# Superfluid densities in neutron-star matter

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The superfluid densities in a mixture of neutron and proton superfluids are discussed within Fermi-liquid theory. There are the usual diagonal densities $\rho_{nn}$ and $\rho_{pp}$ but also off-diagonal densities $\rho_{np}$ and $\rho_{pn}$ which represent coupling between the two species. We express these quantities solely in terms of the Fermi momenta and the Landau parameter $f_1^{pn}$ in a way that explicitly satisfies the Galilean invariance constraints $\rho_{pp}+\rho_{pn}=m_p n_p$ and $\rho_{nn}+\rho_{pn}=m_n n_n$. The results are of astrophysical interest, e.g., in a discussion of the damping of gravitational instability in rapidly rotating neutron stars or of post-glitch relaxation of pulsars. [S0556-2813(96)02911-1]

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## I. INTRODUCTION

Neutron stars have a liquid core of neutrons in beta equilibrium with protons. The number density of protons is found to be a few percent of the number density of neutrons. It is thought that between densities of $2\times10^{14}$ g/cm$^3$ and $\sim5\times10^{14}$ g/cm$^3$ the neutrons are in a $^3P_2$ superfluid state and the protons are in a $^1S_0$ superfluid state [1,2].

In the hydrodynamics of superfluid solutions (or mixtures) the velocity of each species is coupled to the mass current of the other species [3–5]. This coupling of a proton superfluid with a neutron superfluid is of considerable astrophysical interest as it influences the damping of gravitational radiation instability in neutron stars [6], as well as the post-glitch relaxation of pulsars [5]. It is therefore important to have reliable expressions for the diagonal and off-diagonal density coefficients in the mass current expressions. As pointed out in in Ref. [7], it is unclear whether previously derived expressions satisfy the sum rules [see Eq. (4) below] expressing Galilean invariance.

In this paper we present a derivation, within Fermi-liquid theory, of the diagonal and off-diagonal densities in a mixture of neutron and proton superfluids and estimate their numerical values in the neutron-star core.

## II. GENERAL CONSIDERATIONS

The object of this section is to derive the relationship between the mass currents of the two nucleon species and their respective superfluid velocities. In the case of a one-component fluid in a translationally invariant background such as a pure neutron fluid or superfluid $^3$He at zero temperature, Galilean relativity requires that $\vec{g}=nm\vec{v}$, where $\vec{g}$ is the mass current, $n$ is the number density, $m$ is the mass, and $\vec{v}$ is the superfluid velocity. In the interior of a neutron star containing a mixture of neutrons and protons, the flow of neutrons entrains a flow of protons and vice versa. Thus the relationship of momentum and velocity is not as simple as it might at first seem. However, this drag effect is well studied in the context of the helium fluids and the electron liquid in metals. Accordingly, standard methods can be used to derive the relationship.

The superfluid densities are defined by the equations

$$\vec{g}_p=\rho_{pp}\vec{v}_p+\rho_{pn}\vec{v}_n, \tag{1}$$

$$\vec{g}_n=\rho_{np}\vec{v}_p+\rho_{nn}\vec{v}_n. \tag{2}$$

Here $\vec{g}_p$ ($\vec{g}_n$) is the mass current density for the protons (neutrons), and $\vec{v}_p$ ($\vec{v}_n$) is the superfluid velocity. The electrical current density is $e\vec{g}_p/m_p$, where $e$ and $m_p$ are the charge and mass of the proton. The coefficients $\rho$ are the superfluid densities to be calculated. This collection of coefficients is sometimes called a superfluid density tensor. We avoid this terminology, as the coefficients are actually scalars under translations, rotations, and boosts. There is a negligible contribution to the mass current from the normal fluid component in the system under consideration here, since the temperature in neutron stars is much less than the gap energy. We will therefore restrict our attention in this paper to the case of zero temperature. Thus, the normal fluid velocities do not appear in Eqs. (1) and (2). We shall adopt a slightly more compact notation by introducing an isospin index $\sigma(\sigma=n,p)$. Equations (1) and (2) are then written as

$$\vec{g}_\sigma=\sum_{\sigma'}\rho_{\sigma\sigma'}\vec{v}_{\sigma'}. \tag{3}$$

Galilean invariance implies sum rules for the superfluid densities. In the ground state of the combined neutron-proton system, we have $\vec{v}_p=\vec{v}_n=0$. When this state is viewed from a frame moving with velocity $-\vec{v}$, these velocities are $\vec{v}_p=\vec{v}_n=\vec{v}$. In this frame we must have $\vec{g}_p=n_p m_p\vec{v}$, where $n_p$ is the number density of the protons and $m_p$ is their mass. A corresponding statement holds for the neutrons. Comparison with Eq. (3) then yields

$$\sum_{\sigma'}\rho_{\sigma\sigma'}=n_\sigma m_\sigma. \tag{4}$$

 

