Angular-momentum cranking applied to multiphonon anharmonic collective vibrations: Cranked bifurcation theory

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It is shown that the self-consistent angular-momentum cranking technique can be used to generate certain types of collective vibrational solutions of time-dependent mean-field equations. The method is suitable for systems having an equilibrium mean field with an axis of symmetry and is founded on a theorem (cranking bifurcation theorem) proposed in this paper, according to which cranking about an equilibrium axis of symmetry leads to new symmetry-breaking solutions that bifurcate from the axially symmetric solution at the critical cranking frequencies given by $\Omega = \omega_\mu / K_\mu$, where ω_μ is a random-phase-approximation (RPA) frequency for any mode carrying $K_\mu \neq 0$ units of angular momentum along the symmetry axis. The bifurcating solutions correspond to aligned multiphonon excitations including possible large-amplitude anharmonicities. A general heuristic proof of the method is provided, as well as a perturbative demonstration within the framework of the cranked Hartree-Fock (CHF) approximation, which includes a derivation of the RPA. The static CHF approach is then compared to a perturbative treatment of the time-dependent Hartree-Fock equations using the Lindstedt method. It is also shown that the cranking approach may be applied to phenomenological mean-field models to obtain anharmonic corrections to the vibrating potential model. Finally, the calculation of transition matrix elements is briefly discussed. [S0556-2813(96)04507-4]

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I. INTRODUCTION

The cranking method is usually applied to nuclei having an intrinsic state with a static broken rotational symmetry manifested in the mean field. The cranking axis is chosen such that the system is deformed in a plane perpendicular to this axis. The technique then generates the usual rotational bands found in permanently deformed nuclei. Overall, it has been remarkably successful in explaining both qualitatively and even quantitatively a huge body of data up to very high spins [1] at both normal deformations and superdeformations. On the other hand, the common wisdom is that cranking a mean field about an axis of rotational symmetry just carries the field into itself, leading to nothing new, except possibly a redefinition of the Fermi surface (equivalent to introducing a ''tilted Fermi surface''). Therefore, one would not suppose that the cranking method would be of any use in spherical nuclei. However, it will be shown here that this common wisdom is shortsighted. More specifically, it will be demonstrated that when a mean field is cranked about a symmetry axis, it is possible that at certain critical cranking frequencies new solutions that *break* the symmetry *bifurcate* from the symmetric one. In most cases, such a bifurcation corresponds to a band of *multiphonon vibrational states* rather than an ordinary rotational band. In view of the recent experimental ground swell of interest in multiphonon states, the cranking approach is worthwhile pursuing.

Collective vibrations of the one-phonon type have been long identified throughout the nuclear periodic table. All phenomenological collective models, and many microscopic ones, also predict the existence of multiphonon excitations, the study of which is crucial to an evaluation of the importance of anharmonicity. Here, the data have been more elusive, being primarily limited to quadrupole excitations of near-magic spherical nuclei such as isotopes of Cd [2]. With the advent of improved detectors, there has recently been a considerable resurgence of interest in multiphonon excitations sparked by new experimental results. For example, levels in 17 nuclei have been proposed as candidates for $K=4$ two-phonon γ vibrations of deformed rare-earth nuclei [3]. Very recently, persuasive evidence has been reported for such two-phonon γ vibrations in the isotope ^{106}Mo by Guessous *et al.* [4]. In addition, evidence has been cited for two-phonon octupole levels in ^{148}Gd [5], ^{208}Pb [6], ^{144}Ne, and ^{146}Sm [7], and for a mixed octupole-γ vibration in several deformed nuclei [8]. Two-phonon giant dipole [9] and quadrupole [10] resonances have also been observed in various spherical nuclei. From the perspective of systematics, Casten and collaborators [11] have succeeded in correlating data over vast stretches of the periodic table using a simple anharmonic-vibrator formula for quadrupole excitations having two or three *universal* parameters.

A nonexhaustive inventory of the theorist's tool kit of microscopic approaches that can or have been applied to multiphonon anharmonic excitations includes the method of generator coordinates [12], the (fermion) multiphonon method [13], higher random-phase approximations [14], variants of boson-expansions [15], the Dubna quasiparticle-phonon model [16], and specialized techniques for solving time-dependent mean-field (TDMF) theories, such as the self-consistent coordinate method [17], and various formulations of the theory of adiabatic large-amplitude collective motion [18]. Each of these methods has its pros and cons. For example, the multiphonon method, which takes into account the Pauli principle exactly, but may violate particle-number and angular-momentum conservation (deformed nuclei), apparently cannot give rise to a compressed multiphonon spectrum, which seems to be required by some

of the data on deformed nuclei, and is a feature of the phenomenological interacting boson model. Boson expansions, on the other hand, may partially violate the Pauli principle due to truncation errors, but are capable of fulfilling conservation laws to a given order of approximation [19] and can give rise to compressed multiphonon spectra. The method closest to the one to be discussed in this paper is the TDMF approximation, which includes the time-dependent Hartree, Hartree-Fock (HF) and Hartree-Fock-Bogoliubov (HFB) approximations, in which bound states corresponding to anharmonic collective modes are described by periodic orbits obeying a gauge-invariance condition [20] that is essentially equivalent to Bohr-Sommerfeld quantization. Unfortunately, the construction of periodic solutions of the nonlinear TDMF equations is usually a difficult task that has been carried out only for a few simple models [21]. In lieu of exact periodic solutions, one may use secular perturbation theory [22,23] with the small-oscillation approximation, i.e., the random-phase approximation (RPA), as zeroth order. This method is generally limited to small amplitudes, and in many cases runs into complications involving small resonance denominators. The main purpose of this paper is to call attention to yet another method for finding periodic solutions that is applicable to collective vibrations carrying nonzero angular momentum, namely, the self-consistent cranking model (SCCM), which has heretofore played a central role in elucidating rotational states of high angular momentum [1]. However, the proposal here is to use the SCCM to describe states that are normally regarded as *vibrational* rather than rotational in character with the particular aim of applying it for the first time to deformed nuclei.

The idea of applying cranking to vibrational modes was first proposed by Marshalek and Sabato [24] and, independently, by Kamlah [25] 25 years ago. In Ref. [25], angular-momentum projection was used to derive the connection between cranking and the RPA. In the series of papers [24], the cranking approach to vibrations was derived from the c-number limit of boson expansions. The applications to spherical nuclei went beyond the RPA, including perturbatively the leading-order anharmonic corrections to the excitation energies as well as the static quadrupole moments and BE(2)'s involving the aligned n-phonon quadrupole excitations, which have spins $I=2n$. It was also shown how the parameters of the variable-moment-of-inertia (VMI) model for spherical nuclei could be calculated microscopically [26]. The work was not carried further at the time for a number of reasons, including problems with small resonance denominators in many nuclei, the inadequacy of available computers, and, of course, lack of data (ergo interest) on multiphonon states. Since the small-denominator problem can be avoided using exact diagonalization, with modern desktop computers being easily up to the task and data proliferating, the moment is opportune for a revival on a broader basis. Since the essential ingredient of the original *spherical cranking model* is the presence of an axis of symmetry in the mean field, the model can be extended to axially symmetric deformed nuclei, thereby providing a new tool with some special advantages for calculating, for example, the n-phonon γ-vibrational bandheads with spin projections $K=2n$ on the axis of symmetry, as well as other band heads.

The general aim of this paper is to establish the foundations of the cranking approach for vibrations, which will be applied in future work. The main thesis of this work centers on the relation between bifurcations and vibrational frequencies. More precisely, many bifurcations described as rotating equilibria are really vibrations, i.e., rotating surface waves. This is the case not only for nuclear mean fields, but also for collective models and liquid drops, as will be shown in forthcoming work. This viewpoint provides a justification for the long-standing characterization of certain vibrational states as being ''quasirotational.'' A brief presentation of the basic ideas with applications to some simple systems has been given in Refs. [27–31].

In Sec. II, a general but heuristic proof is given for what will be called the cranking bifurcation theorem (CBT), which provides the justification for cranking about a symmetry axis. It is widely believed that cranking about a symmetry axis of a mean field is an unproductive activity, merely tantamount to an identity mapping. However, according to the theorem this is not necessarily so, for at certain critical cranking frequencies, which are related in a simple way to the RPA frequencies, bifurcations occur that break the rotational symmetry. Each bifurcating solution branch represents a sequence of anharmonic multiphonon excitations associated with an excitation operator that carries angular momentum along the symmetry axis and reduces in the small-amplitude limit to the associated RPA phonon. In Sec. III, a second, more specific proof is given for mean fields using the cranked Hartree-Fock (CHF) approximation in the framework of perturbation theory. The results are compared to those obtained from the Lindstedt method as applied to the time-dependent Hartree-Fock (TDHF) equations, the two treatments turning out to be completely equivalent. However, the CHF approach, unlike that of Lindstedt, is not limited to perturbation theory and can thus be applied to large-amplitude vibrations through exact diagonalization. A brief discussion is also given of the application of the CBT to empirical independent-particle deformed potential models such as the Nilsson model and the relation to the vibrating potential model (VPM). The possibility of calculating transition probabilities and improving upon the mean-field approximation is also touched upon. Illustrative applications of the ideas will be presented in a subsequent paper.

II. CRANKING BIFURCATION THEOREM

Consider a nucleus whose equilibrium shape is either spherical or deformed with axial symmetry in some reference mean-field configuration, designated as the ''vacuum,'' which is normally the ground state but may be an excited state as well. The physical picture of a collective vibration that projects $K\neq 0$ units of angular momentum along a symmetry axis corresponds to a rotating distortion, a surface wave traveling around this axis (see Fig. 1). Since such a mode is degenerate, it is possible to choose linear combinations of normal coordinates that describe a *uniformly rotating wave*. From a quantal viewpoint, these modes may be described in terms of the boson creation operators $B_\mu^\dagger$ carrying K_μ units of angular momentum along the symmetry axis. Since the cranking model is basically a classical approximation, these bosons will be replaced by the corresponding

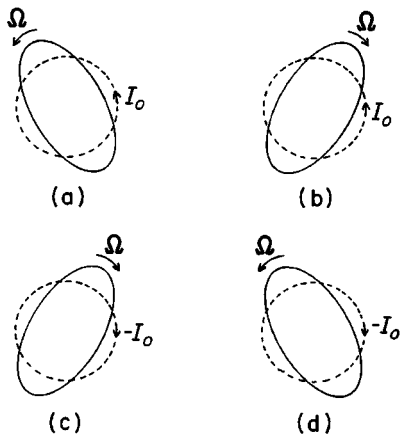

FIG. 1. The rotational motion of the bifurcating solution, represented by the solid ellipse, compared with that of the vacuum, represented by the dashed circle. The rotations may be in the same (a),(c) or opposite (b),(d) senses. The solutions (c),(d) are the time reverses of (a),(b), respectively. Here, $\mathbf{I}_0$ is the angular momentum of the vacuum while $\mathbf{\Omega}$ is the angular velocity of the bifurcating excited state.

c-number complex coordinates β_μ^* obeying the Poisson bracket relations

$$[\beta_\mu, i\beta_\nu^*]_{\mathrm{PB}}=\delta_{\mu,\nu}, \quad [\beta_\mu^*,\beta_\nu^*]_{\mathrm{PB}}=[\beta_\mu,\beta_\nu]_{\mathrm{PB}}=0, \tag{2.1}$$

where the symbol $[\cdot,\cdot]_{\mathrm{PB}}$ denotes the Poisson bracket. If H is the quantal Hamiltonian, then $\langle H\rangle$ will denote the classical Hamiltonian function of β_μ^*, β_μ. This notation serves as a reminder that the classical functions may be regarded as expectation values with respect to suitable coherent states. The equilibrium points of the system relative to a frame rotating uniformly with angular velocity Ω about the three-axis are solutions of the set of equations

$$\frac{\partial\langle H'\rangle}{\partial\beta_\mu^*}=0, \tag{2.2}$$

where

$$\langle H'\rangle\equiv\langle H\rangle-\Omega\langle J_3\rangle \tag{2.3}$$

is the so-called Routhian (Hamiltonian with respect to the rotating frame) with $\langle J_3\rangle$ being the classical angular-momentum component along the symmetry axis (three-axis) given by

$$\langle J_3\rangle\equiv I_0+\sum_\mu K_\mu\beta_\mu^*\beta_\mu. \tag{2.4}$$

(Here and throughout the paper it is assumed that $\hbar=1$.) The diagonal quadratic form for $\langle J_3\rangle$ is, of course, a consequence of designating the three-axis as the symmetry axis. The constant I_0 is the spin of the axially symmetric vacuum. For the ground-state configuration of an even-even nucleus, $I_0=0$, but for an odd-A nucleus, or for an excited state of an even-even nucleus, nonzero values of I_0 may occur, as, for example, in the case of an oblate rotational-band terminus configuration or a high-K isomer [32]. If $I_0=0$, then the excitations occur in time-reversal conjugate pairs corresponding to β_μ^* and $\beta_{-\mu}^*$ with the sum in Eq. (2.4) understood to run over both time-reversal partners and $K_{-\mu}=-K_\mu$. If $I_0\neq0$, then the total spin of a single excited mode is given by $I_0+K_\mu|\beta_\mu|^2$, where K_μ can be positive or negative. In this case, time-reversal symmetry is broken for a fixed choice of vacuum. However, one can also study excitations built on the time-reversed vacuum having spin $-I_0$ to obtain the time-reversed states. The difference between the two cases is illustrated in Fig. 1.

