Validity of certain soft photon amplitudes

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Certain soft photon amplitudes which have been recently suggested as alternatives to the usual Low form of the soft photon approximation are studied and it is demonstrated that problems exist in their relation to the corresponding nonradiative amplitude. The nonradiative amplitude, which is an input to soft photon calculations, is in certain cases required to be evaluated outside of its physical phase space region. Also, for the case of two-body identical particle bremsstrahlung processes, the symmetrized or antisymmetrized form of these soft photon amplitudes cannot be written in terms of the symmetrized or antisymmetrized amplitude for the nonradiative process. It is found that the usual Low form of the soft photon theorem is essentially unaffected by these problems. [S0556-2813(96)06011-6]

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I. INTRODUCTION

Bremsstrahlung processes, particularly proton-proton bremsstrahlung, have long been studied as a method of assessing the importance of off-shell effects in low and intermediate energy hadronic scattering. There have been two main theoretical approaches: nonrelativistic potential models [1–7] which include off-shell effects explicitly, and the soft photon approximation [8–13] which is written in terms of only on-shell information about the nonradiative scattering process. Soft photon amplitudes therefore give information about off-shell effects only through any discrepancy between their prediction for the bremsstrahlung spectrum and experimental measurements, and even then there is an ambiguity in that some of the discrepancy could arise from higher order on-shell effects.

For the proton-proton bremsstrahlung process it had been found in the past that the Low soft photon approximation gave a good description of the older data [14], suggesting that off-shell effects are small. The more recent 280 MeV TRIUMF experiment [15] provided measurements not only of photon spectra but also of polarization observables. These data showed some disagreement with the soft photon prediction [12], indicating for the first time the presence of nontrivial off-shell behavior in the pp elastic scattering process.

Although most soft photon applications have used the Low [8] approach, it is well known that the derivation of the soft photon approximation is not unique. Different choices lead to soft photon amplitudes which differ at $\mathcal{O}(k)$. Recently Liou, Timmermans, and Gibson [13] have suggested an alternate form for the soft photon amplitude. The claim is made that this ''Two-u-Two-t-special'' (TUTTS) amplitude provides better agreement with $pp\gamma$ data at all energies than the traditional Low amplitude. This success is contrasted in that paper with the dramatic failure to describe the data of another alternative soft photon amplitude, the ''Two-s-Two-t-special'' (TSTTS) amplitude.

In this paper we shall investigate two problems which can arise in the application of soft photon amplitudes to particular processes. The first of these, which we shall call the phase space problem, concerns the expression of a soft photon amplitude solely in terms of measurable information about the on-shell nonradiative process. The second, the antisymmetrization problem, concerns the inability to write the correctly antisymmetrized $pp\gamma$ soft photon amplitude in terms of the measured, antisymmetric pp elastic amplitude. The usual Low form for the radiative amplitude will be shown to be immune to these difficulties, while the TUTTS and TSTTS amplitudes fall victim to one or both of the problems.

These problems both have analogs in the case of spinless two-body bremsstrahlung processes. We begin by studying the problems in that algebraically simpler context, deriving the spinless forms of the Low, TUTTS, and TSTTS amplitudes in Sec. II. In Sec. III we consider the phase space problem while in Sec. IV we treat the symmetrization of identical particle spinless bremsstrahlung processes. In Sec. V we extend our results to proton-proton bremsstrahlung where the elastic and radiative amplitudes must be written in antisymmetric form, and present some illustrative examples of the problems discussed.

II. SPIN-0 AMPLITUDES

We begin with two-body spin-zero scattering and first review the derivation of the Low [8] form of the soft photon approximation as well as the ''Two-s-Two-t-special'' (TSTTS) and ''Two-u-Two-t-special'' (TUTTS) forms suggested by Liou and collaborators [13]. The problems in which we are interested may be considered within this algebraically simpler spinless framework, and then carried over with little modification to the more physically interesting case of nucleon-nucleon bremsstrahlung.

We define $A(s,t;p_1^2,p_2^2,p_3^2,p_4^2)$ to be the amplitude describing the nonradiative scattering of particles of mass m_1 and m_2 into a final state composed of masses m_3 and m_4 with s and t the usual Mandelstam variables. The p_i are the four-momenta of the various particles and the variables s and t, and others to be defined later, are considered to be functions of these four-momenta. Contact with the physical nonradiative amplitude, which can be evaluated from measured phase shifts, is made by going to the on-shell limit, $p_i^2=m_i^2$. The Low form of the soft photon amplitude may then be constructed as follows. One writes the contribution of radiation from the external charged particles to the radiative amplitude in terms of off-shell evaluations of the nonradiative scattering amplitude $A()$:

$$\begin{aligned}\mathcal{M}^{\mu}_{\rm ext}\epsilon_{\mu}=&\,eQ_3\frac{p_3\cdot\epsilon}{p_3\cdot k}A[\overline{s}+k\cdot(p_3+p_4),\overline{t}-k\cdot(p_1-p_3);m_1^2,m_2^2,m_3^2+2k\cdot p_3,m_4^2]\\&+eQ_4\frac{p_4\cdot\epsilon}{p_4\cdot k}A[\overline{s}+k\cdot(p_3+p_4),\overline{t}-k\cdot(p_2-p_4);m_1^2,m_2^2,m_3^2,m_4^2+2k\cdot p_4]\\&-eQ_1\frac{p_1\cdot\epsilon}{p_1\cdot k}A[\overline{s}-k\cdot(p_1+p_2),\overline{t}-k\cdot(p_1-p_3);m_1^2-2k\cdot p_1,m_2^2,m_3^2,m_4^2]\\&-eQ_2\frac{p_2\cdot\epsilon}{p_2\cdot k}A[\overline{s}-k\cdot(p_1+p_2),\overline{t}-k\cdot(p_2-p_4);m_1^2,m_2^2-2k\cdot p_2,m_3^2,m_4^2].\end{aligned}\tag{1}$$

Here Q_i are the charges of the various particles and k^μ and ϵ^μ are the photon momentum and polarization vector. The nonradiative amplitude $A()$ is written with each of the charged legs in turn taken off-shell due to the emission of the photon, i.e., we use the same functional form as the on-shell, nonradiative amplitude considered as a function of the p_i, but evaluate the p_i at the radiative point satisfying $p_1+p_2=p_3+p_4+k$. We have chosen to express this off-shell behavior in terms of the average Mandlestam variables $\overline{s}\equiv\frac{1}{2}(p_1+p_2)^2+\frac{1}{2}(p_3+p_4)^2$ and $\overline{t}\equiv\frac{1}{2}(p_1-p_3)^2+\frac{1}{2}(p_2-p_4)^2$. Other choices can be made for the variables and such choices are at this stage entirely equivalent but would later give rise to soft photon amplitudes differing by terms of $\mathcal{O}(k)$.

Following Low [8], the occurrences of the nonradiative amplitude $A()$ in this radiative amplitude are expanded in powers of k^μ about the point with explicit dependencies on k^μ set to zero—in our current example we expand about $A(\overline{s},\overline{t};m_1^2,m_2^2,m_3^2,m_4^2)$. Only the leading two powers in this expansion are retained since it has been shown [11,16,17] that the soft photon approximation is ambiguous in its prediction of higher orders in the power of k^μ expansion due to the ambiguity in choice of expansion point. The truncated expansion of Eq. (1) is

$$\begin{aligned}\mathcal{M}^{\mu}_{\rm ext}\epsilon_{\mu}=&\Bigg[eQ_3\frac{p_3\cdot\epsilon}{p_3\cdot k}\left(1+k\cdot(p_3+p_4)\frac{\partial}{\partial\overline{s}}-k\cdot(p_1-p_3)\frac{\partial}{\partial\overline{t}}+2k\cdot p_3\frac{\partial}{\partial m_3^2}\right)\\&+eQ_4\frac{p_4\cdot\epsilon}{p_4\cdot k}\left(1+k\cdot(p_3+p_4)\frac{\partial}{\partial\overline{s}}-k\cdot(p_2-p_4)\frac{\partial}{\partial\overline{t}}+2k\cdot p_4\frac{\partial}{\partial m_4^2}\right)\\&-eQ_1\frac{p_1\cdot\epsilon}{p_1\cdot k}\left(1-k\cdot(p_1+p_2)\frac{\partial}{\partial\overline{s}}-k\cdot(p_1-p_3)\frac{\partial}{\partial\overline{t}}-2k\cdot p_1\frac{\partial}{\partial m_1^2}\right)\\&-eQ_2\frac{p_2\cdot\epsilon}{p_2\cdot k}\left(1-k\cdot(p_1+p_2)\frac{\partial}{\partial\overline{s}}-k\cdot(p_2-p_4)\frac{\partial}{\partial\overline{t}}-2k\cdot p_2\frac{\partial}{\partial m_2^2}\right)\Bigg]A(\overline{s},\overline{t};m_1^2,m_2^2,m_3^2,m_4^2).\end{aligned}\tag{2}$$

This truncated form is no longer gauge invariant. Gauge invariance may be reimposed by the addition of a term $\mathcal{M}^{\mu}_{\rm int}\epsilon_{\mu}$ which is independent of k and is presumed to have its physical origin in photon emission from internal charged lines in the scattering process. The gauge invariance constraint is $(\mathcal{M}^{\mu}_{\rm ext}+\mathcal{M}^{\mu}_{\rm int})k_{\mu}\equiv 0$, which in this case implies an internal contribution of the form

$$\begin{aligned}\mathcal{M}^{\mu}_{\rm int}\epsilon_{\mu}=-&\Bigg[e(Q_1+Q_2+Q_3+Q_4)(p_3+p_4)\cdot\epsilon\frac{\partial}{\partial\overline{s}}+e(Q_1-Q_2-Q_3+Q_4)(p_1-p_3)\cdot\epsilon\frac{\partial}{\partial\overline{t}}\\&+2e\left(Q_1p_1\cdot\epsilon\frac{\partial}{\partial m_1^2}+Q_2p_2\cdot\epsilon\frac{\partial}{\partial m_2^2}+Q_3p_3\cdot\epsilon\frac{\partial}{\partial m_3^2}+Q_4p_4\cdot\epsilon\frac{\partial}{\partial m_4^2}\right)\Bigg]A(\overline{s},\overline{t};m_1^2,m_2^2,m_3^2,m_4^2).\end{aligned}\tag{3}$$

There is an ambiguity here in the choice of internal radiation contribution since any independently gauge invariant term could also be added to the radiative amplitude at this point.