Furthermore, macroscopic thermodynamics tells us that the energy density change on changing the velocities at zero temperature is

$$d\varepsilon = \sum_{\sigma\sigma'} \rho_{\sigma\sigma'} \vec{v}_{\sigma} \cdot d\vec{v}_{\sigma'} , \tag{5}$$

from which we deduce [4] that

$$\rho_{pn} = \rho_{np} . \tag{6}$$

Any theory of the superfluid densities must satisfy Eqs. (4) and (6).

## III. FERMI-LIQUID THEORY FOR A NEUTRON-PROTON MIXTURE

The energy of a Fermi liquid consisting of two fermion species is given by

$$E([\delta n_s^{\sigma}(\vec{k})]) = E_0 + \sum_{\vec{k}s\sigma} \varepsilon_s^{\sigma}(\vec{k})\,\delta n_s^{\sigma}(\vec{k}) + \frac{1}{2} \sum_{\vec{k}\vec{k}'ss'\sigma\sigma'} f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}')\,\delta n_s^{\sigma}(\vec{k})\,\delta n_{s'}^{\sigma'}(\vec{k}'). \tag{7}$$

This is a simple generalization of a formula used in the theory of heavy nuclei [8]. This formula will form the basis of the theory of superfluid drag, and so we discuss it in some detail.

$E_0$ is the ground state energy, and $\delta n_s^{\sigma}(\vec{k})$ is the occupation function which labels the excited states of the system. Because these states are reached from the corresponding noninteracting states by adiabatic continuation, this label has the same meaning as in the noninteracting case. The energy of the excited state is affected by the interaction. This energy may be parametrized by the functions $\varepsilon_s^{\sigma}(\vec{k})$ and $f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}')$, which are called collectively the Landau parameters.

The function $\varepsilon_s^{\sigma}(\vec{k})$ incorporates the mass renormalization

$$\frac{d\varepsilon^{\sigma}}{dk} = k/m_{\sigma}^{*} , \tag{8}$$

where the derivative is evaluated at the appropriate Fermi surfaces given by the Fermi wave vectors

$$k_{F\sigma} = (3\pi^2 n_{\sigma})^{1/3}. \tag{9}$$

We have chosen units in which $\hbar = 1$ and shall also work in a unit volume. Equation (8) defines the effective mass. The function $f$ determines the interaction between quasiparticles. Since it represents a second partial derivative of the energy with repect to the occupation number, it satisfies symmetry conditions

$$f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}') = f_{s's}^{\sigma\sigma'}(\vec{k},\vec{k}') = f_{ss'}^{\sigma'\sigma}(\vec{k},\vec{k}') = f_{ss'}^{\sigma\sigma'}(\vec{k}',\vec{k}). \tag{10}$$

The expression for the system energy amounts to an expansion valid at low excitation energy. Since the superfluid flows we will discuss represent distortions of the Fermi surface which are small on the scale of the Fermi wave vectors, this expansion is justified. A second, more subtle, point about this expression is that it is usually applied to *normal* Fermi fluids, not the superfluids we discuss here. However, the change in the wave function produced by condensation of a Fermi liquid, judged by the alteration in the momentum distribution, is so minor that Fermi-liquid expressions valid in the normal state may still be applied to the condensed state. This point has been particularly emphasized by Leggett [9] in the context of superfluid $^3$He. Finally we note that this form of the expansion is not capable of taking the spin-orbit interaction into account. For that, a matrix representation of the spin degrees of freedom is necessary.