For simplicity, first consider small oscillations (RPA) about the vacuum configuration. In this limit, the classical Hamiltonian is

$$\langle H\rangle=\langle H_0\rangle\equiv E_0(I_0)+\sum_\mu\omega_\mu\beta_\mu^*\beta_\mu, \tag{2.5}$$

where $E_0(I_0)$ is a constant, the vacuum energy, and β_μ^* has been chosen as a normal-mode coordinate with corresponding frequency ω_μ. The simultaneous diagonal quadratic forms assumed for $\langle J_3\rangle$ and $\langle H\rangle$ can always be achieved since $[\langle H\rangle,\langle J_3\rangle]_{\mathrm{PB}}=0$. From Eq. (2.4), the Routhian (2.3) becomes

$$\langle H'\rangle=E_0(I_0)-\Omega I_0+\sum_\mu(\omega_\mu-\Omega K_\mu)\beta_\mu^*\beta_\mu. \tag{2.6}$$

Then, Eq. (2.2) implies

$$\beta_\mu(\omega_\mu-\Omega K_\mu)=0. \tag{2.7}$$

In general, this set of equations has two kinds of solutions: the trivial solution in which all $\beta_\mu=0$ and nontrivial solutions in which a single coordinate $\beta_{\mu_0}\neq0$, while $\beta_\mu=0$ for $\mu\neq\mu_0$. However, the nontrivial solutions are possible only for a discrete set of cranking frequencies given by

$$\Omega=\Omega_C\equiv\omega_\mu/K_\mu, \quad K_\mu\neq0, \tag{2.8}$$

which is equivalent to the *vanishing* of a vibrational frequency $\omega_\mu-\Omega K_\mu$, in the *rotating* frame. It should be noted that there are no nontrivial solutions for those modes with $K_\mu=0$. The trivial solution just corresponds to the axially symmetric vacuum (unproductive cranking), while the nontrivial solutions are deformed about the three-axis, the magnitude of the deformation depending on β_{μ_0}. Now, Eq. (2.7) does not determine β_{μ_0}, which reflects the independence of the frequency and amplitude for a harmonic oscillator. However, $|\beta_{\mu_0}|$ can be fixed via Eq. (2.4) by prescribing a suitable quantized value $\langle J_3\rangle=I$, which leads to

$$N_{\mu_0}\equiv|\beta_{\mu_0}|^2=(I-I_0)/K_{\mu_0}. \tag{2.9}$$

Of course, the sign of I must be chosen taking into regard that of K_{μ_0} so that the right-hand side (RHS) of (2.9) is positive. The total energy (laboratory frame) for this solution is then given by

$$E=\langle H\rangle=E_0(I_0)+\omega_{\mu_0}|\beta_{\mu_0}|^2=E_0(I_0)+\omega_{\mu_0}(I-I_0)/K_{\mu_0}. \tag{2.10}$$

For example, for quadrupole vibrations about a spherical ground state, one has $I_0=0$ and the cranking procedure engages only those modes with $|K_{\mu_0}|=1,2$. For $|K_{\mu_0}|=2$, the excitation spectrum is given by $\Delta E=\frac{1}{2}\omega_{\mu_0}I=\omega_{\mu_0}n$, where n is the number of phonons. Therefore, the cranking solution selects the sequence of aligned multiphonon excitations with (quantized) spins $I=2n=0,2,4,\ldots$ which, for the lowest frequency ω_{μ_0}, are often yrast states, referred to as quasirotational states. For $|K_{\mu_0}|=1$, the excitation spectrum is given by $\Delta E=\omega_{\mu_0}I=\omega_{\mu_0}n$, corresponding to multiphonon states with spins $I=n=2,3,4,\ldots$. As shown explicitly in Ref. [27] for the U(5) limit of the interacting boson model (IBM) [33], such solutions correspond to rotation about a tilted axis rather than a principal axis of the density distribution, thereby breaking signature symmetry.

Next, consider the more general situation in which $\langle H\rangle$ contains anharmonic corrections to the small-oscillation part in the form of an arbitrary, possibly infinite polynomial in the coordinates β_μ^*, β_μ, which may be the result of a Taylor expansion about the vacuum configuration. The argument is greatly simplified if one chooses the coordinates such that $\langle H\rangle$ is given by the Birkhoff-Gustavson (BG) normal form [34,35]. In this connection, one should distinguish between the nonresonant and the resonant cases. In the nonresonant case, the small-oscillation frequencies are either incommensurate or else the commensurability is the consequence of a global symmetry that forbids resonant coupling terms. Then, as proved by Birkhoff [34], the Hamiltonian can be transmuted by a series of canonical transformations into an infinite polynomial in the action variables N_μ defined by

$$N_\mu\equiv\beta_\mu^*\beta_\mu, \tag{2.11}$$

with the conjugate angle variables being cyclic.[1] With the assumption that the coordinates β_μ^*, β_μ are already chosen as the transformed variables, the most general Birkhoff normal form for $\langle H\rangle$ is given by

$$\begin{aligned}\langle H\rangle=E_0+\sum_\mu\omega_\mu N_\mu+\frac{1}{2}\sum_{\mu_1\mu_2}h^{(4)}(\mu_1,\mu_2)N_{\mu_1}N_{\mu_2}+\cdots\\ +\frac{1}{n!}\sum_{\mu_1\mu_2\cdots\mu_n}h^{(2n)}(\mu_1,\mu_{2,\ldots,}\mu_n)\\ \times N_{\mu_1}N_{\mu_2}\cdots N_{\mu_n}+\cdots,\end{aligned} \tag{2.12}$$

where the numerical coefficients $h^{(2n)}(\mu_1,\mu_{2,\ldots,}\mu_n)$ are completely symmetric in the indices and the N_μ constitute a formal set of constants of motion that are in involution. In the resonant case [35], the normal-mode frequencies ω_μ are connected by r commensurability relations, and $\langle H\rangle$ is a polynomial involving in addition to the N_μ, certain resonant monomials in the β_μ^*, β_μ whose Poisson brackets with the small-oscillation Hamiltonian $\langle H_0\rangle$ vanish [36]. In this case, the n action variables N_μ cannot all be constants of motion, but instead one can find $n-r$ formal constants of motion that are linear combinations of the N_μ, and others, such as global constants of motion, must be found by independent means. Since the treatment of the resonant monomials, which is analogous to degenerate perturbation theory in the quantal case [37], is rather complicated and difficult to demonstrate in full generality, only the nonresonant case will be dealt with here for the sake of clarity.

From Eqs. (2.2)–(2.4) and (2.12), the stationary points in the rotating frame are solutions of the set of equations

$$\begin{aligned}\beta_\mu\Big[\omega_\mu-\Omega K_\mu+\sum_{\mu_1}h^{(4)}(\mu,\mu_1)N_{\mu_1}+\cdots\\ +\frac{1}{(n-1)!}\sum_{\mu_1\cdots\mu_{n-1}}h^{(2n)}(\mu,\mu_{1,\ldots,}\mu_{n-1})\\ \times N_{\mu_1}\cdots N_{\mu_{n-1}}+\cdots\Big]=0.\end{aligned} \tag{2.13}$$

As in the harmonic case, one retains the trivial solution in which every $\beta_\mu=0$. In addition, there are the nontrivial solutions $\beta_\mu=\beta_{\mu_0}\delta_{\mu,\mu_0}$with a single nonzero coordinate β_{μ_0}. Thus, equating to zero the corresponding term in brackets in Eqs. (2.13) with $N_\mu=N_{\mu_0}\delta_{\mu,\mu_0}$ gives the following series in N_{μ_0}:

$$\begin{aligned}\Omega-\Omega_C=h^{(4)}(\mu_0,\mu_0)K_{\mu_0}^{-1}N_{\mu_0}+\cdots+\frac{1}{(n-1)!}\\ \times h^{(2n)}(\mu_0,\ldots,\mu_0)K_{\mu_0}^{-1}(N_{\mu_0})^{n-1}+\cdots,\end{aligned} \tag{2.14}$$

which can be reverted to give a series solution for N_{μ_0} in powers of $\Omega-\Omega_C$. Thus, in the presence of anharmonicity, a nontrivial solution can exist for a continuous range of rotational frequencies Ω in some neighborhood of each critical frequency Ω_C given by Eq. (2.8), and these critical frequencies are points at which an anharmonic branch *bifurcates* from the vacuum solution. It is also interesting to note that if $\Omega_C>0$, which is most frequently the case, and $h^{(4)}(\mu_0,\mu_0)<0$ while the higher orders are small, then the solution backbends in some neighborhood of Ω_C.

The energy corresponding to such an anharmonic branch may be obtained trivially by substituting the solution $N_\mu=N_{\mu_0}\delta_{\mu,\mu_0}$ in Eq. (2.12), which merely limits the dynamics to a particular submanifold in phase space, *ipso facto* proving that the cranking technique gives the correct energy. When the action variable is expressed in terms of the prescribed angular momentum through Eq. (2.9), the energy takes the form of an expansion in powers of $I-I_0$:

$$\begin{aligned}E=\langle H\rangle=E_0(I_0)+\omega_{\mu_0}\left(\frac{I-I_0}{K_{\mu_0}}\right)+\frac{1}{2}h^{(4)}(\mu_0,\mu_0)\left(\frac{I-I_0}{K_{\mu_0}}\right)^2\\ +\cdots+\frac{1}{n!}h^{(2n)}(\mu_0,\ldots,\mu_0)\left(\frac{I-I_0}{K_{\mu_0}}\right)^n+\cdots.\end{aligned} \tag{2.15}$$

Then, Eq. (2.14) is easily seen to be equivalent to the familiar relation

[1]The obvious quantum analog is the perturbative diagonalization of a boson Hamiltonian by successive unitary transformations to produce a function of the boson number operators.

$$\Omega = \frac{\partial E}{\partial I}. \tag{2.16}$$

In summary, a heuristic argument has been given for the following proposition.

If a system of coupled anharmonic oscillators has an axially symmetric equilibrium configuration (*vacuum*), *then self-consistent cranking about this axis yields families of symmetry-breaking solutions that bifurcate from the trivial or vacuum solution at the critical rotational frequencies* $\Omega_C = \omega_\mu / K_\mu$, *where* ω_μ *is the small-oscillation (RPA) frequency (in the laboratory frame) of each mode carrying* $K_\mu \neq 0$ *units of angular momentum along the symmetry axis. Each family describes the dynamics of the system on submanifolds of phase space characterized by the single nonvanishing action* $N_\mu = (I - I_0)/K_\mu$, *where* I *is the angular momentum and* I_0 *that of the vacuum.*

One should note that the location of the bifurcation points is equivalently given by the condition that one of the RPA frequencies in the rotating frame vanish. It should also be emphasized that all of the bifurcations discussed here branch off from the same vacuum, which has been assumed to be independent of Ω, the situation normally prevailing for nuclear mean fields. However, one may also contemplate more general systems in which the vacuum and, therefore, the vibrational frequencies depend on the rotational frequency Ω, as, for example, in fluids with intrinsic vorticity. In such a case, the above argument still goes through formally as long as the vacuum has an axis of rotational symmetry. But then the condition for the critical bifurcation frequencies, $\omega_\mu(\Omega) - K_\mu \Omega = 0$, may become a nonlinear equation for the critical angular frequency Ω. In this situation, unlike the case when the ω_μ are independent of Ω, the existence of a solution cannot be guaranteed in advance; each system has to be examined separately. Examples of such more general bifurcations may be found in fluid dynamics [38].