The soft photon amplitude is the sum of the external and internal contributions:

$$\mathcal{M}^{\mu}_{\rm soft}\epsilon_{\mu}\equiv(\mathcal{M}^{\mu}_{\rm ext}+\mathcal{M}^{\mu}_{\rm int})\epsilon_{\mu}$$

$$=\Bigg\{eQ_3\frac{p_3^{\mu}}{p_3\cdot k}+eQ_4\frac{p_4^{\mu}}{p_4\cdot k}-eQ_1\frac{p_1^{\mu}}{p_1\cdot k}-eQ_2\frac{p_2^{\mu}}{p_2\cdot k}+eQ_3\left[\frac{p_4\cdot k}{p_3\cdot k}p_3^{\mu}-p_4^{\mu}\right]\frac{\partial}{\partial\overline{s}}-eQ_3\left[\frac{p_1\cdot k}{p_3\cdot k}p_3^{\mu}-p_1^{\mu}\right]\frac{\partial}{\partial\overline{t}}$$

$$+eQ_4\left[\frac{p_3\cdot k}{p_4\cdot k}p_4^{\mu}-p_3^{\mu}\right]\frac{\partial}{\partial\overline{s}}-eQ_4\left[\frac{p_2\cdot k}{p_4\cdot k}p_4^{\mu}-p_2^{\mu}\right]\frac{\partial}{\partial\overline{t}}+eQ_1\left[\frac{p_2\cdot k}{p_1\cdot k}p_1^{\mu}-p_2^{\mu}\right]\frac{\partial}{\partial\overline{s}}-eQ_1\left[\frac{p_3\cdot k}{p_1\cdot k}p_1^{\mu}-p_3^{\mu}\right]\frac{\partial}{\partial\overline{t}}$$

$$+eQ_2\left[\frac{p_1\cdot k}{p_2\cdot k}p_2^{\mu}-p_1^{\mu}\right]\frac{\partial}{\partial\overline{s}}-eQ_2\left[\frac{p_4\cdot k}{p_2\cdot k}p_2^{\mu}-p_4^{\mu}\right]\frac{\partial}{\partial\overline{t}}\Bigg\}\epsilon_{\mu}A(\overline{s},\overline{t},m_1^2,m_2^2,m_3^2,m_4^2)\,. \tag{4}$$

This amplitude is usually referred to as the Low choice, although it differs slightly from the construction used in Low's original paper [8]. It is distinguished by the choice of a single expansion point for the nonradiative amplitude.

Recently Liou, Timmermans, and Gibson [13] have considered choices of expansion point which limit the explicit k^{μ} dependence of the nonradiative amplitude $A()$ in Eq. (1) to its invariant mass arguments. This removes the derivatives with respect to s-type and t-type variables from the resulting soft photon amplitude. The authors of [13] suggest that this property makes the soft photon approximation more suitable for application to processes thought to be dominated by s or t channel resonances. To obtain their result, one is required to use a different pair of radiative variables for expansion of the external radiation contribution of each charged particle. Equation (1) now takes the form

$$\mathcal{M}^{\mu}\epsilon_{\mu}=eQ_3\frac{p_3\cdot\epsilon}{p_3\cdot k}A(s_{12},t_{24};m_1^2,m_2^2,m_3^2+2k\cdot p_3,m_4^2)$$
$$+eQ_4\frac{p_4\cdot\epsilon}{p_4\cdot k}A(s_{12},t_{13};m_1^2,m_2^2,m_3^2,m_4^2+2k\cdot p_4)$$
$$-eQ_1\frac{p_1\cdot\epsilon}{p_1\cdot k}A(s_{34},t_{24};m_1^2-2k\cdot p_1,m_2^2,m_3^2,m_4^2)$$
$$-eQ_2\frac{p_2\cdot\epsilon}{p_2\cdot k}A(s_{34},t_{13};m_1^2,m_2^2-2k\cdot p_2,m_3^2,m_4^2), \tag{5}$$

where we have defined $s_{12}\equiv(p_1+p_2)^2$, $s_{34}\equiv(p_3+p_4)^2$, $t_{13}\equiv(p_1-p_3)^2$, $t_{24}\equiv(p_2-p_4)^2$ and where again $A()$ is considered an implicit function of the four-momenta p_i. By expanding Eq. (5) about the point $k^{\mu}=0$, i.e., expanding in the explicit k^{μ} dependence, and truncating after the leading two terms in k^{μ}, and reimposing gauge invariance we have the result

$$\mathcal{M}'_{\rm soft}\cdot\epsilon=eQ_3\frac{p_3\cdot\epsilon}{p_3\cdot k}A(s_{12},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$+eQ_4\frac{p_4\cdot\epsilon}{p_4\cdot k}A(s_{12},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$-eQ_1\frac{p_1\cdot\epsilon}{p_1\cdot k}A(s_{34},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$-eQ_2\frac{p_2\cdot\epsilon}{p_2\cdot k}A(s_{34},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)-B^{\mu}\epsilon_{\mu} \tag{6}$$

where, due to gauge invariance, B^{μ} must satisfy the constraint

$$B^{\mu}k_{\mu}=eQ_3A(s_{12},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$+eQ_4A(s_{12},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$-eQ_1A(s_{34},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$-eQ_2A(s_{34},t_{13};m_1^2,m_2^2,m_3^2,m_4^2). \tag{7}$$

In order to obtain the "Two-s-Two-t-special" (TsTts) amplitude of [13] we must choose B^{μ} itself to have the form

$$B^{\mu}\equiv\frac{(p_1+p_2)^{\mu}}{(p_1+p_2)\cdot k}[eQ_3A(s_{12},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$+eQ_4A(s_{12},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$-eQ_1A(s_{34},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)$$
$$-eQ_2A(s_{34},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)]. \tag{8}$$

The TsTts amplitude is then

$$\mathcal{M}_{\text{TsTts}}\cdot\epsilon=eQ_3\left(\frac{p_3\cdot\epsilon}{p_3\cdot k}-\frac{(p_3+p_4)\cdot\epsilon}{(p_3+p_4)\cdot k}\right)A(s_{12},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)+eQ_4\left(\frac{p_4\cdot\epsilon}{p_4\cdot k}-\frac{(p_3+p_4)\cdot\epsilon}{(p_3+p_4)\cdot k}\right)$$
$$\times A(s_{12},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)-eQ_1\left(\frac{p_1\cdot\epsilon}{p_1\cdot k}-\frac{(p_1+p_2)\cdot\epsilon}{(p_1+p_2)\cdot k}\right)$$
$$\times A(s_{34},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)-eQ_2\left(\frac{p_2\cdot\epsilon}{p_2\cdot k}-\frac{(p_1+p_2)\cdot\epsilon}{(p_1+p_2)\cdot k}\right)A(s_{34},t_{13};m_1^2,m_2^2,m_3^2,m_4^2) \quad (9)$$

where we have employed the relation

$$\frac{(p_3+p_4)\cdot\epsilon}{(p_3+p_4)\cdot k}=\frac{(p_1+p_2)\cdot\epsilon}{(p_1+p_2)\cdot k}.$$

The form of B^μ in Eq. (8) is troubling since it appears to have a $1/k^\mu$ dependence and yet is assumed to represent terms which would arise in a perturbative treatment due to radiation from internal charged lines. Such internal radiation is known from perturbation theory arguments to give contributions regular in k^μ as $k^\mu\to 0$ [18]. By expansion of the occurrences of the nonradiative amplitude $A()$ in Eq. (8) about a common point, say $A(s_{12},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)$, one can show that so long as the charge condition $Q_1=Q_3$, $Q_2=Q_4$ is satisfied the apparent $1/k^\mu$ dependence vanishes. This charge condition holds for the elastic scattering processes in which we are interested. For processes where the charge condition is not satisfied the TsTts amplitude would contain unphysical terms in its internal radiation part, and so would be ill defined.

The remaining soft photon amplitude which we shall later use as an example is the ''Two-u-Two-t-special'' (TuTts) amplitude of Ref. [13]. In constructing it we first note that the nonradiative amplitude may be parametrized in terms of the Mandelstam variables u and t, rather than s and t. We define a function $A'(u,t)\equiv A(s,t)$ subject to the constraint $s+t+u=\Sigma_{i=1}^4 m_i^2$. Thus $A'(u,t)$ is just $A(s,t)$ with s replaced by $\Sigma_{i=1}^4 m_i^2-u-t$. For the on-shell elastic process, $A'(u,t)$ is of course identical to $A(s,t)$. However it is a different function of the p_i and so has a different value than $A(s,t)$ when they are evaluated using the radiative p_i instead of the nonradiative ones.

The off-shell external radiation amplitude of Eqs. (1) and (5) which forms the starting point of the soft photon approximation may be written in terms of this function as

$$\mathcal{M}^\mu\epsilon_\mu=eQ_3\frac{p_3\cdot\epsilon}{p_3\cdot k}A'(u_{14},t_{24};m_1^2,m_2^2,m_3^2+2k\cdot p_3,m_4^2)$$
$$+eQ_4\frac{p_4\cdot\epsilon}{p_4\cdot k}A'(u_{23},t_{13};m_1^2,m_2^2,m_3^2,m_4^2+2k\cdot p_4)$$
$$-eQ_1\frac{p_1\cdot\epsilon}{p_1\cdot k}A'(u_{23},t_{24};m_1^2-2k\cdot p_1,m_2^2,m_3^2,m_4^2)$$
$$-eQ_2\frac{p_2\cdot\epsilon}{p_2\cdot k}A'(u_{14},t_{13};m_1^2,m_2^2-2k\cdot p_2,m_3^2,m_4^2), \quad (10)$$

where, as in the TsTts case, a choice of radiative variables has been made which limits the explicit k^μ dependence in $A'()$ to the invariant mass arguments. We have defined $u_{14}\equiv(p_1-p_4)^2$ and $u_{23}\equiv(p_2-p_3)^2$ in the above. To arrive at the corresponding soft photon amplitude one follows an analogous procedure to that used for the TsTts amplitude—the result is

$$\mathcal{M}_{\text{TuTts}}\cdot\epsilon=eQ_3\left(\frac{p_3\cdot\epsilon}{p_3\cdot k}-\frac{(p_2-p_3)\cdot\epsilon}{(p_2-p_3)\cdot k}\right)A'(u_{14},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)+eQ_4\left(\frac{p_4\cdot\epsilon}{p_4\cdot k}-\frac{(p_1-p_4)\cdot\epsilon}{(p_1-p_4)\cdot k}\right)$$
$$\times A'(u_{23},t_{13};m_1^2,m_2^2,m_3^2,m_4^2)-eQ_1\left(\frac{p_1\cdot\epsilon}{p_1\cdot k}-\frac{(p_1-p_4)\cdot\epsilon}{(p_1-p_4)\cdot k}\right)$$
$$\times A'(u_{23},t_{24};m_1^2,m_2^2,m_3^2,m_4^2)-eQ_2\left(\frac{p_2\cdot\epsilon}{p_2\cdot k}-\frac{(p_2-p_3)\cdot\epsilon}{(p_2-p_3)\cdot k}\right)A'(u_{14},t_{13};m_1^2,m_2^2,m_3^2,m_4^2). \quad (11)$$

During the derivation the constraint $Q_1=Q_3$, $Q_2=Q_4$ once again arises when we disallow unphysical contributions to the internal radiation part of the amplitude.