We wish to evaluate the general expression in Eq. (7) in a relatively special situation, that of superfluid flow. The momentum distribution for species $\sigma$ is that of a Fermi sea translated by a vector $\vec{q}_{\sigma}$ so that

$$\delta n_s^{\sigma}(\vec{k}) = \vec{q}_{\sigma} \cdot \hat{k}\,\delta(k - k_{F\sigma}). \tag{11}$$

For example, if the proton superfluid velocity is $\vec{v}$, then the proton Fermi surface is translated by $\vec{q}_p = m_p \vec{v}$, but the neutron Fermi surface is unchanged, and

$$\delta E = E - E_0 = 2 \sum_{\vec{k}} \varepsilon^{p}(\vec{k})\,\delta n^{p}(\vec{k}) + \frac{1}{2} \sum_{\vec{k}\vec{k}'ss'} f_{ss'}^{pp}(\vec{k},\vec{k}')\,\vec{q}_p \cdot \hat{k}\,\vec{q}_p \cdot \hat{k}'\,\delta(k - k_{Fp}) \times \delta(k' - k_{Fp}). \tag{12}$$

This equation may be simplified by defining the spin-symmetrized interaction $f^{\sigma\sigma'} = \frac{1}{2}[f_{++}^{\sigma\sigma'} + f_{+-}^{\sigma\sigma'}]$, and by noting that the form of the momentum sum picks out only one of the components in the expansion of the interaction function in Legendre polynomials. Thus we define

$$f^{\sigma\sigma'}(\vec{k},\vec{k}') = \sum_{\ell} f_{\ell}^{\sigma\sigma'} P_{\ell}(\cos(\theta)), \tag{13}$$

where $\theta$ is the angle between $\vec{k}$ and $\vec{k}'$; rotational invariance implies that $f$ depends only on this angle. The energy then simplifies to

$$\delta E = 2 \sum_{\vec{k}} \varepsilon^{p}(\vec{k})\,\delta n^{p}(\vec{k}) + 2 \sum_{\vec{k}\vec{k}'} f_1^{pp} \hat{k} \cdot \hat{k}'\,\delta(k' - k_{Fp}), \tag{14}$$

and the sums may be performed to yield

$$\delta E = n_p \frac{q^2}{2m_p^{*}} + \frac{2}{(2\pi)^6} \left(\frac{4\pi}{3}\right)^2 f_1^{pp} k_{Fp}^4 q^2. \tag{15}$$

In the second term in the result, the first numerical factor comes from the sum over spin and phase space density, and the second factor comes from the angular averages. It is conventional at this stage to define a dimensionless interaction parameter. First we note that the density of quasiparticle states at the Fermi surface is given by

$$D_{\sigma} = m_{\sigma}^{*} k_{F\sigma} / \pi^2, \tag{16}$$

and then we set

$$F_1^{\sigma\sigma} = D_\sigma f_1^{\sigma\sigma}. \tag{17}$$

This leads to the relation $k_{F\sigma}^4 = 3\pi^4 D_\sigma n_\sigma / m_\sigma^*$, and finally

$$\delta E = n_p \frac{q_p^2}{2m_p^*}(1 + F_1^{pp}/3). \tag{18}$$

The energy of this state may also be expressed as

$$\delta E = n_p \frac{q_p^2}{2m_p} + \delta V, \tag{19}$$

where the contributions of kinetic and potential energies have been explicitly separated. The same considerations applied to a state of neutron superflow give the analogous equations

$$\delta E = n_n \frac{q_n^2}{2m_n^*}(1 + F_1^{nn}/3) \tag{20}$$

and

$$\delta E = n_n \frac{q_n^2}{2m_n} + \delta V. \tag{21}$$

Since $\delta V$ must be the same in both cases (the two are related by a Galilean boost), we have the relation

$$n_p m_p^2 \left[ \frac{1}{m_p} - \frac{1}{m_p^*}(1 + F_1^{pp}/3) \right] = n_n m_n^2 \left[ \frac{1}{m_n} - \frac{1}{m_n^*}(1 + F_1^{nn}/3) \right]. \tag{22}$$

This introductory calculation gives us the tools we need to derive the restrictions imposed on the Landau parameters by Galilean relativity. In the case of a one-component Fermi liquid, this principle gives the well-known identity

$$m^*/m = 1 + F_1/3. \tag{23}$$

However, in the case of a mixture of Fermi liquids this identity does *not* hold—the effective mass of a proton is affected by its interaction with the neutrons as well as its interaction with other protons. To obtain the generalization to a mixture, we again consider the ground state as viewed from a frame moving with a velocity $-\vec{v}$ [10]. The two Fermi surfaces are displaced by slightly different amounts: $\vec{q}_\sigma = m_\sigma \vec{v}$. The calculation of the energy change proceeds by substituting Eq. (11) into Eq. (7) to give

$$\delta E = n_p \frac{q_p^2}{2m_p^*}(1 + F_1^{pp}/3) + n_n \frac{q_n^2}{2m_n^*}(1 + F_1^{nn}/3) + k_{Fp}^2 k_{Fn}^2 f_1^{np} q_p q_n / 9\pi^4. \tag{24}$$