Neither the CBT nor this paper deals with the separate question of *stability*. This is a complicated issue (especially when one considers that there are more than 50 stability criteria in the literature) which is best left to specific applications in the future.

The line of proof advanced here for the CBT may be considered heuristic since it is based solely on the manipulation of formal power series, without regard for convergence. Indeed, it has been proven that the BG series has a zero radius of convergence for a nonintegrable Hamiltonian [39], although it may give an accurate asymptotic expansion in regions of phase space where the invariant tori remain intact. Remarkably, Kaluža and Robnik [40] have recently recalculated the BG normal form through 14th order for the Hénon-Heiles Hamiltonian, obtaining excellent results for the regular regions in the Poincaré section plots. Of course, the normal form is expected to lose all meaning in those regions of phase space dominated by chaotic trajectories.

For low-energy trajectories in the neighborhood of the equilibrium point, the cranking principle is expected to hold, correctly predicting the bifurcation points, as verified by various models, and providing the correct energies possibly up to some cutoff point that cannot be determined from the formal Taylor expansion. This raises a very important question, namely, that of how many times a collective mode can be repeated. If an excitation mode has a pure particle-hole or two-quasiparticle character, then even a double excitation is ruled out by the Pauli principle. Thus, collectivity is a necessary condition for the repeatability of an excitation. A bifurcating family of solutions labeled by a semiclassically quantized N_μ (or $\langle J_3 \rangle$) corresponds to the repeated application of a boson operator $B_\mu^\dagger$ in the quantum case. The possible disappearance of a continuous trajectory for some value of the continuous parameter N_μ, would then imply an upper limit on the repeatability of the corresponding collective mode. While this possibility cannot be explored with a Taylor expansion, the nonlinearity of the cranking equations suggests that it may be realized in nonperturbative cranking calculations.

Another point that should be emphasized is that the BG method was introduced here solely to facilitate the justification of the cranking technique, not as an end in itself. It should be obvious from the foregoing that if the normal form could be calculated in nuclei, then cranking would become superfluous. In fact, the normal form was calculated through fourth order for a relatively simple nuclear model by Williams and Koonin [41] and further exploration of this approach may be worthwhile. However, the method has the intrinsic drawback of being perturbative. The cranking method, on the other hand, can be applied nonperturbatively as well as perturbatively with the use of any set of coordinates, not necessarily canonical ones. However, to see more clearly the relevance of the foregoing proof to nuclei, one should first recall how canonical coordinates can be introduced into nuclear mean fields. The most straightforward route is to first perform a generalized Holstein-Primakoff (GHP) boson mapping of the nuclear many-body system [15]. Then, one may appeal to a theorem [42–44] according to which the replacement of the boson operators by classical c-numbers results in a canonical parametrization of the one-body density matrix. (Of course, one must choose the proper ordering of operators prior to the c-number replacement). Depending on the system and the type of GHP mapping chosen, one can obtain in this way any of the standard TDMF approximations, with the Hamiltonian (energy) being a functional of the c-number canonical variables (which may be identified with the original β_μ^*, β_μ). While this can be the starting point for calculating a BG normal form, it is generally preferable to work with the standard forms of cranked mean-field equations, using, for example, the noncanonical density matrix elements themselves in the variational procedure. At a later stage, of course, the system is requantized by invoking a Bohr-Sommerfeld condition. The CBT applies not only to microscopic mean fields, but also to phenomenological collective models, such as the IBM [33] and the Bohr-Mottelson model [45] as well as some liquid-drop models. Since these models are bosonized from the outset, the classical approximation is obtained by simply replacing the bosons by c-numbers in the Hamiltonian or, what is essentially equivalent, averaging in a coherent state, and then invoking Eq. (2.2). Examples of such applications to the IBM are given in Refs. [27] and [29].

III. PERTURBATIVE APPLICATIONS IN THE HARTREE-FOCK APPROXIMATION

Because of the heuristic nature of the general proof given for the CBT in the previous section, it would be reassuring to

have an independent argument specifically for mean fields. Such an argument is provided in this section for the TDHF equations. A similar treatment could also be carried out for the time-dependent Hartree-Fock-Bogoliubov (TDHFB) [46] equations with only a little more complexity. The quantal many-body Hamiltonian is taken as the standard form

$$H=\sum_{ab} e_{ab}a_a^\dagger a_b+\frac{1}{4}\sum_{abcd}\mathcal{V}_{ab,cd}a_a^\dagger a_b^\dagger a_d a_c, \tag{3.1}$$

where $a_a^\dagger$ (a_a) is a fermion creation (destruction) operator, the indices a,b,c,d referring to a complete set of nucleon single-particle states, the matrix e represents either the bare kinetic energy or a model independent-particle Hamiltonian, and $\mathcal{V}_{ab,cd}$ is an antisymmetrized two-body matrix element. In the TDHF approximation, the classical Hamiltonian may be taken as the energy functional of the one-body density matrix ρ:

$$\langle H\rangle=E[\rho]=\mathrm{Tr}e\rho+\frac{1}{2}\mathrm{Tr}\ u\rho, \tag{3.2}$$

where ρ is Hermitian and for a system of A interacting nucleons must obey the conditions

$$\rho^2=\rho \tag{3.3}$$

and

$$\mathrm{Tr}\,\rho=A, \tag{3.4}$$

while the matrix u corresponds to the self-consistent potential, defined by

$$u_{ab}\equiv\sum_{cd}\mathcal{V}_{ac,bd}\rho_{dc}, \tag{3.5}$$

which can be written in the shorthand notation $u_1=\mathrm{Tr}_2\,\mathcal{V}_{12}\rho_2$. The TDHF dynamics are governed by the equation of motion

$$i\frac{d\rho}{dt}=[h,\rho], \tag{3.6}$$

where

$$h\equiv e+u. \tag{3.7}$$

In this section, the CHF approximation for a system with an axis of symmetry will be examined in self-consistent perturbation theory through third-order corrections to the density matrix. This is tantamount to calculating the fourth-order BG normal form for the Hamiltonian. In the first step, the RPA will be derived by the purely static cranking technique. Finally, a brief comparison will be made with a more conventional time-dependent perturbation treatment based on the Lindstedt method.

A. Cranked Hartree-Fock approximation

The CHF approximation can be derived using the approach of Thouless and Valatin [47], who transform the TDHF equation (3.6) from the laboratory frame to a frame rotating with uniform angular velocity Ω about the three-axis. Thus, the transformation of the density matrix is given by

$$\rho=e^{-i\Omega t j_3}\hat{\rho}e^{i\Omega t j_3}, \tag{3.8}$$

where $\hat{\rho}$, the density matrix with respect to the rotating frame, obeys Eqs. (3.3) and (3.4) with ρ replaced by $\hat{\rho}$, and the matrix j_3 corresponds to the three-component of the angular momentum. The requirement that $\hat{\rho}$ be time independent then leads to the CHF equations

$$[\hat{h}-\Omega j_3,\hat{\rho}]=0, \tag{3.9}$$

where $\hat{h}\equiv e+\hat{u}$ and

$$\hat{u}_1\equiv\mathrm{Tr}_2\mathcal{V}_{12}\hat{\rho}_2 \tag{3.10}$$

is the mean-field potential in the rotating frame. The energy (3.2) (with ρ replaced by $\hat{\rho}$) may be regarded as the classical Hamiltonian expressed in terms of the noncanonical coordinates $\hat{\rho}_{ab}$ and Eq. (3.9) is the counterpart of Eq. (2.2) for equilibrium in the rotating frame subject to a prescribed quantized value of the angular-momentum component:[2]

$$\langle J_3\rangle=\mathrm{Tr}\,\hat{\rho}j_3=I. \tag{3.11}$$

It will be assumed that the zeroth-order solution of Eq. (3.9) corresponds to a mean-field Hamiltonian $\hat{h}^{(0)}=h^{(0)}$ that is rotationally invariant about the three-axis (including the possibility of spherical symmetry). A single-particle basis can then be chosen that simultaneously diagonalizes $h^{(0)}$, $\rho^{(0)}$, and j_3. Furthermore, it will be convenient to decompose the complete set of single-particle states (denoted by indices $a,b,c,d,...$) into occupied states, labeled by the greek indices $\alpha,\beta,\gamma,\delta,\ldots$, and empty states, labeled by the latin indices $i,j,k,l,\ldots$. Thus

$$h_{ab}^{(0)}=\varepsilon_a\delta_{a,b},\quad (j_3)_{ab}=K_a\delta_{a,b},\quad \rho_{ab}^{(0)}=n_a\delta_{a,b},$$
$$n_a=1,\quad n_i=0. \tag{3.12}$$

The problem of finding perturbative solutions of Eq. (3.9) can now be tackled. The situation here is fundamentally different from ordinary cranking in which the zeroth-order system is already deformed about the axis of rotation and thus responds to an infinitesimal cranking impulse. Instead, from the discussion in Sec. II, one expects a response only for certain discrete threshold frequencies Ω_C. This would seem to suggest that a perturbation expansion in powers of $\Omega-\Omega_C$ would be appropriate. Indeed, that is what one obtains for N_{μ_0} by reverting the expansion (2.14). On the other

[2]Based on the study of simple models, it appears that a quantization rule like $\langle J_3\rangle=[I(I+1)]^{1/2}$, which is often used in the conventional cranking model, is not valid when applied to vibrations. A more appropriate rule is of the form $\langle J_3\rangle=I+\alpha$, where α is a properly chosen Maslov index. However, the approach favored by the author is to simply use $\langle J_3\rangle=I$ in the CHF approximation, and afterwards to pick up quantal zero-point corrections from a quantized RPA about the CHF equilibrium solution [58].

hand, this implies for β_{μ_0} an expansion in powers of $|\Omega-\Omega_C|^{1/2}$, apart from an arbitrary phase factor. In fact, one finds that the latter choice is the correct one for the perturbation treatment of Eq. (3.9). Therefore, defining the formal perturbation parameter ξ as

$$\xi \equiv |\Omega-\Omega_C|^{1/2}, \tag{3.13}$$

one may introduce into Eq. (3.9) the decomposition

$$\Omega = \Omega_C + \theta\xi^2, \tag{3.14}$$

where $\theta = \mathrm{sgn}(\Omega-\Omega_C)$, along with the expansions

$$\hat{\rho} = \rho^{(0)} + \hat{\rho}^{(1)} + \hat{\rho}^{(2)} + \hat{\rho}^{(3)} + \cdots + \hat{\rho}^{(n)} + \cdots \tag{3.15}$$

and

$$\hat{h} = h^{(0)} + \hat{u}^{(1)} + \hat{u}^{(2)} + \hat{u}^{(3)} + \cdots + \hat{u}^{(n)} + \cdots, \tag{3.16}$$

where $\hat{\rho}^{(n)} \propto \xi^n$, implying that $\hat{u}^{(n)} \propto \xi^n$, since, from Eq. (3.10),

$$\hat{u}_1^{(n)} \equiv \mathrm{Tr}_2 \mathcal{V}_{12} \hat{\rho}_2^{(n)}. \tag{3.17}$$

The orders in Eq. (3.9) can now be separated, starting with the zeroth-order equation

$$[h^{(0)} - \Omega_C j_3, \rho^{(0)}] = 0, \tag{3.18}$$

which is satisfied automatically through Eq. (3.12). The important point to note here is that the unperturbed single-particle energies are given by

$$(h^{(0)} - \Omega_C j_3)_{ab} = (\varepsilon_a - \Omega_C K_a)\delta_{a,b}. \tag{3.19}$$

The general nth-order equation is

$$[h^{(0)} - \Omega_C j_3, \hat{\rho}^{(n)}] + \sum_{k=1}^{n} [\hat{u}^{(k)}, \hat{\rho}^{(n-k)}] - \theta\xi^2 [j_3, \hat{\rho}^{(n-2)}](1-\delta_{n,1}) = 0, \quad n \geq 1. \tag{3.20}$$

In addition, the requirements (3.3) and (3.4) must also be satisfied order by order for the density matrix $\hat{\rho}$. This is automatic in zeroth order, while in nth order the condition (3.3) becomes

$$\sum_{k=0}^{n} \hat{\rho}^{(k)} \hat{\rho}^{(n-k)} = \hat{\rho}^{(n)}, \quad n \geq 1. \tag{3.21}$$

Equation (3.4), which requires that $\mathrm{Tr}\,\hat{\rho}^{(n)} = 0$ for $n \geq 1$, can be shown to be automatically satisfied. In fact, Eq. (3.20) determines the particle-hole (ph) matrix elements $\hat{\rho}_{i\alpha}^{(n)}$, while (3.21) determines the particle-particle (pp) matrix elements $\hat{\rho}_{ij}^{(n)}$ and the hole-hole (hh) matrix elements $\hat{\rho}_{\alpha\beta}^{(n)}$, in both cases in terms of lower-order particle-hole matrix elements. The special cases $n = 1,2,3$ will be examined in more detail next.

