Ambiguities in the partial-wave analysis of pseudoscalar-meson photoproduction

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Ambiguities in pseudoscalar-meson photoproduction, arising from incomplete experimental data, have analogs in pion-nucleon scattering. Amplitude ambiguities have important implications for the problems of amplitude extraction and resonance identification in partial-wave analysis. The effect of these ambiguities on observables is described. We compare our results with those found in earlier studies. [S0556-2813(96)03809-5]

I. INTRODUCTION

Our empirical knowledge of the N ($S=0, I=1/2$) and Δ ($S=0, I=3/2$) baryons is mainly based upon data from the scattering and photoproduction of pseudoscalar mesons. Resonance positions and couplings have generally come from partial-wave analyses of the existing (incomplete) sets of observables [1]. The lack of sufficient experimental data implies that the transition amplitudes cannot be uniquely determined. Barring further theoretical input, multiple sets of valid amplitudes exist.

Typical analyses employ the additional constraints of unitarity and analyticity, which reduce the range of potential ambiguities [2]. Amplitudes are expected to be ''smooth'' with the possible exception of threshold cusps. Born terms are usually added, either diagrammatically or through the use of dispersion relations. The Carnegie-Mellon–Berkeley (CMB) [3] and Karlsruhe-Helsinki (KH) [4] groups used a wide range of dispersion relation constraints [5] in their analyses. As a result, these independent studies produced results which were qualitatively very similar. However, recent spin-rotation data [6] are in marked disagreement with the prediction of these analyses. No data of this type were available (in the resonance region) when these analyses were performed.

We will concentrate on the ambiguities which can arise in the partial-wave analysis of pseudoscalar meson photoproduction data. There is a close analogy between the ambiguities found in the photoproduction and the elastic scattering of pseudoscalar mesons. This is particularly evident if one adopts the method of Dean and Lee [7]. In a previous work [8] we considered the problems encountered in constructing a complete experiment. The present study is more general. Here we will show how amplitude ambiguities can alter the angular structure of observables and these results will be compared to some earlier findings of Omelaenko [9]. We will also mention how these results are related to the study of nodal trajectories [10].

II. CONJUGATION SYMMETRIES

As suggested in the Introduction, the ambiguities associated with pseudoscalar meson photoproduction are most easily described in analogy with elastic meson-nucleon scattering. To that end, we will first define the elastic scattering amplitudes. Following the notation of Ref. [7], the transition matrix is given by

$$T = F + iG\hat{n} \cdot \vec{\sigma}, \qquad (1)$$

where $\hat{n}$ is the normal to the scattering plane. The spin-flip (G) and spin-nonflip (F) amplitudes can be decomposed into partial-wave amplitudes

$$F(\theta) = \sum_l \, [(l+1)f_{l+} + lf_{l-}]P_l(\cos\theta), \qquad (2)$$

$$G(\theta) = \sum_l \, (f_{l+} - f_{l-})\sin\theta P_l'(\cos\theta), \qquad (3)$$

where the subscript $l\pm$ gives the J value, $J = l \pm 1/2$, and θ is the center-of-mass scattering angle.

In terms of these amplitudes, the differential cross section ($d\sigma/d\Omega$) and polarization (P) are

$$\frac{d\sigma}{d\Omega} = |F|^2 + |G|^2, \qquad (4)$$

$$P\frac{d\sigma}{d\Omega} = -2\,\mathrm{Im}F^*G. \qquad (5)$$

We will first consider a transformation

$$\begin{pmatrix} F \\ G \end{pmatrix} \rightarrow \begin{pmatrix} -F^* \\ G^* \end{pmatrix}, \qquad (6)$$

which preserves both the cross section and polarization. Therefore experimental information on the differential cross section and polarization alone is insufficient to determine F and G.

The photoproduction amplitude can be similarly divided into spin-single-flip (S_1, S_2), spin-nonflip (N), and spin-double-flip (D) pieces [11]. A transformation analogous to Eq. (6) is ambiguity IV of Ref. [8]:

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \rightarrow \begin{pmatrix} -S_1^* \\ -S_2^* \end{pmatrix} \text{ and } \begin{pmatrix} N \\ D \end{pmatrix} \rightarrow \begin{pmatrix} N^* \\ D^* \end{pmatrix}, \qquad (7)$$

which is a symmetry of the cross section, single-polarization observables, and half of the double-polarization observables listed in Ref. [11].