In Secs. III and IV we shall consider certain problems which arise in the application of soft photon amplitudes. The expressions derived in this section—the Low-$(\overline{s,t})$ amplitude of Eq. (4), the TsTts amplitude of Eq. (9), and the TuTts amplitude of Eq. (11)—will serve as instructive examples which demonstrate how these problems arise and in which circumstances they may be avoided.

III. PHASE SPACE PROBLEM

The soft photon approximation is useful in that it provides a relatively simple link between the low energy part of a measured photon spectrum and the measured cross section for the corresponding nonradiative process.

It is therefore reasonable to insist that a useful soft photon amplitude must not require evaluations of the nonradiative cross section at unphysical, unmeasurable points. Unfortunately, as we shall show in this section, this condition is not satisfied by certain soft photon theorems in the literature. Whether the condition is upheld or not depends both upon the choice of radiative phase space variables one uses to parameterize the nonradiative amplitude during the construction of the soft photon amplitude, and upon the masses of the particles involved in the scattering.

The crucial step in the construction of a soft photon amplitude is the expansion of any off-mass-shell nonradiative amplitudes about points where the kinematic variables have had all explicit dependence on photon momentum k^μ removed, i.e., k^μ has been set to zero wherever it appears. We will show that even after such an expansion the value of the nonradiative amplitude may still be required at points outside of the region where it is measurable by experiment.

For example, a particular off-shell nonradiative amplitude appearing in the derivation of the TsTts soft photon amplitude of Eq. (5) is

$$A(s_{34},t_{24};m_1^2-2k\cdot p_1,m_2^2,m_3^2,m_4^2),$$

where we are considering radiation from particle 1. The soft photon prescription states that we make a Taylor series expansion about the point where explicit dependence on k^μ has been set to zero. For our example this point would be

$$A(s_{34},t_{24};m_1^2,m_2^2,m_3^2,m_4^2).$$

This point can be termed on-shell because it is evaluated with $p_1^2=m_1^2$. However, the function $A(s,t;m_1^2,m_2^2,m_3^2,m_4^2)$ is only physically measurable within the region of the (s,t) plane defined by nonradiative kinematics. We have no guarantee that the region (s_{34},t_{24}) obtained by evaluating s_{34},t_{24} at values of the p_i satisfying radiative phase space constraints is contained within this measurable area. Indeed, for most choices of radiative variable pairs and for most sets of masses m_1,m_2,m_3,m_4 defining phase space, we find that the soft photon amplitude does indeed require evaluations of the nonradiative amplitude at points which are not physically measurable.

Since the arguments of this section will depend only on kinematic constraints and not on the spin structure of the scattering process, we employ the spinless formalism of the previous section though we shall be discussing the kinematics applicable to the interactions $\pi^- p\to\pi^- p\gamma$ and $pp\to pp\gamma$.

We begin with a simple example, choosing the masses $m_1=m_3=m_{\pi^-}$ and $m_2=m_4=m_p$, and considering the TsTts soft photon amplitude of Eq. (9) for a laboratory pion energy of 298 MeV, corresponding to a typical experiment [19]. The hashed region of Fig. 1 shows the physically accessible part of the (s,t) plane—the amplitude $A(s,t)$ would be known over this region if the elastic process $\pi^- p\to\pi^- p$ had been measured at all scattering angles and for interaction energies up to $\sqrt{s}\approx 1.35$ GeV. The TsTts soft photon amplitude calls for the evaluation of the nonradiative amplitude at four points, one of these being $A(s_{34},t_{24})$. In order to calculate this soft photon amplitude for initial state pion kinetic energy of 298 MeV in the laboratory frame and for all allowed energies and orientations of the final state particles, it turns out that we require the function $A()$ for values of s_{34},t_{24} corresponding to the outlined region shown in Fig. 1. This clearly extends far outside of the region where the nonradiative amplitude $A()$ is measurable. Thus, for certain kinematics, the TsTts soft photon approximation to the bremsstrahlung amplitude will not be calculable unless one is prepared to make a model-dependent extrapolation of the nonradiative amplitude $A()$ outside of its measurable region. The introduction of any such model dependence would remove the usefulness of the soft photon approximation as an unambiguous method of relating the $\pi^- p\to\pi^- p$ process to the radiative process $\pi^- p\to\pi^- p\gamma$. The evaluation point (s_{34},t_{13}) also suffers from this problem. The remaining two points in the TsTts amplitude, (s_{12},t_{13}) and (s_{12},t_{24}), may be shown to lie inside the measurable region of nonradiative phase space for any elastic scattering process.

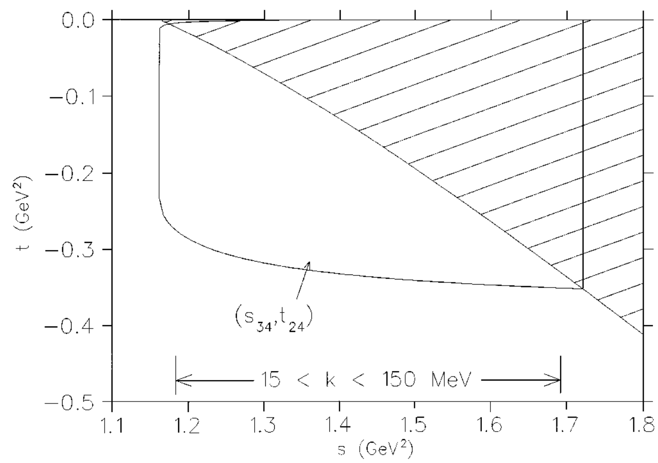

FIG. 1. The hashed region shows the physical region of phase space for the elastic process of $\pi^- p$ scattering at pion beam kinetic energy 298 MeV (i.e., $m_1=m_3=m_{\pi^-}$, $m_2=m_4=m_{\text{proton}}$). The outlined area shows the region covered by the radiative phase space point (s_{34},t_{24}). This area extends far outside of the physical elastic region. The range of s_{34} marked $15<k<150$ MeV is the approximate region of radiative phase space studied in the $\pi^- p\gamma$ experiment of Ref. [19].

We can see intuitively how this problem arises. The quantities s_{12} and s_{34} are related by

$$s_{34}=s_{12}-k\cdot(p_1+p_2+p_3+p_4).$$

As photon energy increases s_{34} becomes progressively smaller than s_{12}. Even though a range (s_{12},t_{24}) as defined by radiative kinematics might lie within the nonradiative region for $s=s_{12}$, if we take $s=s_{34}$ we find the allowed range of the nonradiative variable t to be much smaller. The points (s_{34},t_{24}) may not be contained within this nonradiative physical region.

This problem is by no means isolated to the one example shown above. For the case of proton-proton scattering we can also make the same comparison between the physical elastic region of phase space and the regions mapped out by

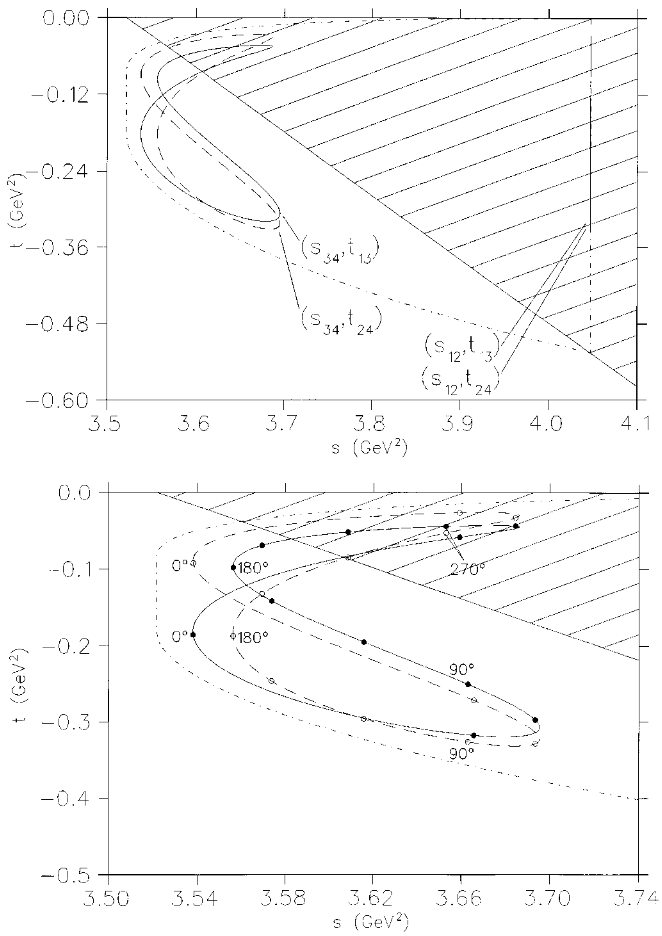

FIG. 2. The hashed area is the physical region of nonradiative phase space for proton-proton elastic scattering. The area enclosed by the dotted line is the region mapped out by the points (s_{34},t_{13}) and (s_{34},t_{24}) for the corresponding bremsstrahlung process, operating at proton beam energy of 280 MeV—this corresponds to the beam energy in the experiment of Ref. [15]. The particular regions of radiative phase space studied in that experiment are also shown—to reproduce the experiment's kinematics we fix the outgoing protons at angles of 12.4° and 12° to the beamline in the laboratory frame. The points (s_{34},t_{13}) and (s_{34},t_{24}) are seen to be outside of the measurable region of elastic phase space for most photon angles. The lower plot is an expansion of the upper and shows the photon angle measured in the target frame for the regions (s_{34},t_{13}) and (s_{34},t_{24}). To guide the eye points have been marked at 30° intervals in photon lab angle along these trajectories.

the radiative variable pairs needed to evaluate the TSTTS soft photon amplitude. The results of this comparison are shown in Fig. 2 for a typical set of kinematics corresponding to the TRIUMF experiment [15]. In this case also, the parts of the soft photon amplitude employing the expansion points $A(s_{12},t_{13})$ or $A(s_{12},t_{24})$ would require only measurable information about the nonradiative, elastic amplitude. The parts of the radiative amplitude using the points $A(s_{34},t_{13})$ or $A(s_{34},t_{24})$ would, however, require unphysical information and would be incalculable unless one resorted to model-dependent extrapolations of the elastic amplitude.

