

Approximate treatment of electron Coulomb distortion in quasielastic (e, e') reactions

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In this paper we address the adequacy of various approximate methods of including Coulomb distortion effects in (e, e') reactions by comparing to an exact treatment using Dirac-Coulomb distorted waves. In particular, we examine approximate methods and analyses of (e, e') reactions developed by Traini *et al.* using a high energy approximation of the distorted waves and phase shifts due to Lenz and Rosenfelder. This approximation has been used in the separation of longitudinal and transverse structure functions in a number of (e, e') experiments including the newly published $^{208}\text{Pb}(e, e')$ data from Saclay. We find that the assumptions used by Traini and others are not valid for typical (e, e') experiments on medium and heavy nuclei, and hence the extracted structure functions based on this formalism are not reliable. We describe an improved approximation which is also based on the high energy approximation of Lenz and Rosenfelder and the analyses of Knoll and compare our results to the Saclay data. At each step of our analyses we compare our approximate results to the exact distorted wave results and can therefore quantify the errors made by our approximations. We find that for light nuclei, we can get an excellent treatment of Coulomb distortion effects on (e, e') reactions just by using a good approximation to the distorted waves, but for medium and heavy nuclei simple additional *ad hoc* factors need to be included. We describe an explicit procedure for using our approximate analyses to extract so-called longitudinal and transverse structure functions from (e, e') reactions in the quasielastic region. [S0556-2813(96)01711-6]

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I. INTRODUCTION

Medium and high energy electron scattering has long been acknowledged as a useful tool in the investigation of nuclear structure and nuclear properties, especially in the quasielastic region. In the plane-wave Born approximation (PWBA), where electrons are described as Dirac plane waves, the cross section for inclusive quasielastic (e, e') processes can be written simply as

$$\frac{d^2\sigma}{d\Omega_e d\omega} = \sigma_M \left[\frac{q_\mu^4}{q^4} S_L(q, w) + \left(\tan^2 \frac{\theta_e}{2} - \frac{q_\mu^2}{2q^2} \right) S_T(q, w) \right], \quad (1)$$

where $q_\mu^2 = \omega^2 - \mathbf{q}^2$ is the four-momentum transfer, σ_M is the Mott cross section given by $\sigma_M = (\alpha/2E)^2 [\cos^2(\theta/2)/\sin^4(\theta/2)]$, and S_L and S_T are the longitudinal and transverse structure functions which depend only on the momentum transfer q and the energy transfer ω . By keeping the momentum and energy transfers fixed while varying the electron energy E and scattering angle θ_e , it is possible to extract the two structure functions with two measurements. However, when the electron wave functions are not Dirac plane waves, but rather are distorted by the static Coulomb field of the target nucleus, such a simple formulation as given in Eq. (1) is no longer possible and, in general, the cross section does not separate into the sum of longitudinal and transverse structure functions with coefficients which only depend on the electron kinematics. Clearly the size of the Coulomb distortion effects depends on the charge of the target nucleus and on the energy of the electrons. In general, Coulomb effects are not too large near the peaks of cross sections, but can have greatly magnified effects when one

extracts ‘‘structure functions’’ by subtracting one cross section from another. For example, in a recent distorted wave calculation [1] of $^{16}\text{O}(e, e'p)$ in the quasielastic region, we found Coulomb effects on the extracted spectroscopic factors to be approximately 3%, while the effects on the extracted fourth structure function was approximately 15%.

However, some sort of extraction of structure functions, albeit approximate, is still very appealing since structure functions are sensitive to different aspects of the underlying knockout process or the final state interaction. Furthermore, (e, e') reactions in the quasielastic region are particularly appealing in that the cross section is the sum of many knockout processes and some of the various Coulomb effects may be partially averaged out. In fact, this is the case. A rather simple approximation known as the effective momentum approximation (EMA) where the electron momenta $p_{i,f}$ are modified by the value of the Coulomb potential at the center of the nucleus goes quite far in reproducing the Coulomb distortion effects for light nuclei. However, the EMA for heavy nuclei does not adequately reproduce the DWBA cross section in the quasielastic region.

During the past decade, (e, e') cross sections have been measured [2–4] for a number of nuclei in the quasielastic region and either plane wave or EMA was used to extract longitudinal and transverse structure functions. These extracted structure functions were compared to the predictions of a simple Fermi-gas model, and in some cases, there appeared to be large suppression (up to about 40%) of the longitudinal structure functions. There were also disagreements with the extracted transverse structure functions and the predictions of the Fermi-gas model, but these were expected since exchange currents, pion production, and other

processes that are primarily induced by transverse photons were not included in the Fermi-gas model. It should be noted that the Fermi-gas model is a rather crude description of a nucleus. In particular, the shape of the structure functions at fixed momentum transfer plotted as a function of energy transfer is not well described. On the other hand, we found that a ‘‘single-particle’’ relativistic model using relativistic Hartree wave functions coupled with the full DWBA treatment of Coulomb distortion for the electrons was in good agreement with the measured cross sections in the quasielastic region for ^{40}Ca [5] and in reasonably good agreement for ^{238}U . Furthermore, the longitudinal structure function extracted for ^{40}Ca was well produced in magnitude and shape by this model [6].

More recently, Jourdan [7] has examined the world data set for inclusive quasielastic scattering on ^{12}C , ^{40}Ca , and ^{56}Fe at momentum transfers of 300, 380, and 570 MeV/c, and has evaluated the Coulomb sum rule. No evidence of suppression is found for the highest q value (570 MeV/c) where the sum rule is most model independent. Jourdan used our Coulomb corrections in arriving at this conclusion. Thus, on the basis of our direct comparison with the measured cross sections of ^{40}Ca , and of Jourdan’s results, the published analyses [8] for $^{208}\text{Pb}(e, e')$ which claim up to 50% suppression of the longitudinal structure function is surprising. However, the approximate treatment of Coulomb distortion used in the analysis is not accurate, and leads to doubts about the extracted structure functions. As a separate matter, we question the nuclear model used in making the claim of suppression.

In this paper we investigate the possibility of including Coulomb distortion effects in (e, e') reactions in the quasielastic region for medium and heavy nuclei in an approximate way. We have an advantage as compared to previous workers in that we have an exact treatment of the static Coulomb distortion of the target nucleus via a distorted wave Born approximation (DWBA) calculation to which we can compare [9]. In Sec. II we will discuss various approximations that permit a ‘‘plane-wave-like’’ approach to the treatment of Coulomb distortion and compare the approximate results to the exact DWBA results in a step by step way. We obtain an approximate potential due to the electron current which describes Coulomb distortion quite well. In Sec. III, we apply this potential with further approximations to the particular case of inclusive quasielastic processes. Finally we compare our calculations to the Saclay data [8] for the inclusive reaction $^{208}\text{Pb}(e, e')$.

In addition, we give an explicit procedure for extracting longitudinal and transverse structure functions from inclusive cross section data in the quasielastic region from medium and heavy nuclei, and make some general conclusions.