As this is a boosted ground state there is no change in potential energy and this energy may also be written as

$$\delta E = n_p \frac{q_p^2}{2m_p} + n_n \frac{q_n^2}{2m_n}. \tag{25}$$

Combining these two equations yields a second identity

$$n_p \frac{m_p}{2} + n_n \frac{m_n}{2} = n_p \frac{m_p^2}{2m_p^*}(1 + F_1^{pp}/3) + n_n \frac{m_n^2}{2m_n^*}(1 + F_1^{nn}/3) + k_{Fp}^2 k_{Fn}^2 f_1^{np} m_p m_n / 9\pi^4. \tag{26}$$

The two identities, Eq. (22) and Eq. (26), replace the effective mass identity of the one-component system. They are generalizations of the identities given by Sjöberg [10] to the case where the two component species have different bare masses. The cross coupling of the two superfluids arises from the last term in Eq. (22). This cross coupling is often referred to as superfluid drag although it involves no dissipation of energy. It is interesting to observe that the cross coupling exists just to the extent that the one-component effective mass relationship fails to hold.

## IV. DERIVATION OF THE SUPERFLUID DENSITIES

We start from the basic fact that the number of nucleons of each species is conserved, and therefore the densities separately satisfy the continuity equation

$$m_\sigma \frac{\partial n_\sigma}{\partial t} + \nabla \cdot \vec{g}_\sigma = 0. \tag{27}$$

This equation defines the mass current $g_\sigma$. In order to obtain an explicit expression for $g_\sigma$, we consider the flow of quasiparticles in phase space. The distribution function satisfies

$$\frac{\partial n_s^\sigma(\vec{k},\vec{r})}{\partial t} = -\nabla_{\vec{r}} n_s^\sigma(\vec{k},\vec{r}) \cdot \nabla_{\vec{k}} \tilde{\varepsilon}_s^\sigma(\vec{k},\vec{r}) + \nabla_{\vec{k}} n_s^\sigma(\vec{k},\vec{r}) \cdot \nabla_{\vec{r}} \tilde{\varepsilon}_s^\sigma(\vec{k},\vec{r}), \tag{28}$$

where

$$\tilde{\varepsilon}_s^\sigma(\vec{k},\vec{r}) \equiv \varepsilon_s^\sigma(\vec{k},\vec{r}) + \sum_{\vec{k}' s' \sigma'} f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}')\, \delta n_{s'}^{\sigma'}(\vec{k}',\vec{r}) \tag{29}$$

is the *local* quasiparticle energy [11]. We now linearize this in $\delta n$ to obtain a Landau-Boltzmann transport equation

$$\frac{\partial \delta n_s^\sigma(\vec{k},\vec{r},t)}{\partial t} + \nabla_{\vec{r}} \delta n_s^\sigma(\vec{k},\vec{r},t) \cdot \vec{v}_s^\sigma(\vec{k}) - \nabla_{\vec{k}} \Theta(k - k_{F\sigma}) \cdot \sum_{\vec{k}', s', \sigma'} f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}') \times \nabla_{\vec{r}} \delta n_{s'}^{\sigma'}(\vec{k}',\vec{r},t) = 0. \tag{30}$$

In order to display the similarity of this equation to the continuity equation, we use the fact that $\nabla_{\vec{k}} \Theta(k - k_{F\sigma}) = \vec{v}(\vec{k})\, \delta(\varepsilon_{\vec{k}} - \varepsilon_{F\sigma})$ to rewrite it as

$$\frac{\partial \delta n_s^\sigma(\vec{k},\vec{r},t)}{\partial t} + \vec{v}_s^\sigma(\vec{k}) \cdot \nabla_{\vec{r}} \left[ \delta n_s^\sigma(\vec{k},\vec{r},t) + \delta(\varepsilon_{\vec{k}} - \varepsilon_{F\sigma}) \times \sum_{\vec{k}', s', \sigma'} f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}')\, \delta n_{s'}^{\sigma'}(\vec{k}',\vec{r},t) \right] = 0. \tag{31}$$

