Shapes and stability within the interacting boson model: Dynamical symmetries

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For algebraic models the coherent states are appropriate trial wave functions to study the energy surfaces of the system. The equilibrium configurations of these functions are classified by means of the separatrix of the catastrophe formalism, which is defined by the bifurcation and Maxwell sets. The bifurcation sets correspond to curves in the parameter space associated to degenerate critical points while the Maxwell sets constitute the locus of points for which the energy surface takes the same value in two or more critical points. As an example we study the energy surfaces associated to the dynamical symmetries of the interacting boson model in the essential parameter space. [S0556-2813(96)05211-9]

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I. INTRODUCTION

The time dependent variational principle (TDVP) is a formulation of the time dependent Schrödinger equation through a variation of an action functional [1]. The Schrödinger equation is obtained by requiring that the action functional be stationary under free variations of the time dependent state.

The Hamiltonian nature of the equations of motion arising from this principle was pointed out [2] and leads to a symplectic structure, that is, a Hamiltonian function, which we are going to call it energy surface, plus generalized Poisson brackets. Thus through this procedure one can do static or dynamic studies of the system. In the present work we will be studying only the static case, that is, the expectation value of the Hamiltonian with respect to a trial wave function. It is important to emphasize that, depending on the generality of the trial wave function that one chooses, the bigger or smaller region of the solution space that one can study. For this reason, if the Hamiltonian has an algebraic structure, it is convenient to choose as trial wave functions its associated coherent states.

We are going to analyze the energy surfaces (ES) within the catastrophe theory formalism, which is based on important mathematical results on functional analysis [3]. Therefore, if we have an energy surface depending on n variables and r essential control parameters, the first step is to find its critical points and determine which are Morse points and which are not. In the Morse points the energy surface can be approximated by a local quadratic form, while for the non-Morse points the energy surface can be written in terms of the catastrophe function, which is constructed by a germ plus a perturbation. For a number of control parameters less or equal to five, without special symmetry conditions, a list of canonical catastrophe functions are known [3].

A connection [4] between the interacting boson model [5] and the geometrical approach of Bohr-Mottelson [6] was done by expressing the IBA-1 Hamiltonian in terms of shape variables. This can be achieved by means of the corresponding coherent states [7]. Analysis of shape and shape phase transitions in this model have been realized in Refs. [8,9]. In this work we apply the procedure introduced in Ref. [8] to the interacting boson model, but for the general Hamiltonian of one and two-body central interactions involving s and d bosons and determining its associated *separatrix*. We show that the equilibrium configurations can be classified by means of two parameters and are enough to describe the most general energy surface. These ES are organized by the *separatrix*, which is useful to know: (i) how many equilibrium configurations yield the system and (ii) if the behavior of the model around the critical points may or may not be approximated by an harmonic oscillator. This analysis generalizes those presented previously, in which only transitions between pairs of exact SU(5), O(6), and SU(3) symmetries are considered [10]. In the past decade, effective Hamiltonians of the IBA-1 have been used to describe energy spectra and transition probabilities of chains of isotopes and isotones [11–13]. The analysis of the stability and shape transitions properties for this last case requires the generalization treated in this work.

In the second section a brief summary of the interacting boson model, in its simpler version is given [5], and their ES are established. By means of the catastrophe theory, in the third section, the number of essential parameters of the ES and separatrix are determined. In the fourth section the behavior of the IBA-1 dynamical symmetries in the essential parameter space is described, together with the kind and order of the shape transitions that they can yield. In the last section a summary of the main stability and shape characteristics of the IBA-1 are indicated.

II. SUMMARY OF THE IBA-1

First a brief review of the interacting boson model (IBA-1) is presented. This model was introduced in 1975 [5] to describe the properties of even-even nuclei through the interactions of two types of bosons: one with angular momentum $L=0$ (s boson) and another with angular momentum

$L=2$ (d boson). These bosons are considered formed by correlated pair of fermions and generate a U(6) group structure. Thus the nuclei are described as a system of s and d bosons, whose number is determined by half of the valence nucleons while the core remains inert.

To construct the Hamiltonian, single and two boson interactions are considered that preserve the total number of bosons and are also rotational scalars. For one body terms, s and d boson number operators result naturally, while for the two body case one gets

$$H_{\text{IBA-1}}=\epsilon_s N_s+\epsilon_d N_d+\sum_{L=0,2,4}\frac{c_L}{2}\sqrt{2L+1}[[\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[L]}\times[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[L]}]^{[0]}+\frac{v_2}{\sqrt{2}}([[\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[2]}\times\tilde{\mathbf{d}}]^{[0]}s+s^\dagger[\mathbf{d}^\dagger\times[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[2]}]^{[0]})$$
$$+\frac{v_0}{2}([\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[0]}s^2+s^{\dagger 2}[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[0]})+u_2 s^\dagger s[\mathbf{d}^\dagger\times\tilde{\mathbf{d}}]^{[0]}+\frac{u_0}{2}s^{\dagger 2}s^2, \tag{2.1}$$

where the boson operators satisfy the commutation relations

$$[s,s^\dagger]=1, \quad [d_\mu,d_\nu^\dagger]=\delta_{\mu\nu}. \tag{2.2}$$

The other possible commutation relations are zero and the coefficients in front of the interaction terms are adjustable parameters that indicate their intensities.

One of the nicest features of the IBA-1 is that the Hamiltonian can be expressed in terms of the Casimir operators of the chain of groups [14]

$$\mathrm{U}(6)\supset\mathrm{U}(5)\supset\mathrm{O}(5)\supset\mathrm{O}(3), \tag{2.3a}$$

$$\mathrm{U}(6)\supset\mathrm{O}(6)\supset\mathrm{O}(5)\supset\mathrm{O}(3), \tag{2.3b}$$

$$\mathrm{U}(6)\supset\mathrm{SU}(3)\supset\mathrm{O}(3), \tag{2.3c}$$

that is

$$H=k_1N_d+k_2N_d^2+k_3NN_d+k_4L^2+k_5\Lambda^2+k_6P^2+k_7Q^2$$
$$+k_8N+k_9N^2, \tag{2.4}$$

where N_d is the d boson number operator, L^2 is the square of the angular momentum, Λ^2 is the Casimir operator of O(5), and P^2 is related to the Casimir operator of O(6) through $P^2=\frac{1}{4}(N^2+4N-\mathcal{L}^2)$. Finally the quadrupole-quadrupole interaction is related to the SU(3) Casimir operator G through the expression $Q^2=\frac{1}{2}G-\frac{3}{8}L^2$. It is important to emphasize that the Hamiltonian would be diagonal if it can be written in terms of the Casimir operators associated to one of the chain of groups (2.3).

In this contribution we are going to study only the static properties of the IBA-1, that is the behavior of the Hamiltonian function or energy surface. For this case we can consider the reality condition to the complex variables [15].