II. APPROXIMATIONS

A. Approximation of the electron potential

The four-potential arising from the electron current j_μ is simply given in terms of the retarded Green’s function by

$$A_\mu(\mathbf{r}) = \int G(\mathbf{r}_e, \mathbf{r}) j_\mu(\mathbf{r}_e) d\mathbf{r}_e, \quad (2)$$

where

$$G(\mathbf{r}_e, \mathbf{r}) = \frac{e^{i\omega|\mathbf{r}_e - \mathbf{r}|}}{|\mathbf{r}_e - \mathbf{r}|}$$

and the electron current is given in terms of the initial and final electron wave functions by

$$j_\mu(\mathbf{r}_e) = \bar{\psi}_f(\mathbf{r}_e) \gamma_\mu \psi_i(\mathbf{r}_e). \quad (3)$$

For scattering processes this current extends over all space and thus the integral in Eq. (2) is not straightforward.

Knoll proposed [10] a way of replacing the integral in Eq. (2) by a series of differential operators by using the transformation

$$\int V(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') d\mathbf{r}' = e^{i\mathbf{q}' \cdot \mathbf{r}} \tilde{V}(-\mathbf{q}' + i\nabla) e^{-i\mathbf{q}' \cdot \mathbf{r}} f(\mathbf{r}) \quad (4)$$

between the function $V(\mathbf{r} - \mathbf{r}')$ and its Fourier transform $\tilde{V}(\mathbf{q}')$. Applying this result to Eq. (2) and making a Taylor series expansion, we obtain the following series expansion for the potential:

$$A_\mu(\mathbf{r}) = \frac{4\pi}{q'^2 - \omega^2} \left[1 + \left(\frac{q'^2 + \nabla^2}{q'^2 - \omega^2} \right) + \left(\frac{q'^2 + \nabla^2}{q'^2 - \omega^2} \right)^2 + \dots \right] j_\mu(\mathbf{r}). \quad (5)$$

Note that while the momentum variable q' is arbitrary, the choice affects the convergence of the series. In particular, for the case of Dirac plane waves for the electron the only dependence of the electron current on \mathbf{r} is simply $e^{i\mathbf{q}' \cdot \mathbf{r}}$ and choosing $\mathbf{q}' = \mathbf{q}$ results in the vanishing of all the terms except the first, and Eq. (5) reduces to the well-known Møller potential,

$$A_\mu(\mathbf{r}) = \frac{4\pi}{q^2 - \omega^2} \bar{u}(\mathbf{p}_f) \gamma_\mu u(\mathbf{p}_i) e^{i\mathbf{q} \cdot \mathbf{r}} = a_\mu e^{i\mathbf{q} \cdot \mathbf{r}},$$

where u is the familiar Dirac plane-wave spinor.

As a test of this approximate procedure for calculating the potential, we calculated the electron charge distribution in the presence of the static Coulomb potential arising from the ground-state charge distribution of ^{208}Pb using the partial wave solutions of the Dirac equation and evaluated the zeroth, zeroth plus first, and zeroth plus first and second terms in Eq. (5). Using these potentials we calculated the inelastic scattering cross section induced by a surface nuclear charge transition density

$$\rho_n^{if}(\mathbf{r}) = \frac{1}{R_n^2} \delta(r - R_n) Y_L^M(\hat{r}), \quad (6)$$

where R_n is the nuclear radius. The ground-state density of the nucleus was described by a Fermi distribution of radius $R = 6.65$ fm and total charge $Z = 82$. The cross sections were calculated at initial energy $E_i = 400$ MeV and final energy $E_f = 300$ MeV with energy transfer $\omega = 100$ MeV. In agreement with Knoll [10], we found that the first and second correction terms are sufficient to fill up the minima and the

contribution of the second correction term was less than 2% for momentum transfer $q \geq 350$ MeV/c. We conclude that this high momentum approximation provides an alternative procedure for calculating the potential arising from inelastic electron scattering processes. However, this procedure does require the numerical solution of the Dirac equation using partial waves which does take some computational time. One advantage, however, is that the radial functions only have to be calculated out to about three times the nuclear radius.

We also applied this procedure to various approximate solutions of the Dirac equation and noted that the first- and second-order terms were not well controlled if one approximates the electron current. Clearly one should not approximate a function and then differentiate it. In the following when we examine approximate electron wave functions, and hence approximate electron currents, we will only use the zeroth term in Eq. (5), but will be sensitive to the choice of q' .

B. High energy wave function approximation

The incoming Coulomb distorted electron scattering wave function that satisfies appropriate boundary conditions for an electron with spin s_i can be expressed in the form of a partial wave sum by [11]

$$\Psi_i^{s_i}(\mathbf{r}) = 4\pi \sqrt{\frac{E_i + m}{2E_i}} \sum_{\kappa, \mu} e^{i\delta_{\kappa}} C_{\mu-s_i}^l \frac{1}{2} \frac{j}{s_i} Y_l^{\mu-s_i}(\hat{\mathbf{p}}_i) \psi_{\kappa}^{\mu}(\mathbf{r}), \quad (7)$$

where the spinor ψ_{κ}^{μ} is an eigenstate with angular momentum quantum numbers κ and μ given explicitly by

$$\psi_{\kappa}^{\mu}(\mathbf{r}) = \begin{bmatrix} f_{\kappa}(r) \chi_{\kappa}^{\mu}(\hat{\mathbf{r}}) \\ i g_{\kappa}(r) \chi_{-\kappa}^{\mu}(\hat{\mathbf{r}}) \end{bmatrix}, \quad (8)$$

where the spin-angle functions are

$$\chi_{\kappa}^{\mu}(\hat{\mathbf{r}}) = \sum_s C_{\mu-s}^l \frac{1}{2} \frac{j}{s} Y_l^{\mu-s}(\hat{\mathbf{r}}) \chi_s. \quad (9)$$

The Dirac quantum number κ determines the angular momentum labels for both l and j . Note that if we ignore the mass of the electron (valid for scattering angles away from extreme forward and backward angles), the electron helicity is a good quantum number, and we only require positive κ solutions. The radial functions f_{κ} and g_{κ} and their corresponding phase factors δ_{κ} are obtained by numerically solving the Dirac radial equation for a finite spherically symmetric nuclear charge distribution. The outgoing distorted wave function ψ_f^s is found from Eq. (7) by making the replacements $(i\delta_{\kappa} \rightarrow -i\delta_{\kappa})$, $(s_i \rightarrow s_f)$, $(E_i \rightarrow E_f)$, and $(\mathbf{p}_i \rightarrow \mathbf{p}_f)$. Note that by setting the phases δ_{κ} to zero and replacing f_{κ} and g_{κ} by the spherical Bessel functions $j_l(pr)$ and $\text{sgn}(\kappa)j_{\bar{l}}(pr)$, where $\bar{l} = l(-\kappa)$, the partial wave sum in Eq. (8) can be summed to give the Dirac plane-wave solution,

$$\Psi_i(\mathbf{r}) = \sqrt{\frac{E_i + m}{2E_i}} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \chi_s \end{pmatrix} e^{i\mathbf{p} \cdot \mathbf{r}}, \quad (10)$$

where the electron spin label has been suppressed. Thus, one way of obtaining a plane-wave-like Coulomb distorted wave function is to approximate the radial functions f_{κ} and g_{κ} by spherical Bessel functions, and to approximate the scattering phase shifts by an operator that can be pulled out of the partial wave sum. Using these two ideas, Lenz and Rosenfelder [12] obtained an approximate scattering wave function for high energy electrons which includes Coulomb distortion for a finite nucleus in an approximate way more than 20 years ago.