1. First order: The RPA

For $n = 1$, Eq. (3.21) becomes

$$(1 - n_a - n_b)\rho_{ab}^{(1)} = 0, \tag{3.22}$$

which implies that

$$\hat{\rho}_{ij}^{(1)} = \hat{\rho}_{\alpha\beta}^{(1)} = 0. \tag{3.23}$$

Therefore, the only nonvanishing matrix elements are of the ph type, $\hat{\rho}_{i\alpha}^{(1)}$. Taking the ph matrix elements of Eq. (3.20) while noting Eqs. (3.17) and (3.19) one obtains the result

$$[\varepsilon_i - \varepsilon_\alpha - \Omega_C(K_i - K_\alpha)]\hat{\rho}_{i\alpha}^{(1)} + \sum_{j\beta} (\mathcal{V}_{i\beta,\alpha j}\hat{\rho}_{j\beta}^{(1)} + \mathcal{V}_{ij,\alpha\beta}\hat{\rho}_{j\beta}^{(1)*}) = 0. \tag{3.24}$$

Since the two-body interaction commutes with J_3, it follows that

$$\mathcal{V}_{ab,cd}(K_a + K_b - K_c - K_d) = 0, \tag{3.25}$$

which implies that the only contributing terms in the first sum on the left in (3.24) are those satisfying $K_j - K_\beta = K_i - K_\alpha$ and, in the second, those with $K_j - K_\beta = -(K_i - K_\alpha)$. Consequently, Eq. (3.24) can be factored into separate blocks, each labeled by a constant value $K \equiv K_i - K_\alpha$. Of course, a further factorization of the blocks may be possible based on other constants of motion, such as parity, and, for spherical nuclei, total angular momentum, but these possibilities will not be explicitly exhibited.

The corresponding density matrix elements will be denoted by $\hat{\rho}_{i\alpha}^{(1)}[K]$, indicating that $K_i - K_\alpha = K$. In general, $\hat{\rho}_{ab}^{(n)}[K]$ denotes the nth-order correction to a density matrix element such that $K_a - K_b = K$. Correspondingly, Eq. (3.24), together with the equation obtained by replacing K by $-K$ and then taking the complex conjugate, is

$$\begin{aligned}(\varepsilon_i - \varepsilon_\alpha - \Omega_C K)\hat{\rho}_{i\alpha}^{(1)}[K] + \sum_{j\beta} (\mathcal{V}_{i\beta,\alpha j}\hat{\rho}_{j\beta}^{(1)}[K] + \mathcal{V}_{ij,\alpha\beta}\hat{\rho}_{j\beta}^{(1)*}[-K]) = 0,\\ (\varepsilon_i - \varepsilon_\alpha + \Omega_C K)\hat{\rho}_{i\alpha}^{(1)*}[-K] + \sum_{j\beta} (\mathcal{V}_{i\beta,\alpha j}^{*}\hat{\rho}_{j\beta}^{(1)*}[-K] + \mathcal{V}_{ij,\alpha\beta}^{*}\hat{\rho}_{j\beta}^{(1)}[K]) = 0.\end{aligned} \tag{3.26}$$

If one defines $\omega_K^{(0)} \equiv \Omega_C K$, then Eqs. (3.26) can be written in the compact supermatrix form

$$\begin{pmatrix} \mathcal{A}[K] & \mathcal{B}[K] \\ \mathcal{B}^*[-K] & \mathcal{A}^*[-K] \end{pmatrix} \begin{pmatrix} \underline{\hat{\rho}}^{(1)}[K] \\ \underline{\hat{\rho}}^{(1)*}[-K] \end{pmatrix} = \omega_K^{(0)} \begin{pmatrix} \underline{\hat{\rho}}^{(1)}[K] \\ -\underline{\hat{\rho}}^{(1)*}[-K] \end{pmatrix}, \tag{3.27}$$

where in general the underscored object $\underline{\hat{\rho}}^{(n)}[K]$ is defined as a column vector formed from the ph matrix elements $\hat{\rho}_{i\alpha}^{(n)}[K]$, and the matrices $\mathcal{A}$ and $\mathcal{B}$ are defined by

$$\mathcal{A}_{i\alpha,j\beta}[K]\equiv(\varepsilon_i-\varepsilon_\alpha)\delta_{i,j}\delta_{\alpha,\beta}+\mathcal{V}_{i\beta,\alpha j}$$
$$\text{when } K_i-K_\alpha=K_j-K_\beta=K,$$

$$\mathcal{B}_{i\alpha,j\beta}[K]\equiv\mathcal{V}_{ij,\alpha\beta} \text{ when } K_i-K_\alpha=K_\beta-K_j=K. \tag{3.28}$$

Equation (3.27) is easily identified as the RPA eigenfrequency-eigenvector equation[3] for an excitation mode that carries K units of angular momentum along the symmetry axis. The implications are now clear: A nontrivial solution for $\hat{\rho}^{(1)}$ is possible in general only for a discrete set of cranking frequencies given by

$$\Omega_C=\omega_K^{(0)}/K, \quad K\neq 0, \tag{3.29}$$

where $\omega_K^{(0)}$ is any one of the RPA frequencies. If $K=0$, the RHS of (3.27) vanishes, so that in general only the trivial solution exists. Once a particular mode is chosen as ''active,'' the corresponding density matrix $\hat{\rho}^{(1)}[\pm K]$ has nonzero elements, but the density matrices for all other modes vanish.

The normalization of the RPA eigenvectors can be determined by considering the angular momentum. It will be assumed that in zeroth order the angular momentum is given by

$$\langle J_3\rangle^{(0)}=\mathrm{Tr}\,\hat{\rho}^{(0)}j_3=I_0, \tag{3.30}$$

where I_0 can have either sign in principle. For the ground state of an even-even nucleus $I_0=0$, in which case time-reversal conjugate single-particle levels are filled pairwise. The nonzero values occur, for example, for rotational-band terminus states and other axially symmetric large-K states, and also for states in odd nuclei, when the time-reversal conjugate single-particle levels are not all filled pairwise. From Eqs. (3.12) and (3.23) it follows that the first-order change in the angular momentum vanishes:

$$\langle J_3\rangle^{(1)}=\mathrm{Tr}\,\hat{\rho}^{(1)}j_3=0. \tag{3.31}$$

The second-order change $\langle J_3\rangle^{(2)}$ requires the diagonal matrix elements of $\hat{\rho}^{(2)}$, which can be obtained from the pp and hh matrix elements as given by Eq. (3.21) for $n=2$ as follows:

$$\hat{\rho}_{ij}^{(2)}[0]=\sum_\alpha\,(\hat{\rho}_{i\alpha}^{(1)}[K]\hat{\rho}_{j\alpha}^{(1)*}[K]+\hat{\rho}_{i\alpha}^{(1)}[-K]\hat{\rho}_{j\alpha}^{(1)*}[-K]),$$

$$\hat{\rho}_{ij}^{(2)}[\pm 2K]=\sum_\alpha\,\hat{\rho}_{i\alpha}^{(1)}[\pm K]\hat{\rho}_{j\alpha}^{(1)*}[\mp K],$$

$$\hat{\rho}_{\alpha\beta}^{(2)}[0]=-\sum_i\,(\hat{\rho}_{i\alpha}^{(1)*}[K]\hat{\rho}_{i\beta}^{(1)}[K]+\hat{\rho}_{i\alpha}^{(1)*}[-K]\hat{\rho}_{i\beta}^{(1)}[-K]),$$

$$\hat{\rho}_{\alpha\beta}^{(2)}[\pm 2K]=-\sum_i\,\hat{\rho}_{i\alpha}^{(1)*}[\mp K]\hat{\rho}_{i\beta}^{(1)}[\pm K]. \tag{3.32}$$

To satisfy the angular-momentum constraint (3.11), it follows from (3.32) that

$$\langle J_3\rangle^{(2)}=\mathrm{Tr}\,\hat{\rho}^{(2)}j_3=K\sum_{i\alpha}\,(\hat{\rho}_{i\alpha}^{(1)}[K]\hat{\rho}_{i\alpha}^{(1)*}[K]-\hat{\rho}_{i\alpha}^{(1)}[-K]\hat{\rho}_{i\alpha}^{(1)*}[-K])=I-I_0. \tag{3.33}$$

This establishes the normalization of the RPA eigenvectors as $(I-I_0)/K$, which is just the number of phonons, each carrying K units, required to change the angular momentum by $I-I_0$ units.

The total energy (3.2) (laboratory frame) is straightforward to evaluate in the RPA order. Since the CHF solution is stationary, $\langle H\rangle$ is a quadratic form in the $\hat{\rho}_{i\alpha}^{(1)}[K]$, $\hat{\rho}_{i\alpha}^{(1)*}[K]$, in fact just the RPA Hamiltonian if the density matrix elements are interpreted as bosons. Therefore, through the RPA order

$$\begin{aligned}\langle H\rangle&=E_0(I_0)+(\underline{\hat{\rho}}^{(1)\dagger}[K]\quad\underline{\hat{\rho}}^{(1)T}[-K])\\&\times\begin{pmatrix}\mathcal{A}[K]&\mathcal{B}[K]\\\mathcal{B}^*[-K]&\mathcal{A}^*[-K]\end{pmatrix}\begin{pmatrix}\underline{\hat{\rho}}^{(1)}[K]\\\underline{\hat{\rho}}^{(1)*}[-K]\end{pmatrix}\\&=E_0(I_0)+\omega_K^{(0)}(\underline{\hat{\rho}}^{(1)\dagger}[K]\quad\underline{\hat{\rho}}^{(1)T}[-K])\\&\times\begin{pmatrix}\underline{\hat{\rho}}^{(1)}[K]\\-\underline{\hat{\rho}}^{(1)*}[-K]\end{pmatrix}\\&=E_0(I_0)+(\omega_K^{(0)}/K)(I-I_0),\end{aligned} \tag{3.34}$$

where $E_0(I_0)$ is the zero-order energy, $\underline{\hat{\rho}}^{(1)T}$ is the row vector obtained by transposing the column vector $\underline{\hat{\rho}}^{(1)}$, and Eq. (3.27) was used to obtain the second equation and Eq. (3.33) to obtain the third. The final result agrees perfectly with Eq. (2.10).