Associated to the IBA-1 model is an intrinsic geometry structure [7,16], determined by its coherent states, which has been discussed in Refs. [16,17]. These can be written in the form

$$|N\alpha_\mu\rangle=(s^\dagger+\Sigma_\mu\alpha_\mu d_\mu^\dagger)^N|0\rangle. \tag{2.5}$$

This yields the energy surface

$$E(\beta,\gamma)=\frac{\langle N,\beta\gamma|H|N,\beta\gamma\rangle}{\langle N,\beta\gamma|N,\beta\gamma\rangle}$$
$$=\frac{N\epsilon\beta^2}{(1+\beta^2)}+\frac{N(N-1)}{(1+\beta^2)^2}\left(a_1\beta^4+a_2\beta^3\cos 3\gamma\right.$$
$$\left.+a_3\beta^2+\frac{u_0}{2}\right), \tag{2.6}$$

where $\epsilon\equiv\epsilon_d-\epsilon_s$; the constant $N\epsilon_s$ was substracted and the variables α_μ were expressed in terms of β, γ, and the Euler angles. As the energy surface is a rotational scalar, there is no dependence in the Euler angles and the parameters a_1, a_2, and a_3 are defined by

$$a_1=\frac{c_0}{10}+\frac{c_2}{7}+\frac{9c_4}{35}, \tag{2.7a}$$

$$a_2=-\frac{2}{\sqrt{35}}v_2, \tag{2.7b}$$

$$a_3=\frac{1}{\sqrt{5}}(v_0+u_2). \tag{2.7c}$$

III. ENERGY SURFACES AND CATASTROPHE THEORY

The energy surface of the IBA-1 is a function depending on the variables (β,γ), and the five parameters: ϵ, u_0, a_1, a_2, and a_3. We will consider the energy surface of the IBA-1 as an example of how to use the catastrophe formalism to determine the shapes and stability of the model [3,16]. The procedure is the following:

(i) The essential control parameters and the germ of the energy surface are determined. This is accomplished by obtaining the equilibrium configurations, that is the critical points of the Eq. (2.6). From them, the critical point with maximum degeneracy is selected and it is called the fundamental root. Then a Taylor series expansion of the energy surface around the fundamental root is done. The germ of the IBA-1 is the first term of the expansion which cannot be canceled by an arbitrary selection of the parameter values.

(ii) One constructs the bifurcation sets of the energy surfaces of the IBA-1. A bifurcation set is the locus of points in the space of essential control parameters at which a transition occurs from one local minimum to another. The bifurcation sets are obtained from the condition det $\mathcal{H}=0$, where $\mathcal{H}$ is the matrix of second derivatives of the energy surface evaluated at the critical points, i.e.,

$$\mathcal{H}_{ij}=\left.\frac{\partial^2\epsilon(x_k)}{\partial x_i\partial x_j}\right|_{(x_1^c,x_2^c)} \tag{3.1}$$

with $x_1=\beta$, and $x_2=\gamma$ and the superindex c in the variables x_1 and x_2 denotes that they are critical points. The gradient of the energy surface is used to get information about its critical points, while the matrix $\mathcal{H}$ defines its nature if det $\mathcal{H}\neq 0$, that is, if they are maxima, minima, or saddle points. For this reason the matrix $\mathcal{H}$ is called the stability matrix. However, if det $\mathcal{H}=0$ the nature and number of critical points change, meaning that (x_1^c,x_2^c) is at least double degenerate. Therefore in a bifurcation set the qualitative nature of the energy surface changes because equilibria are either created or destroyed. The determinant of the stability matrix is equal to the product of its eigenvalues and they are associated to the variables of the energy surface β_c and γ_c. An eigenvalue zero indicates that its associated variable does not behave well, meaning that the energy surface in that critical point cannot be approximated by a quadratic expression in that variable.

(iii) The Maxwell sets are determined; these sets constitute the locus of points in the essential parameter space for which the energy surface takes the same value in two or more critical points. They can be found through the Clausius-Clapeyron equations. For the IBA-1, there are only two essential parameters, denoted by (r_2,r_1), as we shall prove. Then if we assume that for (r_2^0,r_1^0) there are p critical points $x_1,x_2,\dots,x_p$ in which the energy surface has degenerate critical values, the Clausius-Clapeyron equations are defined by

$$\epsilon^{(1)}=\epsilon^{(2)}=\epsilon^{(3)}=\cdots=\epsilon^{(p)}, \tag{3.2a}$$

$$\sum_\alpha\left(\frac{\partial\epsilon^{(k)}}{\partial r_\alpha^0}-\frac{\partial\epsilon^{(k+1)}}{\partial r_\alpha^0}\right)\delta r_\alpha^0=0, \tag{3.2b}$$

where we used the notation $\epsilon^{(k)}=\epsilon(x_k,r_1^0,r_2^0)$ with x_k denoting a point in the variable space, that is (β,γ).

(iv) Finally the separatrix of the IBA-1 is constructed by the union of the bifurcation and Maxwell sets. This divides the parameter space into shape stability regions and identifies the locus of points where there are shape transitions, together with their order.

Next we follow the procedure indicated above. We start by evaluating the critical points of the energy surface, that is $\nabla E(\beta,\gamma)=0$. It is straightforward to find that the critical points are localized along the lines $\gamma_c=0$ (prolate case) and $\gamma_c=\pi/3$ (oblate case). Of course these are repeated by adding to these values multiples of $2\pi/3$ because the energy surface has a C_{3v} symmetry. According to this symmetry all the critical points are located on the Cartesian X axis. For points on the positive side of the X axis, prolate nuclei are described while points on the negative part correspond to oblate nuclei.

For the variable β, $\gamma_c=0$, one finds the algebraic equation

$$\beta\left[a_2\beta^3+2\left(a_3-2a_1-\frac{\epsilon}{N-1}\right)\beta^2-3a_2\beta+2\left(u_0-a_3-\frac{\epsilon}{N-1}\right)\right]=0, \tag{3.3}$$

to determine the critical points. The case $\gamma_c=\pi/3$ is included allowing negative values for β. From this expression, it is immediate that the $\beta_c=0$ is a critical point for any values of the parameters of the energy surface; for this reason it is the fundamental root. The Taylor series expansion of the energy surface $E(\beta)=E(\beta,\gamma_c=0)$ around this fundamental root is given by

$$E(\beta)=\frac{N(N-1)u_0}{2}+N(N-1)\left[(a_3-u_0+w)\beta^2+a_2\beta^3+\left(a_1-2a_3+\frac{3u_0}{2}-w\right)\beta^4\right]+O(5), \tag{3.4}$$

where $w\equiv\epsilon/(N-1)$ was defined and the symbol $O(5)$ indicates terms of the order β^5 or higher. From the last expression, obviously the linear term in β is not appearing because $\beta_c=0$ is a critical point. The quadratic term in β is eliminated by choosing

$$a_3-u_0+w=0. \tag{3.5}$$

This implies that $\beta_c=0$ is double degenerate because also the second derivative in the expansion is canceled. This fundamental root is triple degenerate if we choose (3.5) and that

$$a_2=0. \tag{3.6}$$

Next we try to remove the fourth derivative in the expansion; this implies

$$a_1-2a_3+\frac{3u_0}{2}-w=0. \tag{3.7}$$

However, if the parameters of the model satisfy Eqs. (3.5), (3.6), and (3.7), the energy surface (2.6) becomes independent of β and γ and takes a constant value equal to

$$E=\epsilon N+a_1N(N-1), \tag{3.8}$$

which, by means of the expressions (3.5), (3.6), and (3.7), is equivalent to $E=N(N-1)u_0/2$ as it can be seen directly from (3.4). Of course the higher powers in β that appear in the Taylor series expansion, (3.4), are canceled if the expressions (3.5), (3.6), and (3.7) are satisfied.