This difficulty might be avoided by considering only certain experimental kinematics for the bremsstrahlung process. This is clearly unsatisfactory, however, particularly when we note that most modern experiments cover kinematic ranges for which the points (s_{34},t_{13}) or (s_{34},t_{24}) lie outside the region accessible in the nonradiative process. For example the $\pi^- p\gamma$ experiment of Ref. [19] covered the majority of radiative phase space, with photon energy in the range 15–150 MeV being measured. This region is shown in Fig. 1. The 200 MeV pp bremsstrahlung experiment of Ref. [14] measured the photon spectrum as a function of angle with outgoing proton angles fixed at 16.4° on either side of the beam axis in the laboratory frame and with all particles coplanar. For these kinematics the result is analogous to that of Fig. 2 with the resulting trajectories through radiative phase space of the points $A(s_{34},t_{13})$ and $A(s_{34},t_{24})$ falling outside of the physical region of phase space for the elastic pp process.

In contrast to the TSTTS soft photon amplitude the Low-$(\overline{s},\overline{t})$ soft photon amplitude of Eq. (4) relies on a single evaluation of the nonradiative amplitude, at $A(\overline{s},\overline{t})$. For elastic scattering processes such as $\pi^- p \rightarrow \pi^- p$ our numeric studies have found that the physical region of the radiative variables $(\overline{s},\overline{t})$ can fall slightly outside of the measurable nonradiative region of phase space. For practical purposes, however, only a very tiny region of radiative phase space must be excluded in one's model-independent calculation of the bremsstrahlung process when using the Low-$(\overline{s},\overline{t})$ amplitude.

For the special case of identical particle scattering the radiative $(\overline{s},\overline{t})$ region is entirely contained within the physical nonradiative (s,t) region. This is due to the fact that the Mandelstam variables $\overline{s},\overline{t},\overline{u}$ of a radiative identical particle scattering process satisfy the same phase space constraints as the s,t,u of the corresponding nonradiative process. For the nonradiative process we have the familiar constraints for equal mass, two-body elastic scattering

$$s+t+u=4m^2,$$

$$s \geqslant 4m^2,$$

$$0 \geqslant t,u \geqslant -(s-4m^2). \qquad (12)$$

For the radiative process we can use four-momentum conservation to write

$$(k^\mu)^2=(p_1^\mu+p_2^\mu-p_3^\mu-p_4^\mu)^2 \Rightarrow \overline{s}+\overline{t}+\overline{u}=4m^2, \qquad (13)$$

where $\overline{u} \equiv \frac{1}{2}(p_1-p_4)^2+\frac{1}{2}(p_2-p_3)^2$. We also have the constraints,

$$\overline{s} \geqslant 4m^2,$$

$$0 \geqslant \overline{t},\overline{u} \geqslant -(\overline{s}-4m^2). \qquad (14)$$

The threshold condition on $\overline{s}$ is clear; however, the $\overline{t},\overline{u}$ constraints require some explanation. For identical particle scattering it may be shown, by considering the appropriate rest frames, that the variables t_{13}, and symmetrically t_{24}, have zero as their upper bounds. Thus the average $\overline{t}=\frac{1}{2}(t_{13}+t_{24})$ is also bounded above by zero. Putting $\overline{t} \leqslant 0$ into Eq. (13) we find the lower bound for $\overline{u}$; $\overline{u} \geqslant -(\overline{s}-4m^2)$. Finally, noting that the constraints on $\overline{t}$ and on $\overline{u}$ must be the same for identical particle scattering through the symmetry of the kinematics under the interchange of final state particles, we have the result of Eq. (14). Our conclusion is that for identical particle scattering, all points $(\overline{s},\overline{t})$ defined by radiative phase space constraints lie within the allowed (s,t) region of the corresponding nonradiative process. Numeric studies of the phase space regions confirm this conclusion.

From numeric studies of the kinematics for the interactions $\pi^- p \to \pi^- p$, $pp \to pp$ and the corresponding bremsstrahlung processes, it appears that the points used in the TuTts soft photon amplitude lie inside the measurable nonradiative region, at least for the kinematic conditions relevant to existing experiments. This implies that, for these interactions, the TuTts amplitude does not suffer from the phase space problem.

IV. SYMMETRIZATION PROBLEM

In Sec. V we will show that a problem exists with correctly antisymmetrizing the spin-$\frac{1}{2}$ TsTts and TuTts amplitudes of Ref. [13]. To illustrate the source of this problem we consider in this section the quite analogous and algebraically simpler symmetrization of the spin-0 TsTts and TuTts amplitudes.

The spin-0 amplitudes given in Sec. II would have to be explicitly symmetrized if applied to the case of identical particle scattering. Upon attempting to symmetrize the TsTts and TuTts amplitudes we find that the connection to measurable nonradiative scattering data is lost. More specifically, the symmetrized radiative TsTts and TuTts amplitudes cannot be written in terms of the symmetrized nonradiative amplitudes, and thus cannot be evaluated directly from experimental information on the nonradiative process. This problem is quite independent of the phase space problem discussed previously. The Low amplitude, employing a single choice of Taylor expansion point at $(\overline{s},\overline{t})$, is also treated for comparison. It is found that the symmetrized $(\overline{s},\overline{t})$ soft photon amplitude may be written in terms of the measurable, symmetrized nonradiative scattering amplitude. This is due to a special property of the $(\overline{s},\overline{t})$ variables.

We denote the unsymmetrized nonradiative scattering amplitude by $A(s,t)$, where we have suppressed the invariant mass arguments of this function. The symmetrized amplitude, which we obtain by adding in the amplitude with $p_3 \leftrightarrow p_4$, is then

$$A^S(s,t) \equiv A(s,t) + A(s,u) = A(s,t) + A(s,4m^2 - s - t). \tag{15}$$

The unsymmetrized Low-$(\overline{s},\overline{t})$ amplitude may be written [10]

$$\begin{aligned}\mathcal{M}_{(\overline{s},\overline{t})}\cdot\epsilon = eQ\Bigg\{&\left[\frac{p_3\cdot\epsilon}{p_3\cdot k} + \mathcal{D}^\mu(p_3)\frac{\partial}{\partial p_3^\mu}\right]A(\overline{s},\overline{t}) + \left[\frac{p_4\cdot\epsilon}{p_4\cdot k} + \mathcal{D}^\mu(p_4)\frac{\partial}{\partial p_4^\mu}\right]A(\overline{s},\overline{t})\\ &- \left[\frac{p_1\cdot\epsilon}{p_1\cdot k} - \mathcal{D}^\mu(p_1)\frac{\partial}{\partial p_1^\mu}\right]A(\overline{s},\overline{t}) - \left[\frac{p_2\cdot\epsilon}{p_2\cdot k} - \mathcal{D}^\mu(p_2)\frac{\partial}{\partial p_2^\mu}\right]A(\overline{s},\overline{t})\Bigg\}\end{aligned} \tag{16}$$

where, following Ref. [10], we introduce the notation

$$\mathcal{D}^\mu(p) \equiv \frac{p\cdot\epsilon}{p\cdot k}k^\mu - \epsilon^\mu. \tag{17}$$

This is easily related to the usual form of Eq. (4) for the amplitude by noting that

$$\frac{\partial}{\partial p_1^\mu} = \frac{\partial\overline{s}}{\partial p_1^\mu}\frac{\partial}{\partial\overline{s}} + \frac{\partial\overline{t}}{\partial p_1^\mu}\frac{\partial}{\partial\overline{t}} = (p_{1\mu} + p_{2\mu})\frac{\partial}{\partial\overline{s}} + (p_{1\mu} - p_{3\mu})\frac{\partial}{\partial\overline{t}} \tag{18}$$

with similar expressions for the other derivatives.

The symmetrized amplitude is then $\mathcal{M}^S\cdot\epsilon \equiv \mathcal{M}_{(12\to34)}\cdot\epsilon + \mathcal{M}_{(12\to43)}\cdot\epsilon$,

$$\begin{aligned}\mathcal{M}^S_{(\overline{s},\overline{t})}\cdot\epsilon = eQ\Bigg\{&\left[\frac{p_3\cdot\epsilon}{p_3\cdot k} + \mathcal{D}^\mu(p_3)\frac{\partial}{\partial p_3^\mu}\right]A(\overline{s},\overline{t}) + \left[\frac{p_4\cdot\epsilon}{p_4\cdot k} + \mathcal{D}^\mu(p_4)\frac{\partial}{\partial p_4^\mu}\right]A(\overline{s},\overline{u}) + \left[\frac{p_4\cdot\epsilon}{p_4\cdot k} + \mathcal{D}^\mu(p_4)\frac{\partial}{\partial p_4^\mu}\right]A(\overline{s},\overline{t})\\ &+ \left[\frac{p_3\cdot\epsilon}{p_3\cdot k} + \mathcal{D}^\mu(p_3)\frac{\partial}{\partial p_3^\mu}\right]A(\overline{s},\overline{u}) - \left[\frac{p_1\cdot\epsilon}{p_1\cdot k} - \mathcal{D}^\mu(p_1)\frac{\partial}{\partial p_1^\mu}\right]A(\overline{s},\overline{t}) - \left[\frac{p_1\cdot\epsilon}{p_1\cdot k} - \mathcal{D}^\mu(p_1)\frac{\partial}{\partial p_1^\mu}\right]A(\overline{s},\overline{u})\\ &- \left[\frac{p_2\cdot\epsilon}{p_2\cdot k} - \mathcal{D}^\mu(p_2)\frac{\partial}{\partial p_2^\mu}\right]A(\overline{s},\overline{t}) - \left[\frac{p_2\cdot\epsilon}{p_2\cdot k} - \mathcal{D}^\mu(p_2)\frac{\partial}{\partial p_2^\mu}\right]A(\overline{s},\overline{u})\Bigg\}.\end{aligned} \tag{19}$$