We sum this equation over the momenta and spins and use

$$\frac{\partial n_\sigma}{\partial t} = \sum_{\vec{k},s} \frac{\partial \delta n_s^\sigma(\vec{k})}{\partial t}, \tag{32}$$

which yields an expression

$$\vec{g}_\sigma = m_\sigma \sum_{\vec{k},s} \vec{v}_s^\sigma(\vec{k}) \delta \widetilde{n}_s^\sigma(\vec{k}) \tag{33}$$

for the mass current. Here the definition

$$\delta \widetilde{n}_s^\sigma(\vec{k}) \equiv \delta n_s^\sigma + \delta(\varepsilon_{\vec{k}} - \varepsilon_{F\sigma}) \sum_{\vec{k}',s',\sigma'} f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}') \delta n_{s'}^{\sigma'}(\vec{k}') \tag{34}$$

has been made.

This is now an explicit expression for the mass current. To obtain the superfluid densities, it only remains to evaluate it when a supercurrent is flowing. We therefore take $\vec{v}_\sigma = \vec{q}_\sigma / m_\sigma$, and use Eq. (11) for the change in the distribution function.

Since it is the second term in Eq. (34) which is responsible for superfluid cross coupling, we show some of the steps involved in its simplification:

$$\delta(\varepsilon_{\vec{k}} - \varepsilon_{F\sigma}) \sum_{\vec{k}',s',\sigma'} f_{ss'}^{\sigma\sigma'}(\vec{k},\vec{k}') \delta n_{s'}^{\sigma'}(\vec{k}')$$

$$= 2 \frac{m_\sigma}{k_{F\sigma}} \delta(k - k_{F\sigma}) \sum_{\vec{k}'\sigma'} f_1^{\sigma\sigma'} \hat{k} \cdot \hat{k}' \vec{q}_{\sigma'} \cdot \hat{k}' \delta(k' - k_{F\sigma}) \tag{35}$$

$$= 2 \frac{m_\sigma}{k_{F\sigma}} \delta(k - k_{F\sigma}) \frac{4\pi}{3} \frac{1}{2\pi^3} \sum_{\sigma'} f_1^{\sigma\sigma'} \vec{q}_{\sigma'} \cdot \hat{k} k_{F\sigma'}^2. \tag{36}$$

With these results, Eq. (33) becomes

$$\vec{g}_\sigma = m_\sigma \sum_{\vec{k}\sigma'} \frac{\vec{k}}{m_\sigma^*} \vec{q}_{\sigma'} \cdot \hat{k} \delta(k - k_{F\sigma}) \left[ \delta_{\sigma\sigma'} + \frac{m_\sigma^* k_{F\sigma'}^2 f_1^{\sigma\sigma'}}{3\pi^2 k_{F\sigma}} \right]. \tag{37}$$

Evaluation of the sums leads to

$$\vec{g}_\sigma = \frac{m_\sigma^2}{m_\sigma^*} n_\sigma \vec{v}_\sigma + \frac{m_\sigma^2 k_{F\sigma}}{3\pi^2} f_1^{\sigma\sigma} n_\sigma \vec{v}_\sigma + \frac{1}{9\pi^4} m_\sigma m_{-\sigma} k_{F\sigma}^2 k_{F-\sigma}^2 f_1^{np} \vec{v}_{-\sigma}. \tag{38}$$

By comparison with Eqs. (1) and (2), we now identify the superfluid densities as

$$\rho_{pp} = \frac{m_p^2}{m_p^*} n_p (1 + F_1^{pp}/3), \tag{39}$$

$$\rho_{nn} = \frac{m_n^2}{m_n^*} n_n (1 + F_1^{nn}/3), \tag{40}$$

$$\rho_{np} = \rho_{pn} = \frac{1}{9\pi^4} m_n m_p k_{Fn}^2 k_{Fp}^2 f_1^{np}. \tag{41}$$

These are the superfluid densities expressed entirely in terms of calculable quantities, the Landau parameters of the neutron-proton mixture. They manifestly satisfy the symmetry rule, Eq. (6). Earlier results [12,5] for the superfluid densities omitted the factor of $1 + F_1^{pp}/3$ and $1 + F_1^{nn}/3$ in $\rho_{pp}$ and $\rho_{nn}$, respectively.