Therefore we conclude that the energy surface is overdetermined if we fix three parameters, and for this reason, one can only satisfy Eqs. (3.5) and (3.6). Then we have that the first term different from zero is β^4, which characterizes the germ of the system, and we introduce the essential control parameters

$$r_1=\frac{a_3-u_0+w}{2a_1+w-a_3},\quad r_2=-\frac{2a_2}{2a_1+w-a_3}. \tag{3.9}$$

The energy surface can be rewritten in terms of these parameters in the form

$$\epsilon(\beta,\gamma)\equiv\frac{E(\beta,\gamma)-N(N-1)u_0/2}{\epsilon_0}$$
$$=\frac{1}{(1+\beta^2)^2}\,[\beta^4+r_1\beta^2(\beta^2+2)-r_2\beta^3\,\cos 3\gamma], \tag{3.10}$$

where we have defined $\epsilon_0=N(N-1)(2a_1+w-a_3)/2$. Notice that if ϵ_0 is zero we have a trivial case and the energy surface is a constant given by the Eq. (3.8), so the interesting cases occur when ϵ_0 is different from zero.

In summary, through the Taylor series expansion around the fundamental root $\beta_c=0$, it was possible to determine the essential parameters and the germ of the system. In terms of these new parameters the critical points are found, in the Appendix, by solving the algebraic equation

$$\beta(r_2\beta^3+4\beta^2-3r_2\beta+4r_1)=0. \tag{3.11}$$

Following step (ii) of the procedure, we find the next bifurcation sets.

Bifurcation set r_2 axis. The critical point $\beta_c=0$ is degenerate if and only if the condition $r_1=0$ is satisfied; therefore in this critical point the energy surface is always stable when $r_1\neq0$, and presents degeneracy all along the r_2 axis in both variables.

Bifurcation set r_1 axis with $r_1<0$. As it was shown in the Appendix, for this set we have the critical point $(\beta_c,\gamma_c)=(\sqrt{-r_1},\gamma)$. It presents degeneracy in the variable γ and it is a non-Morse point. This is proved by evaluating, in this point, the stability matrix and finding that the eigenvalue associated to the variable γ is equal to zero. This set is associated with γ-unstable nuclei.

Bifurcation sets r_{11} and r_{12}. To find the degeneracy of the critical points that fall outside the r_2 and r_1 axes, the criteria of the stability matrix requires solving algebraic equations of fifth degree. Then it is simpler to use an equivalent procedure that involves a mapping between the critical and $\mathfrak{R}^2$ parameter manifolds [3]. The critical manifold $\beta_c\neq0$ is defined by

$$r_2\beta_c^3+4\beta_c^2-3r_2\beta_c+4r_1=0. \tag{3.12}$$

The coordinates of any point on this manifold embedded in the three-dimensional space are

$$(\beta_c;r_2,r_1)=(\lambda_1;\lambda_2,-\tfrac{1}{4}[\lambda_2\lambda_1^3+4\lambda_1^2-3\lambda_2\lambda_1]), \tag{3.13}$$

where r_1 was obtained from Eq. (3.12). Now we consider the projection mapping of the two-dimensional manifold (3.12) down onto the two-dimensional essential control parameters plane $\mathfrak{R}^2$, defined by

$$\chi:(\beta_c,r_2,r_1)\rightarrow(r_2,r_1),$$
$$r_2=\lambda_2,$$
$$r_1=-\tfrac{1}{4}\,(\lambda_2\lambda_1^3+4\lambda_1^2-3\lambda_2\lambda_1). \tag{3.14}$$

The previous expression defines a singular mapping if the Jacobian of the transformation

$$\det\begin{pmatrix}\dfrac{\partial r_2}{\partial\lambda_1} & \dfrac{\partial r_2}{\partial\lambda_2}\\ \dfrac{\partial r_1}{\partial\lambda_1} & \dfrac{\partial r_1}{\partial\lambda_2}\end{pmatrix}$$
$$=\det\begin{pmatrix}0 & 1\\ -\tfrac{3}{4}\lambda_2\lambda_1^2-2\lambda_1+\tfrac{3}{4}\lambda_2 & -\tfrac{1}{4}\lambda_1^3+\tfrac{3}{4}\lambda_1\end{pmatrix}$$
$$=-\frac{3}{4}\,\lambda_2(1-\lambda_1^2)+2\lambda_1 \tag{3.15}$$

is zero. Imposing this condition, one gets the expression

$$\lambda_2=\frac{8\lambda_1}{3(1-\lambda_1^2)}. \tag{3.16}$$

Then in general the mapping is invertible except for the curve in the three-dimensional space

$$(\beta_c;r_2,r_1)=\left(\lambda_1;\,\frac{8\lambda_1}{3(1-\lambda_1^2)},\,\frac{\lambda_1^2}{3(1-\lambda_1^2)}\,(\lambda_1^2+3)\right), \tag{3.17}$$

because it indicates the set of points at which the tangent plane to the manifold (3.12) is vertical, meaning that they have associated the same essential control parameters. Then the projection (3.14) takes the form

$$\chi:(\beta_c,r_2,r_1)\rightarrow(r_2,r_1),$$
$$r_2=\frac{8\lambda_1}{3(1-\lambda_1^2)},$$
$$r_1=\frac{\lambda_1^2}{3(1-\lambda_1^2)}\,(\lambda_1^2+3), \tag{3.18}$$

which is denoting the parametric equations for the bifurcation sets that fall outside the r_1 and r_2 axes. Eliminating the parameter λ_1 one obtains

$$r_{11}=-\frac{(9r_2^2+16)^{3/2}}{54r_2^2}-\frac{32}{27r_2^2}-1, \tag{3.19a}$$

$$r_{12}=\frac{(9r_2^2+16)^{3/2}}{54r_2^2}-\frac{32}{27r_2^2}-1. \tag{3.19b}$$

Following the calculations made in the Appendix the critical points onto the bifurcation sets have $D=0$. Thus they are obtained by replacing $r_1\rightarrow r_{11}$ into Eq. (A7) and the result into (A10)

$$\beta_{11}=-\frac{4}{3r_2}+2\frac{\sqrt{9r_2^2+16}}{3r_2}, \tag{3.20a}$$

$$\beta_{12}=-\frac{4}{3r_2}-\frac{\sqrt{9r_2^2+16}}{3r_2}. \tag{3.20b}$$

In similar form, the critical points onto the set r_{12} are found if we replace $r_1\rightarrow r_{12}$

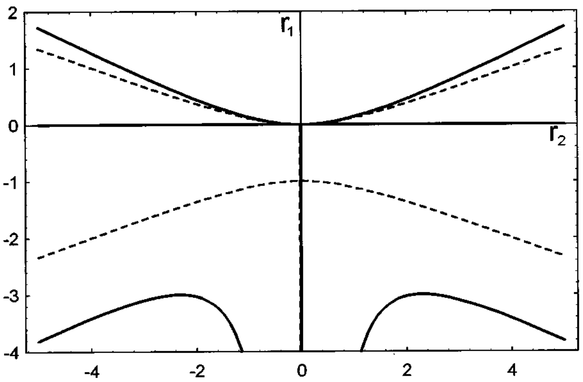

FIG. 1. The separatrix of the IBA-1 is shown. For positive r_1 values, the continuous line is indicating the bifurcation set r_{12} and the dashed one the Maxwell set r_{13}^+. For negative r_1 values, the bifurcation sets r_{11} and r_1 semiaxis are displayed with continuous lines, while the Maxwell sets r_{13}^- and r_1 semiaxis are shown with dashed lines.

$$\beta_{21}=-\frac{4}{3r_2}-2\frac{\sqrt{9r_2^2+16}}{3r_2}, \tag{3.21a}$$

$$\beta_{22}=-\frac{4}{3r_2}+\frac{\sqrt{9r_2^2+16}}{3r_2}. \tag{3.21b}$$

From the critical points (3.20) and (3.21), we proved that β_{12} and β_{22} are degenerate critical points by determining the eigenvalues of the stability matrix. In this analysis it was found that β is the bad behaved variable. In Fig. 1 the constructed bifurcation sets r_2 axis, negative r_1 semiaxis, r_{11}, and r_{12} are illustrated with continuous lines.

