Pauli principle in the soft-photon approach to proton-proton bremsstrahlung

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(Received 11 April 1996)

A relativistic and manifestly gauge-invariant soft-photon amplitude, which is consistent with the soft-photon theorem and satisfies the Pauli principle, is derived for the proton-proton bremsstrahlung process. This soft-photon amplitude is the first two-*u*-two-*t* special amplitude to satisfy all theoretical constraints. The conventional Low amplitude can be obtained as a special case. It is demonstrated that previously proposed amplitudes for this process, both the (*u,t*) and (*s,t*) classes, violate the Pauli principle at some level. The origin of the Pauli principle violation is shown to come from two sources: (i) For the (*s,t*) class, the two-*s*-two-*t* amplitude transforms into the two-*s*-two-*u* amplitude under the interchange of two initial-state (or final-state) protons. (ii) For the (*u,t*) class, the use of an internal emission amplitude determined from the gauge-invariance constraint alone, without imposition of the Pauli principle, causes a problem. The resulting internal emission amplitude can depend upon an electromagnetic factor which is not invariant under the interchange of the two protons. [S0556-2813(96)05909-2]

PACS number(s): 13.75.Cs, 13.40.−f

I. INTRODUCTION

It has been known since the early work of Low [1] that the soft-photon theorem applies to all nuclear bremsstrahlung processes. This theorem states that, when the total bremsstrahlung amplitude is expanded in powers of the photon momentum (energy) K, the coefficients of the two leading terms are independent of off-shell effects. Therefore, the theorem implies that a soft-photon approximation (an on-shell approximation based upon the first two terms) should provide a good description of any bremsstrahlung process, including proton-proton bremsstrahlung ($pp\gamma$). The open question has been how to construct a soft-photon amplitude which satisfies all theoretical constraints.

During the past three decades, a variety of soft-photon amplitudes have been proposed to describe the $pp\gamma$ process. Although most of these amplitudes are relativistic, gauge invariant, and consistent with the soft-photon theorem, they violate the Pauli principle at some level. The requirement of fully satisfying the Pauli principle was heretofore neglected. The purpose of this paper is to provide a derivation of a soft-photon amplitude that not only is consistent with the soft-photon theorem, is valid relativistically, is manifestly gauge invariant, but also satisfies the Pauli principle.

Recently, a prescription to generate two classes of soft-photon amplitudes was discussed: (1) the two-u-two-t special ($TuTts$) amplitudes from the class expressed in terms of the (u,t) Mandelstam variables and (2) the two-s-two-t special ($TsTts$) amplitudes from the class expressed in terms of the (s,t) Mandelstam variables [2]. In Ref. [2], simple cases were used to demonstrate basic ideas and methods. The two particles involved in the scattering were assumed to be spinless and to have different masses and charges. The elastic scattering amplitude was defined as the sum of a direct amplitude and an exchange amplitude. Under these assumptions, the derived amplitudes are applicable to a description of bremsstrahlung processes involving the scattering of two bosons, but not two fermions. Because the proton is a spin-1/2 particle and the two-proton amplitude must obey the Pauli principle, the pp elastic amplitude must be antisymmetric under interchange of the protons. That is, for the pp case the scattering amplitude should be obtained as the direct amplitude minus (not plus) the exchange amplitude. Therefore, the $TuTts$ amplitude derived in Ref. [2] is not a proper representation of the $pp\gamma$ process, even though the argument regarding why the $TuTts$-type amplitude should be used to describe the pp bremsstrahlung process is correct. Moreover, there is an additional problem which is related to the ambiguity in determining the internal emission amplitude. Without imposing the fermion antisymmetry requirement, the gauge invariant condition alone does not yield a unique expression for the internal amplitude. This important point, emphasized here, was not imposed in Ref. [2]. As a result, the internal amplitude obtained in Ref. [2] for the nonidentical particles considered is not a proper choice for bremsstrahlung processes involving two identical nucleons. For the case of $pp\gamma$ the violation of the Pauli principle for the *TuTts* amplitude introduced in Ref. [2] is not serious since such violation is found only in the term of order K.

A more realistic $TuTts$ amplitude for the $pp\gamma$ process was proposed recently [3]. That amplitude is relativistic, gauge invariant, and consistent with the soft-photon theorem. However, it does not obey the Pauli principle at the K^1 order in the expansion in terms of K. The problem arises from the internal amplitude. It involves an electromagnetic factor which is not invariant under the interchange of the two initial-state (incoming) or two final-state (outgoing) protons. As we demonstrate below, this factor is but one of two possible choices that can be obtained by imposing gauge invariance. The second choice for the invariant factor was missed in Ref. [3], because the requirement that the Pauli principle be satisfied was not imposed in the derivation.

Except for the $TuTts$ amplitude discussed in Ref. [3], almost all $pp\gamma$ soft-photon amplitudes considered in the literature belong to the (s,t) class. These amplitudes depend upon the pp elastic amplitude, which is evaluated at the square of the total center-of-mass energy s and the square of the momentum transfer t. In fact, in most cases the average s and the average t were used. The amplitudes obtained by Nyman [4] and Fearing [5] are two well-known examples. Such amplitudes are classified as Low amplitudes. Except for the Low amplitudes, all other amplitudes in the (s,t) class violate the Pauli principle for the following reason: If one interchanges the two initial-state (or final-state) protons, then one converts the (s,t) class of amplitudes into the (s,u) class of amplitudes. Because the $pp\gamma$ process involves a half-off-shell amplitude (not an elastic amplitude), the (s,u) amplitude obtained by this procedure is completely different from the original (s,t) amplitude. Therefore, it is impossible to regain the original (s,t) amplitude with just a sign change after interchanging the two protons.

This paper is structured as follows. In Sec. II we define the pp elastic scattering amplitude which will be used as input to generate the bremsstrahlung amplitudes for the $pp\gamma$ process. We use the amplitude introduced by Goldberger, Grisaru, MacDowell, and Wong (GGMW) [6], but without incorporating the Fierz transformation. In Sec. III we derive a relativistic $TuTts$ amplitude by imposing gauge invariance and the Pauli principle. In deriving the amplitude, a straightforward and rigorous approach, slightly different from that employed in Ref. [2], is utilized. We verify that the resulting $TuTts$ amplitude is consistent with the soft-photon theorem. Finally, a variety of other amplitudes, which violate the Pauli principle, are discussed in Sec. IV.