In the high energy limit with good helicity ($E \gg m_e$), the approximate wave function of Lenz and Rosenfelder [12], and Knoll [10] can be written as

$$\Psi^{(\pm)}(\mathbf{r}) = \eta(r) e^{\pm i\delta(\mathbf{J}^2)} e^{i\mathbf{p} \cdot \mathbf{r}} \eta(r) u_p \quad (11)$$

where u_p is the Dirac plane-wave spinor and

$$\eta(r) = \frac{1}{pr} \int_0^r [p - V(r')] dr' + \text{correction term}. \quad (12)$$

Lenz and Rosenfelder approximated the dependence of the phases on the operator \mathbf{J}^2 by $\delta(\mathbf{J}^2) = \delta_{1/2} + b(\mathbf{J}^2 - 3/4)$ where b depends on the Coulomb potential. This particular approximation for the phase shifts is not necessary since any function of the operator \mathbf{J}^2 still allows the partial wave sum to be carried out. Lenz and Rosenfelder claimed that this equation is valid for $(|V(r)|/p) \ll 1$ and $j + 1/2 \ll pR$. The second condition being primarily determined by the expression used for the phase shifts.

The approximate radial functions are given by spherical Bessel functions with modified argument,

$$f_{\kappa}(r) = \frac{x}{pr} j_l(x), \quad (13)$$

$$g_{\kappa}(r) = \frac{x}{pr} \text{sgn}(\kappa) j_{\bar{l}}(x),$$

where $x = p'(r)r$ and

$$p'(r) = p - \frac{1}{r} \int_0^r V(r) dr + \frac{\Delta}{r}, \quad (14)$$

where $V(r)$ is the spherically symmetric Coulomb potential of the target nucleus and Δ is a correction term in the argument of order $\kappa^2/(pr)^2$ and is normally neglected.

To avoid having an r -dependent momentum, Traini *et al.* [14] made the further approximation that $p'(r) \cong p'(0) = p - V(0)$ where $V(0)$ is the static Coulomb potential evaluated at the origin. This approximation of the radial wave function leads to what is known as the effective momentum approximation (EMA). An approximation to the radial wave function where we keep the r dependence in the momentum but still neglect Δ will be referred to as the local effective momentum approximation (LEMA). In looking at

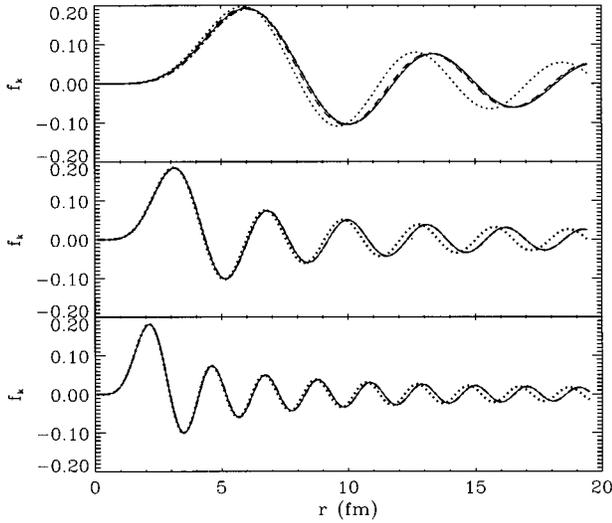


FIG. 1. Radial wave functions in ^{208}Pb for $\kappa=5$. The top is for energy $E=200$ MeV, the middle for energy $E=400$ MeV and the bottom for energy $E=600$ MeV. The solid line is the exact DW wave function while the dash-dotted line is LEMA+ Δ , the dashed line is LEMA and the dotted line is EMA.

these approximate wave functions, one should keep in mind that it is the spatial region around the nuclear surface that contributes most significantly to electron-induced transitions. That is, partial waves with angular momenta of order pR where R is the nuclear radius play a large role in the transition amplitude. Thus, the approximation that the angular momentum is significantly less than pr is not valid. With this point in mind, we sought an *ad hoc* approximation for the correction Δ . We found that the following expression describes the radial wave function at larger radial distances quite well:

$$\Delta(\alpha Z, E, \kappa^2) = -\alpha Z \left(\frac{16\kappa}{E} \right)^2 = a\kappa^2.$$

The parameter $a = -\alpha Z (16/E)^2$ where the number 16 is given in MeV and was determined by comparing to the exact distorted radial wave function. We will label this case LEMA + Δ . Note that we have chosen the parametrization of Δ so that it contains κ^2 which can be expressed in terms of the operator J^2 so that we will still be able to sum the partial wave series.

To investigate these approximations we compare the radial wave functions calculated using EMA, LEMA, and LEMA + Δ to the exact Coulomb distorted waves (DW) for various angular momentum states for electron scattering from ^{208}Pb . The radius of ^{208}Pb is approximately 6.5 fm. Figures 1 and 2 show the comparison between three approximate radial wave functions and the exact distorted radial wave function for different energies ($E=200$ MeV, 400 MeV, and 600 MeV) and different κ values ($\kappa=5$ and $\kappa=15$). The EMA wave function is acceptable at small radial distances, particularly much less than the nuclear radius, but at larger radii the approximate radial wave function is shifted too much to the left indicating the potential $V(0)$ at the origin is too large. The LEMA wave function is a much better approximation to the exact wave function than EMA

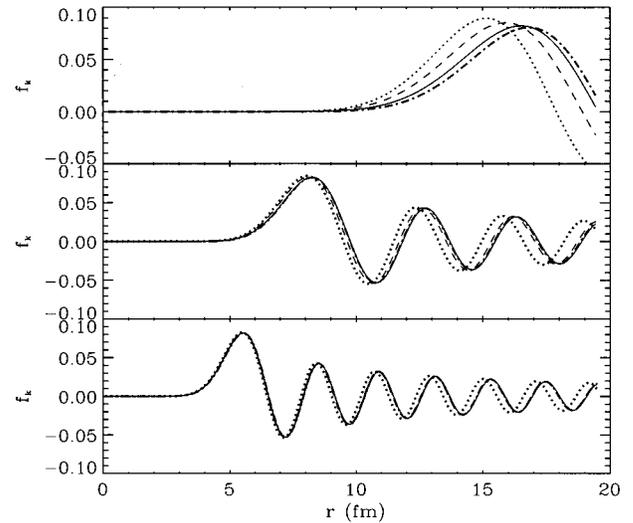


FIG. 2. Radial wave functions in ^{208}Pb for $\kappa=15$. The same as Fig. 1.

for medium and high energies out to radii beyond the nuclear radius, but at lower energies (less than 250 MeV) it also deviates significantly from the DW result. The wave function with the LEMA + Δ correction agrees with the distorted one almost perfectly above 400 MeV.