2. Second order

Since the pp and hh matrix elements of $\hat{\rho}^{(2)}$ have already been found from the condition $\hat{\rho}^2=\hat{\rho}$ and [Eqs. (3.32)], the ph matrix elements may be found from Eq. (3.20). It is easily shown that Eq. (3.20) is also consistent with the pp and hh matrix elements as given by Eqs. (3.32) for the case $n=2$. It should be noted that the term in (3.20) proportional to $\theta\xi^2$ vanishes since $[j_3,\rho^{(0)}]=0$. One then readily obtains in matrix form the following two inhomogeneous equations for the ph matrix of $\hat{\rho}^{(2)}$:

$$\begin{pmatrix}\mathcal{A}[0]&\mathcal{B}[0]\\\mathcal{B}^*[0]&\mathcal{A}^*[0]\end{pmatrix}\begin{pmatrix}\underline{\hat{\rho}}^{(2)}[0]\\\underline{\hat{\rho}}^{(2)*}[0]\end{pmatrix}=\begin{pmatrix}\underline{R}^{(2)}[0]\\\underline{R}^{(2)*}[0]\end{pmatrix} \tag{3.35}$$

and

[3] In a more familiar notation, $\hat{\rho}_{i\alpha}^{(1)}[K]\propto X_{i\alpha}[K]$ and $\hat{\rho}_{i\alpha}^{(1)*}[-K]\propto Y_{i\alpha}[-K]$.

$$\begin{pmatrix} \mathcal{A}[2K]-2\omega_K^{(0)}\mathcal{I} & \mathcal{B}[2K] \\ \mathcal{B}^*[-2K] & \mathcal{A}^*[-2K]+2\omega_K^{(0)}\mathcal{I} \end{pmatrix} \begin{pmatrix} \underline{\hat{\rho}}^{(2)}[2K] \\ \underline{\hat{\rho}}^{(2)*}[-2K] \end{pmatrix} = \begin{pmatrix} \underline{R}^{(2)}[2K] \\ \underline{R}^{(2)*}[-2K] \end{pmatrix}, \tag{3.36}$$

where $\mathcal{I}$ is an identity matrix and $\underline{R}^{(2)}[0]$ and $\underline{R}^{(2)}[2K]$ are column vectors constructed from the following ph matrix elements:

$$R_{i\alpha}^{(2)}[0] \equiv [\hat{\rho}^{(1)}[K],\hat{u}^{(1)}[-K]]_{i\alpha}+[\hat{\rho}^{(1)}[-K],\hat{u}^{(1)}[K]]_{i\alpha} - \hat{u}_{i\alpha}^{\prime(2)}[0] \tag{3.37}$$

and

$$R_{i\alpha}^{(2)}[2K] \equiv [\hat{\rho}^{(1)}[K],\hat{u}^{(1)}[K]]_{i\alpha}-\hat{u}_{i\alpha}^{\prime(2)}[2K], \tag{3.38}$$

where [.,.] refers to the commutator of two matrices. The matrix $\hat{u}^{(1)}[K]$, the first-order change in the self-consistent field [Eq. (3.17)], is explicitly given by

$$\hat{u}_{i\alpha}^{(1)}[K]=\sum_{j\beta}\ (\mathcal{V}_{i\beta,\alpha j}\hat{\rho}_{j\beta}^{(1)}[K]+\mathcal{V}_{ij,\alpha\beta}\hat{\rho}_{j\beta}^{(1)*}[-K]), \tag{3.39}$$

while $\hat{u}_{i\alpha}^{\prime(2)}$ is the part of $\hat{u}_{i\alpha}^{(2)}$ arising from the pp and hh matrix elements of $\hat{\rho}^{(2)}$, which for any $n\geqslant 2$ and any integer K is defined by

$$\hat{u}_{i\alpha}^{\prime(n)}[K]=\sum_{jk}\ \mathcal{V}_{ik,\alpha j}\hat{\rho}_{jk}^{(n)}[-K]+\sum_{\beta\gamma}\ \mathcal{V}_{i\gamma,\alpha\beta}\hat{\rho}_{\beta\gamma}^{(n)}[-K]. \tag{3.40}$$

The second-order density matrix is especially useful for determining the leading-order corrections to static electromagnetic moments. For example, in the HFB analog it has been applied to the problem of calculating the static quadrupole moments of one-phonon quadrupole vibrations in spherical nuclei, for which the solution of Eq. (3.35) suffices [22,26]. However, to continue to higher orders, one must also solve Eq. (3.36). It should then be noticed that because of the presence of the submatrix $\mathcal{A}[2K]-2\omega_K^{(0)}\mathcal{I}$, the inversion of the matrix in Eq. (3.36) may give rise to resonance denominators. This is just the problem of commensurate frequencies mentioned in Sec. II reappearing now in a different guise. The solution of this problem within the framework of degenerate perturbation theory will not be discussed here. The simpler solution ultimately is to exactly diagonalize the CHF equations rather than use perturbation theory.

3. Third-order–leading-order corrections to the RPA energy

The pp and hh matrix elements of $\hat{\rho}^{(3)}$ are obtained from Eq. (3.21) for $n=3$, the nonvanishing ones being of the following types:

$$\hat{\rho}_{ij}^{(3)}[\pm K]=\sum_{\alpha}\ (\hat{\rho}_{i\alpha}^{(1)}[\pm K]\hat{\rho}_{j\alpha}^{(2)*}[0]+\hat{\rho}_{i\alpha}^{(2)}[0]\hat{\rho}_{j\alpha}^{(1)*}[\mp K]+\hat{\rho}_{i\alpha}^{(1)}[\mp K]\hat{\rho}_{j\alpha}^{(2)*}[\mp 2K]+\hat{\rho}_{i\alpha}^{(2)}[\pm 2K]\hat{\rho}_{j\alpha}^{((1)*}[\pm K]),$$

$$\hat{\rho}_{\alpha\beta}^{(3)}[\pm K]=-\sum_{i}\ (\hat{\rho}_{i\alpha}^{(1)*}[\pm K]\hat{\rho}_{i\beta}^{(2)}[\pm 2K]+\hat{\rho}_{i\alpha}^{(2)*}[\mp 2K]\hat{\rho}_{i\beta}^{(1)}[\mp K]+\hat{\rho}_{i\alpha}^{(1)*}[\mp K]\hat{\rho}_{i\beta}^{(2)}[0]+\hat{\rho}_{i\alpha}^{(2)*}[0]\hat{\rho}_{i\beta}^{(1)}[\pm K]), \tag{3.41}$$

$$\hat{\rho}_{ij}^{(3)}[\pm 3K]=\sum_{\alpha}\ (\hat{\rho}_{i\alpha}^{(1)}[\pm K]\hat{\rho}_{j\alpha}^{(2)*}[\mp 2K]+\hat{\rho}_{i\alpha}^{(2)}[\pm 2K]\hat{\rho}_{j\alpha}^{(1)*}[\mp K]),$$

$$\hat{\rho}_{\alpha\beta}^{(3)}[\pm 3K]=-\sum_{i}\ (\hat{\rho}_{i\alpha}^{(1)*}[\mp K]\hat{\rho}_{i\beta}^{(2)}[\pm 2K]+\hat{\rho}_{i\alpha}^{(2)*}[\mp 2K]\hat{\rho}_{i\beta}^{(1)}[\pm K]).$$

This immediately implies that all diagonal matrix elements of $\hat{\rho}^{(3)}$ vanish and therefore that

$$\langle J_3\rangle^{(3)}=\mathrm{Tr}\hat{\rho}^{(3)}j_3=0, \tag{3.42}$$

guaranteeing that the condition $\langle J_3\rangle=I$, fixed by Eqs. (3.30) – (3.33), continues to hold through third order. The ph matrix elements of $\hat{\rho}^{(3)}$ are then obtained by taking the ph matrix elements of Eq. (3.20) with $n=3$. In this way one finds two inhomogeneous matrix equations, the first being

$$\begin{pmatrix} \mathcal{A}[K]-\omega_K^{(0)}\mathcal{I} & \mathcal{B}[K] \\ \mathcal{B}^*[-K] & \mathcal{A}^*[-K]+\omega_K^{(0)}\mathcal{I} \end{pmatrix} \begin{pmatrix} \underline{\hat{\rho}}^{(3)}[K] \\ \underline{\hat{\rho}}^{(3)*}[-K] \end{pmatrix} = \begin{pmatrix} \underline{R}^{(3)}[K] \\ \underline{R}^{(3)*}[-K] \end{pmatrix} + (\Omega-\Omega_C)K \begin{pmatrix} \underline{\hat{\rho}}^{(1)}[K] \\ -\underline{\hat{\rho}}^{(1)*}[-K] \end{pmatrix} \tag{3.43}$$

and the second

$$\begin{pmatrix} \mathcal{A}[3K]-3\omega_K^{(0)}\mathcal{I} & \mathcal{B}[3K] \\ \mathcal{B}^*[-3K] & \mathcal{A}^*[-3K]+3\omega_K^{(0)}\mathcal{I} \end{pmatrix} \begin{pmatrix} \underline{\hat{\rho}}^{(3)}[3K] \\ \underline{\hat{\rho}}^{(3)*}[-3K] \end{pmatrix} = \begin{pmatrix} \underline{R}^{(3)}[3K] \\ \underline{R}^{(3)*}[-3K] \end{pmatrix}, \tag{3.44}$$

where $\underline{R}^{(3)}[K]$ is a column vector composed of the ph matrix elements

$$R^{(3)}_{i\alpha}[K]=[\hat{\rho}^{(2)}[0],\hat{u}^{(1)}[K]]_{i\alpha}+[\hat{\rho}^{(2)}[2K],\hat{u}^{(1)}[-K]]_{i\alpha}+[\hat{\rho}^{(1)}[K],\hat{u}^{(2)}[0]]_{i\alpha}+[\hat{\rho}^{(1)}[-K],\hat{u}^{(2)}[2K]]_{i\alpha}-u'^{(3)}_{i\alpha}[K], \tag{3.45}$$

while $\underline{R}^{(3)}[3K]$ is a column vector composed of the ph matrix elements,

$$R^{(3)}_{i\alpha}[3K]\equiv[\hat{\rho}^{(2)}[2K],\hat{u}^{(1)}[K]]_{i\alpha}+[\hat{\rho}^{(1)}[K],\hat{u}^{(2)}[2K]]_{i\alpha}-u'^{(3)}_{i\alpha}[3K], \tag{3.46}$$

with $u'^{(3)}_{i\alpha}[K]$, $u'^{(3)}_{i\alpha}[3K]$ defined by Eq. (3.40). The term explicitly proportional to $\Omega-\Omega_C$ in (3.43), the role of which will be discussed momentarily, corresponds to the term proportional to $\theta\xi^2$ in Eq. (3.20), which contributes for the first time in the present order.

If one wishes only to calculate the correction to the energy, then it is not necessary to actually solve for $\hat{\rho}^{(3)}$ or to even consider Eq. (3.44). Instead, one may multiply both sides of Eq. (3.43) by the row vector $(\underline{\hat{\rho}}^{(1)\dagger}[K]\ \ \underline{\hat{\rho}}^{(1)T}[-K])$. The LHS of the result vanishes, as implied by the H.c. of the RPA Eq. (3.27), leaving

$$(\underline{\hat{\rho}}^{(1)\dagger}[K]\quad \underline{\hat{\rho}}^{(1)T}[-K])\begin{pmatrix}\underline{R}^{(3)}[K]\\ \underline{R}^{(3)*}[-K]\end{pmatrix}+(\Omega-\Omega_C)K(\underline{\hat{\rho}}^{(1)\dagger}[K]\quad \underline{\hat{\rho}}^{(1)T}[-K])\begin{pmatrix}\underline{\hat{\rho}}^{(1)}[K]\\ -\underline{\hat{\rho}}^{(1)*}[-K]\end{pmatrix}=0. \tag{3.47}$$

But from the RPA normalization condition (3.33), this result may be written as

$$\Omega-\Omega_C=-\left(\frac{1}{I-I_0}\right)(\underline{\hat{\rho}}^{(1)\dagger}[K]\quad \underline{\hat{\rho}}^{(1)T}[-K])\begin{pmatrix}\underline{R}^{(3)}[K]\\ \underline{R}^{(3)*}[-K]\end{pmatrix}, \tag{3.48}$$

which just corresponds to Eq. (2.14) of the BG method. Now, one expects the energy, including the leading-order correction to the RPA energy (3.34), to have the form of an expansion in powers of $(I-I_0)/K$:

$$E=E_0(I_0)+\omega^{(0)}_K\left(\frac{I-I_0}{K}\right)+\frac{1}{2}\alpha_K\left(\frac{I-I_0}{K}\right)^2. \tag{3.49}$$

From Eq. (2.16), which is implied by the Hellmann-Feynman theorem for the CHF approximation, it then follows that

$$\Omega-\Omega_C=\alpha_K\left(\frac{I-I_0}{K}\right) \tag{3.50}$$

to the given order of approximation. Comparison with (3.48) then establishes the coefficient α_K as

$$\alpha_K=-\frac{K}{(\mathrm{Tr}\,\hat{\rho}^{(2)}j_3)^2}(\underline{\hat{\rho}}^{(1)\dagger}[K]\quad \underline{\hat{\rho}}^{(1)T}[-K])\begin{pmatrix}\underline{R}^{(3)}[K]\\ \underline{R}^{(3)*}[-K]\end{pmatrix}, \tag{3.51}$$

with the use of Eq. (3.33). From the discussion in Sec. II, the result (3.49) is equivalent to that obtained from cranking the BG normal form including quartic anharmonic terms.