II. THE PROTON-PROTON ELASTIC SCATTERING AMPLITUDE

The Feynman amplitude F for pp elastic scattering,

$$p(q_i^\mu)+p(p_i^\mu)\to p(\overline{q_f^\mu})+p(\overline{p_f^\mu}), \tag{1}$$

can be written as [6]

$$\begin{aligned}F&=F_1(G_1-\widetilde{G}_1)+F_2(G_2+\widetilde{G}_2)+F_3(G_3-\widetilde{G}_3)\\&\quad+F_4(G_4+\widetilde{G}_4)+F_5(G_5-\widetilde{G}_5)\\&=\sum_{\alpha=1}^{5}F_\alpha[G_\alpha+(-1)^\alpha\widetilde{G}_\alpha],\end{aligned} \tag{2}$$

where

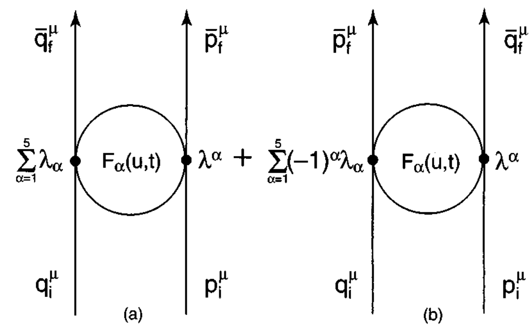

FIG. 1. Schematic representation of the proton-proton elastic scattering process: (a) corresponds to a sum over the five direct amplitudes; (b) corresponds to a sum over the five exchange amplitudes multiplied by the sign factor $(-1)^\alpha$.

$$\begin{aligned}G_\alpha&=\overline{u(q_f)}\lambda_\alpha u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i),\\\widetilde{G}_\alpha&=\overline{u(p_f)}\lambda_\alpha u(q_i)\overline{u(q_f)}\lambda^\alpha u(p_i),\end{aligned} \tag{3}$$

and we define

$$(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)\equiv\left(1,\frac{\sigma_{\mu\nu}}{\sqrt{2}},i\gamma_5\gamma_\mu,\gamma_\mu,\gamma_5\right),$$

$$(\lambda^1,\lambda^2,\lambda^3,\lambda^4,\lambda^5)\equiv\left(1,\frac{\sigma^{\mu\nu}}{\sqrt{2}},i\gamma_5\gamma^\mu,\gamma^\mu,\gamma_5\right).$$

Note that λ_α and λ^α are tensors. For example, $\lambda^2\lambda_2=\lambda_2\lambda^2=\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$, where the summation over μ and ν is implied. In Eq. (2), F_α $(\alpha=1,\ldots,5)$ are invariant functions of the Mandelstam variables s, t, and u,

$$\begin{aligned}s&=(q_i+p_i)^2=(\overline{q_f}+\overline{p_f})^2,\\t&=(\overline{p_f}-p_i)^2=(\overline{q_f}-q_i)^2,\\u&=(\overline{p_f}-q_i)^2=(\overline{q_f}-p_i)^2.\end{aligned} \tag{4}$$

Because of energy-momentum conservation,

$$q_i^\mu+p_i^\mu=\overline{q_f^\mu}+\overline{p_f^\mu}, \tag{5}$$

s, t, and u satisfy the following relation,

$$s+t+u=4m^2, \tag{6}$$

so that only two of them are independent. (Here m is the proton mass.) The optimal choice of these two independent variables will depend on the fundamental diagrams (or the dominant tree diagrams) of a given process. In our case, guided by a meson-exchange theory of the NN interaction, we choose u and t to be the two independent variables, and we write $F_\alpha=F_\alpha(u,t)$. In Eq. (2), $\Sigma_{\alpha=1}^5F_\alpha(u,t)G_\alpha$ represents a sum over the five direct amplitudes, while $\Sigma_{\alpha=1}^5(-1)^\alpha F_\alpha(u,t)\widetilde{G}_\alpha$ represents a sum over the five exchange amplitudes multiplied by the sign factor arising for two nucleons. The five direct amplitudes are depicted in Fig. 1(a) and the five exchange amplitudes are exhibited in Fig.

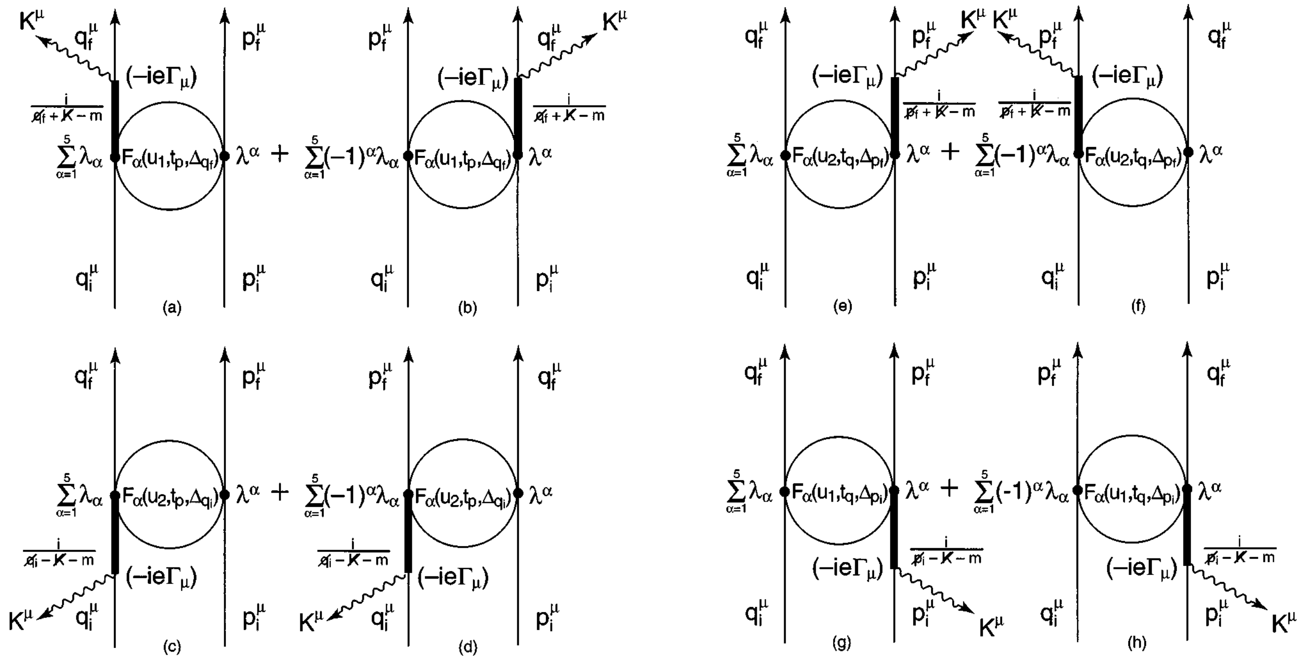

FIG. 2. The external bremsstrahlung diagrams generated from Fig. 1: (a) and (b) represent photon emission from the q_f leg; (c) and (d) from the q_i leg; (e) and (f) from the p_f leg; (g) and (h) from the p_i leg.

1(b). These ten elastic-scattering diagrams will be used as source graphs to generate bremsstrahlung diagrams.

The Pauli principle imposes some restrictions on $F_\alpha(u,t)$. For isotopic triplet states, we require that

$$F_\alpha(u,t)=(-1)^{\alpha+1}F_\alpha(t,u). \tag{7}$$

If we interchange $\overline{q_f^\mu}$ with $\overline{p_f^\mu}$ (or q_i^μ with p_i^μ), then (i) u is interchanged with t; (ii) G_α is interchanged with $\widetilde{G}_\alpha$; and (iii) the direct amplitude $F_\alpha(u,t)G_\alpha$ will be interchanged with the exchange amplitude $[-(-1)^\alpha F_\alpha(u,t)\widetilde{G}_\alpha]$ but with opposite sign. Thus, the amplitude F given by Eq. (2) changes sign, and the Pauli principle is therefore satisfied.

III. PROTON-PROTON BREMSSTRAHLUNG AMPLITUDES

A. External amplitudes

We can use Figs. 1(a) and 1(b) as source graphs to generate external emission pp bremsstrahlung diagrams.

