\underline{G}^{C}(E)]^{-1}-\underline{v}\}=0, \tag{33}$$

where the matrices have the same block structure as before in Eq. (21). The important point is that $\underline{v}$ contains only matrix elements of the short-range interactions; in fact they are completely equivalent to Eq. (22).

For the calculation of the matrix elements of the Coulomb-like Green's operator we proceed along the lines of the two-potential formalism [16]. First we rewrite it using the resolvent equation

$$G^{C}_{\alpha}(E)=\widetilde{G}^{C}_{\alpha}(E)+\widetilde{G}^{C}_{\alpha}(E)(v^{C}_{\beta}+v^{C}_{\gamma}-u^{C}_{\alpha})G^{C}_{\alpha}(E) \tag{34}$$

with $\widetilde{G}^{C}$ defined by

$$\widetilde{G}^{C}_{\alpha}(E)=(E-H^{0}-v^{s}_{\alpha}-v^{C}_{\alpha}-u^{C}_{\alpha})^{-1}. \tag{35}$$

Here we have introduced the auxiliary potential u^{C}_{α}, which is required to have the asymptotic form

$$u^{C}_{\alpha}\sim\frac{Z_{\alpha}(Z_{\beta}+Z_{\gamma})}{\eta_{\alpha}} \tag{36}$$

as $\eta_{\alpha}\to\infty$. It may be viewed as the effective Coulomb potential between the center of mass of the subsystem α (with charge $Z_{\beta}+Z_{\gamma}$) and the third particle (with charge Z_{α}). The important role of the potential u^{C}_{α} is that asymptotically it compensates the Coulomb tail of the long-range potentials $v^{C}_{\beta}+v^{C}_{\gamma}$ in Eq. (34). Thus the combination $U=v^{C}_{\beta}+v^{C}_{\gamma}-u^{C}_{\alpha}$ can be subject to a separable expansion and effectively be treated as a short-range potential. With the help of the formal solution of Eq. (34) we may now express the inverse matrix $[\underline{G}^{C}_{\alpha}(E)]^{-1}$ as

$$[\underline{G}^{C}_{\alpha}(E)]^{-1}=[\underline{\widetilde{G}}^{C}_{\alpha}(E)]^{-1}-\underline{U}, \tag{37}$$

where $\underline{U}$ is constructed from the matrix elements

$$\underline{U}_{l_{\alpha}\lambda_{\alpha}n\nu,l'_{\alpha}\lambda'_{\alpha}n'\nu'}={}_{\alpha}\langle n\nu l\lambda|(v^{C}_{\beta}+v^{C}_{\gamma}-u^{C}_{\alpha})|n'\nu'l'\lambda'\rangle_{\alpha} \tag{38}$$

resulting from the CS expansion. The matrix elements of the Green's operator $\widetilde{G}^{C}_{\alpha}(E)$ can again be calculated by the contour integral as before. We are then left with the integral

$$\underline{\widetilde{G}}^{C}_{l_{\alpha}\lambda_{\alpha}n\nu,l'_{\alpha}\lambda'_{\alpha}n'\nu'}(E)$$
$$=\frac{1}{2\pi i}\oint_{C}d\epsilon_{\alpha}\langle\widetilde{nl}|(E-\epsilon-h_{\xi_{\alpha}})^{-1}|\widetilde{n'l'}\rangle_{\alpha\alpha}$$
$$\times\langle\widetilde{\nu\lambda}|(\epsilon-h^{C}_{\eta_{\alpha}})^{-1}|\widetilde{\nu'\lambda'}\rangle_{\alpha}, \tag{39}$$

where $h^{C}_{\eta_{\alpha}}=h^{0}_{\eta_{\alpha}}+u^{C}_{\alpha}$ and $h_{\xi_{\alpha}}=h^{0}_{\xi_{\alpha}}+v^{s}_{\alpha}+v^{C}_{\alpha}$.

The calculation of the Faddeev amplitude $|\Psi_{\alpha}\rangle$ in Eq. (31) is completely analogous to the short-range case of the previous section. Only in Eq. (29) the Green's operator $G_{\alpha}(E)$ must now be replaced by the Coulomb Green's operator $G^{C}_{\alpha}(E)$ of Eq. (32).

IV. TESTS OF THE METHOD

In this section we demonstrate the performance of the method in calculations of various three-body bound states. We have selected cases for which benchmark results are already available in the literature. The comparisons will prove the efficiency of our method, especially in the situation when Coulomb forces are present.