Exploratory calculations of α_K were made for quadrupole vibrations ($K=2$) of spherical nuclei in the Sn region in Ref. [26] using the HFB counterpart of Eq. (3.51) with the pairing plus quadrupole-quadrupole interaction. Extensive modern calculations of this parameter for quadrupole excitations built on the ground state ($I_0=0$) would be highly desirable in view of the recent claim of a ''universal'' value for spherical nuclei by Casten and collaborators [11].

B. Transformation to the laboratory frame and the Lindstedt method

Although all physical quantities of interest can be calculated with the aid of $\hat{\rho}$, the density matrix in the rotating frame, it is instructive to transform it back to the laboratory frame, using Eq. (3.8). Since the angular-momentum matrix j_3 is diagonal, so is $e^{\pm i\Omega t j_3}$, which then trivially leads to the result

$$\rho_{ab}=\hat{\rho}_{ab}e^{i\Omega t(K_b-K_a)} \tag{3.52}$$

for the density matrix in the laboratory frame. If $K_b-K_a=nK$, where n is a positive or negative integer, then

$$\rho_{ab}[nK]=\hat{\rho}_{ab}[nK]e^{in\omega_K t}, \tag{3.53}$$

where

$$\omega_K\equiv K\Omega, \tag{3.54}$$

Ω, being defined by Eq. (3.48) in lowest order, is the exact nonlinear frequency, not the RPA frequency $\omega^{(0)}_K$ defined by Eq. (3.29). The density matrix in the laboratory frame can then be decomposed as follows:

$$\rho(t)=\rho^{(0)}+\rho^{(1)}+\rho^{(2)}+\rho^{(3)}+\cdots, \tag{3.55}$$

where

$$\rho^{(1)}=\hat{\rho}^{(1)}[K]e^{-i\omega_K t}+\hat{\rho}^{(1)}[-K]e^{i\omega_K t}, \tag{3.56}$$

$$\rho^{(2)}=\hat{\rho}^{(2)}[0]+\hat{\rho}^{(2)}[2K]e^{-i2\omega_K t}+\hat{\rho}^{(2)}[-2K]e^{i2\omega_K t}, \tag{3.57}$$

and

$$\rho^{(3)}=\hat{\rho}^{(3)}[K]e^{-i\omega_K t}+\hat{\rho}^{(3)}[-K]e^{i\omega_K t}+\hat{\rho}^{(3)}[3K]e^{-i3\omega_K t}+\hat{\rho}^{(3)}[-3K]e^{i3\omega_K t}. \tag{3.58}$$

Here it is understood that the matrix elements $\hat{\rho}^{(m)}_{ab}[nK]$ are zero unless $K_a-K_b=nK$. Also, since $\rho^{(m)}$ is Hermitian,

$$\hat{\rho}^{(m)}[-nK]=\hat{\rho}^{(m)\dagger}[nK]. \tag{3.59}$$

Thus, the time-dependent solution in the laboratory frame is obtained as a Fourier expansion.

One can interpret Eqs. (3.55)–(3.58) as a boson expansion by first introducing the BG coordinates with the time dependence given by

$$\beta_K\equiv\left(\frac{I-I_0}{K}\right)^{1/2}e^{-i\omega_K t},\quad \beta_K^*=\left(\frac{I-I_0}{K}\right)^{1/2}e^{i\omega_K t}, \tag{3.60}$$

so that, for example, $e^{i2\omega_K t}\rightarrow[K/(I-I_0)](\beta_K^*)^2$, etc., while constant terms such as $\hat{\rho}^{(2)}[0]$ are multiplied by $[K/(I-I_0)]$ $\beta_K^*\beta_K=[K/(I-I_0)]N_K$. In this way, one obtains an expansion of the noncanonical density matrix elements in terms of the canonical variables for a particular excitation mode. Upon quantization, the density matrix elements map into fermion-pair operators and the canonical variables into boson operators. Of course, such a treatment has the usual problems of operator ordering which may not correctly reproduce quantal contraction terms that can be computed by working with a quantized boson mapping from the outset.

As an alternative to the cranking approach, one may treat the TDHF equation (3.6) by direct dynamical perturbation theory as in the work of Meyer [22] and of Abada and Vautherin [23]. A convenient way to carry out this program in a way that avoids the problem of secular terms is to use the method of Lindstedt [48]. The first step in this method is the introduction of the dimensionless time variable τ given by

$$\tau\equiv\omega t, \tag{3.61}$$

whereupon the TDHF equation (3.6) becomes

$$i\omega\frac{d\rho}{d\tau}=[h,\rho]. \tag{3.62}$$

The next step involves the expansion of ρ, h, and the frequency ω in powers of the amplitude:

$$\rho=\rho^{(0)}+\rho^{(1)}+\rho^{(2)}+\rho^{(3)}+\cdots+\rho^{(n)}+\cdots, \tag{3.63}$$

$$h=h^{(0)}+u^{(1)}+u^{(2)}+\cdots+u^{(n)}+\cdots, \tag{3.64}$$

$$\omega=\omega^{(0)}+\omega^{(1)}+\omega^{(2)}+\cdots+\omega^{(n-1)}+\cdots. \tag{3.65}$$

Upon separation of orders, one obtains of course in the first order the RPA equations of motion, including modes with $K=0$, as well as those treated earlier, and $\omega^{(0)}$ is identified with one of the RPA frequencies. In each higher order, an inhomogeneous linear differential equation must be solved involving the RPA kernal. The frequency expansion (3.65) is then exploited to eliminate secular terms in the solution. In second order, this is easily accomplished by letting $\omega^{(1)}=0$, tantamount to ignoring the frequency renormalization. In third order, however, a nonzero value of $\omega^{(2)}$ is required, which is just proportional to the frequency shift in Eq. (3.48) for a mode with $K\neq 0$.

To make a long story short, for modes with $K\neq 0$, the Lindstedt solution is identical with the cranking solution referred to the laboratory frame, as given by Eqs. (3.55) – (3.58), which is, of course, what one expects. The Lindstedt method has one advantage over perturbed cranking, namely, that it can describe modes with $K=0$, although otherwise it is a little more cumbersome. On the other hand, the Lindstedt method is limited to perturbation theory, whereas the self-consistent cranking equations can be diagonalized exactly, which is a major advantage for modes with $K\neq 0$.

C. CBT and the vibrating potential model

The vibrating potential model (VPM) [49,50] is a purely phenomenological TDMF approximation. Whereas the TDHF approximation is based on an *ab initio* calculation of the fluctuating mean field from a given internucleon interaction, the VPM begins directly with a time-dependent mean field derived from an empirically successful static deformed potential model such as the Nilsson model by allowing the deformation parameters to acquire a time dependence. The empirical mean field can be used in reverse to derive an effective internucleon interaction [51–53], which necessarily has many-body components. In effect, this means that the VPM can be regarded as an application of the time-dependent Hartree[4] or Hartree-Bogoliubov (when the pairing potential is included) approximation. In the VPM, the mean-field Hamiltonian for A nucleons $H_{\rm MF}(\underline{\alpha}(t))$, where $\underline{\alpha}(t)$ denotes a set of time-dependent deformation parameters, is defined so that the total energy $E=\langle H_{\rm MF}(\underline{\alpha}(t))\rangle$, the expectation value with respect to the TDMF state vector,[5] which is a solution of the time-dependent Schrödinger equation corresponding to the Hamiltonian $H_{\rm MF}(\underline{\alpha}(t))$. The deformation parameters $\underline{\alpha}(t)$ are defined to guarantee nuclear incompressibility in some manner, usually by requiring volume conservation, the condition that the volumes of equipotential surfaces be independent of deformation. In addition, at each instant, the shape consistency between the potential and density distribution is secured by imposing the set of conditions $\langle\partial H_{\rm MF}(\underline{\alpha}(t))/\partial\alpha^i\rangle=0$, which also guarantee that E is a constant of motion.

The proposal here is to replace the time-dependent treatment of the VPM with the static cranking model (CVPM). Thus, if $|\Phi_\Omega(\underline{\alpha})\rangle$ is an eigenvector of the Routhian

$$H'_{\rm MF}(\underline{\alpha},\Omega)\equiv H_{\rm MF}(\underline{\alpha})-\Omega J_3, \tag{3.66}$$

one must seek solutions satisfying the extremum conditions

$$\left\langle\Phi_\Omega(\underline{\alpha})\left|\frac{\partial H'_{\rm MF}(\underline{\alpha},\Omega)}{\partial\alpha^i}\right|\Phi_\Omega(\underline{\alpha})\right\rangle =\left\langle\Phi_\Omega(\underline{\alpha})\left|\frac{\partial H_{\rm MF}(\underline{\alpha})}{\partial\alpha^i}\right|\Phi_\Omega(\underline{\alpha})\right\rangle=0. \tag{3.67}$$

Of course, since these equations also include ordinary cranking solutions, one must locate those that bifurcate from axi-

[4]The exchange term is not included in the VPM.

[5]This means that $H_{\rm MF}(\underline{\alpha}(t))$ already contains the corrections to prevent overcounting of internucleon interactions.

ally symmetric configurations. It will now be demonstrated that these bifurcation points are determined by the RPA equations for the VPM.

Consider first the case of the axially symmetric ground-state configuration of an even-even nucleus, denoted by $|0\rangle$, with orthogonal excited configurations denoted by $|n\rangle$. With the three-axis designated as the symmetry axis, one has

$$J_3|0\rangle=0, \quad J_3|n\rangle=K_n. \tag{3.68}$$

Since $|0\rangle$ is a self-consistent solution of Eqs. (3.67) for $\Omega=0$, corresponding to the equilibrium deformation parameter set $\underline{\alpha}(0)$, it satisfies the conditions

$$\left\langle 0\left|\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha^i}\right)_0\right|0\right\rangle=0, \tag{3.69}$$

with the notation $(\partial H_{\mathrm{MF}}/\partial\alpha^i)_0\equiv(\partial H_{\mathrm{MF}}/\partial\alpha^i)_{\underline{\alpha}=\underline{\alpha}(0)}$. For simplicity, it will first be assumed that cranking changes only one of the deformation parameters, denoted simply by α, which is assumed to be real and defined so that

$$\left[J_3,\left[J_3,\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha}\right)_0\right]\right]=K^2\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha}\right)_0; \tag{3.70}$$

i.e., the operator $(\partial H_{\mathrm{MF}}/\partial\alpha)_0$ transfers $\pm K$ units of angular momentum along the three-axis. In the neighborhood of a bifurcation point, the Routhian (3.66) may be expanded as

$$H'_{\mathrm{MF}}(\underline{\alpha},\Omega)\simeq H_{\mathrm{MF}}(\underline{\alpha}(0))-\Omega J_3+\delta\alpha\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha}\right)_0 \tag{3.71}$$

to first order in the deformation change $\delta\alpha$. Since $|0\rangle$ is the ground state of the zero-order Routhian $H_{\mathrm{MF}}(\underline{\alpha}(0))-\Omega J_3$, the perturbed ground state to first order is given by

$$\Phi_\Omega(\underline{\alpha})\simeq|0\rangle-\delta\alpha\sum_{n\neq0}\frac{\langle n|(\partial H_{\mathrm{MF}}/\partial\alpha)_0|0\rangle}{E_n-E_0-\Omega K_n}|n\rangle, \tag{3.72}$$

where $H_{\mathrm{MF}}(\underline{\alpha}(0))|0\rangle=E_0|0\rangle$ and $H_{\mathrm{MF}}(\underline{\alpha}(0))|n\rangle=E_n|n\rangle$. The self-consistency condition (3.67) may be evaluated to first order in $\delta\alpha$ using the expansion

$$\left\langle\Phi_\Omega(\underline{\alpha})\left|\frac{\partial H_{\mathrm{MF}}(\underline{\alpha})}{\partial\alpha}\right|\Phi_\Omega(\underline{\alpha})\right\rangle$$

$$\simeq\left\langle\Phi_\Omega(\underline{\alpha})\left|\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha}\right)_0+\delta\alpha\left(\frac{\partial^2H_{\mathrm{MF}}}{\partial\alpha^2}\right)_0\right|\Phi_\Omega(\underline{\alpha})\right\rangle=0, \tag{3.73}$$

which, with the help of (3.72) and the invariance of $|0\rangle$ under time reversal, leads to the following result:

$$\delta\alpha\left[\left\langle 0\left|\left(\frac{\partial^2H_{\mathrm{MF}}}{\partial\alpha^2}\right)_0\right|0\right\rangle-2\sum_{n\neq0}\frac{|\langle n|(\partial H_{\mathrm{MF}}/\partial\alpha)_0|0\rangle|^2(E_n-E_0)}{(E_n-E_0)^2-(\Omega K)^2}\right]=0. \tag{3.74}$$

Therefore, the nontrivial solution $\delta\alpha\neq0$ is possible only if $K\neq0$ and $\omega\equiv\Omega K$ is a solution of the equation

$$2\sum_{n\neq0}\frac{|\langle n|(\partial H_{\mathrm{MF}}/\partial\alpha)_0|0\rangle|^2(E_n-E_0)}{(E_n-E_0)^2-\omega^2}=\left\langle 0\left|\left(\frac{\partial^2H_{\mathrm{MF}}}{\partial\alpha^2}\right)_0\right|0\right\rangle, \tag{3.75}$$

which is just the RPA dispersion formula derived from the VPM and originally applied to γ vibrations of even-even deformed nuclei [49].