Next, we proceed to build the Maxwell sets by means of Eqs. (3.2). For the IBA-1, it is found that there are three Maxwell sets, associated to the branches $\beta_c=0$, and $\beta_c=\sqrt{-r_1}$.

For the basic branch, $\beta_c=0$, the energy surface has value zero, then to find the Maxwell set, the Eq. (3.10) is equated to zero, with $\gamma_c=0$ and $\gamma_c=\pi/3$. After simplifications it is reduced to

$$(r_1+1)\beta^2\mp r_2\beta+2r_1=0. \tag{3.22}$$

This equation gives us two intersections of the energy surface with the x axis, then there is always an extreme point between them. Therefore, when the solutions are degenerated we find the Maxwell sets associated to the basic branch. The solutions of the last expression are degenerated when the discriminant is zero, that is

$$r_2^2-8r_1(r_1+1)=0. \tag{3.23}$$

Solving r_1 as function of r_2 we get two sets, which will be denoted by $r_{13}^{\pm}$ and are given by

$$r_{13}^{\pm}=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1+r_2^2/2}. \tag{3.24}$$

The value r_{13}^+ corresponds to the Maxwell set where the minima of the energy surface degenerate, while r_{13}^- is associated to maxima. There are four critical points onto these bifurcation sets. However, those associated with the degeneration of maxima are $\beta_c=0$ and β_{13}^- while the minima occur for $\beta_c=0$ and β_{13}^+. The $\beta_{13}^{\pm}$ critical points can be calculated by solving Eq. (3.22) under the replacement of $r_1\to r_{13}^{\pm}$ and they are given by

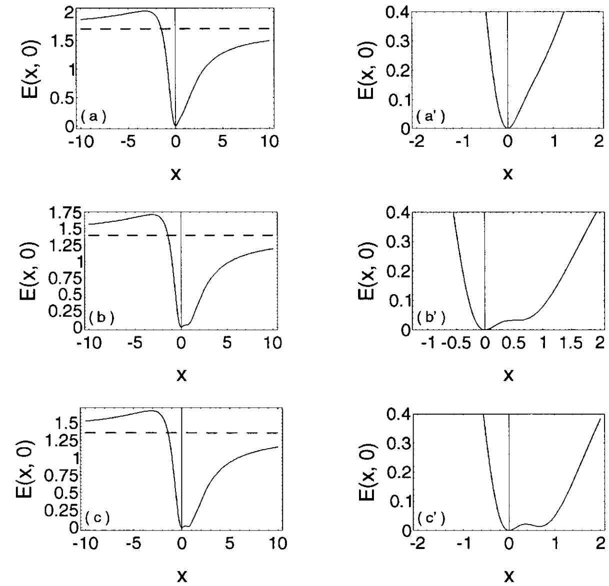

FIG. 2. Energy surfaces cuts $E(x,0)$, with $x=\beta\cos\gamma$, are displayed. In (a) for the region $r_1>r_{12}$, in (b) for $r_1=r_{12}$, and in (c) for the domain $r_{13}^+<r_1<r_{12}$. The plots (a′), (b′), and (c′) are closer views to the minima for the corresponding curves, and they show the appearance of an excited prolate local minimum.

$$\beta_{13}^{\pm}=\frac{r_2}{\pm\sqrt{1+r_2^2/2}+1}, \tag{3.25}$$

where the positive values of $\beta_{13}^{\pm}$ correspond to the prolate nuclei while the negative ones correspond to the oblate case.

For the branch $\beta_c=\sqrt{-r_1}$, the Maxwell set is associated to the locus of points of the negative r_1 semiaxis. The ES in this branch presents minima of the same depth for any value of the variable γ. In Fig. 1, the Maxwell sets $r_{13}^{\pm}$ and the negative r_1 semiaxis are displayed by dashed lines.

In the second and third steps we have constructed the bifurcation and Maxwell sets. Thus the separatrix in the IBA-1 is defined by the curves in the parameter space (r_2,r_1), which result from the bifurcation and Maxwell sets associated to the critical points. In the bifurcation sets the qualitative nature of the energy surface changes because equilibria are either created or destroyed. In Fig. 1, the separatrix of the IBA-1 is displayed. The typical shapes of the ES are stable within the six regions divided by the separatrix. Therefore the qualitative aspect of the surfaces can be determined by exhibiting a cut valid for $\gamma=0$ and $\gamma=\pi/3$ of the energy surface defined in Eq. (3.10). The ES are shown in Figs. 2–5 in all the regions when one moves along the line $r_2=4\sqrt{2}/3$. In these figures one can appreciate the following: for $r_1>r_{12}$ the shape of the ground state band is spherical; for values of the control parameters between the bifurcation r_{12} and the Maxwell r_{13}^+ sets, the ES have a second prolate minimum. For nuclei described by ES of this region, the shape coexistence phenomena for excited states among

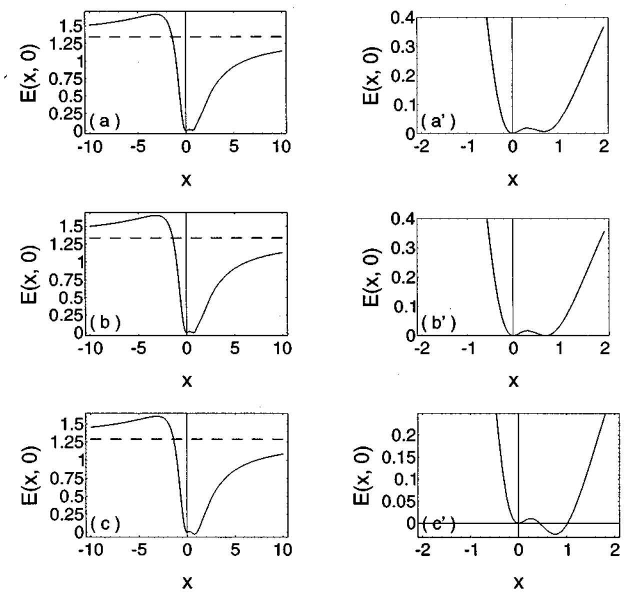

FIG. 3. Cuts $E(x,0)$ are shown. In (a) for $r_{13}^{+}<r_1<r_{12}$, in (b) for $r_1=r_{13}^{+}$, and in (c) for the region $0<r_1<r_{13}^{+}$. The corresponding closer views are displaying the transition from spherical to deformed shapes.