A. Illustration of the convergence of the CS expansion

Before presenting the final results, let us demonstrate the convergence of the results for the three-body bound-state energies. For this purpose we take the example of the Ali-Bodmer (AB) potential [17] between α particles of mass M

$$v^{s}(r)=500\ \exp[-(0.7r)^{2}]-130\ \exp[-(0.475r)^{2}] \tag{40}$$

without and

$$v(r)=v^{s}(r)+4e^{2}/r \tag{41}$$

with Coulomb interaction. We use units such that $\hbar^{2}/M=10.366\ 75$ MeV fm^{2} and $e^{2}=1.44$ MeV fm.

Evidently the quality of the results will depend on the number of terms employed in the separable expansion of the (short-range) potential. We quote the values of the binding energies from calculations without (Table I) and with (Table II) Coulomb forces, taking into account different numbers of channels $n_{\rm ch}=1,2,3,4$ (corresponding to angular momentum states up to $l=\lambda=0$, $l=\lambda=2$, $l=\lambda=4$, and $l=\lambda=6$ employed). In all cases it is observed that convergence up to five significant digits is comfortably achieved with $N=20$ terms applied for n and ν in the separable expansion. Remarkably, the speed of the convergence is everywhere similar, irrespective of how many angular momentum channels are included and whether or not Coulomb forces are present. We note especially for the Coulomb case that the satisfactory convergence stems from reliable separable expansions of the potentials v^{s} and U, which—from the point of view of scattering theory—are both short-range potentials; the falloff of U is much slower than that of v^{s}, though.

TABLE I. Convergence of the binding energy of a three-boson system interacting via the Ali-Bodmer potential, Eq. (40), with increasing basis for the separable expansion. N denotes the maximum number of basis states employed for n and ν in Eq. (19).

N	Number of channels n_{ch}			
	1	2	3	4
12	4.13899	5.11756	5.17699	5.17888
13	4.14092	5.11911	5.17862	5.18057
14	4.14044	5.11897	5.17846	5.18043
15	4.14046	5.11898	5.17850	5.18047
16	4.14065	5.11917	5.17871	5.18069
17	4.14069	5.11926	5.17880	5.18079
18	4.14069	5.11926	5.17881	5.18080
19	4.14069	5.11926	5.17881	5.18080
20	4.14070	5.11927	5.17882	5.18081
21	4.14071	5.11928	5.17884	5.18083
22	4.14071	5.11929	5.17884	5.18083
23	4.14071	5.11929	5.17884	5.18083
24	4.14071	5.11929	5.17884	5.18083

In principle, the convergence may also depend on the (range) parameter b of the Coulomb-Sturmian functions. We found, however, that the dependence is weak in a relatively large interval of possible choices, just as was established in the two-body case [8,9].

The convergence is practically of the same quality in the case of the other potentials considered below.

B. Results for various three-body bound states

We now present our converged results for the Malfliet-Tjon (MT) [18] and AB [17] potentials and compare them to other benchmark calculations. For the MT potential

$$v^s(r) = V_r \exp(-3.11r)/r - V_a \exp(-1.55r)/r \quad (42)$$

between two nucleons of mass m we consider two cases:

TABLE II. Same as in Table I for the Ali-Bodmer plus Coulomb potential, Eq. (41).

N	Number of channels n_{ch}			
	1	2	3	4
12	1.90151	2.81629	2.86703	2.86839
13	1.90473	2.81833	2.86912	2.87054
14	1.90368	2.81796	2.86871	2.87014
15	1.90373	2.81797	2.86875	2.87019
16	1.90404	2.81819	2.86899	2.87044
17	1.90397	2.81824	2.86904	2.87049
18	1.90401	2.81824	2.86905	2.87050
19	1.90400	2.81824	2.86905	2.87050
20	1.90401	2.81825	2.86906	2.87051
21	1.90402	2.81826	2.86907	2.87053
22	1.90402	2.81827	2.86908	2.87053
23	1.90402	2.81827	2.86908	2.87053
24	1.90402	2.81827	2.86908	2.87053

TABLE III. Binding energies in the case of the MTVa potential for a system of three bosons.