If the photon is emitted from the q_f-leg, then we obtain Figs. 2(a) and 2(b). The amplitudes corresponding to these two diagrams can be written as

$$M_\mu^{q_f}(u_1,t_p,\Delta_{q_f})=e\sum_{\alpha=1}^{5}F_\alpha(u_1,t_p,\Delta_{q_f})\Bigg[\overline{u(q_f)}\Gamma_\mu\frac{1}{\not{q}_f+\not{K}-m}\lambda_\alpha u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i)$$
$$+(-1)^\alpha\overline{u(p_f)}\lambda_\alpha u(q_i)\overline{u(q_f)}\Gamma_\mu\frac{1}{\not{q}_f+\not{K}-m}\lambda^\alpha u(p_i)\Bigg], \tag{8}$$

where

$$u_1=(p_f-q_i)^2=(p_i-q_f-K)^2,$$

$$t_p=(p_f-p_i)^2=(q_i-q_f-K)^2,$$

$$\Delta_{q_f}=(q_f+K)^2=m^2+2q_f\cdot K,$$

and

$$\Gamma_\mu=\gamma_\mu-i\frac{\kappa}{2m}\sigma_{\mu\nu}K^\nu \tag{9}$$

is the electromagnetic vertex. Here $e>0$ is the proton charge, κ is the anomalous magnetic moment of the proton, and we have used three-body energy-momentum conservation for the $pp\gamma$ process,

$$q_i^\mu+p_i^\mu=q_f^\mu+p_f^\mu+K^\mu. \tag{10}$$

It is easy to show that

$$\overline{u(q_f)}\Gamma_\mu \frac{1}{\not{q}_f+\not{K}-m}=\overline{u(q_f)}\left(\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K}\right), \tag{11a}$$

where

$$R_\mu^{q_f}=\tfrac{1}{4}[\gamma_\mu,\not{K}]+\frac{\kappa}{8m}\{[\gamma_\mu,\not{K}],\not{q}_f\}, \tag{11b}$$

and we have used $[A,B]\equiv AB-BA$ and $\{A,B\}\equiv AB+BA$. If we expand $F_\alpha(u_1,t_p,\Delta_{q_f})$ about $\Delta_{q_f}=m^2$,

$$F_\alpha(u_1,t_p,\Delta_{q_f})=F_\alpha(u_1,t_p)+\left.\frac{\partial F_\alpha(u_1,t_p,\Delta_{q_f})}{\partial \Delta_{q_f}}\right|_{\Delta_{q_f}=m^2} \times(2q_f\cdot K)+\cdots, \tag{12}$$

where

$$F_\alpha(u_1,t_p)\equiv F_\alpha(u_1,t_p,m^2),$$

then Eq. (8) becomes

$$M_\mu^{q_f}(u_1,t_p,\Delta_{q_f})=e\sum_{\alpha=1}^{5}\left[F_\alpha(u_1,t_p)+(2q_f\cdot K)\left.\frac{\partial F_\alpha(u_1,t_p,\Delta_{q_f})}{\partial \Delta_{q_f}}\right|_{\Delta_{q_f}=m^2}+\cdots\right] \times\left[\overline{u(q_f)}\left(\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K}\right)\lambda_\alpha u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i)+(-1)^\alpha\overline{u(p_f)}\lambda_\alpha u(q_i)\overline{u(q_f)}\left(\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K}\right)\lambda^\alpha u(p_i)\right]. \tag{13}$$

If the photon is emitted from the q_i leg, then we get Figs. 2(c) and 2(d), and the corresponding amplitudes have the form

$$M_\mu^{q_i}(u_2,t_p,\Delta_{q_i})=e\sum_{\alpha=1}^{5}F_\alpha(u_2,t_p,\Delta_{q_i})\left[\overline{u(q_f)}\lambda_\alpha\frac{1}{\not{q}_i-\not{K}-m}\Gamma_\mu u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i) +(-1)^\alpha\overline{u(p_f)}\lambda_\alpha\frac{1}{\not{q}_i-\not{K}-m}\Gamma_\mu u(q_i)\overline{u(q_f)}\lambda^\alpha u(p_i)\right], \tag{14}$$

where

$$u_2=(q_f-p_i)^2=(q_i-p_f-K)^2,$$

and

$$\Delta_{q_i}=(q_i-K)^2=m^2-2q_i\cdot K.$$

If we use the relation

$$\frac{1}{\not{q}_i-\not{K}-m}\Gamma_\mu u(q_i)=-\left(\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K}\right)u(q_i), \tag{15}$$

where $R_\mu^{q_i}$ is given by the same expression as Eq. (11b) but with q_f replaced by q_i, and expand $F_\alpha(u_2,t_p,\Delta_{q_i})$ about $\Delta_{q_i}=m^2$,

$$F_\alpha(u_2,t_p,\Delta_{q_i})=F_\alpha(u_2,t_p)+\left.\frac{\partial F_\alpha(u_2,t_p,\Delta_{q_i})}{\partial \Delta_{q_i}}\right|_{\Delta_{q_i}=m^2}(-2q_i\cdot K)+\cdots, \tag{16}$$

where

$$F_\alpha(u_2,t_p)\equiv F_\alpha(u_2,t_p,m^2),$$

we obtain from Eq. (14)

$$M_\mu^{q_i}(u_2,t_p,\Delta_{q_i})=-e\sum_{\alpha=1}^{5}\left[F_\alpha(u_2,t_p)-(2q_i\cdot K)\frac{\partial F_\alpha(u_2,t_p,\Delta_{q_i})}{\partial\Delta_{q_i}}\bigg|_{\Delta_{q_i}=m^2}+\cdots\right]\left[\overline{u(q_f)}\lambda_\alpha\left(\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K}\right)u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i)\right.$$
$$\left.+(-1)^\alpha\overline{u(p_f)}\lambda_\alpha\left(\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K}\right)u(q_i)\overline{u(q_f)}\lambda^\alpha u(p_i)\right]. \tag{17}$$

Similarly, if the photon is emitted from the p_f leg and p_i leg, then we obtain Figs. 2(e) and 2(f) and Figs. 2(g) and 2(h), respectively. The amplitudes corresponding to these figures have the following expressions:

$$M_\mu^{p_f}(u_2,t_q,\Delta_{p_f})=e\sum_{\alpha=1}^{5}\left[F_\alpha(u_2,t_q)+(2p_f\cdot K)\frac{\partial F_\alpha(u_2,t_q,\Delta_{p_f})}{\partial\Delta_{p_f}}\bigg|_{\Delta_{p_f}=m^2}+\cdots\right]\left[\overline{u(q_f)}\lambda_\alpha u(q_i)\overline{u(p_f)}\left(\frac{p_{f\mu}+R_\mu^{p_f}}{p_f\cdot K}\right)\lambda^\alpha u(p_i)\right.$$
$$\left.+(-1)^\alpha\overline{u(p_f)}\left(\frac{p_{f\mu}+R_\mu^{p_f}}{p_f\cdot K}\right)\lambda_\alpha u(q_i)\overline{u(q_f)}\lambda^\alpha u(p_i)\right], \tag{18}$$

and

$$M_\mu^{p_i}(u_1,t_q,\Delta_{p_i})=-e\sum_{\alpha=1}^{5}\left[F_\alpha(u_1,t_q)-(2p_i\cdot K)\frac{\partial F_\alpha(u_1,t_q,\Delta_{p_i})}{\partial\Delta_{p_i}}\bigg|_{\Delta_{p_i}=m^2}+\cdots\right]\left[\overline{u(q_f)}\lambda_\alpha u(q_i)\overline{u(p_f)}\lambda^\alpha\left(\frac{p_{i\mu}+R_\mu^{p_i}}{p_i\cdot K}\right)u(p_i)\right.$$
$$\left.+(-1)^\alpha\overline{u(p_f)}\lambda_\alpha u(q_i)\overline{u(q_f)}\lambda^\alpha\left(\frac{p_{i\mu}+R_\mu^{p_i}}{p_i\cdot K}\right)u(p_i)\right]. \tag{19}$$