At the lower energy this wave function is a bit to the right of the distorted wave function, but the discrepancy is acceptably small. Our conclusion is that LEMA is much better than EMA and may be acceptable for some reactions. The approximation LEMA + Δ furnishes quite a good description of the distorted wave radial functions out to several nuclear radii.

The other important ingredient in Coulomb distortion considerations are the phase shifts δ_κ . In order to sum the partial wave series in Eq. (7), the phase shifts in Eq. (11) were expressed as a function of the total angular momentum operator J^2 . The eigenvalues for this operator are $j(j+1)$ when operating on the partial waves, so the question is how well are the exact phase shifts reproduced by the expression [10]

$$\delta_j = \delta_{1/2} + b \left(j(j+1) - \frac{3}{4} \right) = \delta_{1/2} + b[\kappa^2 - 1], \quad (15)$$

where we used the relation $j = |\kappa| - 1/2$. For a uniform charge distribution of radius R ,

$$\delta_{1/2} = Z\alpha \left(\frac{4}{3} - \ln 2pR \right) + b \quad (16)$$

and

$$b = -\frac{3Z\alpha}{4p(0)^2 R^2}. \quad (17)$$

We investigated this κ^2 approximation for the phase shifts for the Coulomb potential with ^{208}Pb and 400 MeV electrons. The phase shifts are in good agreement with the exact phase shifts for small κ values, but for large κ values, the approximate phase shifts are much too large in magnitude.

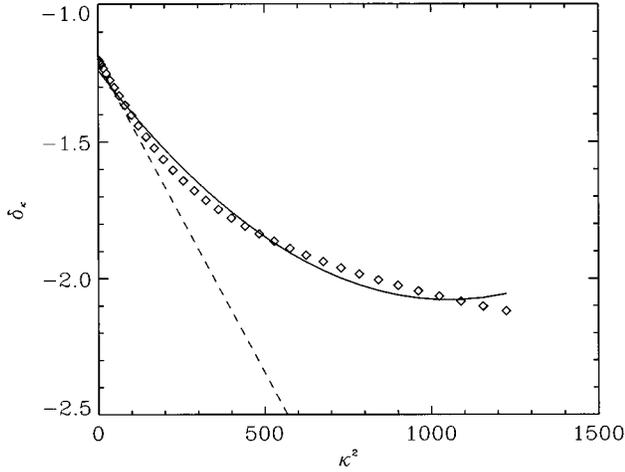


FIG. 3. Comparison of the exact and $\delta(\kappa^2)$ fits to the phases in ^{208}Pb for the energy $E=400$ MeV and $\kappa_{\text{max}}=35$. The diamonds are the exact phases and the dashed line is the κ^2 fit to the phases and the solid line includes the κ^4 term in the fit.

The breakdown of the approximation occurs for $\kappa \approx pR$ as expected. However, as noted earlier, it is these orbitals that play a dominant role in electron induced reactions from the nucleus. In order to avoid this violation, we assume that the approximate phase shifts contain a κ^4 term expressed as

$$\begin{aligned} \delta_\kappa &= a_0 + a_2 \kappa^2 + a_4 \kappa^4 \\ &= b_0 + b_2 \left(j(j+1) - \frac{3}{4} \right) + b_4 \left(j(j+1) - \frac{3}{4} \right)^2, \end{aligned} \quad (18)$$

where $b_0 = a_0 + a_2 + a_4$, $b_2 = a_2 + 2a_4$, and $b_4 = a_4$. Unfortunately, we do not have a simple analytical expression for the coefficients, so a_0 , a_2 , and a_4 are fitted to the exact phase shifts calculated with a distorted wave code. As is evident in Fig. 3, including the κ^4 terms leads to a much better description of the exact phases.

Clearly higher powers of κ^2 could also be included if needed. Of course, using the description of the phases given in Eq. (18) rather than the one in Eq. (15) requires calculation of the exact Coulomb phases for both the incoming and outgoing electron energy and then determining the coefficients a_n by a fitting procedure. However, the solution for the phases is straightforward and very rapid for modern computers, so this poses no real practical problem.

We have calculated various multipoles of the scalar potential in the partial wave formalism and confirmed that the approximate radial function and the approximate phases are in close agreement with the exact potential. These two approximations permit the summation of the partial wave series of Eq. (7) so that the electron wave function for incoming or outgoing waves can be written as

$$\Psi^{(\pm)}(\mathbf{r}) = \frac{p'(r)}{p} e^{\pm i \delta(\mathbf{J}^2)} e^{i[1 + \Delta/(p'(r)r)]} \mathbf{p}'(r) \cdot \mathbf{r} u_p. \quad (19)$$

The operator \mathbf{J}^2 is the square of the total angular momentum operator ($\mathbf{J} = \mathbf{L} + \mathbf{S}$), and previous workers [13,14] have made a power series expansion of the phase term and applied successive terms to the plane-wave-like part of the wave

function. The problem with this procedure is that the phase shifts are greater than one for κ values of importance and this series is very slowly converging. It is straightforward to check that keeping only the first three terms in the exponential expansion leads to significant errors. We choose not to make this expansion, but to approximate J^2 by the orbital angular momentum squared L^2 and further to replace L^2 by its classical value $[\mathbf{r} \times \mathbf{p}(r)]^2$. Clearly we are neglecting the spin dependence of the phase shifts by this classical approximation, but since the processes we are interested in are dominated by angular momentum values around 10 or more, we expect the spin dependence to be negligible. We confirmed this estimate by comparisons of our partial wave calculation of $(e, e'p)$ where the full spin dependence is included to a three-dimensional numerical integration using the classical approximation. In like manner, we also replace the κ^2 dependence in Δ by $[\mathbf{r} \times \mathbf{p}(r)]^2$. Finally, the approximate Coulomb distorted wave function of Eq. (19) with the Δ correction is given explicitly by [15]

$$\begin{aligned} \Psi^{(\pm)}(\mathbf{r}) &= \frac{p'(r)}{p} e^{\pm i \delta\{[\mathbf{r} \times \mathbf{p}'(r)]^2\}} e^{i a [\hat{p}'(r) \cdot \hat{r}] [\mathbf{r} \times \mathbf{p}'(r)]^2} \\ &\quad \times e^{i \mathbf{p}'(r) \cdot \mathbf{r}} u_p. \end{aligned} \quad (20)$$