To show that the densities satisfy the sum rule, Eq. (4), we form the sum

$$\rho_{pp} + \rho_{pn} = \frac{m_p^2}{m_p^*} n_p (1 + F_1^{pp}/3) + \frac{1}{9\pi^4} m_n m_p k_{Fn}^2 k_{Fp}^2 f_1^{np}. \tag{42}$$

However, referring back to the two Fermi-liquid identities, Eqs. (22) and (26), and eliminating $n_n$ from them leads to an expression for the neutron-proton interaction term:

$$k_{Fp}^2 k_{Fn}^2 f_1^{np} m_p m_n / 9\pi^4 = n_p m_p \left[ 1 - \frac{m_p}{m_p^*}(1 + F_1^{pp}/3) \right]. \tag{43}$$

Substitution of Eq. (43) into Eq. (42) then leads to

$$\rho_{pp} + \rho_{pn} = m_p n_p, \tag{44}$$

and similar reasoning leads to

$$\rho_{nn} + \rho_{np} = m_n n_n, \tag{45}$$

so that the expression for the superfluid densities is consistent with Galilean relativity.

## V. ESTIMATION OF THE DENSITY COEFFICIENTS

Since the density coefficients satisfy the constraints (44) and (45), the expressions (39) and (40) can be simplified:

$$\rho_{pp} = m_p n_p - \rho_{pn}, \tag{46}$$

$$\rho_{nn} = m_n n_n - \rho_{pn}, \tag{47}$$

with $\rho_{pn}$ given by Eq. (41). It is then clear that for a given choice of the Fermi momenta $k_{Fp}$ and $k_{Fn}$, the diagonal densities $\rho_{pp}$ and $\rho_{nn}$, just like the off-diagonal density $\rho_{pn}$, can be expressed in terms of just one Landau parameter $f_1^{pn}$. In this section we estimate the value of $\rho_{pn}$ in the neutron-star core.

The Landau parameters are functions of the proton and neutron densities or, equivalently, of the total nucleon density $n = n_p + n_n$ and of the asymmetry parameter $\alpha = (m_n n_n - m_p n_p)/(m_n n_n + m_p n_p)$. For nuclear matter $\alpha = 0$, while for pure neutron matter $\alpha = 1$. The nucleon density is conveniently expressed in terms of an ''average'' Fermi momentum $k_F = (1.5\pi^2 n)^{1/3}$. A fair approximation of the dependence on $\alpha$ is given by [16]

$$f_1^{np}(k_F, \alpha) = c(k_F), \tag{48}$$

$$f_1^{pp}(k_F, \alpha) = a(k_F) - b(k_F)\alpha, \tag{49}$$

$$f_1^{nn}(k_F, \alpha) = a(k_F) + b(k_F)\alpha. \tag{50}$$

Thus we will take the value of $f_1^{np}$ to be independent of the asymmetry parameter while the other two parameters depend on the asymmetry in a linear fashion.

The theoretical value of $f_1^{pn}$, like that of other Landau parameters, is model dependent. To get an idea of the uncertainty involved we look at the expressions for the effective mass. From Eq. (41), we have

$$\frac{m_p^*}{m_p}=1+\frac{1}{3}D_p\left[f_1^{pp}+\left(\frac{k_{Fn}}{k_{Fp}}\right)^2 f_1^{pn}\frac{m_n}{m_p}\right] \tag{51}$$

and

$$\frac{m_n^*}{m_n}=1+\frac{1}{3}D_n\left[f_1^{nn}+\left(\frac{k_{Fp}}{k_{Fn}}\right)^2 f_1^{pn}\frac{m_p}{m_n}\right]. \tag{52}$$

Our expressions differ from those of Ref. [10] by the insignificant factor of $m_n/m_p$ in the last term of each equation. Henceforth we take $m_n=m_p\equiv m$. Reference [13] tabulates the values of $m^*/m$ for nuclear matter and for neutron matter. In the range of densities of interest to us (2/3 nuclear to $\sim 3\times$ nuclear) the values for the effective mass vary by a factor of 2 between models. For any given model, $m^*$ varies by no more than a factor of 2 as the density varies from 0.15 fm$^{-3}$ to 0.60 fm$^{-3}$. Thus the theoretical uncertainty in the values of Landau parameters seems to be larger than the range of variability of the same parameters in the superfluid core of the neutron star.