The associated change in partial-wave amplitudes is clear if we first introduce the helicity amplitudes and helicity elements of Walker [12]:

$$S_1 = \frac{1}{\sqrt{2}}\sin\theta\cos\frac{1}{2}\theta\sum_{l=1}^{\infty}(B_{l+}-B_{(l+1)-})(P_l''-P_{l+1}''), \quad (8)$$

$$D = \frac{1}{\sqrt{2}}\sin\theta\sin\frac{1}{2}\theta\sum_{l=1}^{\infty}(B_{l+}+B_{(l+1)-})(P_l''+P_{l+1}''), \quad (9)$$

$$N = \sqrt{2}\cos\frac{1}{2}\theta\sum_{l=0}^{\infty}(A_{l+}-A_{(l+1)-})(P_l'-P_{l+1}'), \quad (10)$$

$$S_2 = \sqrt{2}\sin\frac{1}{2}\theta\sum_{l=0}^{\infty}(A_{l+}+A_{(l+1)-})(P_l'+P_{l+1}'). \quad (11)$$

The transformation given in Eq. (7) is then equivalent to an exchange of helicity elements:

$$B_{l+}\leftrightarrow B_{(l+1)-}^* \quad \text{and} \quad A_{l+}\leftrightarrow -A_{(l+1)-}^*. \quad (12)$$

It should be noted that this transformation is only pertinent above the $\pi\pi N$ threshold. At lower energies it violates unitarity in the form of Watson's theorem [13].

III. CONTINUOUS SYMMETRIES

As discussed in Ref. [7], the polarization and cross section for elastic scattering are also invariant under rotations of the F and G amplitudes:

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}. \quad (13)$$

Here ϕ is a parameter which can vary with the energy and scattering angle. While this transformation does not preserve elastic unitarity, it has implications for resonance identification above the inelastic threshold [7]. As noted in Ref. [7], if this rotation (with $\phi = -\theta$) is composed with the conjugation operation given in Eq. (6), the Minami ambiguity [14]

$$f_{l\pm}\to -f_{(l\pm 1)\mp}^* \quad (14)$$

results. This transformation, applied to the partial-wave amplitudes, preserves elastic unitarity along with the cross section and polarization.

The above rotation also has an analog in terms of photoproduction amplitudes. For example, ambiguity III of Ref. [8] is given by

$$\begin{pmatrix} S_1 \\ D \end{pmatrix}\to\begin{pmatrix} D \\ -S_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} N \\ S_2 \end{pmatrix}\to\begin{pmatrix} S_2 \\ -N \end{pmatrix}, \quad (15)$$

which is a special case ($\phi = \pi/2$) of the more general transformation

$$\begin{pmatrix} S_1' \\ D' \\ N' \\ S_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} S_1 \\ D \\ N \\ S_2 \end{pmatrix}. \quad (16)$$

Here also ϕ depends on the energy and scattering angle. Ambiguities I and II of Ref. [8] can be generalized in a similar way. While a constant value of ϕ was chosen in Ref. [8], the choice $\phi = \phi(\theta)$ is more interesting. (In fact, ϕ *must* vary with the scattering angle θ [15].) The simplest choice $\phi = \theta$ was shown [16] to confuse the identification of resonances in elastic scattering. The choice [7] $\phi = \epsilon\sin\theta$, for a small (angle-independent) parameter ϵ, is also interesting as it illustrates a case where solutions may be continuously varied with ϵ.

The cross-section, single-polarization, and beam-target double-polarization observables are invariant under the above transformation. The beam-recoil and target-recoil observables are not.

IV. FITTING ANGULAR DISTRIBUTIONS

So far, we have not explicitly considered the problems which arise in fitting angular distributions. Here one generally adopts the methods of Barrelet [17] or Gersten [18] in order to write the transversity amplitudes as factorized polynomials in some function of the scattering angle. The case of πN elastic scattering has been reviewed by Höhler [4]. Here we will concentrate on photoproduction, following the treatment given by Omelaenko [9].