with $\delta\{[\mathbf{r} \times \mathbf{p}'(r)]^2\} = b_0 + b_2 [\mathbf{r} \times \mathbf{p}'(r)]^2 + b_4 [\mathbf{r} \times \mathbf{p}'(r)]^4$. Using this wave function and the first term of Eq. (5) we obtain the following four-potential, which includes in an approximate way, the Coulomb distortion of the target nucleus

$$\begin{aligned} A^\mu(\mathbf{r}) &= \frac{4\pi}{4p_i p_f \sin^4(\theta_e/2)} e^{i(\delta_i\{[\mathbf{r} \times \mathbf{p}'_i(r)]^2\} + \delta_f\{[\mathbf{r} \times \mathbf{p}'_f(r)]^2\})} \\ &\quad \times e^{i(\Delta_i - \Delta_f)} e^{i \mathbf{q}'(r) \cdot \mathbf{r}} \bar{u}_f \gamma^\mu u_i \end{aligned} \quad (21)$$

where

$$\Delta = a [\mathbf{r} \times \mathbf{p}'(r)]^2 [\hat{r} \cdot \hat{p}'(r)]$$

and

$$\mathbf{q}'(r) = \mathbf{p}'_i(r) - \mathbf{p}'_f(r).$$

The approximate potential of Eq. (21) is similar to the plane-wave result except for the phase factors and the radial dependence in the momentum transfer. Unfortunately, the spatial dependence in the phase factors makes a multipole decomposition of this potential impractical. However, since it is an analytical function it is straightforward to calculate interaction matrix elements by performing the three-dimensional integration over \mathbf{r} numerically.

III. APPLICATION TO THE INCLUSIVE PROCESS

For the inclusive cross section (e, e') , the longitudinal and transverse structure functions in Eq. (1) are bilinear products of the Fourier transform of the components of the nuclear transition current density integrated over outgoing nucleon angles. Furthermore, it is the Dirac structure of the Møller potential which leads to the cross section containing one term with only longitudinal components of the current and a second containing only transverse components. However, the Dirac structure of the approximate potential in Eq.

(21) is the same as the plane-wave result. Therefore, even with Coulomb distribution included, albeit in an approximate way, the separation of the cross section into a longitudinal term and a transverse term persists. Explicitly, the structure functions for knocking out nucleons from a shell with angular momentum j_b are given by

$$S_L(q, \omega) = \sum_{\mu_b s_p} \frac{\rho_P}{2(2j_b + 1)} \int |N_0|^2 d\Omega_P, \quad (22)$$

$$S_T(q, \omega) = \sum_{\mu_b s_p} \frac{\rho_P}{2(2j_b + 1)} \int (|N_x|^2 + |N_y|^2) d\Omega_P, \quad (23)$$

where the nucleon density of states $\rho_P = pE_p / (2\pi)^2$, the z axis is taken to be along \mathbf{q} , and μ_b and s_p are the z components of the angular momentum of the bound and continuum state particles. The Fourier transfer of the nuclear current $J^\mu(\mathbf{r})$ is simply,

$$N^\mu = \int J^\mu(\mathbf{r}) e^{i\mathbf{q} \cdot \mathbf{r}} d^3r. \quad (24)$$

The continuity equation has been used to eliminate the z component (N_z) via the equation $N_z = -(\omega/q)N_0$. Note that when we use the approximate electron four-potential along with current conservation to eliminate the z component of the current we run into a problem since the momentum transfer \mathbf{q}' depends on r both in magnitude and direction. In addition, the phase factors depend on \mathbf{r} . To avoid generating additional terms we assume the direction of $\mathbf{q}'(r)$ is along the asymptotic momentum transfer \mathbf{q} which defines the \hat{z} axis, and neglect the dependence on \mathbf{r} in the phases and in $\mathbf{q}'(r)$, when taking the divergence of \mathbf{N} . With this further approximation, current conservation implies $\omega N_0 + \mathbf{q}'(r) \cdot \mathbf{N} = 0$. Using the approximate potential of Eq. (21), the cross section for the inclusive reaction (e, e') can be written as

$$\frac{d^2\sigma}{d\Omega_e d\omega} = \sigma_M \left[\frac{q_\mu^4}{q^4} S_L(q', \omega) + \left(\tan^2 \frac{\theta_e}{2} - \frac{q_\mu^2}{2q^2} \right) S_T(q', \omega) \right] \quad (25)$$

and the transform of the transition nuclear current elements which appears in S_L and S_T are given by

$$N_0 = \int \left(\frac{q'_\mu(r)}{q_\mu} \right)^2 \left(\frac{q}{q'(r)} \right)^2 e^{i\delta_f \{[\mathbf{r} \times \mathbf{p}'_i(r)]^2\}} e^{i\delta_f \{[\mathbf{r} \times \mathbf{p}'_f(r)]^2\}} \times e^{i(\Delta_i - \Delta_f)} e^{i\mathbf{q}'(r) \cdot \mathbf{r}} J_0(\mathbf{r}) d^3r, \quad (26)$$

$$\mathbf{N}_T = \int e^{i\delta_i \{[\mathbf{r} \times \mathbf{p}'_i(r)]^2\}} e^{i\delta_f \{[\mathbf{r} \times \mathbf{p}'_f(r)]^2\}} \times e^{i(\Delta_i - \Delta_f)} e^{i\mathbf{q}'(r) \cdot \mathbf{r}} J_T(\mathbf{r}) d^3r. \quad (27)$$

As noted earlier, a multipole expansion of the approximate potential is not practical, and since the inclusive reaction (e, e') requires a sum over all occupied neutron and proton shells, numerical integration is very time consuming. In order to have a more practical procedure we choose to make additional approximations. First, we neglect all of the

phase factors (δ and Δ) in Eqs. (26) and (27) but retain the r dependence in \mathbf{q}' . This returns us to the approximation we call LEMA. For light to medium nuclei this approximation is in good agreement with the full DWBA result regarding the shape as a function of energy transfer, but has a small discrepancy in magnitude. The magnitude is corrected with an overall factor of $[p'_i(0)/p_i]^2$ in the cross section. However, for heavier nuclei, we noticed that LEMA with the magnitude factor was a very good approximation to the DWBA results for large electron scattering angle where the transverse term dominates, but deviated significantly for forward electron angles where the longitudinal term has a significant contribution. Thus it appears that the phase factors play a significant role in the longitudinal term for large Coulomb distortion.