Here,

$$t_q=(q_f-q_i)^2=(p_i-p_f-K)^2,$$
$$\Delta_{p_f}=(p_f+K)^2=m^2+2p_f\cdot K,$$
$$\Delta_{p_i}=(p_i-K)^2=m^2-2p_i\cdot K, \tag{20}$$

and $R_\mu^{p_f}$ and $R_\mu^{p_i}$ are given by the same expressions as $R_\mu^{q_f}$ in Eq. (11b) but with q_f replaced by p_f and p_i, respectively.

The external emission process is the sum of emission processes from the four proton legs. Therefore, the external bremsstrahlung amplitude, M_μ^E, can be written as

$$M_\mu^E=M_\mu^{q_f}(u_1,t_p,\Delta_{q_f})+M_\mu^{q_i}(u_2,t_p,\Delta_{q_i})+M_\mu^{p_f}(u_2,t_q,\Delta_{p_f})+M_\mu^{p_i}(u_1,t_q,\Delta_{p_i}). \tag{21}$$

B. Internal amplitudes

The internal bremsstrahlung amplitude, M_μ^I, can be obtained from the gauge-invariance condition,

$$(M_\mu^E+M_\mu^I)K^\mu=0. \tag{22}$$

However, this condition alone cannot give a unique expression for the amplitude M_μ^I. The ambiguity can be removed if the additional requirement of satisfying the Pauli principle is also imposed. Because R_μ^Q $(Q=q_f,p_f,q_i,p_i)$ are separately gauge invariant, *viz.* $R_\mu^Q K^\mu=0$, we find

$$M_\mu^I K^\mu=-M_\mu^E K^\mu$$
$$=-e\sum_{\alpha=1}^{5}\left[F_\alpha(u_1,t_p)-F_\alpha(u_2,t_p)+F_\alpha(u_2,t_q)-F_\alpha(u_1,t_q)+(2q_f\cdot K)\frac{\partial F_\alpha(u_1,t_p,\Delta_{q_f})}{\partial\Delta_{q_f}}\bigg|_{\Delta_{q_f}=m^2}\right.$$

$$+(2q_i\cdot K)\frac{\partial F_\alpha(u_2,t_p,\Delta_{q_i})}{\partial\Delta_{q_i}}\bigg|_{\Delta_{q_i}=m^2}+(2p_f\cdot K)\frac{\partial F_\alpha(u_2,t_q,\Delta_{p_f})}{\partial\Delta_{p_f}}\bigg|_{\Delta_{p_f}=m^2}+(2p_i\cdot K)\frac{\partial F_\alpha(u_1,t_q,\Delta_{p_i})}{\partial\Delta_{p_i}}\bigg|_{\Delta_{p_i}=m^2}+\cdots\Bigg]$$
$$\times[\overline{u(q_f)}\lambda_\alpha u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i)+(-1)^\alpha\overline{u(p_f)}\lambda_\alpha u(q_i)\overline{u(q_f)}\lambda^\alpha u(p_i)]. \tag{23}$$

Let us define

$$I_\alpha\equiv F_\alpha(u_1,t_p)-F_\alpha(u_2,t_p)+F_\alpha(u_2,t_q)-F_\alpha(u_1,t_q) \tag{24a}$$

$$=\tfrac{1}{2}\{[F_\alpha(u_1,t_p)-F_\alpha(u_2,t_p)]-[F_\alpha(u_1,t_q)-F_\alpha(u_2,t_q)]+[F_\alpha(u_1,t_p)-F_\alpha(u_1,t_q)]-[F_\alpha(u_2,t_p)-F_\alpha(u_2,t_q)]\}. \tag{24b}$$

The choice of the expression given by Eq. (24b) is guided by the requirement that the Pauli principle be satisfied. Using the kinematic identities

$$u_1-u_2=2(q_f-p_i)\cdot K=2(q_i-p_f)\cdot K,$$

$$t_p-t_q=2(q_f-q_i)\cdot K=2(p_i-p_f)\cdot K, \tag{25}$$

and the mean-value theorem, we obtain

$$F_\alpha(u_1,t_p)-F_\alpha(u_2,t_p)=2(q_i-p_f)\cdot K\frac{\partial F_\alpha(u_m,t_p)}{\partial u_m}, \tag{26a}$$

$$F_\alpha(u_1,t_q)-F_\alpha(u_2,t_q)=2(q_i-p_f)\cdot K\frac{\partial F_\alpha(u_m',t_q)}{\partial u_m'}, \tag{26b}$$

$$F_\alpha(u_1,t_p)-F_\alpha(u_1,t_q)=2(p_i-p_f)\cdot K\frac{\partial F_\alpha(u_1,t_m)}{\partial t_m}, \tag{26c}$$

$$F_\alpha(u_2,t_p)-F_\alpha(u_2,t_q)=2(p_i-p_f)\cdot K\frac{\partial F_\alpha(u_2,t_m')}{\partial t_m'}, \tag{26d}$$

where u_m and u_m' lie between u_1 and u_2, and t_m and t_m' lie between t_p and t_q. Inserting Eqs. (26a)–(26d) into Eq. (24b), we get

$$I_\alpha=(q_i-p_f)\cdot K\frac{\partial F_\alpha(u_m,t_p)}{\partial u_m}-(q_i-p_f)\cdot K\frac{\partial F_\alpha(u_m',t_q)}{\partial u_m'}+(p_i-p_f)\cdot K\frac{\partial F_\alpha(u_1,t_m)}{\partial t_m}-(p_i-p_f)\cdot K\frac{\partial F_\alpha(u_2,t_m')}{\partial t_m'}. \tag{27}$$

The expression for M_μ^I can now be generated if we substitute Eq. (27) into Eq. (23). We find

$$M_\mu^I=-e\sum_{\alpha=1}^{5}\Bigg[(q_i-p_f)_\mu\frac{\partial F_\alpha(u_m,t_p)}{\partial u_m}-(q_i-p_f)_\mu\frac{\partial F_\alpha(u_m',t_q)}{\partial u_m'}+(p_i-p_f)_\mu\frac{\partial F_\alpha(u_1,t_m)}{\partial t_m}-(p_i-p_f)_\mu\frac{\partial F_\alpha(u_2,t_m')}{\partial t_m'}$$
$$+2q_{f\mu}\frac{\partial F_\alpha(u_1,t_p,\Delta_{q_f})}{\partial\Delta_{q_f}}\bigg|_{\Delta_{q_f}=m^2}+2q_{i\mu}\frac{\partial F_\alpha(u_2,t_p,\Delta_{q_i})}{\partial\Delta_{q_i}}\bigg|_{\Delta_{q_i}=m^2}+2p_{f\mu}\frac{\partial F_\alpha(u_2,t_q,\Delta_{p_f})}{\partial\Delta_{p_f}}\bigg|_{\Delta_{p_f}=m^2}$$
$$+2p_{i\mu}\frac{\partial F_\alpha(u_1,t_q,\Delta_{p_i})}{\partial\Delta_{p_i}}\bigg|_{\Delta_{p_i}=m^2}+\cdots\Bigg][\overline{u(q_f)}\lambda_\alpha u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i)+(-1)^\alpha\overline{u(p_f)}\lambda_\alpha u(q_i)\overline{u(q_f)}\lambda^\alpha u(p_i)]. \tag{28}$$