A common factor in this expression is

$$A(\overline{s},\overline{t}) + A(\overline{s},\overline{u}).$$

In order to have the soft photon amplitude solely a function of the measurable, symmetric part of the nonradiative amplitude we must be able to write this factor in terms of the symmetric function $A^S()$. From Eq. (15) it is clear that since the relation $\overline{s}+\overline{t}+\overline{u}=4m^2$ holds among the radiative variables we can write

$$A(\overline{s},\overline{t}) + A(\overline{s},\overline{u}) = A(\overline{s},\overline{t}) + A(\overline{s},4m^2 - \overline{s} - \overline{t}) = A^S(\overline{s},\overline{t}). \tag{20}$$

The symmetrized radiative amplitude now takes on the form of the unsymmetrized amplitude, but with $A(\overline{s,t})$ replaced by the measurable, symmetrized nonradiative amplitude $A^S(\overline{s,t})$:

$$\mathcal{M}^S_{(\overline{s,t})}\cdot\epsilon=eQ\Bigg\{\left[\frac{p_3\cdot\epsilon}{p_3\cdot k}+\mathcal{D}^\mu(p_3)\frac{\partial}{\partial p_3^\mu}\right]A^S(\overline{s,t})+\left[\frac{p_4\cdot\epsilon}{p_4\cdot k}+\mathcal{D}^\mu(p_4)\frac{\partial}{\partial p_4^\mu}\right]A^S(\overline{s,t})-\left[\frac{p_1\cdot\epsilon}{p_1\cdot k}-\mathcal{D}^\mu(p_1)\frac{\partial}{\partial p_1^\mu}\right]A^S(\overline{s,t})-\left[\frac{p_2\cdot\epsilon}{p_2\cdot k}-\mathcal{D}^\mu(p_2)\frac{\partial}{\partial p_2^\mu}\right]A^S(\overline{s,t})\Bigg\}. \qquad (21)$$

This procedure of correctly symmetrizing the radiative amplitude by simply replacing $A()\to A^S()$ works only for the Low-$(\overline{s,t})$ case, due to the relationship $\overline{s}+\overline{t}+\overline{u}=4m^2$ which holds only for this specific Low choice of variables. We will now show explicitly that such a replacement in the TsTts or TuTts amplitudes does not work and that these amplitudes cannot be expressed in terms of symmetrized nonradiative amplitudes.

For the radiative process, the unsymmetrized TsTts amplitude of Eq. (9) is

$$\mathcal{M}^\mu_{\rm TsTts}=eQ\Bigg(\left[\frac{p_3^\mu}{p_3\cdot k}-\frac{(p_3+p_4)^\mu}{(p_3+p_4)\cdot k}\right]A(s_{12},t_{24})+\left[\frac{p_4^\mu}{p_4\cdot k}-\frac{(p_3+p_4)^\mu}{(p_3+p_4)\cdot k}\right]A(s_{12},t_{13})-\left[\frac{p_1^\mu}{p_1\cdot k}-\frac{(p_1+p_2)^\mu}{(p_1+p_2)\cdot k}\right]A(s_{34},t_{24})-\left[\frac{p_2^\mu}{p_2\cdot k}-\frac{(p_1+p_2)^\mu}{(p_1+p_2)\cdot k}\right]A(s_{34},t_{13})\Bigg). \qquad (22)$$

We define the symmetrized amplitude $\mathcal{M}^{S\mu}_{\rm TsTts}\equiv\mathcal{M}^\mu_{(12\to34)}+\mathcal{M}^\mu_{(12\to43)}$

$$\mathcal{M}^{S\mu}_{\rm TsTts}=eQ\Bigg(\left[\frac{p_3^\mu}{p_3\cdot k}-\frac{(p_3+p_4)^\mu}{(p_3+p_4)\cdot k}\right][A(s_{12},t_{24})+A(s_{12},u_{14})]+\left[\frac{p_4^\mu}{p_4\cdot k}-\frac{(p_3+p_4)^\mu}{(p_3+p_4)\cdot k}\right][A(s_{12},t_{13})+A(s_{12},u_{23})]-\left[\frac{p_1^\mu}{p_1\cdot k}-\frac{(p_1+p_2)^\mu}{(p_1+p_2)\cdot k}\right][A(s_{34},t_{24})+A(s_{34},u_{23})]-\left[\frac{p_2^\mu}{p_2\cdot k}-\frac{(p_1+p_2)^\mu}{(p_1+p_2)\cdot k}\right][A(s_{34},t_{13})+A(s_{34},u_{14})]\Bigg). \qquad (23)$$

We define symmetrized functions in analogy to the nonradiative case:

$$A^{S_1}(s_{34},t_{24},u_{23})\equiv A(s_{34},t_{24})+A(s_{34},u_{23}),$$
$$A^{S_2}(s_{34},t_{13},u_{14})\equiv A(s_{34},t_{13})+A(s_{34},u_{14}),$$
$$A^{S_3}(s_{12},t_{24},u_{14})\equiv A(s_{12},t_{24})+A(s_{12},u_{14}),$$
$$A^{S_4}(s_{12},t_{13},u_{23})\equiv A(s_{12},t_{13})+A(s_{12},u_{23}). \qquad (24)$$

Notice that these functions are not the same as the symmetric nonradiative function $A^S(s,t)$ which would be measured in the nonradiative process. This is because an internal constraint similar to that for the nonradiative phase space variables, $s+t+u=4m^2$, does not hold for s_{34}, t_{24}, and u_{23}, or for the other sets of radiative variables which appear as arguments in the A^{S_i}. Instead one has relations like $s_{34}+t_{24}+u_{23}=4m^2-2k\cdot p_1$. Thus direct replacement of, for example, $A^{S_1}(s_{34},t_{24},u_{23})$ by $A^S(s_{34},t_{24})$ will give an error of the form

$$A^{S_1}(s_{34},t_{24},u_{23})-A^S(s_{34},t_{24})=-2p_1\cdot k\frac{\partial}{\partial u}A(s_{34},u)\big|_{u=u_{23}}+\mathcal{O}(k^2). \qquad (25)$$

One would naively expect this $\mathcal{O}(k)$ error which is being introduced to give rise to an $\mathcal{O}(k/k)$ or $\mathcal{O}(1)$ error in the amplitude $\mathcal{M}^{S\mu}_{\rm TsTts}$. Due to a cancellation in this leading order between the four $A^{S_i}()$ terms appearing in $\mathcal{M}^{S\mu}_{\rm TsTts}$ the error introduced by these replacements is instead only of $\mathcal{O}(k)$. Thus in this case the error is of the same order as other terms dropped in the derivation of the amplitude.

For the TuTts amplitude another difficulty arises in trying to symmetrize. The unsymmetrized amplitude is given by Eq. (11):

$$\mathcal{M}^\mu_{\rm TuTts}=eQ\Bigg(\left[\frac{p_3^\mu}{p_3\cdot k}-\frac{(p_2-p_3)^\mu}{(p_2-p_3)\cdot k}\right]A'(u_{14},t_{24})+\left[\frac{p_4^\mu}{p_4\cdot k}-\frac{(p_1-p_4)^\mu}{(p_1-p_4)\cdot k}\right]A'(u_{23},t_{13})-\left[\frac{p_1^\mu}{p_1\cdot k}-\frac{(p_1-p_4)^\mu}{(p_1-p_4)\cdot k}\right]A'(u_{23},t_{24})-\left[\frac{p_2^\mu}{p_2\cdot k}-\frac{(p_2-p_3)^\mu}{(p_2-p_3)\cdot k}\right]A'(u_{14},t_{13})\Bigg) \qquad (26)$$

and we define the symmetrized amplitude as $\mathcal{M}^{S\mu}_{\rm TuTts}\equiv\mathcal{M}^\mu_{(1,2\to3,4)}+\mathcal{M}^\mu_{(1,2\to4,3)}$ so that

$$\mathcal{M}^{S\mu}_{\mathrm{TuTTs}}=eQ\Bigg(\left[\frac{p_3^\mu}{p_3\cdot k}-\frac{(p_2-p_3)^\mu}{(p_2-p_3)\cdot k}\right]A'(u_{14},t_{24})+\left[\frac{p_3^\mu}{p_3\cdot k}-\frac{(p_1-p_3)^\mu}{(p_1-p_3)\cdot k}\right]A'(t_{24},u_{14})+\left[\frac{p_4^\mu}{p_4\cdot k}-\frac{(p_1-p_4)^\mu}{(p_1-p_4)\cdot k}\right]A'(u_{23},t_{13})$$
$$+\left[\frac{p_4^\mu}{p_4\cdot k}-\frac{(p_2-p_4)^\mu}{(p_2-p_4)\cdot k}\right]A'(t_{13},u_{23})-\left[\frac{p_1^\mu}{p_1\cdot k}-\frac{(p_1-p_4)^\mu}{(p_1-p_4)\cdot k}\right]A'(u_{23},t_{24})-\left[\frac{p_1^\mu}{p_1\cdot k}-\frac{(p_1-p_3)^\mu}{(p_1-p_3)\cdot k}\right]A'(t_{24},u_{23})$$
$$-\left[\frac{p_2^\mu}{p_2\cdot k}-\frac{(p_2-p_3)^\mu}{(p_2-p_3)\cdot k}\right]A'(u_{14},t_{13})-\left[\frac{p_2^\mu}{p_2\cdot k}-\frac{(p_2-p_4)^\mu}{(p_2-p_4)\cdot k}\right]A'(t_{13},u_{14})\Bigg). \quad (27)$$

Consistent with our previous definition of $A'()$ we have, for example, $A'(t_{24},u_{14})=A(\Sigma_i m_i^2-t_{24}-u_{14},u_{14})$ where $A(s,t)$ is the nonsymmetrized amplitude for the nonradiative process. Again there is difficulty because the radiative amplitude is not expressed in terms of the symmetrized nonradiative amplitude, which is all that can be measured. The terms in the square brackets of the form $p_i/k\cdot p_i$ can in fact be factored leaving a correctly symmetrized combination of the A' as an overall factor. However, the problem arises with the $(p_i-p_j)^\mu/k\cdot(p_i-p_j)$ terms, which were added to make the amplitude gauge invariant. The momenta are such that these terms cannot be factored leaving just the symmetrized amplitude.