We also noticed that for forward electron angles the low energy side of the DWBA quasielastic peak looks similar to the plane-wave result. Thus, it appears that in the longitudinal term the phase factors are partially cancelling the effect of the effective momentum $\mathbf{q}'(r)$ on the low ω side of the quasielastic peak. We examined a number of simple *ad hoc* modifications to the longitudinal term as described by LEMA and the magnitude factor in order to reproduce the DWBA forward angle results for (e, e') reactions on ^{208}Pb in the quasielastic region. Based on a number of trials we propose the following ‘‘Fourier’’ transform for the charge component of the current:

$$N_0^{\text{LEMA}'} = \left(\frac{p'_i(0)}{p_i} \right) \int e^{i\mathbf{q}''(r) \cdot \mathbf{r}} J_0(\mathbf{r}) d^3r, \quad (28)$$

where $[p'_i(0)/p_i]$ is the magnitude enhancement and $\mathbf{q}''(r) = \mathbf{p}'_i(r) - \mathbf{p}'_f(r)$, $p''(r) = p - (\lambda/r) \int_0^r V(r') dr'$ and the factor λ , which depends on the energy transfer ω , is given by $\lambda = (\omega/\omega_0)^2$ with $\omega_0 = q^2/1.4M$. Clearly for small ω , \mathbf{q}'' approaches the asymptotic value, while for ω past the quasielastic peak which is located approximately at ω_0 in the cross section, the effective momentum differs significantly from the asymptotic value. We have tested this *ad hoc* prescription for $0.3 \leq \omega/\omega_0 \leq 2.0$ for a range of energies and nuclei and find excellent agreement with the DWBA result. The transverse ‘‘Fourier’’ transform only contains the normalization factor and is given by

$$\mathbf{N}_T^{\text{LEMA}'} = \left(\frac{p'_i(0)}{p_i} \right) \int e^{i\mathbf{q}'(r) \cdot \mathbf{r}} J_T(\mathbf{r}) d^3r. \quad (29)$$

We will refer to the cross section for the inclusive reaction calculated with these two structure functions as the LEMA' result. Clearly $N_0^{\text{LEMA}'}$ and $\mathbf{N}_T^{\text{LEMA}'}$ represent a modified Fourier transform of the nuclear transition current. The approximation known as the EMA replaces $\mathbf{q}'(r)$ with $\mathbf{q}'(0)$ wherever it appears in Eq. (26) and Eq. (27) for N_0 and \mathbf{N}_T and the phases are neglected as usual. We find that for light nuclei the EMA is adequate, but it leads to large errors for nuclei as heavy as ^{208}Pb .

In our analyses of quasielastic scattering, we use relativistic bound and continuum single-particle wave functions. For the inclusive reaction we use continuum solutions for the outgoing, but unobserved, nucleons which are in the same

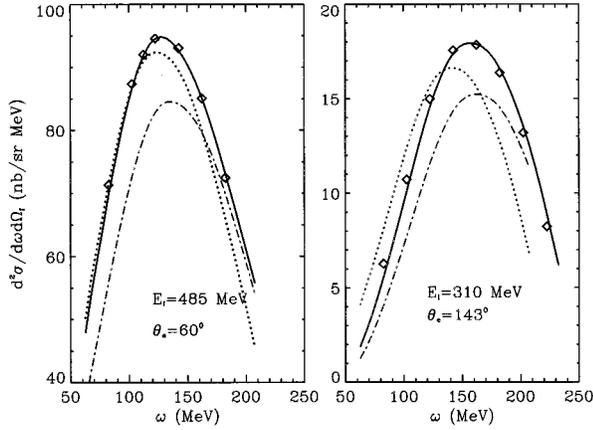


FIG. 4. The differential cross section for $^{208}\text{Pb}(e, e')$ at two different electron energies and scattering angles. The dotted line is the PWBA result, the diamonds are the DWBA result, the dash-dotted line is the EMA result, and the solid line is the LEMA' result.

Hartree potential as the bound state orbitals [5]. This choice ensures charge conservation and gauge invariance. Thus, in this relativistic single-particle model the nuclear current matrix element is

$$J^\mu(\mathbf{r}) = e \bar{\Psi}_p \hat{J}^\mu \Psi_b, \quad (30)$$

where we use the free nucleon current operator

$$\hat{J}^\mu = F_1 \gamma^\mu + F_2 \frac{i \mu_T}{2m_N} \sigma^{\mu\nu} q_\nu, \quad (31)$$

where μ_T is the nucleon anomalous magnetic moment (for proton $\mu_T = 1.793$ and for neutron $\mu_T = -1.91$). The form factor F_1 and F_2 are related to the electric and magnetic form factors G_E and G_M by

$$G_E = F_1 + \frac{\mu_T q_\mu^2}{4M^2} F_2, \quad (32)$$

$$G_M = F_1 + \mu_T F_2. \quad (33)$$

We choose the standard result [16]

$$G_E = G_M / (\mu_T + 1) = (1 - q_\mu^2 / 0.71)^{-2}, \quad (34)$$

where in this formula q_μ^2 is in units of GeV^2 .

In Fig. 4, we compare various approximations to the DWBA result as a function of the energy transfer ω for two cases, incident electron energy $E_i = 310$ MeV, scattering angle $\theta_e = 143^\circ$ and $E_i = 485$ MeV, scattering angle $\theta_e = 60^\circ$ data sets. The dotted line is the PWBA result, the diamonds are the DWBA result, the dash-dotted line is the EMA result, and the solid line is the LEMA' result. We notice that the EMA result is always lower than the DWBA result although the peak position is approximately in the right place. This lack of agreement with the EMA is not too surprising based on our previous examination of the wave function. Replacing the average value of the Coulomb potential between the origin and the position r by the value at the origin is too large an error for r near the nuclear surface

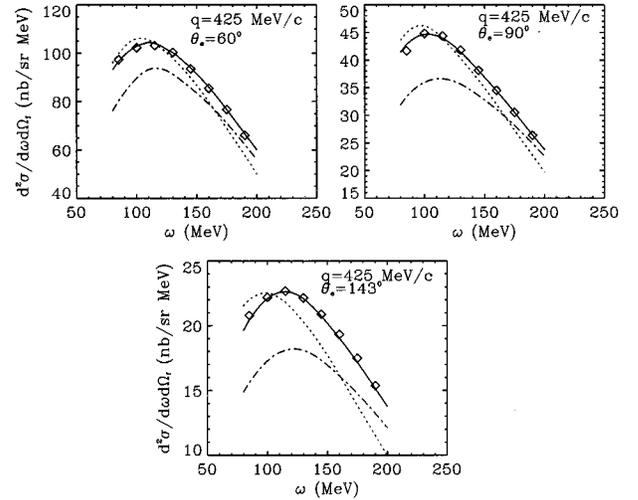


FIG. 5. The differential cross section for ^{208}Pb at constant momentum transfer $q = 425$ MeV/c, but three different electron scattering angles.

where most of the interaction takes place. Since the EMA is such a bad approximation to the full DWBA result, it should not be used as a basis for including the Coulomb phase terms in (e, e') or $(e, e'p)$ reactions. Over the whole region, the LEMA' result is in excellent agreement with the DWBA result apart from the extreme wings of the quasielastic peak. The difference around the peak is less than 2% and side parts are about 5%.

We illustrate our approximations for other kinematics by calculating the cross section at fixed momentum transfer $q = 425$ MeV/c with three different electron scattering angles ($\theta_e = 60^\circ, 90^\circ$, and 143°) as shown Fig. 5. The EMA result is always lower than the DWBA and the LEMA' results and is shifted toward large energy transfer by about 10 MeV. LEMA' again reproduces the DWBA cross sections quite well.