	Angular momentum channels			
	1	2	3	4
This work	8.04251	8.22953	8.24978	8.25215
Faddeev [19]	8.0424	8.228	8.249	8.251
Faddeev [20]		8.25273		
ATMS [21]		8.26(1)		
GFMC [22]		8.26(1)		
IDEA [23]		8.25		
SVM [24]		8.2527		

MTVa: $V_r = 1458.0470$, $V_a = 578.0890$,
MTVb: $V_r = 1438.4812$, $V_a = 570.3316$

(here we use units such that $\hbar^2/m = 41.47$ MeV fm^2). For MTVa we may compare to the results of the Los Alamos–Iowa [19] and Groningen [20] groups, both of which are obtained from a direct solution of the Faddeev equations in configuration space, and in addition to results from an ATMS (amalgamation of two-body correlations into multiple scattering process) [21] calculation, a Green's function Monte Carlo (GFMC) calculation [22], an integro-differential-equation approach (IDEA) [23], and a stochastic variational method (SVM) [24]. From Table III it is evident that our method provides very accurate predictions for binding energies in all cases. This is true for the channel-by-channel comparison with the Los Alamos–Iowa calculation and likewise for the comparison with the best results from the other works. With respect to the best results quoted in the lower rows of Table III we note that for the corresponding calculations the number of angular momentum channels employed is either not definitely known or not specified in a scheme like ours.

In order to demonstrate that in addition to binding energies our method also provides accurate three-body wave functions, we calculated the root-mean-square (rms) radius $\langle r^2 \rangle^{1/2}$. Corresponding results are given in Table IV for the MTVb potential, in which case we can compare to the calculations of the Los Alamos–Iowa group [19]. For both the binding energy and the rms radius the channel-by-channel comparison indicates perfect agreement.

In the case of the MTI-III potential for a system of three fermions, acting in singlet and triplet states, we may compare to the two-channels calculation of the Los Alamos–Iowa group. For the MTI-III potential as parametrized in Ref. [25],

TABLE IV. Binding energies and root-mean-square radii for the MTVb potential for a system of three bosons.

		Angular momentum channels			
		1	2	3	4
$-E_B$	This work	7.5398	7.7147	7.7338	7.7361
	Faddeev [19]	7.540	7.714	7.733	7.735
$\langle r^2 \rangle^{1/2}$	This work	1.7265	1.7117	1.7098	1.7095
	Faddeev [19]	1.727	1.711	1.710	1.710

TABLE V. Binding energies and root-mean-square radii in the case of the Ali-Bodmer and Ali-Bodmer plus Coulomb potentials [Eqs. (40) and (41), respectively] for a system of three bosons.

Ali-Bodmer		Without v^C	With v^C
$-E_B$	This work	5.181	2.871
	ATMS [21]	5.18	
	SVM [24]	5.18	2.872
$\langle r^2\rangle^{1/2}$	This work	2.434	2.517
	ATMS [21]	2.43	
	SVM [24]	2.43	2.517

we obtain a binding energy (converged result) of $E=8.5358$ in comparison to $E=8.536$ calculated by the Los Alamos–Iowa collaboration.

Finally we come to the comparison of the results for the binding energy of a system of three bosons interacting via the AB and AB plus Coulomb potential (Table V). We may compare our four-channels result to calculations with the ATMS method [21] (uncharged case only) and the SVM [24,26]. We do not know of any Faddeev results in this case. Again we show predictions for the binding energies and rms radii. All the values quoted in Table V show a convincing agreement of our results with the ones from the other approaches. We specially stress the agreement of the result for the case of rigorously including the Coulomb interaction with the rather reliable answer from the SVM.

V. CONCLUSION

We have suggested a separable expansion scheme, relying on Coulomb-Sturmian basis functions, for solving the three-body problem. The method is especially suited to the case when Coulomb-like interactions are present in one or all subsystems. It allows one to solve the three-body integral equations by expanding only the short-range part of the interaction in a separable form while keeping the effect of the long-range part in an exact manner via a proper integral representation of the three-body Coulomb Green's operator. As a consequence the method has good convergence properties and can in practice be made arbitrarily accurate by employing an increasing number of terms in the separable expansion. The usage of the Coulomb-Sturmian basis is essential to allow for the accurate evaluation of the matrix elements of the Coulomb Green's operator.

Beyond the studies of the method in systems with two-body asymptotics conducted before [8–11], we have now demonstrated its convergence properties and efficiency in (benchmark) calculations of the three-body bound-state problem without and with Coulomb interactions. In both cases the solution of the Faddeev equations shows a rapid convergence, and, whenever a comparison is possible to existing results in the literature, correct predictions for the binding energies and wave functions are achieved.

The method is capable of treating any kind of short-range interactions, even in the case when Coulomb-like forces are present. The solution of the three-body bound-state problem was carried out here. However, the method is also applicable for scattering problems. In this regard it has been proven useful already in the two-body case [9–11]. To solve the corresponding problem for a (charged) three-body system with the Faddeev equations some technical details in connection with the evaluation of the then occurring matrix elements still need to be worked out.