C. The two-*u*-two-*t* special amplitude $M_\mu^{TuTts}(u_1,u_2;t_p,t_q)$

The amplitude M_μ^{TuTts} can be obtained if we combine the amplitude M_μ^E given by Eq. (21) with the amplitude M_μ^I given by Eq. (28),

$$\begin{aligned} M_\mu^T &= M_\mu^E + M_\mu^I \\ &= M_\mu^{TuTts} + \mathcal{O}(K). \end{aligned} \tag{29}$$

We observe that all off-shell derivative terms cancel precisely. The derivatives of F_α with respect to u_m, u'_m, t_m, and t'_m can be replaced by the finite differences by using Eqs. (26a)–(26b). For example, Eq. (26a) gives

$$\frac{\partial F_\alpha(u_m,t_p)}{\partial u_m} = \frac{F_\alpha(u_1,t_p) - F_\alpha(u_2,t_p)}{2(q_i-p_f)\cdot K}.$$

If we use the finite differences and the following relations:

$$\frac{(q_f-p_i)\cdot \varepsilon}{(q_f-p_i)\cdot K} = \frac{(q_i-p_f)\cdot \varepsilon}{(q_i-p_f)\cdot K},$$

$$\frac{(p_i-p_f)\cdot \varepsilon}{(p_i-p_f)\cdot K} = \frac{(q_f-q_i)\cdot \varepsilon}{(q_f-q_i)\cdot K}, \tag{30}$$

the amplitude M_μ^{TuTts} can be written as

$$\begin{aligned} M_\mu^{TuTts} = e\sum_{\alpha=1}^{5} [&\overline{u(q_f)} X_{\alpha\mu} u(q_i) \overline{u(p_f)} \lambda^\alpha u(p_i) \\ &+ \overline{u(q_f)} \lambda_\alpha u(q_i) \overline{u(p_f)} Y_\mu^\alpha u(p_i) \\ &+ \overline{u(p_f)} \lambda^\alpha u(q_i) \overline{u(q_f)} Z_{\alpha\mu} u(p_i) \\ &+ \overline{u(p_f)} T_\mu^\alpha u(q_i) \overline{u(q_f)} \lambda_\alpha u(p_i)], \end{aligned} \tag{31}$$

where

$$\begin{aligned} X_{\alpha\mu} = &F_\alpha(u_1,t_p)\left[\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K} - V_\mu\right]\lambda_\alpha \\ &- F_\alpha(u_2,t_p)\lambda_\alpha\left[\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K} - V_\mu\right], \end{aligned}$$

$$\begin{aligned} Y_\mu^\alpha = &F_\alpha(u_2,t_q)\left[\frac{p_{f\mu}+R_\mu^{p_f}}{p_f\cdot K} - V_\mu\right]\lambda^\alpha \\ &- F_\alpha(u_1,t_q)\lambda^\alpha\left[\frac{p_{i\mu}+R_\mu^{p_i}}{p_i\cdot K} - V_\mu\right], \end{aligned}$$

$$\begin{aligned} Z_{\alpha\mu} = &(-1)^\alpha F_\alpha(u_1,t_p)\left[\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K} - V_\mu\right]\lambda_\alpha \\ &- (-1)^\alpha F_\alpha(u_1,t_q)\lambda_\alpha\left[\frac{p_{i\mu}+R_\mu^{p_i}}{p_i\cdot K} - V_\mu\right], \end{aligned}$$

$$\begin{aligned} T_\mu^\alpha = &(-1)^\alpha F_\alpha(u_2,t_q)\left[\frac{p_{f\mu}+R_\mu^{p_f}}{p_f\cdot K} - V_\mu\right]\lambda^\alpha \\ &- (-1)^\alpha F_\alpha(u_2,t_p)\lambda^\alpha\left[\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K} - V_\mu\right], \end{aligned} \tag{32}$$

with

$$\begin{aligned} V_\mu = &\frac{(q_f-p_i)_\mu}{2(q_f-p_i)\cdot K} + \frac{(q_f-q_i)_\mu}{2(q_f-q_i)\cdot K} = \frac{(q_i-p_f)_\mu}{2(q_i-p_f)\cdot K} \\ &+ \frac{(p_i-p_f)_\mu}{2(p_i-p_f)\cdot K}. \end{aligned}$$

It is easy to verify that M_μ^{TuTts} is gauge invariant; that is, one can demonstrate that $M_\mu^{TuTts} K^\mu = 0$.

If p_i is interchanged with q_i, or if q_f is interchanged with p_f, we find

$$\begin{aligned} X_{\alpha\mu} &\underset{q_i\leftrightarrow p_i}{\leftrightarrow} -Z_{\alpha\mu}, \\ Y_\mu^\alpha &\underset{q_i\leftrightarrow p_i}{\leftrightarrow} -T_\mu^\alpha, \\ X_{\alpha\mu} &\underset{q_f\leftrightarrow p_f}{\leftrightarrow} -T_{\alpha\mu}, \\ Y_\mu^\alpha &\underset{q_f\leftrightarrow p_f}{\leftrightarrow} -Z_\mu^\alpha. \end{aligned} \tag{33}$$

Equation (33) assures one that the amplitude M_μ^{TuTts} will change sign if $q_i \leftrightarrow p_i$ or $q_f \leftrightarrow p_f$. Hence the Pauli principle is still satisfied.

The amplitude M_μ^{TuTts} given by Eq. (31) can be separated into an external contribution $M_\mu^{TuTts}(E)$ and an internal contribution $M_\mu^{TuTts}(I)$,

$$M_\mu^{TuTts} = M_\mu^{TuTts}(E) + M_\mu^{TuTts}(I), \tag{34}$$

where

$$\begin{aligned} M_\mu^{TuTts}(E) = e\sum_{\alpha=1}^{5} [&\overline{u(q_f)} X_{\alpha\mu}(E) u(q_i) \overline{u(p_f)} \lambda^\alpha u(p_i) \\ &+ \overline{u(q_f)} \lambda_\alpha u(q_i) \overline{u(p_f)} Y_\mu^\alpha(E) u(p_i) \\ &+ \overline{u(p_f)} \lambda^\alpha u(q_i) \overline{u(q_f)} Z_{\alpha\mu}(E) u(p_i) \\ &+ \overline{u(p_f)} T_\mu^\alpha(E) u(q_i) \overline{u(q_f)} \lambda_\alpha u(p_i)]. \end{aligned} \tag{35}$$

and

$$\begin{aligned} M_\mu^{TuTts}(I) = -e\sum_{\alpha=1}^{5} V_\mu[&F_\alpha(u_1,t_p) - F_\alpha(u_2,t_p) + F_\alpha(u_2,t_q) \\ &- F_\alpha(u_1,t_q)][G_\alpha + (-1)^\alpha \widetilde{G}_\alpha] \end{aligned} \tag{36a}$$

$$\begin{aligned} = -eV_\mu[&F(u_1,t_p) - F(u_2,t_p) + F(u_2,t_q) \\ &- F(u_1,t_q)]. \end{aligned} \tag{36b}$$