IV. SEPARATION PROCEDURE AND STRUCTURE FUNCTIONS

Since the LEMA' cross section for (e, e') has the same structure as the plane-wave result, a Rosenbluth-type separation can be used to extract a longitudinal and a transverse contribution even in the presence of large Coulomb distortions. In fact, experimental evidence for this was given in one of the early papers on quasielastic scattering [4] where cross section measurements on ^{238}U at three different electron scattering angles, but with the same energy and momentum transfer, fell on a straight line when a Rosenbluth plot was made. Our LEMA' approximation gives a theoretical explanation for this observation. Of course the separated structure functions are no longer bilinear products of the simple Fourier transforms of the current components integrated over outgoing nucleon directions. Inclusion of Coulomb distortion within LEMA' results in an r -dependent Fourier momentum variable which differs for the longitudinal and transverse case. We have only been able to check LEMA' for our particular model of the quasielastic process, so we cannot prove that it applies to other models. However, based on previous work we know our model describes

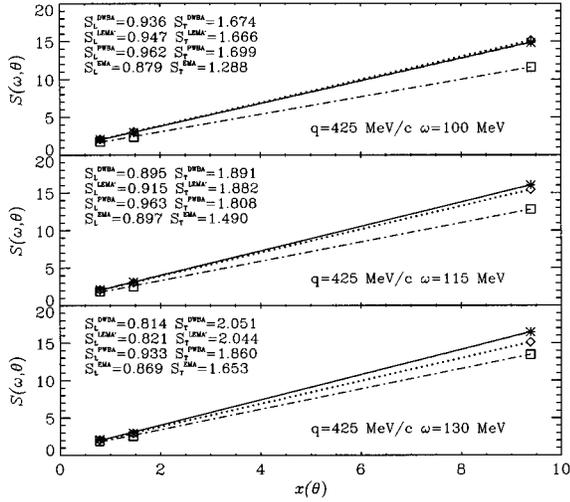


FIG. 6. Rosenbluth separation plot of the cross section for ^{208}Pb at $q=425$ MeV/c. The solid line is the DWBA result, the dashed line is for the LEMA', the dotted line is for the PWBA, and the dash-dotted line is for the EMA. The unit of the total structure function is MeV^{-1} .

$(e, e'p)$ and (e, e') quite well in the quasielastic region and its treatment of the spatial dependence in the nuclear charge and current is very realistic. Thus we believe that the Coulomb corrections that we calculate will be appropriate for any realistic nuclear model. We conclude that LEMA' is a good approximation for the inclusive cross section (e, e') in the quasielastic region.

To illustrate the ‘‘Rosenbluth’’ separation we write $S = S_L + xS_T$ where S is the total structure function given by

$$S = \left(\frac{q}{q_\mu}\right)^4 \frac{1}{\sigma_M} \frac{d^2\sigma}{d\Omega_e d\omega} \quad (35)$$

and $x = [\tan^2(\theta_e/2) - q_\mu^2/2q^2]/(q_\mu^4/q^4)$.

In Fig. 6, we compare the structure functions extracted using our calculations of (e, e') on ^{208}Pb using various treatments of Coulomb distortion for a momentum transfer of $q=425$ MeV/c and three different values of energy transfer around the peak of the cross section. In each case we find a very good fit to a straight line. Furthermore, we see that the intercept and slope extracted using the full DWBA and LEMA' calculations agree within 2% in all cases. The EMA and plane wave (PWBA) results clearly are in disagreement with the DWBA results.

Given this result there are several ways to analyze experimental (e, e') data. If one has a model for the process under investigation, one can calculate the r -dependent Fourier transforms of Eqs. (28) and (29) and compare to the experimentally measured cross section. Or, one could make a Rosenbluth separation to obtain the LEMA' structure functions as a function of energy transfer ω and asymptotic momentum transfer q . Although these structure functions clearly have some dependence on electron energy and scattering angle in addition to their dependence on q and ω , in our model of the quasielastic process this dependence is not very strong and we recommend that in comparing a theoretic

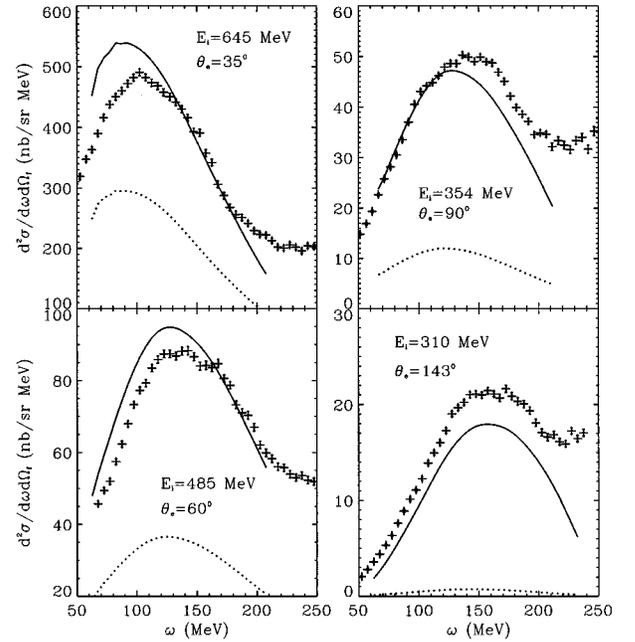


FIG. 7. The differential cross section for $^{208}\text{Pb}(e, e')$. The solid line is the LEMA' result and the dotted line shows the longitudinal contribution. The data are from [8].

cal model with the extracted LEMA' structure functions that electron energies and angles in the same range as used in the experiment be used.

Lacking a model for the process being measured, one could take a simple model with suitable geometry and calculate the ratio of the cross section calculated with PWBA to one calculated with LEMA' and renormalize the measured cross section data by multiplying by this ratio. The resulting pseudo-PWBA ‘‘data’’ could then be separated by a Rosenbluth procedure and would produce structure functions in terms of the Fourier transforms of the current components. Clearly this procedure introduces some error if the model amounts of longitudinal and transverse contributions are not close to the amounts in the process being measured since LEMA' treats them somewhat differently. However, this difference is not too large and seems to be the best one can do.

Finally one could assume that the theoretical model has the correct kinematics and spatial dependence, but that the magnitude of the longitudinal and/or transverse portions of the model is not correct. That is, one could use the model to calculate the longitudinal and transverse contributions to the cross section and then multiply each by a scale factor to be determined by making a least squares fit to the cross section data.