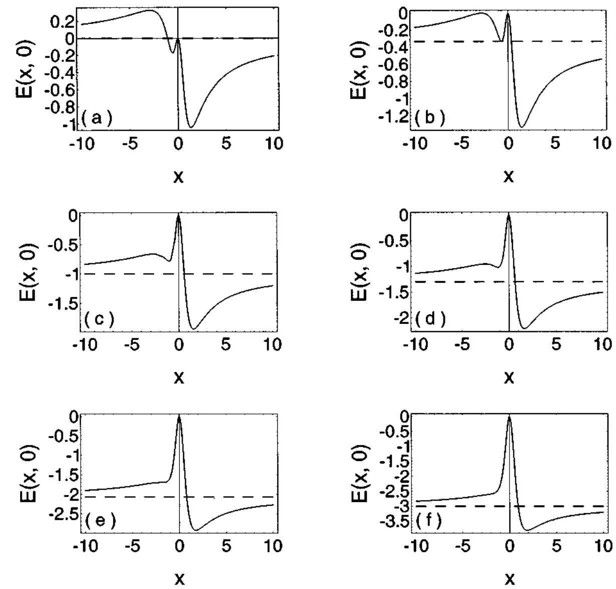

FIG. 5. Cuts $E(x,0)$ are shown. In (a) for $r_{13}^{-}<r_1<0$, in (b) for $r_1=r_{13}^{-}$, in (c) and (d) the curves are shown for two values of r_1 in the region $r_{11}<r_1<r_{13}^{-}$, in (e) for $r_1=r_{11}$, and in (f) for $r_1<r_{11}$. Notice that the oblate region becomes inaccessible afterwards the maxwell set r_{13}^{-}.

spherical and prolate shapes are present. For $0<r_1<r_{13}^{+}$ the shape of the ground state band is prolate and excited bands are spherical, indicating again the presence of the shape coexistence phenomena. For $r_{13}^{-}<r_1<0$ the spherical excited band disappears and the system has a very well defined prolate minimum characteristic of rotational nuclei. In Figs. 4 and 5 there is an apparent oblate local minimum; however, it is a saddle point because it is unstable with respect to the variable γ. For $r_1<r_{13}^{-}$ there is only a prolate minimum (see Figs. 4 and 5) and for even smaller values of r_1 the system is less and less bound, as is indicated in all the figures by the dashed line. It represents the asymptotic value, $E\rightarrow r_1+1$, of the energy surface when $\beta\rightarrow\infty$.

In Fig. 6 the corresponding ES are illustrated when one moves along the line $r_2=0$ in the parameter space. For $r_1>0$ one has a spherical shape, while for $r_1<0$ we have the presence of γ-unstable nuclei. The ES are shown in Fig. 7,

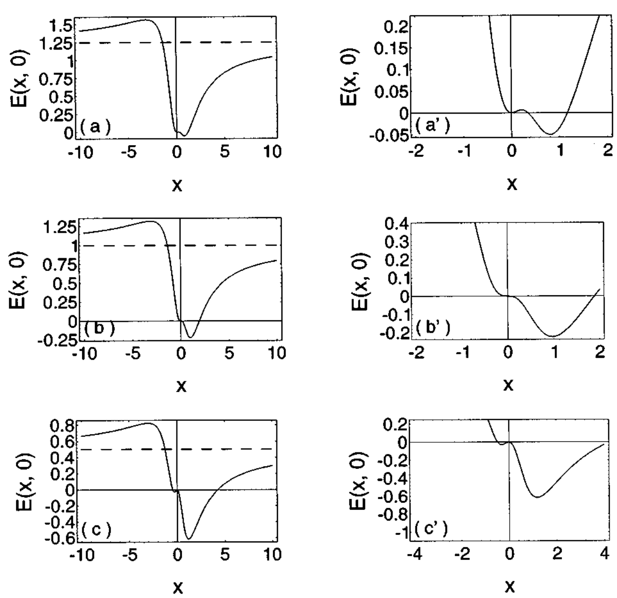

FIG. 4. Cuts $E(x,0)$ are shown. In (a) for $0<r_1<r_{13}^{+}$, in (b) for $r_1=0$, and in (c) for $r_{13}^{-}<r_1<0$. The closer views are displaying the disappearance of the excited spherical states.

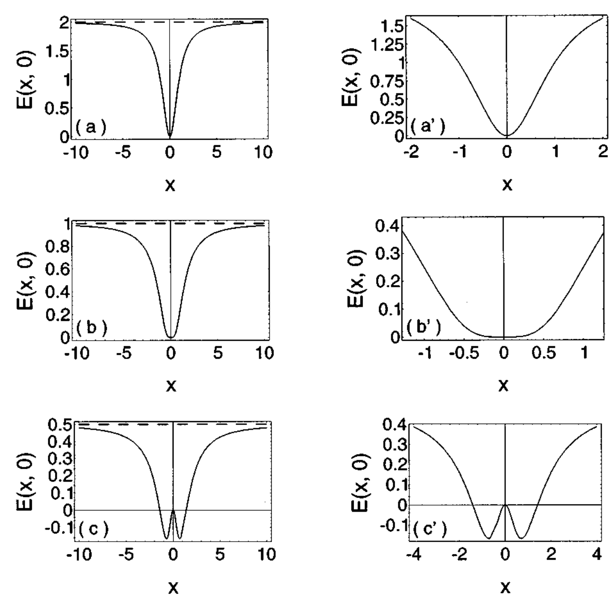

FIG. 6. Cuts $E(x,0)$ are shown for $r_2=0$. In (a) for $r_1=1$, in (b) for $r_1=0$, and in (c) for $r_1=-1/2$. The plots at the right indicate closer views at the minima, displaying the transition from a spherical to deformed γ-unstable shapes. In particular, in (b′) the catastrophe germ of the IBA-1 is shown.

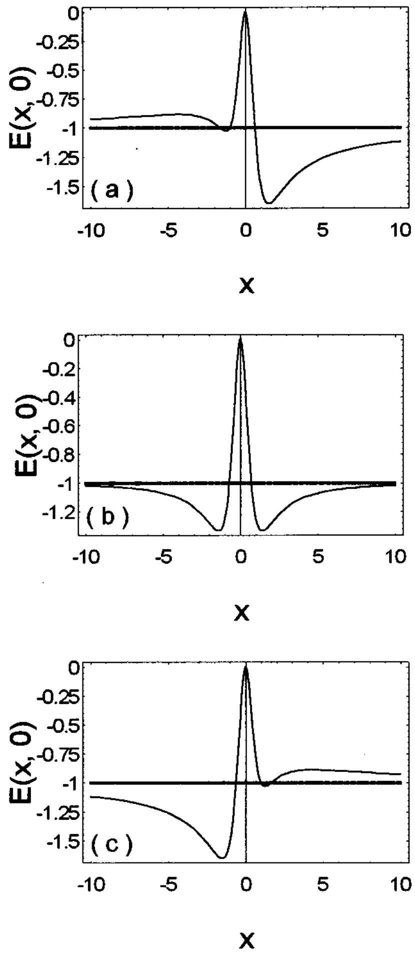

FIG. 7. Cuts $E(x,0)$ are shown for $r_1=-2$. In (a) for $r_2=1$, in (b) for $r_2=0$, and in (c) for $r_2=-1$. The plots are displaying the transition from a prolate to oblate shapes, through a γ unstable.

along the $r_1=-2$ straight line. For $r_2<0$ the ES characterize oblate shapes plus a prolate saddle critical point, while for positive r_2 values describe prolate shapes with an oblate saddle critical point. Notice that the possible shapes present in the left-hand side of the parameter space can be obtained by making a reflection ($\beta\rightarrow-\beta$) in the ones plotted in the Figs. 2–5.

Shape transitions

The classical theory of phase transitions can be applied to the catastrophe formalism [3]. We will say that a shape transition occurs when the point (β,γ) describing the state of a physical system jumps from one locally stable critical branch to another. This will be happening when the control parameters (r_2,r_1) are varied and crossings through the separatrix of the system are taking place.

Usually the control parameters are assumed to depend on only one parameter, which is used to describe a curve in the parameter space; thus in general we have

$$r_1\rightarrow r_1(s), \tag{3.26a}$$

$$r_2\rightarrow r_2(s), \tag{3.26b}$$

$$(\beta_c,\gamma_c)\rightarrow(\beta_c(s),\gamma_c(s)), \tag{3.26c}$$

$$E_c(\beta_c,\gamma_c)\rightarrow E_c(\beta_c(s),\gamma_c(s)). \tag{3.26d}$$

It is convenient to classify the shape transitions by their order, then a transition is of the order n if

$$\lim_{\epsilon\rightarrow 0}\frac{\partial^i E_c(s)}{\partial s^i}\bigg|_{s_0-\epsilon}=\lim_{\epsilon\rightarrow 0}\frac{\partial^i E_{c'}(s)}{\partial s^i}\bigg|_{s_0+\epsilon}, \tag{3.27}$$

for $i=0,1,2,3,\ldots,(n-1)$, but it is not satisfied for $i=n$. The parameter s_0 denotes a point onto the separatrix.