The procedure followed in Ref. [13] is to take for the symmetrized radiative amplitude the original unsymmetrized TuTTs amplitude with the functions $A'()$ replaced by their counterparts from the symmetrized elastic process. This prescription would give the result

$$eQ\Bigg(\left[\frac{p_3^\mu}{p_3\cdot k}-\frac{(p_2-p_3)^\mu}{(p_2-p_3)\cdot k}\right]A'^S(u_{14},t_{24})$$
$$+\left[\frac{p_4^\mu}{p_4\cdot k}-\frac{(p_1-p_4)^\mu}{(p_1-p_4)\cdot k}\right]A'^S(u_{23},t_{13})$$
$$-\left[\frac{p_1^\mu}{p_1\cdot k}-\frac{(p_1-p_4)^\mu}{(p_1-p_4)\cdot k}\right]A'^S(u_{23},t_{24})$$
$$-\left[\frac{p_2^\mu}{p_2\cdot k}-\frac{(p_2-p_3)^\mu}{(p_2-p_3)\cdot k}\right]A'^S(u_{14},t_{13})\Bigg) \quad (28)$$

where, for example,

$$A'^S(u_{23},t_{24})=A'(u_{23},t_{24})+A'(t_{24},u_{23})$$
$$=A(4m^2-u_{23}-t_{24},t_{24})$$
$$+A(4m^2-u_{23}-t_{24},u_{23})$$
$$=A^S(4m^2-u_{23}-t_{24},t_{24}). \quad (29)$$

This expresses the result in terms of measurable nonradiative amplitudes, but is not completely symmetric under the interchange of particle labels 3 and 4 because of the kinematic factors multiplying A^S.

A detailed comparison between the symmetrized amplitude of Eq. (27) and the form of Eq. (28) shows them to be unequal. They differ in this case by terms of $\mathcal{O}(k/k)$. Since the $\mathcal{O}(k/k)$ terms are uniquely determined in a soft photon approach and can come only from the diagrams with radiation from external legs [18], this must mean that this prescription in effect adds in some $\mathcal{O}(k/k)$ terms which are not allowed by the analyticity requirements of the soft photon approach. We have found no way to express the correct expression, Eq. (27), solely in terms of the measurable symmetrized nonradiative amplitude, $A^S(s,t)$, other than simply expanding about $\overline{u},\overline{t}$ and thus recovering the original Low amplitude Eq. (4), up to corrections of $\mathcal{O}(k)$.

Since the problem arises with the $(p_i-p_j)^\mu/k\cdot(p_i-p_j)$ terms which were added to make the amplitude gauge invariant, an alternative approach would be to try to find some different gauge term to add which does not suffer from these problems [20]. To explore this possibility rewrite Eq. (26) as

$$\overline{\mathcal{M}}^{\mu}_{\mathrm{TuTTs}}=eQ\Bigg(\left[\frac{p_3^\mu}{p_3\cdot k}-\Delta_3^\mu\right]A'(u_{14},t_{24})+\left[\frac{p_4^\mu}{p_4\cdot k}-\Delta_4^\mu\right]$$
$$\times A'(u_{23},t_{13})-\left[\frac{p_1^\mu}{p_1\cdot k}-\Delta_1^\mu\right]A'(u_{23},t_{24})$$
$$-\left[\frac{p_2^\mu}{p_2\cdot k}-\Delta_2^\mu\right]A'(u_{14},t_{13})\Bigg). \quad (30)$$

Here the Δ_i^μ are structures of the general form $V^\mu/k\cdot V$, where V is some vector, and originate in the term added to ensure gauge invariance. In accord with general principles this gauge term must be $\mathcal{O}(1)$ and cannot contain terms $\mathcal{O}(k/k)$. To determine the conditions imposed on the Δ_i^μ by this analyticity requirement we expand the $A'()$ about the single point $\overline{s},\overline{t},\overline{u}$. The result is that all the Δ_i^μ, or more precisely, all $\epsilon\cdot\Delta_i$, must be equal. One can see from Eq. (26) that for the $\mathcal{M}^{\mu}_{\mathrm{TuTTs}}$ amplitude this is satisfied.

Next we define Δ_i' to be Δ_i with $p_3\leftrightarrow p_4$ and symmetrize the amplitude by adding in a piece with $p_3\leftrightarrow p_4$ as was done in going from Eq. (26) to Eq. (27). Then the requirement that we be able to express the full symmetrized amplitude in terms of the symmetrized elastic amplitudes of Eq. (29), which are the measurable quantities, requires that $\Delta_1=\Delta_1'$, $\Delta_2=\Delta_2'$, $\Delta_3=\Delta_4'$, and $\Delta_4=\Delta_3'$. This condition is not satisfied by the Δ_i of the symmetrized TuTTs amplitude of Eq. (27) and that is the reason that the TuTTs amplitude cannot be written in a correctly symmetrized form.

Putting these requirements on the Δ_i together, we find that the Δ_i must all be the same and must be symmetric in the interchange $p_3\leftrightarrow p_4$. That is not true for the Δ_i of the TuTTs amplitude, but it is easy to find such Δ_i. For example, consider the following four Δ_i:

$$\frac{p_3^\mu + p_4^\mu}{k\cdot(p_3+p_4)},$$

$$\frac{1}{4}\left(\frac{p_1^\mu}{k\cdot p_1}+\frac{p_2^\mu}{k\cdot p_2}+\frac{p_3^\mu}{k\cdot p_3}+\frac{p_4^\mu}{k\cdot p_4}\right),$$

$$\frac{1}{2}\left(\frac{p_3^\mu - p_4^\mu}{k\cdot(p_3-p_4)}+\frac{p_1^\mu - p_2^\mu}{k\cdot(p_1-p_2)}\right),$$

$$\frac{1}{2}\left(\frac{p_1^\mu - p_4^\mu}{k\cdot(p_1-p_4)}+\frac{p_1^\mu - p_3^\mu}{k\cdot(p_1-p_3)}\right). \tag{31}$$

Any of these, when used in the symmetrized version of Eq. (30), will produce an amplitude $\overline{\mathcal{M}}^{\mu}_{\text{TuTts}}$ which is gauge invariant, satisfies the analyticity and other requirements of a soft photon theorem, is properly symmetric in the interchange of identical particles so that only the measurable symmetrized elastic amplitudes are required, and is expressed in terms of the same kinematic variables as the original $\mathcal{M}^{\mu}_{\text{TuTts}}$. All of these amplitudes will be equivalent, as is generally true of various different soft photon amplitudes, in the sense that they will differ by terms which are $\mathcal{O}(k)$.

The arguments above have shown that the TSTTS and original TUTTS amplitudes for a spin-0 scattering process cannot be made correctly symmetric under interchange of identical particles while maintaining the necessary link to the symmetric nonradiative amplitude. For the TSTTS case we found the failure in symmetrization to arise at $\mathcal{O}(k)$ in the amplitude, but for the TUTTS case it arises at $\mathcal{O}(k/k)$. In the next section we extend the derivation of the Low-$(\overline{s,t})$, TSTTS, and TUTTS amplitudes to spin-$\frac{1}{2}$ identical particle scattering, and consider the calculation of proton-proton bremsstrahlung. We thus show that exactly the same problems which arose in this section in symmetrizing a spin-0 amplitude arise also in antisymmetrizing a spin-$\frac{1}{2}$ amplitude.

V. EXTENSION TO SPIN- $\frac{1}{2}$

In this section we will consider the proton-proton bremsstrahlung process. This requires the extension of the spinless formalism of Sec. II to the scattering of spin- $\frac{1}{2}$ particles. Care must also be taken to write the pp elastic and the pp bremsstrahlung amplitudes such that they are antisymmetric under interchange of final state protons.

The unsymmetrized amplitude for elastic proton-proton scattering may be written

$$A[12\to 34]\equiv \sum_{\alpha=1}^{5} F_\alpha(s,t)[\overline{u_3}t_\alpha u_1][\overline{u_4}t^\alpha u_2], \tag{32}$$

where

$$t_\alpha \equiv \left\{1, \frac{1}{\sqrt{2}}\sigma^{\mu\nu}, i\gamma_5\gamma^\mu, \gamma^\mu, \gamma_5\right\}$$

with summation over the Lorentz indices of the t_α being implied. The antisymmetrized pp elastic amplitude is then defined as

$$\begin{aligned} A^A(s,t) &\equiv A[12\to 34] - A[12\to 43] \\ &= \sum_{\alpha=1}^{5} F_\alpha(s,t)[\overline{u_3}t_\alpha u_1][\overline{u_4}t^\alpha u_2] \\ &\quad - \sum_{\alpha=1}^{5} F_\alpha(s,u)[\overline{u_4}t_\alpha u_1][\overline{u_3}t^\alpha u_2]. \end{aligned} \tag{33}$$

Using the Fierz relation [21]

$$(t_\alpha)_{\phi\sigma}(t^\alpha)_{\tau\nu} = \sum_{\beta=1}^{5} \mathcal{C}_{\alpha\beta}(t_\beta)_{\phi\nu}(t^\beta)_{\tau\sigma} \tag{34}$$

with

$$\mathcal{C}_{\alpha\beta} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & -2 & 0 & 0 & 6 \\ 4 & 0 & -2 & 2 & -4 \\ 4 & 0 & 2 & -2 & -4 \\ 1 & 1 & -1 & -1 & 1 \end{pmatrix}, \tag{35}$$

the antisymmetrized expression $A^A(s,t)$ can be put back into the form of the unsymmetrized $A[12\to 34]$,

$$\begin{aligned} A^A(s,t) = \sum_{\alpha=1}^{5}\left(F_\alpha(s,t) - \sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_\beta(s,u)\right)[\overline{u_3}t_\alpha u_1] \\ \times[\overline{u_4}t^\alpha u_2]. \end{aligned} \tag{36}$$

The correctly antisymmetrized amplitude is then identical to the unsymmetrized amplitude but with $F_\alpha(s,t)$ replaced by

$$F_\alpha^A(s,t) \equiv F_\alpha(s,t) - \sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_\beta(s,u), \tag{37}$$

where $s+t+u=4m^2$. It is these functions, $F_\alpha^A(s,t)$, and not the unsymmetrized $F_\alpha(s,t)$ which are experimentally accessible through study of pp elastic scattering. We must therefore ensure that the antisymmetrized forms of our soft photon approximations to the proton-proton bremsstrahlung amplitude can be expressed purely in terms of the $F_\alpha^A(s,t)$, rather than the $F_\alpha(s,t)$.