In Eq. (35), $X_{\alpha\mu}(E)$, $Y^{\alpha}_{\mu}(E)$, $Z_{\alpha\mu}(E)$, and $T^{\alpha}_{\mu}(E)$ are given by the following expressions:

$$X_{\alpha\mu}(E)=F_{\alpha}(u_1,t_p)\left(\frac{q_{f\mu}+R^{q_f}_{\mu}}{q_f\cdot K}\right)\lambda_{\alpha}-F_{\alpha}(u_2,t_p)\lambda_{\alpha}\left(\frac{q_{i\mu}+R^{q_i}_{\mu}}{q_i\cdot K}\right),$$

$$Y^{\alpha}_{\mu}(E)=F_{\alpha}(u_2,t_q)\left(\frac{p_{f\mu}+R^{p_f}_{\mu}}{p_f\cdot K}\right)\lambda^{\alpha}-F_{\alpha}(u_1,t_q)\lambda^{\alpha}\left(\frac{p_{i\mu}+R^{p_i}_{\mu}}{p_i\cdot K}\right),$$

$$Z_{\alpha\mu}(E)=(-1)^{\alpha}F_{\alpha}(u_1,t_p)\left(\frac{q_{f\mu}+R^{q_f}_{\mu}}{q_f\cdot K}\right)\lambda_{\alpha}-(-1)^{\alpha}F_{\alpha}(u_1,t_q)\lambda_{\alpha}\left(\frac{p_{i\mu}+R^{p_i}_{\mu}}{p_i\cdot K}\right),$$

$$T^{\alpha}_{\mu}(E)=(-1)^{\alpha}F_{\alpha}(u_2,t_q)\left(\frac{p_{f\mu}+R^{p_f}_{\mu}}{p_f\cdot K}\right)\lambda^{\alpha}-(-1)^{\alpha}F_{\alpha}(u_2,t_p)\lambda^{\alpha}\left(\frac{q_{i\mu}+R^{q_i}_{\mu}}{q_i\cdot K}\right). \tag{37}$$

Note that we have used the definition of the elastic amplitude, Eq. (2) (with $\overline{p_f}\to p_f$ and $\overline{q_f}\to q_f$), to obtain Eq. (36b) from Eq. (36a).

The amplitude $M^{TuTts}_{\mu}(I)$ does not vanish in general. If we use the following expansion,

$$F_{\alpha}(u_1,t_p)-F_{\alpha}(u_2,t_p)+F_{\alpha}(u_2,t_q)-F_{\alpha}(u_1,t_q)=+[2(q_i-p_f)\cdot K][2(p_i-p_f)\cdot K]\times\frac{\partial^2F_{\alpha}(u_1,t_q)}{\partial t_q\partial u_1}+\mathcal{O}(K^3), \tag{38}$$

then Eq. (36a) can be written as

$$M^{TuTts}_{\mu}(I)=-e\sum_{\alpha=1}^{5}[2(p_i-p_f)\cdot K(q_i-p_f)_{\mu}+2(q_i-p_f)\cdot K(p_i-p_f)_{\mu}]\frac{\partial^2F_{\alpha}(u_1,t_q)}{\partial t_q\partial u_1}\times[G_{\alpha}+(-1)^{\alpha}\widetilde{G}_{\alpha}]+\mathcal{O}(K^2), \tag{39}$$

which shows that $M^{TuTts}_{\mu}(I)$ is order of K. This feature, together with the fact that M^{TuTts}_{μ} is free of off-shell derivatives, proves that the amplitude M^{TuTts}_{μ} is consistent with the soft-photon theorem. In order to obey charge conservation, the internal amplitude $M^{TuTts}_{\mu}(I)$ must vanish at the tree level. To see this, let us consider a one-boson-exchange (OBE) model. For any OBE model, the elastic amplitudes $F(u_i,t_j)$ can be expressed as follows:

$$F(u_i,t_j)=F_D(t_j)-F_E(u_i)\quad(i=1,2;j=p,q). \tag{40}$$

Here, $F_D(t_j)$ and $F_E(u_i)$ represent all direct amplitudes and all exchange amplitudes, respectively. Inserting Eq. (40) into Eq. (36b), we find

$$M^{TuTts}_{\mu}(I)=0. \tag{41}$$

IV. DISCUSSION

A. The two-*u*-two-*t* special amplitudes

The amplitude M^{TuTts}_{μ} given by Eq. (31) is not the only two-u-two-t special amplitude which can be constructed from the external amplitude Eq. (21) by imposing gauge invariance. In Eq. (23), the expression for $M^{I}_{\mu}K^{\mu}$ involves a factor I_{α} defined by Eq. (24a). If we rewrite I_{α} in the form shown in Eq. (24b), and use the formulas given by Eqs. (26a)–(26d), we obtain Eq. (27). It is this expression for I_{α} that gives us the amplitude M^{TuTts}_{μ}. This amplitude has many good features: it is relativistic, gauge invariant, consistent with the soft-photon theorem, and it satisfies the Pauli principle.

However, Eq. (27) is not a unique expression for I_{α}. If we substitute Eqs. (26a) and (26b) into Eq. (24a), we obtain

$$\overline{I_{\alpha}}=2(q_i-p_f)\cdot K\frac{\partial F_{\alpha}(u_m,t_p)}{\partial u_m}-2(q_i-p_f)\cdot K\frac{\partial F_{\alpha}(u'_m,t_p)}{\partial u'_m}, \tag{42}$$

which is different from Eq. (27). If Eq. (42) were used, we would obtain a new amplitude, $\overline{M^{TuTts}_{\mu}}$, which is given by the same expression as M^{TuTts}_{μ} defined in Eqs. (31) and (32) but with V_{μ} replaced by $\overline{V_{\mu}}$, with

$$\overline{V_{\mu}}=\frac{(q_i-p_f)_{\mu}}{(q_i-p_f)\cdot K}=\frac{(q_f-p_i)_{\mu}}{(q_f-p_i)\cdot K}. \tag{43}$$

If p_i is interchanged with q_i (or if q_f is interchanged with p_f), then we have

$$\overline{V_{\mu}}\underset{q_i\leftrightarrow p_i}{\longrightarrow}\frac{(p_i-p_f)_{\mu}}{(p_i-p_f)\cdot K}=\frac{(q_f-q_i)_{\mu}}{(q_f-q_i)\cdot K}\neq\overline{V_{\mu}}, \tag{44a}$$

while, on the other hand,

$$V_{\mu}\underset{q_i\leftrightarrow p_i}{\longrightarrow}V_{\mu}. \tag{44b}$$

Thus, there is an important difference between the two amplitudes, M^{TuTts}_{μ} and $\overline{M^{TuTts}_{\mu}}$, because M^{TuTts}_{μ} does satisfy the Pauli principle, while $\overline{M^{TuTts}_{\mu}}$ violates it. This is the main reason why the amplitude M^{TuTts}_{μ}, not $\overline{M^{TuTts}_{\mu}}$, should be used for the $pp\gamma$ process.