The use of a single deformation parameter may be adequate in the RPA order if it is judiciously chosen, but generally not in higher orders. For example, if the axial asymmetry parameter $\alpha=\gamma$ is chosen to describe γ vibrations in the RPA order, then in higher orders there will be a change $\delta\beta$ in the deformation parameter β due to anharmonic couplings between γ and β vibrations. Even in the RPA order, several deformation parameters α^i_K may be associated with a particular mode carrying K units of angular momentum along the symmetry axis, for example, parameters that take into account the coupling of quadrupole and hexadecapole degrees of freedom with $|K|=2$. The RPA equations for the more general case in which

$$J_3|0\rangle=I_0|0\rangle \tag{3.76}$$

can now readily be derived by the cranking method. The deformation parameters α^i_K, which in general are complex, may be chosen to satisfy

$$\left[J_3,\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha^i_K}\right)_0\right]=-K\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha^i_K}\right)_0, \tag{3.77}$$

where $\alpha^i_{-K}=(-1)^K\alpha^{i*}_K$. This property is most appropriate for mean fields arising from multipole interactions; a somewhat different parametrization would be more suitable for pairing fields with nonzero angular momentum, which can also be accommodated in this framework. The first-order state vector now becomes

$$\Phi_\Omega(\underline{\alpha})\simeq|0\rangle-\sum_{i,K}\delta\alpha^i_K\sum_{n\neq0}\frac{\langle n|(\partial H_{\mathrm{MF}}/\partial\alpha^i_K)_0|0\rangle}{E_n-E_0+\Omega K}|n\rangle. \tag{3.78}$$

The generalization of the first-order self-consistency condition (3.73),

$$\left\langle\Phi_\Omega(\underline{\alpha})\left|\left(\frac{\partial H_{\mathrm{MF}}}{\partial\alpha^i_K}\right)_0+\sum_{j,K'}\delta\alpha^j_{K'}\left(\frac{\partial^2H_{\mathrm{MF}}}{\partial\alpha^i_K\partial\alpha^j_{K'}}\right)_0\right|\Phi_\Omega(\underline{\alpha})\right\rangle=0, \tag{3.79}$$

then leads to the following set of equations for each K (and parity, etc.):

$$\sum_j \delta\alpha_K^j \Bigg[\left\langle 0 \left| \left(\frac{\partial^2 H_{\rm MF}}{\partial\alpha_K^{i*}\partial\alpha_K^j} \right)_0 \right| 0 \right\rangle - \sum_{n\neq 0} \frac{\langle 0|(\partial H_{\rm MF}/\partial\alpha_K^{i*})_0|n\rangle\langle n|(\partial H_{\rm MF}/\partial\alpha_K^j)_0|0\rangle}{E_n - E_0 + \Omega K} - \sum_{n\neq 0} \frac{\langle 0|(\partial H_{\rm MF}/\partial\alpha_K^j)_0|n\rangle\langle n|(\partial H_{\rm MF}/\partial\alpha_K^{i*})_0|0\rangle}{E_n - E_0 - \Omega K} \Bigg] = 0. \tag{3.80}$$

This system of linear homogeneous equations in the $\delta\alpha_K^{j*}$ has nontrivial solutions if and only if its determinant has solutions for $\omega \equiv \Omega K$, which, of course, is possible only if $K \neq 0$. This determinant is just a generalized form of the RPA equation when several modes are coupled, as, for example, quadrupole and hexadecapole modes in an axially symmetric deformed nucleus. In the special case when only two parameters α_K and $\alpha_{-K} = (-1)^K \alpha_K^*$ enter, the determinant simplifies to

$$\sum_{n\neq 0} \frac{|\langle n|(\partial H_{\rm MF}/\partial\alpha_K)_0|0\rangle|^2}{E_n - E_0 + \omega} + \sum_{n\neq 0} \frac{|\langle n|(\partial H_{\rm MF}/\partial\alpha_K^*)_0|0\rangle|^2}{E_n - E_0 - \omega} = \left\langle 0 \left| \left(\frac{\partial^2 H_{\rm MF}}{\partial\alpha_K \partial\alpha_K^*} \right)_0 \right| 0 \right\rangle, \tag{3.81}$$

and a second equation is obtained by allowing $\omega \rightarrow -\omega$, which corresponds physically to the time-reverse obtained by letting $\Omega \rightarrow -\Omega$. However, by allowing both positive and negative solutions for ω, all RPA roots are contained in Eq. (3.81). It should be noted that since the vacuum has the spin projection I_0 along the three-axis, the admixed configurations $|n\rangle$ in the first sum have the spin projections $I_0 + K$, while those in the second sum have the spin projection $I_0 - K$, in accord with Eq. (3.77).

RPA equations similar to (3.81) were first derived by Bès and Chung [54] to describe γ vibrations in odd nuclei and later by Andersson and Krumlinde [32] to describe γ vibrations built on oblate rotational-band termination states, in both cases with the use of the quadrupole-quadrupole (QQ) interaction. Indeed, their equations can be recovered from the preceding ones by defining the mean-field Hamiltonian for the QQ interaction as

$$H_{\rm MF} = H_0 - \chi \sum_{K=-2}^{2} \hat{Q}_K \alpha_K^* + \frac{1}{2}\chi \sum_{K=-2}^{2} \alpha_K^* \alpha_K, \tag{3.82}$$

where H_0 is the part of the mean-field Hamiltonian that is independent of the relevant deformation parameters, $\hat{Q}_K$, $K = \pm 2, \pm 1, 0$, are components of the quadrupole tensor, χ is the strength of the QQ interaction, and the last term on the right is introduced to guarantee that the total energy $E = \langle H_{\rm MF}(\underline{\alpha})\rangle$ and that the VPM self-consistency condition (3.67) is equivalent to the Hartree self-consistency condition

$$\langle \Phi_\Omega(\underline{\alpha})|\hat{Q}_K|\Phi_\Omega(\underline{\alpha})\rangle = \alpha_K. \tag{3.83}$$

In this way, one recovers for $|K| = 2$ the results of Refs. [54,32]. It should also be noted that the QQ force gives results that are very similar to those derived from the VPM potential based on the Nilsson model, the main difference being that in the latter case, the strength χ is determined automatically when evaluating the RHS of Eq. (3.81), whereas in the former case, it is a free parameter. It is also interesting to note that for $|K| = 1$, the nonzero values of the quadrupole moments $\langle \Phi_\Omega(\underline{\alpha})|\hat{Q}_{\pm 1}|\Phi_\Omega(\underline{\alpha})\rangle$ imply that the rotation takes place about an axis that is *tilted* with respect to the principal axes of the mean field. The topic of tilted rotation has been gaining considerable interest lately [55], but the possible connection with vibrational modes has not been widely recognized, although it is discussed in Ref. [27].

The advantage of the cranking approach to the VPM is that one is not limited to the RPA, but can include anharmonicities of all orders, and also cover the case of odd nuclei. In the latter case, the approach of Bès and Chung, which, in contrast to other methods that require coupling of the odd nucleon with the phonons of the even-even core, allows excitations directly from the ground state of the odd nucleus, can be generalized to take into account anharmonicities. Finally, it should be pointed out that the Strutinsky shell-correction method [56] could be applied to the CVPM in much the same way as in the normal cranking of phenomenological potential models. However, some rethinking of this procedure may be required. For example, normalization to a rigidly rotating liquid drop would not be correct for a vibrational band, where the moments of inertia may be significantly smaller than rigid-body values.

IV. TRANSITION PROBABILITIES IN THE CRANKING APPROXIMATION

The discussion thus far has centered on the calculation of energies of multiphonon vibrations using the self-consistent cranking method. The most important remaining question is whether one can also calculate transition rates within this formalism. In short, the answer is yes, provided that the transitions connect states lying along the cranked trajectory and that one is content with a semiclassical estimate. It is well known that the cranked mean-field method in general approximates energies with greater accuracy than wave functions, which carry the added burden of broken symmetries. In particular, the wave functions are not angular-momentum eigenstates. A general remedy is to project out states having good angular momentum as well as other symmetries and to use these states to calculate transition matrix elements. While such an approach is fine, it requires, in its fullest implementation, a formidable amount of computation and takes one a considerable distance away from the relative simplicity of the cranking method. An alternative is provided by the self-consistent cranking plus RPA (SCC+RPA) method [57–59], which has been successfully applied to high-spin states and could just as well be applied to cranked vibrations. Now, the

SCC model, which describes a uniformly rotating system (including rotating waves), generates a ''Regge trajectory'' on an energy (E) vs angular momentum (I) plot. The RPA, in the SCC+RPA, has three principal effects: (i) It displaces the trajectory vertically by a small amount due to the inclusion of the angular-momentum-dependent RPA correlation energy, (ii) it allows for excitations based on the cranked trajectory as the reference state; and (*iii*) it isolates the degrees of freedom associated with broken symmetries. Property (iii) notwithstanding, the SCC+RPA method does not directly generate eigenstates with good quantum numbers owing to the appearance of Goldstone modes corresponding to the constants of motion violated in the SCC step. Nevertheless, as shown in Ref. [58], it is still possible to correctly calculate matrix elements by making a connection between eigenstates and transition operators having good quantum numbers and the corresponding SCC+RPA eigenvectors and transition operators, which depend on the Goldstone degrees of freedom.

Now, the introduction of RPA correlations[6] on top of the cranked SCC trajectory certainly goes beyond the SCC approximation itself, including quantal effects left out in the SCC step. Physically, the excitations represent either waves traveling around the uniformly rotating nucleus or the wobbling motion of the angular momentum relative to a body-fixed frame. The corresponding quantal excitations or deexcitations, as the case may be, take one from the trajectory on the E vs I plot to other points above or below. In Ref. [58], it has been shown, however, that the transition matrix elements connecting successive states lying on the cranked trajectory itself are proportional to expectation values of the transition operator *calculated with respect to the SCC vacuum*. In other words, for such transitions, one need not go beyond the cranked mean-field approximation; the SCC +RPA method is only used to provide the justification. On the other hand, to calculate transitions departing from the cranked trajectory, one must solve for the RPA amplitudes, which is beyond the intended purview of this paper. Since the formal discussion, which depends on connecting the Goldstone modes with an extended Holstein-Primakoff representation of angular momentum, has already been provided in the earlier work on the SCC+RPA, it suffices here to list the most commonly required reduced transition probabilities (RTP's). The electric quadrupole tensor will be denoted by $\mathcal{M}(E2,\mu)$ and the magnetic dipole vector by $\mathcal{M}(M1,\mu)$. The corresponding RTP's between successive points on the cranked trajectory are just given by [58]

$$B(E2;I\rightarrow I-\mu)=|\langle\mathcal{M}(E2,-\mu)\rangle_I|^2,\quad \mu=1,2,$$

$$B(M1;I\rightarrow I-1)=|\langle\mathcal{M}(M1,-1)\rangle_I|^2, \tag{4.1}$$

where $\langle\cdots\rangle_I$ denotes the expectation value of the multipole operator taken with respect to the SCC state for which $\langle J_3\rangle_I=I$. For a normal cranking trajectory that preserves signature symmetry, only the $E2$ transition with $\mu=2$ is applicable since the successive members have $|\Delta I|=2$. However, for a broken-signature trajectory, whose successive members have $|\Delta I|=1$, all of the above transitions are expected to be nonzero, including the $M1$; this would be the situation for tilted rotation. The SCC+RPA gives for the static electric quadrupole moment Q_0 and the static magnetic dipole moment μ_0 for states along the cranked trajectory the expressions that have been traditionally used in cranking calculations, namely,

$$Q_0=\left(\frac{16\pi}{5}\right)^{1/2}\langle\mathcal{M}(E2,0)\rangle_I,\quad \mu_0=\left(\frac{4\pi}{3}\right)\langle\mathcal{M}(M1,0)\rangle_I. \tag{4.2}$$

There are two caveats concerning the above expressions, in particular, the RTP's. First, since the SCC+RPA model assumes that the states connected by a transition lie on a smoothly varying part of the trajectory, application of the above formulas to states lying on different portions of a backbending trajectory and thus having very different structures would be highly questionable [60]. The second caveat is that, strictly speaking, the above expressions, being semiclassical, are valid for large values of the spin I, implying errors of the order of I^{-1}, which could be significant for the usual low-spin vibrations, in particular for the RTP's. In principal, higher-order corrections to the SCC+RPA model, including those of order I^{-1}, could be computed, although that goes beyond the cranked mean-field philosophy. A possible simple way to partially correct for inaccuracies of order I^{-1} in the RTP's when they are important is to multiply the expressions (4.1) by an I-dependent factor that approaches unity for large values of I. The proposal is not to introduce empirical ''fudge factors,'' but rather factors that allow one to smoothly interpolate between the known theoretical quantal behavior of the system at low spins and the cranking result at high spins in the spirit of the correspondence principle.