We now consider the form of the three soft photon amplitudes of Sec. II extended to spin- $\frac{1}{2}$ identical particle scattering. The unsymmetrized pp bremsstrahlung soft photon amplitudes may be written in the form

$$\mathcal{M}^\mu\epsilon_\mu = \sum_{\alpha=1}^{5} eQ[\overline{u_3}X_\mu^\alpha\epsilon^\mu u_1\overline{u_4}t_\alpha u_2 + \overline{u_3}t_\alpha u_1\overline{u_4}Y_\mu^\alpha\epsilon^\mu u_2] \tag{38}$$

with the functions X_μ^α and Y_μ^α taking on a different form for each of the Low-$(\overline{s,t})$, the TSTTS, and the TUTTS amplitudes. Using the notation of Ref. [13]

$$\mathcal{R}(p)\cdot\epsilon = \frac{1}{4}[\not\epsilon,\not k] + \frac{i\kappa_p}{8m}\{[\not\epsilon,\not k],\not p\},$$

where $\kappa=1.79$ and m are the proton anomalous magnetic moment and mass, we may write the X^{α}_{μ} and Y^{α}_{μ} functions for each case as follows:

(i) Low-$(\overline{s},\overline{t})$ amplitude:

$$X^{\alpha}\cdot\epsilon\equiv\left[\frac{p_3\cdot\epsilon+\mathcal{R}(p_3)\cdot\epsilon}{p_3\cdot k}+\mathcal{D}^{\mu}(p_3)\frac{\partial}{\partial p_3^{\mu}}\right]t^{\alpha}F_{\alpha}(\overline{s},\overline{t})$$
$$-t^{\alpha}\left[\frac{p_1\cdot\epsilon+\mathcal{R}(p_1)\cdot\epsilon}{p_1\cdot k}-\mathcal{D}^{\mu}(p_1)\frac{\partial}{\partial p_1^{\mu}}\right]F_{\alpha}(\overline{s},\overline{t}),$$
$$Y^{\alpha}\cdot\epsilon\equiv\left[\frac{p_4\cdot\epsilon+\mathcal{R}(p_4)\cdot\epsilon}{p_4\cdot k}+\mathcal{D}^{\mu}(p_4)\frac{\partial}{\partial p_4^{\mu}}\right]t^{\alpha}F_{\alpha}(\overline{s},\overline{t})$$
$$-t^{\alpha}\left[\frac{p_2\cdot\epsilon+\mathcal{R}(p_2)\cdot\epsilon}{p_2\cdot k}-\mathcal{D}^{\mu}(p_2)\frac{\partial}{\partial p_2^{\mu}}\right]F_{\alpha}(\overline{s},\overline{t}). \quad (39)$$

(ii) TsTts amplitude:

$$X^{\alpha}\cdot\epsilon\equiv\left[\frac{p_3\cdot\epsilon+\mathcal{R}(p_3)\cdot\epsilon}{p_3\cdot k}-\frac{(p_3+p_4)\cdot\epsilon}{(p_3+p_4)\cdot k}\right]t^{\alpha}F_{\alpha}(s_{12},t_{24})$$
$$-t^{\alpha}\left[\frac{p_1\cdot\epsilon+\mathcal{R}(p_1)\cdot\epsilon}{p_1\cdot k}-\frac{(p_1+p_2)\cdot\epsilon}{(p_1+p_2)\cdot k}\right]F_{\alpha}(s_{34},t_{24}),$$
$$Y^{\alpha}\cdot\epsilon\equiv\left[\frac{p_4\cdot\epsilon+\mathcal{R}(p_4)\cdot\epsilon}{p_4\cdot k}-\frac{(p_3+p_4)\cdot\epsilon}{(p_3+p_4)\cdot k}\right]t^{\alpha}F_{\alpha}(s_{12},t_{13})$$
$$-t^{\alpha}\left[\frac{p_2\cdot\epsilon+\mathcal{R}(p_2)\cdot\epsilon}{p_2\cdot k}-\frac{(p_1+p_2)\cdot\epsilon}{(p_1+p_2)\cdot k}\right]F_{\alpha}(s_{34},t_{13}). \quad (40)$$

(iii) TuTts amplitude:

$$X^{\alpha}\cdot\epsilon\equiv\left[\frac{p_3\cdot\epsilon+\mathcal{R}(p_3)\cdot\epsilon}{p_3\cdot k}-\frac{(p_2-p_3)\cdot\epsilon}{(p_2-p_3)\cdot k}\right]t^{\alpha}F'_{\alpha}(u_{14},t_{24})$$
$$-t^{\alpha}\left[\frac{p_1\cdot\epsilon+\mathcal{R}(p_1)\cdot\epsilon}{p_1\cdot k}-\frac{(p_1-p_4)\cdot\epsilon}{(p_1-p_4)\cdot k}\right]F'_{\alpha}(u_{23},t_{24}),$$
$$Y^{\alpha}\cdot\epsilon\equiv\left[\frac{p_4\cdot\epsilon+\mathcal{R}(p_4)\cdot\epsilon}{p_4\cdot k}-\frac{(p_1-p_4)\cdot\epsilon}{(p_1-p_4)\cdot k}\right]t^{\alpha}F'_{\alpha}(u_{23},t_{13})$$
$$-t^{\alpha}\left[\frac{p_2\cdot\epsilon+\mathcal{R}(p_2)\cdot\epsilon}{p_2\cdot k}-\frac{(p_2-p_3)\cdot\epsilon}{(p_2-p_3)\cdot k}\right]F'_{\alpha}(u_{14},t_{13}). \quad (41)$$

For the TuTts expressions we have defined the functions $F'_{\alpha}(u,t)\equiv F_{\alpha}(s,t)$ under the pp elastic phase space constraint $s+t+u=4m^2$.

One can easily see by comparison of Eqs. (39), (40), and (41) with Eqs. (16), (22), and (26), respectively that these results are very similar to those obtained for the spin-0 case. They each contain the extra factor $R(p)$ which arises from the magnetic moment part of the electromagnetic coupling. The scalar amplitudes A of the spin-0 cases are replaced by a sum over terms involving the scalar amplitudes F_{α}, which are functions of the same variables as A, times a momentum independent matrix factor t_{α}. Because of the Dirac structure, one must be careful about the ordering of the t_{α} factors, and of course include the spinors as in Eq. (38). The important point though is that the dependence on kinematic factors and on the scalar variables appearing in the amplitudes is essentially the same as in the spin-0 case.

The phase space problem noted for the spinless case in Sec. III depends only on kinematics, not on the particles' spins, and so carries over directly to proton-proton bremsstrahlung. The physical region of radiative variables pairs such as (s_{34},t_{24}) can still lie outside of the measurable region in the (s,t) plane of nonradiative phase space.

The amplitudes given above must be antisymmetrized if we are to treat identical spin-$\frac{1}{2}$ particle scattering. The antisymmetrization of the spin-$\frac{1}{2}$ amplitudes is no different in principle than the symmetrization of spin-0 amplitudes given in the preceding section and the results are identical: the Low-$(\overline{s},\overline{t})$ amplitude can be successfully antisymmetrized while still being expressed solely in terms of the measured elastic phase shifts; the TsTts and TuTts amplitudes cannot be so expressed. We now show these attempts at antisymmetrization explicitly and demonstrate how the problems arise.

As has been shown previously for example by Fearing [10], one can antisymmetrize the Low-$(\overline{s},\overline{t})$ amplitude using an analogous procedure to that shown above for pp elastic scattering. One would take the amplitude of Eqs. (38) and (39), exchange $p_3^{\mu}\leftrightarrow p_4^{\mu}$, and apply the Fierz manipulation. The result is then in the same form as Eq. (39), but with $F_{\alpha}(\overline{s},\overline{t})$ replaced by $\Sigma_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_{\beta}(\overline{s},\overline{u})$. The final antisymmetrized form is then also identical to Eq. (39) but with $F_{\alpha}(\overline{s},\overline{t})$ replaced by $F_{\alpha}^{A(\mathrm{Low})}(\overline{s},\overline{t})$ where

$$F_{\alpha}^{A(\mathrm{Low})}(\overline{s},\overline{t})\equiv F_{\alpha}(\overline{s},\overline{t})-\sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_{\beta}(\overline{s},\overline{u}), \quad (42)$$

and $\overline{s},\overline{t},\overline{u}$ satisfy the radiative phase space constraint $\overline{s}+\overline{t}+\overline{u}=4m^2$. By their identical definitions we see that $F_{\alpha}^{A(\mathrm{Low})}\equiv F_{\alpha}^{A}$. Operationally, therefore, one only has to take the antisymmetrized F_{α}^{A} from a phase shift analysis of pp elastic scattering data and insert these functions in the unsymmetrized Low-$(\overline{s},\overline{t})$ amplitude of Eq. (39) in order to ensure the correct antisymmetrization of this radiative amplitude.