To find the order of the shape transitions present in the IBA-1 we use the following procedure.

Evaluate the energy surface in two critical points onto the bifurcation set being crossed, with the appropriate parameter values. If they are not equal, one has a zero order transition. If that is not the case, take the first derivative of the energy surface with respect to s taking into account that the function is evaluated in two critical points

$$\frac{dE}{ds}=\frac{\partial E}{\partial r_1}\frac{\partial r_1}{\partial s}+\frac{\partial E}{\partial r_2}\frac{\partial r_2}{\partial s}, \tag{3.28}$$

where $\partial r_k/\partial s$ are the direction cosines. Furthermore E must be evaluated in critical points onto the bifurcation. If they are not equal one has a first order transition and so on.

Next, we study the shape transitions that occur when the control parameters of the energy surface are crossing the separatrix of the IBA-1 model. We start by considering $r_2>0$; when the bifurcation set r_{12} is intersected the ES have a zero order transition between the spherical ground state band and excited prolate states (see Fig. 2). If the set r_{13}^{+} is crossed, there is a first order transition from a spherical to a prolate shape (see Fig. 3). When the bifurcation set $r_1=0$ is intersected there is a zero order transition between excited spherical states and the ground state deformed band of the system (see Figs. 3 and 4), except for $r_2=0$. In this case (see Fig. 6) we have a second order shape transition from a spherical nucleus to a γ-unstable deformed one. Notice in Fig. 4 the appearance of a minimum in the variable β for oblate shapes; however, it is unstable in the variable γ, and therefore a saddle critical point. The bifurcation set negative r_1 semiaxis produces a first order transition when it is crossed from one side to the other (see Fig. 7). For $r_2<0$, the ES are reflected with respect to the vertical axis under the mapping $r_2\rightarrow-r_2$ because it is equivalent to change $\beta\rightarrow-\beta$.

IV. DYNAMICAL SYMMETRIES

The dynamical symmetries of the IBA-1 have played an important role in the developing of the model. For example, their energy spectra and electromagnetic transitions remind us of those obtained with geometrical models. The U(5) dynamical symmetry describes an anharmonic vibrator, the SU(3) is an axial rotor, and finally the O(6) limit is a deformed γ-unstable rotor. Then we consider it important to find the locus of points in the essential parameter space which are associated to these dynamical symmetries.

We start by studying a truncation of the IBA-1 Hamiltonian (2.1), which is comprised of Casimir operators of the U(5) limit, that is

$$H=k_1N_d+k_2N_d^2+k_3NN_d+k_4L^2+k_5\Lambda^2+k_9N^2. \tag{4.1}$$

These parameters k_i are related to the single-boson and two-boson matrix elements through the Eq. (4) of the Ref. [11]. By means of that expression and the Eq. (2.7), it is found

$$\epsilon=k_1+k_2+k_3+6k_4+4k_5, \tag{4.2a}$$

$$u_0=2k_9, \tag{4.2b}$$

$$a_1=k_2+k_3+k_9, \tag{4.2c}$$

$$a_2=0, \tag{4.2d}$$

$$a_3=k_3+2k_9. \tag{4.2e}$$

These five parameters define the energy surface of the vibrational limit and through them and Eq. (3.5), the control parameters of the system are calculated, i.e.,

$$r_2=0, \quad r_1=\frac{b_1+k_2+k_3N}{b_1+(2N-1)k_2+k_3N}, \tag{4.3}$$

where $b_1=k_1+6k_4+4k_5$. Notice that if $k_2=0$, we have $(r_2,r_1)=(0,1)$ in the essential parameter space. In general, one has that the possible ES are characterized by the points on the r_1 axis; the r_1 values depend strongly on the relative intensities and signs of the parameters b_1, k_2, and k_3. For small values of k_2, the ES are in the vicinity of $r_1=1$.

The O(6) algebra is comprised of fifteen generators, ten of them close under the commutation relations of an O(5) algebra. The other five have phase ambiguities due to the fact that the s and d boson operators could undergo a gauge transformation without changing the commutation properties. There are two realizations used in the literature, that is

$$\Lambda_{\mu\mu'}=d^\dagger_\mu\tilde{d}_{\mu'}-d^\dagger_{\mu'}\tilde{d}_\mu, \quad \Lambda_\mu=s^\dagger\tilde{d}_\mu+sd^\dagger_\mu, \tag{4.4a}$$

$$\Lambda_{\mu\mu'}=d^\dagger_\mu\tilde{d}_{\mu'}-d^\dagger_{\mu'}\tilde{d}_\mu, \quad \overline{\Lambda_\mu}=s^\dagger\tilde{d}_\mu-sd^\dagger_\mu. \tag{4.4b}$$

We are going to distinguish these two realizations (4.4a) and (4.4b), following [18], by denoting the corresponding algebras by O(6) and $\underline{\text{O(6)}}$, respectively.

Next we study the O(6) limit, which is defined by the Hamiltonian

$$H=k_4L^2+k_5\Lambda^2+k_6\overline{P^2}+k_9N^2. \tag{4.5}$$

The ES associated to this limit are determined by the set of parameters

$$\epsilon=6k_4+4k_5, \tag{4.6a}$$

$$a_1=\frac{k_6}{4}+k_9, \tag{4.6b}$$

$$a_2=0, \tag{4.6c}$$

$$a_3=\frac{k_6}{2}+2k_9, \tag{4.6d}$$

$$u_0=\frac{k_6}{2}+2k_9. \tag{4.6e}$$

Evaluating the control parameters it is found $(r_2,r_1)=(0,1)$, independently of the intensities of the interactions. It is important to note that the operators L^2 and Λ^2 have one body contributions proportional to N_d. If these are eliminated one arrives at the condition of an energy surface constant. The most general interaction (4.5), without one body contributions, has been discussed previously [19] and is responsible for the collective motion.

For the $\underline{\text{O}}(6)$ case we replace in the Hamiltonian (4.5) the operator $\overline{P^2}$ by

$$P^2=\frac{5}{4}[\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[0]}[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[0]}+\frac{1}{4}s^{\dagger 2}s^2-\frac{\sqrt{5}}{4}([\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[0]}s^2+s^{\dagger 2}[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[0]}). \tag{4.7}$$

Then the associated ES can be determined easily by means of the expression (4) of Ref. [11] and Eq. (2.7). They are characterized by the parameters

$$\epsilon=6k_4+4k_5, \tag{4.8a}$$

$$u_0=\frac{k_6}{2}+2k_9, \tag{4.8b}$$

$$a_1=\frac{k_6}{4}+k_9, \tag{4.8c}$$

$$a_2=0, \tag{4.8d}$$

$$a_3=-\frac{k_6}{2}+2k_9. \tag{4.8e}$$

The corresponding control parameters are evaluated through Eq. (3.5) and it is found that

$$r_2=0, \quad r_1=\frac{\epsilon-k_6(N-1)}{\epsilon+k_6(N-1)}. \tag{4.9}$$

Notice again the influence of the one body terms of L^2 and Λ^2. If these are eliminated one arrives at the values $(r_2,r_1)=(0,-1)$. In this limit depending on the relative strength between ϵ and k_6, that is $\xi=k_6(N-1)/\epsilon$, the ES are associated to the following r_1 values:

$$-1<\xi\leqslant 1\Rightarrow 0<r_1, \tag{4.10a}$$

$$1<\xi<\infty\Rightarrow -1<r_1<0, \tag{4.10b}$$

$$-\infty<\xi\leqslant -1\Rightarrow r_1<-1. \tag{4.10c}$$

Therefore, as we have shown in the previous discussion (see Fig. 6), for $r_1>0$ the energy surface has a minimum in $\beta=0$, while if $r_1<0$ it has a maximum in $\beta=0$ plus two minima of the same depth.