V. COMPARISON WITH EXPERIMENTAL DATA

In Fig. 7, we compare our theoretical results based on the relativistic ‘‘single-particle’’ model [5] calculated with LEMA' to the Saclay data [8] for several electron angles ($\theta_e = 143^\circ, 90^\circ, 60^\circ$, and 35°). The solid line is our model result for the cross section while the dotted line shows the longitudinal contribution to the cross section. Clearly the longitudinal contribution is quite small except for the forward electron scattering angle of 35° where it represents

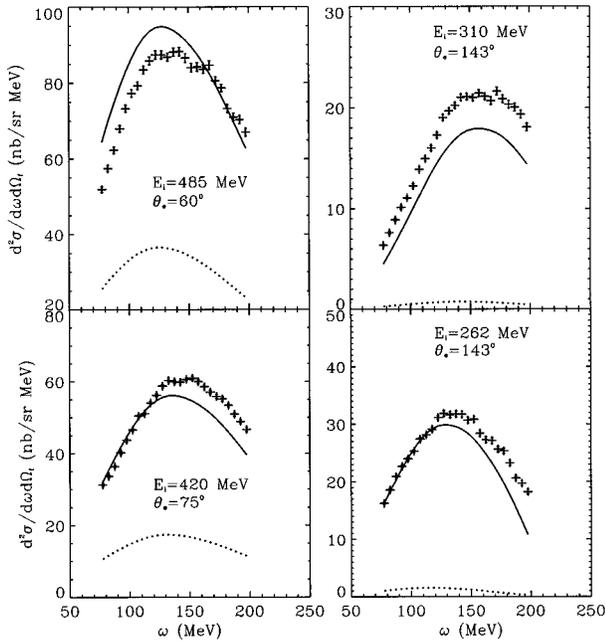


FIG. 8. The comparison of LEMA' and Saclay data.

slightly over 50% of the cross section. Pion production is not included in our model, so the behavior at large ω is not expected to agree with the data. Clearly the agreement between the data and the calculation is not very good, which can be contrasted with quite good agreement between our calculations and the (e, e') data from Bates on ^{40}Ca [6]. Further, these may be a suggestion of longitudinal suppression if coupled with a transverse enhancement.

We selected all of the Saclay data with energy transfer between 100 MeV and 200 MeV which had a clearly defined quasielastic peak in the cross section (218 data points), and used our model with LEMA' to calculate the longitudinal and transverse contributions to the cross section for each kinematical point in the data set. We performed a linear least squares fit by scale factors in front of the longitudinal and transverse contributions. This fit produced a factor in front of the longitudinal term of 0.69 and in front of the transverse term of 1.25. However, the fit is not very good since the χ^2 per data point is 60. The only conclusion we can make is that we do not find a 50% suppression of the longitudinal contribution in ^{208}Pb . Furthermore, it appears to us that the experimental data do not scale quite correctly. In Fig. 8 we show four sets of experimental data along with our model. Two of the sets are at forward angles with quite similar kinematics and two are at backward angles with similar kinematics.

One of the forward angle data sets falls significantly below our model while the other is above. It would be difficult to modify the longitudinal and transverse strength in our model to get such a dramatic shift in behavior. For the larger angle case, the contribution of the longitudinal terms are negligible and again the kinematics do not vary so much between the two cases. The 310 MeV result lies far above our calculation while the 262 MeV result is only slightly above our calculation.

VI. CONCLUSION

We have developed a simple way of including Coulomb distortion in the Møller potential for inelastic electron scattering from medium and heavy nuclei. In this paper we have applied a simplified version of this approximation to (e, e') reactions from medium and heavy nuclei in the quasielastic region. The previously used effective momentum approximation (EMA) disagrees with the distorted wave analysis for nuclei as heavy as ^{208}Pb , while the local effective momentum approximation with an enhancement factor and an *ad hoc* correction to the longitudinal term (LEMA') reproduces the full DWBA calculation very well. Our most important finding is that LEMA' allows the cross section to be separated into a longitudinal and a transverse contribution. However, the resulting structure functions depend on r -dependent Fourier transforms of the transition current components. We recommend several different procedures for using LEMA' for the analysis of experimental data including one where distortion effects can be applied to the experimental data and "plane-wave" structure functions can then be extracted from the corrected data.

We analyzed the quasielastic (e, e') data from Saclay on ^{208}Pb using a relativistic single-particle model and LEMA'. We do not find agreement with the data and even if we vary the longitudinal and transverse contributions in our model do not agree with the data. This is to be contrasted with our excellent agreement [6] with quasielastic data on ^{40}Ca and rough agreement with quasielastic data [4] on ^{238}U . We strongly recommend that additional (e, e') experiments on medium and heavy nuclei in the quasielastic region be carried out. The treatment of Coulomb corrections is no longer a serious hindrance to the analysis of such experiments.

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- [1] Yanhe Jin, D. S. Onley, and L. E. Wright, Phys. Rev. C **50**, 168 (1994).
- [2] Z. E. Meziani *et al.*, Nucl. Phys. A**446**, 113 (1985); Phys. Rev. Lett. **52**, 2130 (1984); **54**, 1233 (1985).
- [3] M. Deady *et al.*, Phys. Rev. C **33**, 1897 (1986); **28**, 631 (1983).

- [4] C. C. Blatchley, J. J. LeRose, O. E. Pruet, P. D. Zimmerman, C. F. Williamson, and M. Deady, Phys. Rev. C **34**, 1243 (1986).
- [5] Yanhe Jin, D. S. Onley, and L. E. Wright, Phys. Rev. C **45**, 1333 (1992).
- [6] T. C. Yates, C. F. Williamson, W. M. Schmitt, M. Osborn, M.

- Deady, Peter D. Zimmerman, C. C. Blatchley, Kamal K. Seth, M. Sarmiento, B. Parker, Yanhe Jin, L. E. Wright, and D. S. Onley, Phys. Lett. B **312**, 382 (1993).
- [7] J. Jourdan, Phys. Lett. B **353**, 189 (1996).
- [8] A. Zghiche *et al.*, Nucl. Phys. **A573**, 513 (1994).
- [9] Yanhe Jin, D. S. Onley, and L. E. Wright, Phys. Rev. C **45**, 1311 (1992); Yanhe Jin, Ph.D. dissertation, Ohio University, 1991.
- [10] J. Knoll, Nucl. Phys. **A201**, 289 (1973); **A223**, 462 (1974).
- [11] H. Überall, *Electron Scattering from Complex Nuclei* (Academic Press, New York, 1971).
- [12] F. Lenz, thesis, University of Freiburg (1971); F. Lenz and R. Rosenfelder, Nucl. Phys. **A176**, 513 (1971).
- [13] M. Traini, S. Turck-Chiéze, and A. Zghiche, Phys. Rev. C **38**, 2799 (1988); Phys. Lett. B **213**, 1 (1988).
- [14] C. Giusti and F. D. Pacati, Nucl. Phys. **A473**, 717 (1987).
- [15] Kyungsik Kim, Ph.D. dissertation, Ohio University, 1996.
- [16] G. Höhler, E. Pietarinen, I. Sabba-Stevanescu, F. Borkowski, G. G. Simon, V. H. Walter, and R. D. Wendling, Nucl. Phys. **B114**, 505 (1976).