If we apply the Fierz transformation,

$$\begin{pmatrix}\widetilde{G}_1\\ \widetilde{G}_2\\ \widetilde{G}_3\\ \widetilde{G}_4\\ \widetilde{G}_5\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&1&1\\ 6&-2&0&0&6\\ 4&0&-2&2&-4\\ 4&0&2&-2&-4\\ 1&1&-1&-1&1\end{pmatrix}\begin{pmatrix}G_1\\ G_2\\ G_3\\ G_4\\ G_5\end{pmatrix}, \tag{45}$$

we can write Eq. (2) in the form

$$F=\sum_{\alpha=1}^{5} F_{\alpha}^{e}(u,t)G_{\alpha}, \tag{46}$$

or in the form

$$F=\sum_{\alpha=1}^{5} F_{\alpha}^{e}(s,t)G_{\alpha}, \tag{47}$$

where

$$\begin{pmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \\ F_5^e \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 6 & -4 & 4 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 6 & 2 & 1 \\ -1 & 0 & -2 & 2 & 1 \\ -1 & 6 & 4 & -4 & 3 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix}. \tag{48}$$

Equation (46) is obtained when we choose (u,t) to be the two independent variables; i.e., we use $F_{\alpha}=F_{\alpha}(u,t)$ in Eq. (2). On the other hand, Eq. (47) is obtained if the two independent variables are (s,t). For the pp elastic case, the expressions for F given by Eqs. (2), (46), and (47) are identical. However, if these expressions are used as input to generate $pp\gamma$ amplitudes, then the constructed amplitudes will be different. The amplitude generated from Eq. (47) will be discussed in the next subsection. Here, we would like to present, without showing the details of derivation, two more two-u-two-t special amplitudes that can be obtained from Eq. (46). We have

$$M_{i\mu}^{TuTts}=e\sum_{\alpha=1}^{5}[\overline{u(q_f)}X_{i\alpha\mu}u(q_i)\overline{u(p_f)}\lambda^{\alpha}u(p_i) + \overline{u(q_f)}\lambda_{\alpha}u(q_i)\overline{u(p_f)}Y_{i\mu}^{\alpha}u(p_i)], \tag{49}$$

where $(i=1,2)$,

$$X_{i\alpha\mu}=F_{\alpha}^{e}(u_1,t_p)\left[\frac{q_{f\mu}+R_{\mu}^{q_f}}{q_f\cdot K}-V_{i\mu}\right]\lambda_{\alpha} - F_{\alpha}^{e}(u_2,t_p)\lambda_{\alpha}\left[\frac{q_{i\mu}+R_{\mu}^{q_i}}{q_i\cdot K}-V_{i\mu}\right],$$

$$Y_{i\mu}^{\alpha}=F_{\alpha}^{e}(u_2,t_q)\left[\frac{p_{f\mu}+R_{\mu}^{p_f}}{p_f\cdot K}-V_{i\mu}\right]\lambda^{\alpha} - F_{\alpha}^{e}(u_1,t_q)\lambda^{\alpha}\left[\frac{p_{i\mu}+R_{\mu}^{p_i}}{p_i\cdot K}-V_{i\mu}\right], \tag{50}$$

with

$$V_{1\mu}=V_{\mu},$$

$$V_{2\mu}=\overline{V_{\mu}},$$

and V_{μ} and $\overline{V_{\mu}}$ are defined by Eqs. (32) and (43), respectively. In deriving $M_{1\mu}^{TuTts}$, we have used Eqs. (24b) and (26a)–(26d). On the other hand, we have used Eqs. (24a), (26a), and (26b) to derive $M_{2\mu}^{TuTts}$. It can be shown that the amplitude $M_{1\mu}^{TuTts}$ is identical to the amplitude M_{μ}^{TuTts}. Let us outline the proof as follows: If we write Eqs. (45) and (48) in the form $\tilde{G}_{\alpha} = \Sigma_{\beta}\underline{C}_{\alpha\beta}G_{\beta}$ and $F_{\alpha}^{e}=\Sigma_{\beta}C_{\alpha\beta}F_{\beta}$, respectively, then $C_{\alpha\beta}$ and $\underline{C}_{\alpha\beta}$ can be defined. The first step is to transform the exchange terms, the third and fourth terms of Eq. (31), into the same form as the direct terms by using the Fierz identity, $(\lambda_{\alpha})_{ab}(\lambda^{\alpha})_{cd} = \Sigma_{\beta=1}^{5}C_{\alpha\beta}(\lambda_{\beta})_{ad}(\lambda^{\beta})_{cb}$. The second step is to combine these transformed direct terms obtained in the first step with the original direct terms of Eq. (31). The amplitude $M_{1\mu}^{TuTts}$ can be easily obtained if we use the identity $\underline{C}_{\alpha\beta} = \delta_{\alpha\beta}+(-1)^{\beta}C_{\beta\alpha}$. Clearly, $M_{1\mu}^{TuTts}$ satisfies the Pauli principle, because it is identical to M_{μ}^{TuTts}. The proof can also be carried out starting directly from $M_{1\mu}^{TuTts}$. This can be accomplished by using another identity, $\Sigma_{\alpha=1}^{5}C_{\alpha\gamma}\underline{C}_{\alpha\beta} = (-1)^{\beta}C_{\gamma\beta}$.

The amplitude $M_{2\mu}^{TuTts}$ ($\equiv \overline{M}_{\mu}^{TuTts}$) has been used in Ref. [3]. This amplitude violates the Pauli principle because its internal amplitude depends upon $\overline{V}_{\mu}$. However, the violation is only of order K. To see this, one need only observe that the internal amplitude for $M_{2\mu}^{TuTts}$ is given by the same expresssion as that in Eq. (36a) but with V_{μ} replaced by $\overline{V}_{\mu}$, F_{α} replaced by F_{α}^{e} and the $\tilde{G}_{\alpha}$ omitted. If one then carries out an expansion similar to that given by Eq. (38), one sees that the internal amplitude contributes only to the term of order K, the third term, in the soft-photon expansion.

The amplitudes M_{μ}^{TuTts} and $M_{2\mu}^{TuTts}$ have been numerically studied. We found that the $pp\gamma$ cross sections calculated from the two amplitudes are not significantly different, except for those cases when both proton scattering angles are very small and the photon angle ψ_{γ} is around 180°. The amplitude M_{μ}^{TuTts} gives the expected symmetric angular distribution for $0\leqslant\psi_{\gamma}\leqslant 180°$ and $180°\leqslant\psi_{\gamma}\leqslant 360°$, while $M_{2\mu}^{TuTts}$ yields angular distributions that are slightly distorted around the point $\psi_{\gamma}=180^{o}$. Otherwise, both calculated cross sections are in good agreement with the experimental data and most of the potential model predictions.

Finally, it should be pointed out that the Low amplitude can be derived from either M_{μ}^{TuTts} or $M_{1\mu}^{TuTts}$, and therefore it satisfies the Pauli principle.