When we attempt the same procedure for the TsTts amplitude a problem arises. The calculation goes just as with the $(\overline{s},\overline{t})$ case except for the definition of the four antisymmetrized F_{α} functions. To obtain the antisymmetrized TsTts amplitude we must replace the F_{α} of the unsymmetrized amplitude of Eq. (40), in analogy with Eqs. (23) and (24), as follows:

$$F_{\alpha}(s_{34},t_{24})\to F_{\alpha}^{A(1)}(s_{34},t_{24},u_{23})$$
$$\equiv F_{\alpha}(s_{34},t_{24})-\sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_{\beta}(s_{34},u_{23}),$$

$$F_{\alpha}(s_{34},t_{13})\to F_{\alpha}^{A(2)}(s_{34},t_{13},u_{14})$$
$$\equiv F_{\alpha}(s_{34},t_{13})-\sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_{\beta}(s_{34},u_{14}),$$

$$F_\alpha(s_{12},t_{24})\rightarrow F_\alpha^{A(3)}(s_{12},t_{24},u_{14})$$
$$\equiv F_\alpha(s_{12},t_{24})-\sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_\beta(s_{12},u_{14}),$$
$$F_\alpha(s_{12},t_{13})\rightarrow F_\alpha^{A(4)}(s_{12},t_{13},u_{23})$$
$$\equiv F_\alpha(s_{12},t_{13})-\sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}F_\beta(s_{12},u_{23}). \quad (43)$$

The various Lorentz variables in these definitions do not satisfy the same internal constraints as do the s,t,u of $F_\alpha^A(s,t)$—for example, in $F_\alpha^{A(3)}(s_{12},t_{24},u_{14})$ we have that

$$s_{12}+t_{24}+u_{14}=4m^2+2k\cdot p_3\neq 4m^2$$

in general. Thus these $F_\alpha^{A(1,2,3,4)}$ cannot be simply replaced by the $F_\alpha^A(s,t)$ of pp elastic scattering as defined in Eq. (37). The functions are not identically defined. To make such a replacement will result in the radiative amplitude having improper symmetry properties. As in the spin-0 case, however, one can calculate the error introduced by such a replacement. Again the naive estimate of the error is too pessimistic, due to cancellation, and the actual error is only of $\mathcal{O}(k)$.

While attempting to antisymmetrize the TUTTS amplitude we find another difficulty, just as we did in the spin-0 case. After replacing $p_3^\mu\leftrightarrow p_4^\mu$ in the unsymmetrized version of TUTTS and applying the Fierz manipulation we find

$$M_\mu^{3\leftrightarrow 4}=eQ_p\sum_{\alpha=1}^{5}\sum_{\beta=1}^{5}\mathcal{C}_{\beta\alpha}\Bigg(F'_\beta(t_{13},u_{23})\overline{u_4}\left[\frac{p_{4\mu}+\mathcal{R}_\mu(p_4)}{p_4\cdot k}-\frac{(p_2-p_4)_\mu}{(p_2-p_4)\cdot k}\right]t_\alpha u_2\overline{u_3}t^\alpha u_1$$
$$-F'_\beta(t_{24},u_{23})\overline{u_4}t_\alpha u_2\overline{u_3}t^\alpha\left[\frac{p_{1\mu}+\mathcal{R}_\mu(p_1)}{p_1\cdot k}-\frac{(p_1-p_3)_\mu}{(p_1-p_3)\cdot k}\right]u_1+F'_\beta(t_{24},u_{14})\overline{u_4}t_\alpha u_2\overline{u_3}\left[\frac{p_{3\mu}+\mathcal{R}_\mu(p_3)}{p_3\cdot k}\right.$$
$$\left.-\frac{(p_1-p_3)_\mu}{(p_1-p_3)\cdot k}\right]t^\alpha u_1-F'_\beta(t_{13},u_{14})\overline{u_4}t_\alpha\left[\frac{p_{2\mu}+\mathcal{R}_\mu(p_2)}{p_2\cdot k}-\frac{(p_2-p_4)_\mu}{(p_2-p_4)\cdot k}\right]u_2\overline{u_3}t^\alpha u_1\Bigg). \quad (44)$$

The full antisymmetrized amplitude is then obtained by subtracting this from the amplitude obtained from Eqs. (38) and (41). The factors $R(p)$ involving the magnetic moments cause no problem. They can be separated, along with the $p_i/k\cdot p_i$ pieces, from the F'_α which can then be put in a form analogous to Eq. (37). The $(p_i-p_j)^\mu/k\cdot(p_i-p_j)$ terms mix up the momenta however, just as for the spin-0 case and prevent the radiative amplitudes from being put into the original form given in Eqs. (38) and (41), except with the antisymmetrized F'_α. Thus it is impossible to express the TUTTS amplitude purely in terms of the antisymmetrized F_α^A functions which are the measurable quantities. The amplitudes TSTTS and TUTTS could be correctly antisymmetrized by computing $M_\mu-M_\mu(p_3\leftrightarrow p_4)$ directly in terms of the unsymmetrized functions $F_\alpha(s,t)$. As stated previously however, the $F_\alpha(s,t)$ cannot be derived from pp elastic scattering data alone. Thus we would lose the direct connection between a process and its radiative counterpart which gives the soft photon theorem its utility. Alternatively, one could modify the original TUTTS amplitude by changing the terms added to give gauge invariance, just as was discussed for the spin-0 case.

From Ref. [13] we see that Liou *et al.* seem to have used the antisymmetric $F_\alpha^A(s,t)$ pp elastic functions directly in their unsymmetrized TSTTS and TUTTS amplitudes. Their results cannot have the correct symmetry properties, and again the error will be of $\mathcal{O}(k)$ for the TSTTS amplitude and of $\mathcal{O}(k/k)$ for the TUTTS.

We shall now illustrate these ideas with some numeric results for the pp bremsstrahlung soft photon spectrum. Low energy proton-proton elastic scattering data are usually parameterized in terms of a phase shift fit. For our calculations we have used as input the recent analysis of the Nijmegen group, Refs. [22–24]. The relationship between these phase shifts and the invariant functions $F_\alpha^A(s,t)$ is straightforward but rather algebraically involved, and is set down explicitly in Ref. [25]. In order to investigate the work of Liou, Timmermans, and Gibson [13] we have inserted these F_α^A in place of the unsymmetrized functions F_α in the soft photon amplitudes of Eqs. (39), (40), and (41). As shown above this will give rise to the correctly antisymmetrized radiative amplitude only for the Low-$(\overline{s,t})$ case of Eq. (39).

In Fig. 3 we have shown the differential spectrum $d^5\sigma/d\Omega_3 d\Omega_4 d\theta_k$ as a function of the laboratory frame photon angle θ_k for one of the kinematic choices studied by the Harvard pp bremsstrahlung experiment of Ref. [26]—that is, with beam kinetic energy of 157 MeV, with final state protons detected at 10° on either side of the beamline direction, and with all particles coplanar. Shown are the Low-$(\overline{s,t})$ and TUTTS soft photon calculations using the Nijmegen phase shift data set as input. From the symmetry of this experimental setup it is clear that the photon angular spectrum must be symmetric under reflection about the beamline axis. This is the case for the Low-$(\overline{s,t})$ spectrum, while the TUTTS spectrum does not have this symmetry property. Since the authors of Ref. [13] give their results only in the region $\theta_k=0°\rightarrow 180°$ this cannot be checked directly from their paper. We obtain results shown in Fig. 3 which agree very well with their results for that range of θ_k but are not mirror symmetric about $\theta_k=180°$, which they should be. This is consistent with an error in the antisymmetrization of the TUTTS amplitude.

Due to the phase space problem described in Sec. II we cannot present a result for the TSTTS amplitude without ex-

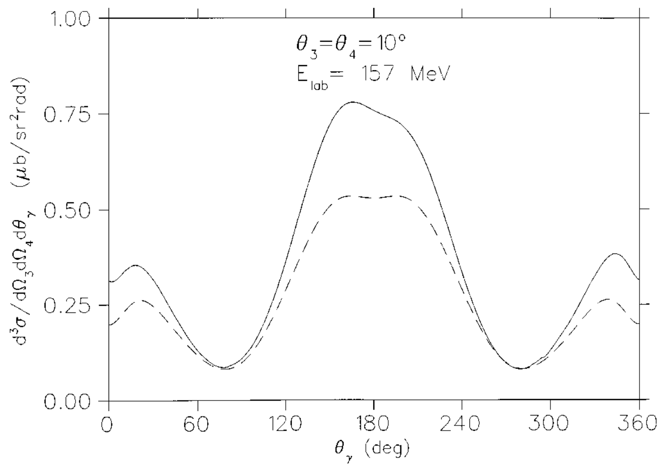

FIG. 3. The solid line shows the TUTTS soft photon approximation for pp bremsstrahlung for the case $T=157$ MeV, $\theta_3=\theta_4=10°$. The Low-$(\overline{s,t})$ soft photon amplitude is shown by the dashed line. Both are computed using the most recent Nijmegen phase shift data set. The spectra should exhibit mirror symmetry in θ_γ about the point $\theta_\gamma=180°$, since the protons are emitted at equal angles on either side of the beam direction. The Low-$(\overline{s,t})$ result is symmetric, but the TUTTS result is clearly not. This is in agreement with our work of Sec. V.

trapolating the function F_α^A far outside of the physical region of nonradiative phase space. In terms of the phase shifts, which are parameterized by center-of-mass frame momentum and scattering angle $\theta_{\mathrm{c.m.}}$, this would correspond to evaluating the nonradiative amplitude for $\cos\theta_{\mathrm{c.m.}}<-1$. The authors of Ref. [13] do present results for the TSTTS amplitude. These spectra differ by large factors from other soft photon calculations, from potential model calculations, and from experimental data. This difference appears in spite of the well known result that the leading two orders of the photon spectrum are uniquely predicted. The authors of Ref. [13] suggest that this large discrepancy is evidence that the TUTTS amplitude should be preferred over the TSTTS amplitude for calculations of identical particle scattering. An alternative explanation might be that this large discrepancy is simply a reflection of the fact that the elastic amplitudes, which are required far outside the physical region, were extrapolated in some unphysical way.

VI. CONCLUSIONS

We have seen that problems arise in the practical application of certain soft photon amplitudes to two-body bremsstrahlung processes, in particular to pp bremsstrahlung. Since the variables $\bar{s}$, $\bar{t}$, and $\bar{u}$ of radiative phase space satisfy the same constraints as the Mandelstam variables s, t, and u of the corresponding nonradiative process we find that the usual Low formulation of the approximation, which expresses all elastic amplitudes at a single point in terms of these variables, is unaffected by the phase space problem and the antisymmetrization problem. In contrast, neither the ''Two-u-Two-t-special'' (TUTTS) nor the ''Two-s-Two-t-special'' (TSTTS) amplitudes suggested in Ref. [13] can be antisymmetrized while being written in terms of the measurable pp elastic amplitude. Additionally the TSTTS amplitude was found to be incalculable unless one makes a model-dependent extrapolation of the pp elastic amplitude outside of its physical, on-shell region. We conclude that the TUTTS and TSTTS soft photon amplitudes, and those other alternative forms for the soft photon approximation which rely on Taylor expansions about radiative variable pairs other than $(\overline{s,t})$ and share the same problems, are not suitable for the soft photon analysis of proton-proton bremsstrahlung.

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