The SU(3) algebra is formed by nine generators, the angular momentum plus the quadrupole operators. There are two currently used versions of the quadrupole operator

$$Q_\mu=(s^\dagger\tilde{d}_\mu+sd^\dagger_\mu)-\frac{\sqrt{7}}{2}[\mathbf{d}^\dagger\times\tilde{\mathbf{d}}]^{[2]}_\mu, \tag{4.11a}$$

$$\overline{Q_\mu}=(s^\dagger\tilde{d}_\mu+sd^\dagger_\mu)+\frac{\sqrt{7}}{2}[\mathbf{d}^\dagger\times\tilde{\mathbf{d}}]^{[2]}_\mu, \tag{4.11b}$$

which together with the angular momentum generate algebras that we denote SU(3) and $\overline{\mathrm{SU}}(3)$, respectively.

We start analyzing the $\overline{\mathrm{SU}}(3)$ limit by considering the Hamiltonian

$$H=k_4L^2+k_7\overline{Q^2}+k_9N^2. \tag{4.12}$$

For this Hamiltonian, we find the parameters

$$\epsilon=6k_4-\tfrac{9}{4}k_7, \tag{4.13a}$$

$$u_0=2k_9, \tag{4.13b}$$

$$a_1=\frac{k_7}{2}+k_9, \tag{4.13c}$$

$$a_2=-2\sqrt{2}k_7, \tag{4.13d}$$

$$a_3=4k_7+2k_9, \tag{4.13e}$$

that define the typical ES within this symmetry. Substituting the last parameters into Eq. (3.9) one gets

$$r_2=\frac{16\sqrt{2}k_7(N-1)}{24k_4+3k_7-12k_7N},\quad r_1=\frac{24k_4-25k_7+16k_7N}{24k_4+3k_7-12k_7N}. \tag{4.14}$$

In the last expression, if the total number of bosons is equal to one the harmonic limit $(r_2,r_1)=(0,1)$ is obtained, while, for $N\gg1$, the two body terms dominate and one gets $(r_2,r_1)=(-4\sqrt{2}/3,-4/3)$. Notice that this is the result one finds when the Hamiltonian is comprised only by the $\overline{\mathrm{SU}}(3)$ Casimir operator.

For the SU(3) case we have to replace the $\overline{Q^2}$ interaction by Q^2, which is defined by the expression

$$Q^2=5N_s+\frac{11}{4}N_d+\frac{7}{4}[\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[0]}[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[0]}-\frac{3\sqrt{5}}{8}[[\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[2]}\times[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[2]}]^{[0]}+\frac{3}{2}[[\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[4]}\times[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[4]}]^{[0]}$$
$$+2\sqrt{5}s^\dagger s[\mathbf{d}^\dagger\times\tilde{\mathbf{d}}]^{[0]}+\sqrt{5}([\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[0]}s^2+s^{\dagger2}[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[0]})-\sqrt{35}([[\mathbf{d}^\dagger\times\mathbf{d}^\dagger]^{[2]}\times\tilde{\mathbf{d}}]^{[0]}s+s^\dagger[\mathbf{d}^\dagger\times[\tilde{\mathbf{d}}\times\tilde{\mathbf{d}}]^{[2]}]^{[0]}). \tag{4.15}$$

The parameters

$$\epsilon=6k_4-\tfrac{9}{4}k_7, \tag{4.16a}$$

$$u_0=2k_9, \tag{4.16b}$$

$$a_1=\frac{k_7}{2}+k_9, \tag{4.16c}$$

$$a_2=2\sqrt{2}k_7, \tag{4.16d}$$

$$a_3=4k_7+2k_9, \tag{4.16e}$$

yield the ES associated to this case, and from them it is straightforward to find

$$r_2=-\frac{16\sqrt{2}k_7(N-1)}{24k_4+3k_7-12k_7N},\quad r_1=\frac{24k_4-25k_7+16k_7N}{24k_4+3k_7-12k_7N}. \tag{4.17}$$

Once more, for $N=1$ one gets the harmonic limit and when $N\gg1$ the control parameters take the values $(4\sqrt{2}/3,-4/3)$. This corresponds to the case of a Hamiltonian defined by the SU(3) Casimir operator.

In summary we have found that the dynamical symmetries described above can yield ES determined by the points on the straight lines

$$r_1=\pm\frac{7\sqrt{2}}{8}r_2+1, \tag{4.18a}$$

$$r_2=0. \tag{4.18b}$$

Therefore we have for the exact limits the following results: For the harmonic case the point (0,1); for the SU(3) two points $(\pm4\sqrt{2}/3,-4/3)$ characterizing nuclei with prolate

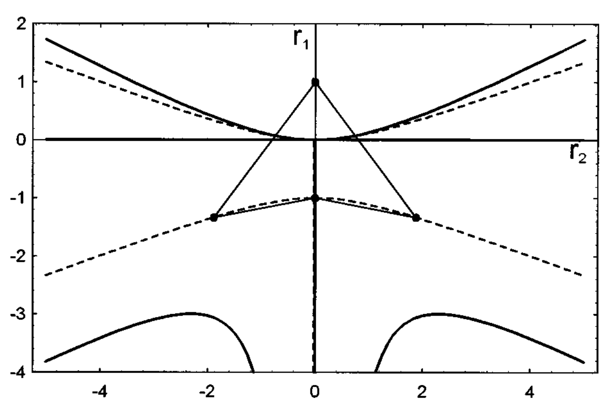

FIG. 8. The IBA-1 dynamical symmetries are described in the essential parameter space, together with the separatrix of the IBA-1. The vertices of the double triangle are indicating the SU(3), SO(6), and U(5) limits. The crossings of the sides of this double triangle and the separatrix show the points where there are shape transitions.

and oblate shapes, respectively; for the γ-unstable limit the point $(0,-1)$. Thus they describe the double triangle indicated in the Fig. 8. The straight lines (4.18a) represent transitions of the ES from spherical to oblate or prolate nuclei while the (4.18b) indicates transitions from spherical to γ-unstable nuclei. Notice also that these results were obtained by considering Hamiltonians associated to the Casimir operators of a single chain of groups (2.3). This happens because the Casimir operators have one body terms in it, which correspond to the harmonic limit.

To have the transition from the γ-unstable to oblate or prolate nuclei, one has to consider Hamiltonians of the form

$$H=\alpha_1\left[\frac{G}{\overline{G}}\right]-\alpha_2\mathcal{L}^2, \tag{4.19}$$

where G and $\overline{G}$ are indicating the Casimir operators of SU(3) and $\overline{\mathrm{SU}}(3)$, respectively. $\mathcal{L}^2$ denotes the Casimir operator of O(6). The ES for these cases are characterized by the parametric equations

$$r_1=\frac{2\eta-4}{-2\eta+3}, \quad r_2=\pm\frac{4\sqrt{2}}{2\eta-3}, \tag{4.20}$$

with $\eta=\alpha_1/\alpha_2$. These represent the following two straight lines:

$$r_1=\pm\frac{\sqrt{2}}{8}r_2-1, \tag{4.21}$$

which are also shown in the Fig. 8.