B. The two-s-two-t special amplitudes $M_{\mu}^{TsTts}(s_i,s_f;t_p,t_q)$

Another class of amplitudes, the two-s-two-t special amplitudes, can be derived if (s,t) are chosen to be the independent variables. The input (elastic-scattering amplitude) used to generate this class of amplitudes can be either Eq. (2) with $F_{\alpha}=F_{\alpha}(s,t)$ (without introducing the Fierz transformation) or Eq. (47), which is obtained from Eq. (2) by applying the Fierz transformation. In other words, two amplitudes can be constructed, but it can be shown that they are identical. The same procedure as outlined in the previous subsection can be followed to obtain the proof. Here, we will just present the expressions for these two amplitudes without derivation, because the procedures for deriving them are very similar to those used in the previous sections for M_{μ}^{TuTts}.

If Eq. (2) with $F_{\alpha}=F_{\alpha}(s,t)$ is used as input, the resulting $pp\gamma$ amplitude assumes the form

$$M_{1\mu}^{TsTts}(s_i,s_f;t_p,t_q)=e\sum_{\alpha=1}^{5}[\overline{u(q_f)}\widetilde{X}_{1\alpha\mu}u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i)+\overline{u(q_f)}\lambda_\alpha u(q_i)\overline{u(p_f)}\widetilde{Y}_{1\mu}^{\alpha}u(p_i)+\overline{u(p_f)}\lambda_\alpha u(q_i)\overline{u(q_f)}\widetilde{Z}_{1\mu}^{\alpha}u(p_i)+\overline{u(p_f)}\widetilde{T}_{1\alpha\mu}u(q_i)\overline{u(q_f)}\lambda^\alpha u(p_i)], \tag{51}$$

where

$$\widetilde{X}_{1\alpha\mu}=F_\alpha(s_i,t_p)\left(\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K}-W_\mu\right)\lambda_\alpha-F_\alpha(s_f,t_p)\lambda_\alpha\left(\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K}-W_\mu\right),$$

$$\widetilde{Y}_{1\mu}^{\alpha}=F_\alpha(s_i,t_q)\left(\frac{p_{f\mu}+R_\mu^{p_f}}{p_f\cdot K}-W_\mu\right)\lambda^\alpha-F_\alpha(s_f,t_q)\lambda^\alpha\left(\frac{p_{i\mu}+R_\mu^{p_i}}{p_i\cdot K}-W_\mu\right),$$

$$\widetilde{Z}_{1\mu}^{\alpha}=(-1)^\alpha F_\alpha(s_i,t_p)\left(\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K}-W_\mu\right)\lambda^\alpha-(-1)^\alpha F_\alpha(s_f,t_q)\lambda^\alpha\left(\frac{p_{i\mu}+R_\mu^{p_i}}{p_i\cdot k}-W_\mu\right),$$

$$\widetilde{T}_{1\alpha\mu}=(-1)^\alpha F_\alpha(s_i,t_q)\left(\frac{p_{f\mu}+R_\mu^{p_f}}{p_f\cdot K}-W_\mu\right)\lambda_\alpha-(-1)^\alpha F_\alpha(s_f,t_p)\lambda_\alpha\left(\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K}-W_\mu\right), \tag{52}$$

with

$$W_\mu=\frac{(p_i+q_i)_\mu}{(p_i+q_i)\cdot K}=\frac{(p_f+q_f)_\mu}{(p_f+q_f)\cdot K}.$$

On the other hand, if Eq. (47) is used to generate the two-s-two-t special amplitude, we obtain

$$M_{2\mu}^{TsTts}(s_i,s_f;t_p,t_q)=e\sum_{\alpha=1}^{5}[\overline{u(q_f)}\widetilde{X}_{2\alpha\mu}u(q_i)\overline{u(p_f)}\lambda^\alpha u(p_i)+\overline{u(q_f)}\lambda_\alpha u(q_i)\overline{u(p_f)}\widetilde{Y}_{2\mu}^{\alpha}u(p_i)], \tag{53}$$

where

$$\widetilde{X}_{2\alpha\mu}=F_\alpha^e(s_i,t_p)\left(\frac{q_{f\mu}+R_\mu^{q_f}}{q_f\cdot K}-W_\mu\right)\lambda_\alpha-F_\alpha^e(s_f,t_p)\lambda_\alpha\left(\frac{q_{i\mu}+R_\mu^{q_i}}{q_i\cdot K}-W_\mu\right),$$

$$\widetilde{Y}_{2\mu}^{\alpha}=F_\alpha^e(s_i,t_q)\left(\frac{p_{f\mu}+R_\mu^{p_f}}{p_f\cdot K}-W_\mu\right)\lambda^\alpha-F_\alpha^e(s_f,t_q)\lambda^\alpha\left(\frac{p_{i\mu}+R_\mu^{p_i}}{p_i\cdot K}-W_\mu\right). \tag{54}$$

Obviously, both amplitudes ($M_{1\mu}^{TsTts}\equiv M_{2\mu}^{TsTts}$) are relativistic, gauge invariant, and consistent with the soft-photon theorem. The most serious theoretical arguments against the use of these two amplitudes to describe the $pp\gamma$ process are that they are quite different from the amplitude constructed from the OBE model and that they violate the Pauli principle. If q_i is interchanged with p_i (or q_f with p_f), then t_p and t_q will be transformed into u_1 and u_2, and one obtains the amplitude $M_{i\mu}^{TsTts}(s_i,s_f;u_1,u_2)$ $(i=1,2)$, which is completely different from the amplitude $-M_{i\mu}^{TsTts}(s_i,s_f;t_p,t_q)$.

We have shown that M_μ^{TuTts} (or $M_{1\mu}^{TuTts}$) is a suitable amplitude to use in describing the $pp\gamma$ process, because it meets all theoretical requirements. As we have noted above, even though the amplitude $M_{2\mu}^{TuTts}$ does not satisfy the Pauli principle at the order K, its numerical predictions are close to the results calculated from the amplitude M_μ^{TuTts}, potential models [7–11], and the OBE model [12], for most cases. That is, the violation of the Pauli principle is not serious, and the amplitude describes the $pp\gamma$ cross sections rather well. The (s,t) class of amplitudes ($M_{1\mu}^{TsTts}\equiv M_{2\mu}^{TsTts}$), on the other hand, *cannot* reproduce the OBE result. The OBE amplitude, in fact, belongs to the (u,t) class of amplitudes. Moreover, the violation of the Pauli principle for the (s,t) class of amplitudes is far more serious than it is for the (u,t) class of amplitudes. This is the most compelling reason why the (u,t) class of amplitudes should be used to describe the $pp\gamma$ process, and why the optimal amplitude is M_μ^{TuTts} given by Eq. (31).

ACKNOWLEDGMENTS

The work of M.K.L. was supported in part by the City University of New York Professional Staff Congress-Board of Higher Education Faculty Research Award Program, while the work of R.T. was included in the research program of the Stichting voor Fundamenteel Onderzoek der Materie (FOM) with financial support from the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO), and the work of B. F. G. was performed under the auspices of the U. S. Department of Energy. We thank A. Korchin and O. Scholten for raising the question of how the Pauli principle should be incorporated in the soft-photon approximation for the case of identical fermions. We thank H. Fearing for his comments regarding the Low amplitude. Helpful discussions with Yi Li are gratefully acknowledged.

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