On the empirical side, values of the Landau parameters have been extracted from the properties of excited states of $^{208}$Pb and of neighboring nuclei [14,15], with the result that $F_1^{np}=-0.5\pm 0.25$. Theoretical calculations for the same parameter have been reported [17,18] for nuclear matter, giving

$$\left(\frac{2k_{F0}m^*}{\pi^2\hbar^2}\right)f_1^{np}(k_{F0},0)=-1.1. \tag{53}$$

We shall adopt the latter value in our estimates below. Here, $n_0=k_{F0}^3/(1.5\pi^2)$ is the density of nuclear matter. Note that the value of $f_1^{np}$ from Eq. (53), although derived for nuclear matter, when applied to pure neutron matter (for which we also take $F_1^{nn}-F_1^{pp}=-0.2$, following Ref. [16]), gives the result $m_n^*/m_n-m_p^*/m_p\approx 0.4$. This is in agreement with values calculated over a wide range of densities for neutron-star matter [10].

Equations (41) and (53) can be combined to give

$$\frac{\rho_{np}}{\rho_n}=\frac{\rho_{np}}{m_n\rho_n}=-1.1\times\frac{1}{3}\left(\frac{k_{Fp}}{k_{Fn}}\right)^2\left[\frac{m_p}{2m^*}\left(\frac{k_{Fn}}{k_{F0}}\right)\times\frac{f_1^{np}(k_{Fn},\alpha)}{f_1^{np}(k_{F0},0)}\right]. \tag{54}$$

Taking into account that for neutron matter of the same density as nuclear matter $k_{Fn}=2^{1/3}k_{F0}$ and taking $m^*/m=0.6$ for nuclear matter, the expression in the square brackets is found to be approximately $(n/n_0)^{1/3}$. Here we assumed that the value of $f_1^{np}$ in Eq. (53) is the correct value for neutron-star matter over the density range of interest, so that $f_1^{np}(k_F,\alpha)\approx f_1^{np}(k_{F0},0)$, as suggested by Eq. (48) and by the discussion following Eq. (53).

Thus, we estimate that, to within a factor of 2, $\rho_{pn}\approx -0.04m_n n_n$. This implies that $\rho_{pp}\approx 2m_p n_p$ in neutron-star matter.

## VI. DISCUSSION

As we have seen, expressions for the diagonal and off-diagonal densities derived in the context of Fermi-liquid theory explicitly satisfy the Galilean invariance constraints. This allows the diagonal densities $\rho^{pp}$ and $\rho^{nn}$ to be expressed in terms of the off-diagonal density $\rho^{np}$. All the densities can then be expressed in terms of just one Landau parameter $f_1^{np}$, and we estimate $\rho_{pn}\approx -0.04m_n n_n$, and hence $\rho_{pp}\approx 2m_p n_p$, in neutron-star matter.

Our results lead to an estimate for the dimensionless density ''determinant'' $(\rho^{pp}\rho^{nn}-\rho_{np}\rho_{pn})/(\rho_p\rho_n)\approx 2$. This value is about the same as one used in a recent calculation of the oscillations of superfluid neutron stars [7].

Knowledge of the value of the off-diagonal density is crucial in estimating the coupling of the superfluid interior to the normal components of the neutron star. As shown in Ref. [5], the Fermi-liquid interaction (expressed by the off-diagonal density) of the neutron and proton superfluids causes the neutron vortex lines (carrying angular momentum) to be magnetized. Of all the interaction processes between the electrons and the neutron superfluid, the one with the shortest time scale yet identified is the scattering of electrons from the induced magnetic fields of the vortex lines. The electrons are not superfluid and their viscous coupling to the crust then provides a mechanism for transfer of angular momentum from the neutron superfluid to the crust. The coupling time $\tau_d$ between the crust and the superfluid from this compound mechanism is therefore directly affected by the value of the off-diagonal densities.

In order to estimate $\tau_d$, we use the formalism developed in Ref. [19], which leads to

$$\tau_d\sim 100\left(\frac{\rho_{pp}}{\rho_{np}}\right)^2 P, \tag{55}$$

where $P$ is the period of the pulsar. Substituting the values given above for $\rho_{pp}$ and $\rho_{np}$, we arrive at $\tau_d\sim 625P$. For the Vela pulsar ($P=0.89$ s), this leads to $\tau_d\sim 56s$. These estimates are similar to those of Refs. [5] and [19]. Since the post-glitch relaxation time of pulsars is generally of order weeks or longer, we may conclude, in agreement with earlier authors, that the coupling between crust and core superfluid is not responsible for the observed relaxation time.

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