ACKNOWLEDGMENTS

The authors are grateful to K. Varga for providing results from his SVM calculation for comparison. The work of Z.P. was supported by OTKA under Contracts No. F4305 and No. T17298. This work received support also from the Austrian-Hungarian Scientific-Technical Cooperation within Project No. A23.

[1] W. Plessas, in *Few-Body Methods: Principles and Applications*, edited by T.-K. Lim, C.-G. Bao, D.-P. Hou, and S. Huber (World Scientific, Singapore, 1986), p. 43.
[2] D. J. Ernst, C. M. Shakin, and R. M. Thaler, Phys. Rev. C **8**, 46 (1973).
[3] Y. Koike, J. Haidenbauer, and W. Plessas, Phys. Rev. C **35**, 396 (1987).
[4] T. Cornelius, W. Glöckle, J. Haidenbauer, Y. Koike, W. Plessas, and H. Witała, Phys. Rev. C **41**, 2538 (1990).
[5] W. Glöckle, H. Kamada, H. Witała, D. Hüber, J. Golak, K. Miyagawa, and S. Ishikawa, Few-Body Syst. Suppl. **8**, 9 (1996).
[6] C. R. Chen, G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. C **39**, 1261 (1989); E. O. Alt, W. Sandhas, and H. Ziegelmann, Nucl. Phys. A **445**, 429 (1985); G. H. Berthold, A. Stadler, and H. Zankel, Phys. Rev. Lett. **61**, 1077 (1988); Phys. Rev. C **41**, 1365 (1990); H.-P. Kotz, Ch. Brandstätter, W. Plessas, and G. Berthold, in *Few-Body Problems in Physics*, edited by F. Gross (AIP, New York, 1995), p. 482.
[7] E. O. Alt and M. Rauh, Few-Body Syst. Suppl. **7**, 160 (1994); Few-Body Syst. **17**, 121 (1994).
[8] Z. Papp, J. Phys. A **20**, 153 (1987).
[9] Z. Papp, Phys. Rev. C **38**, 2457 (1988); Phys. Rev. A **46**, 4437 (1992).
[10] Z. Papp, Comput. Phys. Commun. **70**, 426 (1992); **70**, 435 (1992).
[11] Z. Papp and E. W. Schmid, Few-Body Syst. **13**, 159 (1992).
[12] M. Rotenberg, Ann. Phys. (N.Y.) **19**, 262 (1962); Adv. At. Mol. Phys. **6**, 233 (1970).
[13] R. Balian and E. Brézin, Nuovo Cimento B **2**, 403 (1969).
[14] L. Bianchi and L. Favella, Nuovo Cimento **6**, 6873 (1964).
[15] J. V. Noble, Phys. Rev. **161**, 945 (1967).
[16] Gy. Bencze, Nucl. Phys. A **196**, 135 (1972); Lett. Nuovo Cimento **17**, 91 (1976); Gy. Bencze, G. Cattapan, and V. Vanzani, *ibid.* **20**, 248 (1977); Gy. Bencze, P. Doleschall, C. Chandler, A. G. Gibson, and D. Walliser, Phys. Rev. C **43**, 992 (1991).
[17] S. Ali and A. R. Bodmer, Nucl. Phys. **80**, 99 (1966).

[18] R. A. Malfliet and J. A. Tjon, Nucl. Phys. A **127**, 161 (1969).
[19] J. L. Friar, B. F. Gibson, and G. L. Payne, Phys. Rev. C **24**, 2279 (1981).
[20] N. W. Schellingerhout and L. P. Kok, Nucl. Phys. A **508**, 299c (1990).
[21] Y. Akaishi, in *Models and Methods in Few-Body Physics*, edited by L. S. Ferreira, A. C. Fonseca, and L. Streit, Lecture Notes in Physics Vol. 273 (Springer, Heidelberg, 1987), p. 324; in *Cluster Models and Other Topics*, edited by T. T. S. Kuo and E. Osnes, International Review of Nuclear Physics Vol. 4 (World Scientific, Singapore, 1986), p. 259.
[22] J. G. Zabolitzky, K. E. Schmidt, and M. H. Kalos, Phys. Rev. C **25**, 1111 (1982); J. G. Zabolitzky, Prog. Part. Nucl. Phys. **16**, 103 (1986).
[23] M. Fabre de la Ripelle, H. Fiedeldey, and S. A. Sofianos, Few-Body Syst. **6**, 157 (1989).
[24] K. Varga and Y. Suzuki, Phys. Rev. C **52**, 2885 (1995).
[25] G. L. Payne, J. L. Friar, B. F. Gibson, and I. R. Afnan, Phys. Rev. C **22**, 823 (1980).
[26] K. Varga (private communication).