For example, assume that the critical bifurcation point corresponds to zero spin. Then since, as was demonstrated in Sec. III, the solutions correspond to the RPA in some neighborhood of the critical point, one may choose the factors so as to reproduce the well-known RTP's of the fully quantized RPA. Here, it is important to distinguish between trajectories that describe vibrations in spherical nuclei and those in axially symmetric deformed nuclei, which require different factors. It is straightforward to check that for the case of aligned quadrupole vibrations forming the sequence of n-phonon states with $I=2n$, Eq. (4.1) (only the first for $\mu=2$ is relevant) already reproduces the RPA result as it is, thus requiring no correction in this limit. One way to see this is to directly calculate $\langle\mathcal{M}(E2,-2)\rangle_I=\mathrm{Tr}\ m(E2,-2)\rho^{(1)}$, where $m(E2,-2)$ is the matrix corresponding to the electric quadrupole operator, and then to compare with the conventional quantal calculation of the $B(E2)$ in the RPA. In this calculation, the related Eqs. (3.11) and (3.33) (with $I_0=0$ and $K=2$), which establish the normalization of $\rho^{(1)}$, play an essential role. They guarantee that the transition matrix element is proportional to $\sqrt{n}=\sqrt{I/2}$, which in the quantal calculation arises from the matrix element of the phonon annihilation operator. Another way is provided by the following argument, which also puts into perspective the rela-

[6]This should be distinguished from the RPA diagrams that are summed along with higher-order anharmonicities in the cranked trajectory itself, as discussed earlier.

tion between the cranking and quantal transition matrix elements when anharmonic corrections to the RPA are included. Let $B^{\dagger}_{2\mu}$ denote a quadrupole phonon creation operator carrying μ units of angular momentum along the three-axis. The subspace of aligned phonon states with spins $I=2n$ is generated by the states

$$|nII\rangle=\left[\left(\frac{I}{2}\right)!\right]^{-1/2}(B^{\dagger}_{22})^{I/2}|0\rangle, \tag{4.3}$$

where $|nIM\rangle$ denotes an n-phonon angular momentum eigenstate and $|0\rangle$ the spin-zero vacuum state of an even-even nucleus. The $E2$ operators projected into this subspace then must have the following forms:

$$Q_{22}\equiv P\mathcal{M}(E2,2)P=B^{\dagger}_{22}f(N_2),$$

$$Q_{2-2}\equiv P\mathcal{M}(E2,-2)P=f(N_2)B_{22}, \tag{4.4}$$

P being the projector, $N_2\equiv B^{\dagger}_{22}B_{22}$ the phonon number operator, and f an arbitrary (unknown) function thereof. It will be assumed from now on that the phonons have been chosen such that the Hamiltonian H is diagonal, i.e., a function only of the boson number operators, which implies, incidentally, that PHP is a function only of N_2. Then states (4.3) are exact eigenstates and one immediately has the matrix elements

$$\begin{aligned}\langle n-1\ \ I-2\ \ I-2|Q_{2-2}|nII\rangle&=\langle n-1\ \ I-2\ \ I-2|\mathcal{M}(E2,-2)|nII\rangle\\&=\left(\frac{I}{2}\right)^{1/2}f\left(\frac{I-2}{2}\right).\end{aligned} \tag{4.5}$$

From the Wigner-Eckart theorem and the explicit value of the corresponding Clebsch-Gordan coefficient, one then arrives at the following *exact*, though very general, reduced matrix element:

$$\langle n-1\ \ I-2\|\mathcal{M}(E2)\|nI\rangle=(2I+1)^{1/2}\left(\frac{I}{2}\right)^{1/2}f\left(\frac{I-2}{2}\right). \tag{4.6}$$

The transition from the exact result to the cranked mean-field approximation involves two steps. First, the bosons are replaced by c-numbers as follows: $B_{22}\rightarrow\beta_{22}$, $B^{\dagger}_{22}\rightarrow\beta^{*}_{22}$, and $N_2\rightarrow|\beta_{22}|^2$. Second, the function $f(N_2)\rightarrow\tilde{f}(|\beta_{22}|^2)$, where $\tilde{f}$ differs from f by terms of order $\mathcal{N}^{-1}$, where $\mathcal{N}$ is a relatively large number of the order of the number of interacting particles or available valence levels; i.e., $\mathcal{N}^{-1}$ is the smallness parameter appearing in boson expansions [15]. To be more specific, assume that f is a holomorphic function and therefore can be expanded in the form

$$f(N_2)=\sum_{n=0}^{\infty}c_nN_2^n. \tag{4.7}$$

In general, the coefficients can be expected to have expansions in powers of $\mathcal{N}^{-1}$, so that one may write $c_n=c_n^{(0)}+O(\mathcal{N}^{-1})$. Then the function $\tilde{f}$ is given by

$$\tilde{f}(|\beta_{22}|^2)=\sum_{n=0}^{\infty}c_n^{(0)}|\beta_{22}|^{2n}. \tag{4.8}$$

Similar considerations hold for other operators. In this way, one recovers the BG representation of Sec. II. The Eq. (2.9) with $I_0=0$ and $K_{\mu_0}=2$ implies that $|\beta_{22}|^2=I/2$ and one is free to choose the real value $\beta_{22}=\sqrt{I/2}$. With these two steps, $\mathcal{M}(E2,-2)\rightarrow\langle\mathcal{M}(E2,-2)\rangle_I$, where

$$\langle\mathcal{M}(E2,-2)\rangle_I=\tilde{f}(|\beta_{22}|^2)\beta_{22}=\left(\frac{I}{2}\right)^{1/2}\tilde{f}\left(\frac{I}{2}\right), \tag{4.9}$$

which is to be compared to Eq. (4.5). From Eq. (4.1) it follows that the cranking result agrees with the exact $B(E2)$ to leading order in the two expansion parameters I^{-1} and $\mathcal{N}^{-1}$. A similar argument can be given for the γ vibrations of deformed nuclei using strong-coupling eigenvectors, where the cranking solution describes the sequence of aligned-phonon bandheads with $I=K=2n$, $n=1,2,\ldots$. In this case, the validity of Eq. (4.1) at low spins is improved by multiplying the first of Eqs. (4.1) by the square of a Clebsch-Gordan coefficient, namely, $|\langle I2\ \ I-2|I-2\ \ I-2\rangle|^2=(2I-3)/(2I+1)$.

V. SUMMARY AND CONCLUSIONS

It has been shown that the self-consistent cranking model possesses a class of solutions that has escaped general notice. These solutions bifurcate from a state having at least one axis of C_∞ symmetry, the existence of which is a necessary condition for the symmetry-breaking bifurcation. The possible bifurcation points are related in a simple way to the RPA frequencies for modes carrying nonvanishing angular momentum about the C_∞ axis, as expressed in the cranking bifurcation theorem (CBT). This relationship comes about precisely because the uniformly rotating cranking solutions are actually vibrations, i.e., waves traveling about the cranking axis. It was shown explicitly that such a bifurcation includes anharmonic corrections to the harmonic RPA. An exact (as opposed to perturbative) numerical solution of the cranking equations then sums anharmonicities of all orders in a classical approximation. The solution trajectory, as depicted in an E vs I plot, typically corresponds to a sequence of aligned multiphonon states, as, for example, the sequence $I=2n+$, for n-phonon quadrupole excitations. The method can be implemented either within the context of fully self-consistent cranked mean fields generated by fundamental nucleon-nucleon interactions or within the context of phenomenological mean fields, such as the Nilsson model, in which case the results correspond to an anharmonic extension of the vibrating potential model. It was also noted that transition matrix elements connecting states lying on a trajectory can be calculated within the bounds of the formalism.

The cranking approach to vibrations has a broad range of possible applications, although it also has its limitations. One limitation already discussed is that only excitation modes carrying angular momentum are accessible, therefore excluding the breathing mode and β vibrations. But that still leaves a very wide range of treatable modes. The other limitations are shared with the cranking model for ordinary rotational

states, and stem from the fact that this method is fundamentally a classical one with a superimposed Bohr-Sommerfeld quantization condition on the angular momentum. Thus, additional quantum corrections usually exist, although these may in many cases be adequately treated by the inclusion of the RPA (SCC+RPA). However, one must still be cautious in situations in which quantal tunneling plays an important role. Apart from these caveats, the cranking approach can be a worthy and even superior competitor to adiabatic time-dependent mean-field approximations for bound states. In addition to greater simplicity, the cranking approach has the advantage of not assuming the adiabatic approximation, and therefore can sum collective kinetic-energy as well as potential-energy anharmonicities of all orders. Of course, if the potential exhibits multiple minima so that tunneling is important, then the adiabatic approach with quantization is probably the superior approach.

The cranking approach to vibrations may be able to shed a new light on the old question of the repeatability of phonon excitations. When are multiple excitations of a given vibrational mode to be expected? Since the repeatability of an excitation mode having a pure particle-hole or two-quasiparticle character is ruled out by the Pauli principle, the issue of repeatability is closely tied to the collectivity of the mode and its interplay with single-particle modes. Since the cranking equations are nonlinear, it is possible for a solution trajectory not only to suddenly appear as a bifurcation but just as suddenly to disappear, thereby signaling a cutoff on repeatability. Should a cutoff occur for an amplitude corresponding to less than two quanta, one would have to conclude that the mode in question cannot be repeated. Of course, such behavior can probably be correlated with crossing patterns of the underlying quasiparticle Routhian plots as for ordinary rotations. One reason for believing that the cranking technique can reproduce cutoffs on vibrations when they exist is that it can reproduce cutoffs on ordinary rotational bands in exactly soluble models such as the Elliott model [61], as will be discussed in a subsequent publication.

There are many concrete applications for which the proposed formalism may be suitable. These include various types of multiphonon excitations such as the classical quadrupole and octupole vibrations in spherical nuclei, γ vibrations in deformed nuclei, and dipole and quadrupole giant resonances in both spherical and deformed nuclei. In addition, as briefly pointed out earlier, the cranking approach may provide a new perspective on vibrations in odd nuclei by adding anharmonic corrections to the approximation of Bès and Chung [54]. Of special current interest are octupole vibrations in superdeformed nuclei, which are expected to be very soft. Thus far, these have been treated mainly in the RPA [62]. While the present formalism would certainly provide an improvement through the inclusion of anharmonicities for the $K=1$, 2, 3 octupole bands ($K=0$ is not accessible), it should be kept in mind that there may be important tunneling effects for octupole bands that are not included. Another phenomenon accessible to the cranking approach is that of tilted rotation [55]. The CBT implies the existence of bands of collective tilted states, as distinguished from the those arising through the asymmetries introduced by extra quasiparticles, which have been the main focus of attention. Finally, since the cranking approach to vibrations in perturbation theory leads to angular-momentum expansions of the form of Eq. (3.49), which includes the phenomenological universal quadrupole-vibrator energy expression of Casten and collaborators [11], it is uniquely suited for a microscopic calculation of the parameters.

A paper soon to follow will discuss some illustrative applications of the CBT to simple soluble systems.

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