V. CONCLUSIONS

In this work by means of the coherent states and the catastrophe theory, the separatrix of the IBA-1 general Hamiltonian of one and two body central interactions was constructed. The Maxwell set r_{13}^{+} divides the ES in two regions: For $r_1>r_{13}^{+}$ the ES describe spherical nuclei, while for $r_1<r_{13}^{+}$ they characterize deformed nuclei; prolate when $r_2>0$ and oblate if $r_2<0$. The shape coexistence phenomena between spherical and deformed shapes is present in the ES for r_1 values into the region $0<r_1<r_{12}$, which is stronger in the Maxwell set r_{13}^{+}. Although the ES do not have simultaneously oblate and prolate local minima, for the region $r_{13}^{-}<r_1<0$ the possibility of having fluctuations between oblate and prolate shapes exists. In the IBA-1 model there are at most second order shape transitions according to the thermodynamic Eherenfest classification.

The dynamical symmetries of the IBA-1 are described in the space of control parameters by the double triangle shown in Fig. 8. It was found that the SU(3) limit can describe prolate or oblate shapes, while the groups O(6) and SU(5) appear when $r_2=0$. The intersections between the double triangle and the curves forming the separatrix indicate strong qualitative changes in the ES.

Notice that this formulation allows us to obtain the most general interaction which yields a constant energy surface. This has been used thoroughly in Ref. [18] to separate intrinsic and collective parts of an algebraic Hamiltonian. In particular we found for the O(6) limit that their realizations (4.4a) and (4.4b) produce γ-unstable and constant energy surfaces, respectively. However, we want to emphasize that this is due to the definition used for the coherent states (2.5) and, if a gauge transformation is made in the s and d bosons forming the coherent states, the opposite result could be obtained.

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APPENDIX: CRITICAL POINTS OF THE IBA-1

We separate the critical points and review its behavior in the following regions.

(i) $r_1=0$ and $r_2=0$. In this case, the expression (3.7) becomes $\beta^3=0$ and then one gets that $\beta_c=0$ has triple degeneracy, which as we have shown is the maximum one present in the IBA-1.

(ii) $r_1=0$ and $r_2\neq0$. We have four critical points, $\beta_c=0$ is doubly degenerated for all the possible values of r_2, and

$$\beta_{r_2}^{\pm}=\frac{1}{r_2}\left(-2\pm\sqrt{4+3r_2^2}\right), \tag{A1}$$

where $\beta_{r_2}^{+}$ yields a minimum in the energy surface, while $\beta_{r_2}^{-}$ corresponds to a maximum.

(iii) $r_1\neq0$ and $r_2=0$. This is clearly the γ-unstable case because we have $a_2=0$ and all the dependence in γ in the energy surface disappears. For this case Eq. (3.8) can be immediately solved. We have three critical points, one of them $\beta_c=0$ and the other two are given by

$$\beta_{r_1}^{\pm}=\pm\sqrt{-r_1}. \tag{A2}$$

The $\beta_c=0$ corresponds to a maximum in the energy surface, while the last two to minima.

(iv) $r_1\neq0$ and $r_2\neq0$. For this case, again the $\beta_c=0$ is a critical point and the others are obtained by solving Eq. (3.8) when $\beta\neq0$, which is a cubic equation; that is

$$(r_2\beta^3+4\beta^2-3r_2\beta+4r_1)=0. \tag{A3}$$

By making the change of variable

$$y=\beta+\frac{4}{3r_2}, \tag{A4}$$

one gets the reduced third order algebraic equation

$$y^3+ay+b=0, \tag{A5}$$

where the parameters

$$a=-3-\frac{16}{3r_2^2}, \tag{A6}$$

$$b=4\left(\frac{r_1+1}{r_2}+\frac{32}{27r_2^3}\right). \tag{A7}$$

Notice that the parameters a and b let us to rewrite Eqs. (3.12a) and (3.12b) in the form of the elementary cusp catastrophe.

Following the standard procedure for solving the reduced cubic equation, one defines the discriminant

$$D=4a^3+27b^2, \tag{A8}$$

to find the nature of the roots. If $D>0$ there is only one real root

$$y_1=\left\{\frac{-b}{2}\right\}^{1/3}+\left\{\frac{-b}{2}-\sqrt{D/108}\right\}^{1/3}. \tag{A9}$$

If $D=0$ one has

$$y_1=\left\{\frac{-b}{2}\right\}^{1/3}, \quad y_2=y_3=\left\{\frac{-b}{2}\right\}^{1/3}. \tag{A10}$$

Finally for $D<0$ one has the irreducible case with three real roots

$$y_1=2\sqrt{-a/3}\cos(\phi/3), \tag{A11a}$$

$$y_2=-2\sqrt{-a/3}\cos\left(\frac{\pi-\phi}{3}\right), \tag{A11b}$$

$$y_3=-2\sqrt{-a/3}\cos\left(\frac{\pi+\phi}{3}\right), \tag{A11c}$$

where ϕ is defined by the expression

$$\tan\phi=-\frac{\sqrt{-D}}{\sqrt{27}b}. \tag{A12}$$

[1] P. W. Langhoff, S. T. Epstein, and M. Karplus, Rev. Mod. Phys. **44**, 602 (1972), and references therein.

[2] A. K. Kerman and S. E. Koonin, Ann. Phys. (N.Y.) **100**, 332 (1976).

[3] R. Gilmore, *Catastrophe Theory for Scientists and Engineers* (Wiley, New York, 1981).

[4] J. N. Ginocchio and M. W. Kirson, Phys. Rev. Lett. **44**, 1744 (1980); Nucl. Phys. **A350**, 31 (1980).

[5] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).

[6] J. M. Eisenberg and W. Greiner, *Nuclear Theory, Vol. 1, Nuclear Models* (North-Holland, Amsterdam, 1987).

[7] R. Gilmore, J. Math. Phys. **20**, 891 (1979).

[8] D. H. Feng, R. Gilmore, and S. R. Deans, Phys. Rev. C **23**, 1254 (1981).

[9] A. E. L. Dieperink, O. Scholten, and F. Iachello, Phys. Rev. Lett. **44**, 1747 (1980).

[10] A. E. L. Dieperink and O. Scholten, Nucl. Phys. **A346**, 125 (1980).

[11] O. Castaños, P. Federman, A. Frank, and S. Pittel, Nucl. Phys. **A379**, 61 (1982).

[12] A. Frank, Phys. Rev. C **39**, 652 (1989).

[13] E. L. González, Bachelor thesis, Universidad Nacional Autónoma de México, 1990.

[14] O. Castaños, E. Chacón, A. Frank, and M. Moshinsky, J. Math. Phys. **20**, 35 (1979).

[15] R. L. Hatch and S. Levit, Phys. Rev. C **25**, 614 (1982); E. López-Moreno and O. Castaños, Rev. Mex. Fis. (to be published).

[16] R. Gilmore and Da Hsuan Feng, in *Geometric and Dynamical Properties of the Interacting Boson Model*, edited by F. Iachello (Plenum, New York, 1981), p. 149, and references therein.

[17] O. Castaños, A. Frank, P. O. Hess, and M. Moshinsky, Phys. Rev. C **24**, 1367 (1981).

[18] A. Leviatan, Ph.D. thesis, Weizmann Institute of Science, Rehovot, Israel, 1986; Ann. Phys. (N.Y.) **179**, 201 (1987).

[19] M. W. Kirson and A. Leviatan, Phys. Rev. Lett. **55**, 2846 (1985).