The use of transversity amplitudes

$$b_1 = \tfrac{1}{2}[(S_1+S_2)+i(N-D)], \quad (17)$$

$$b_2 = \tfrac{1}{2}[(S_1+S_2)-i(N-D)], \quad (18)$$

$$b_3 = \tfrac{1}{2}[(S_1-S_2)-i(N+D)], \quad (19)$$

$$b_4 = \tfrac{1}{2}[(S_1-S_2)+i(N+D)], \quad (20)$$

allows the problem to be stated very simply. Measurements of the differential cross-section and single-polarization observables determine only the moduli of b_1-b_4, not their phases. This leaves four undetermined phases. However, one overall phase is not observable, and so there remain three unknown phases. These three unknowns correspond to the first three ambiguities of Ref. [8], which when expressed in the b_i basis and generalized for arbitrary angle ϕ as in Eq. (16) become

$$
\text{(I)} \quad
\begin{pmatrix} b_1' \\ b_2' \\ b_3' \\ b_4' \end{pmatrix}
=
\begin{pmatrix}
e^{-i\phi} & 0 & 0 & 0 \\
0 & e^{-i\phi} & 0 & 0 \\
0 & 0 & e^{i\phi} & 0 \\
0 & 0 & 0 & e^{i\phi}
\end{pmatrix}
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix},
$$

$$
\text{(II)} \quad
\begin{pmatrix} b_1' \\ b_2' \\ b_3' \\ b_4' \end{pmatrix}
=
\begin{pmatrix}
e^{-i\phi} & 0 & 0 & 0 \\
0 & e^{i\phi} & 0 & 0 \\
0 & 0 & e^{i\phi} & 0 \\
0 & 0 & 0 & e^{-i\phi}
\end{pmatrix}
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix},
$$

$$
\text{(III)} \quad
\begin{pmatrix} b_1' \\ b_2' \\ b_3' \\ b_4' \end{pmatrix}
=
\begin{pmatrix}
e^{i\phi} & 0 & 0 & 0 \\
0 & e^{-i\phi} & 0 & 0 \\
0 & 0 & e^{i\phi} & 0 \\
0 & 0 & 0 & e^{-i\phi}
\end{pmatrix}
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.
$$

$$(21)$$

Since the differential cross-section and single-polarization observables give no information about the phases of b_1–b_4, it would appear that the angles ϕ above are completely arbitrary. However, this is not so. The form of the multipole expansion [9] requires that

$$
b_1(\theta) = -b_2(-\theta) \quad \text{and} \quad b_3(\theta) = -b_4(-\theta), \quad (22)
$$

which restricts the dependence of ϕ on θ. In ambiguity I, ϕ must be an even function of θ, while, in ambiguities II and III, ϕ must be an odd function of θ.

The constraint given in Eq. (22) allowed Omelaenko [9] to parametrize the four transversity amplitudes in terms of two functions [18]

$$
b_1 = c a_{2L} \frac{e^{i\theta/2}}{(1+x^2)^L} \prod_{i=1}^{2L} (x - \alpha_i), \quad (23)
$$

$$
b_3 = -c a_{2L} \frac{e^{i\theta/2}}{(1+x^2)^L} \prod_{i=1}^{2L} (x - \beta_i), \quad (24)
$$

with $x = \tan\theta/2$. Ambiguities result from the fact that complex conjugation of the roots (α_i and β_i) alters the relative phases (but not the moduli) of the transversity amplitudes. One further condition

$$
\prod_{i=1}^{2L} \alpha_i = \prod_{i=1}^{2L} \beta_i \quad (25)
$$

restricts this freedom. The most simple case (all roots conjugated) is equivalent to the composition of the two transformations IV [Eq. (7)] and III with $\phi = \theta$. The choice of a finite cutoff in L further restricts the values of $\phi(\theta)$ appearing in Eq. (21). This is because if, for example, $b_1' = b_1 e^{i\theta}$, and b_1' is to be reexpressed in the form of Eq. (23), the product must go to $i = 2(L+1)$. Therefore, if the product is restricted to $i = 2L$, this transformation is ruled out. In this case, the only indeterminacy is the freedom to conjugate the roots.

The effect of root conjugation was demonstrated in Fig. 5 of Ref. [9]. Some of the double-polarization observables changed dramatically. This transformation was originally applied, however, to pion photoproduction amplitudes in the first resonance region, where Watson's theorem and the threshold energy dependence can be used to resolve the ambiguities. It would be interesting to examine the effect of the ambiguities at higher energies, where such constraints do not exist. The region with a center-of-mass energy near 1.9 GeV seems promising. This is the energy at which the recent ITEP-PNPI spin-rotation measurements were made. Here there are many overlapping resonance candidates and we are well separated from the threshold region.

We should also mention a recent study where these ambiguities could have important consequences. The nodal trajectory method [10] is concerned with the number (and energy dependence) of nodes found in photoproduction observables. Observables are split into ''Legendre classes'' having similar nodal structure. However, this grouping of observables is not respected by the transformations we have discussed.

As a test case, we chose the target-recoil observable L_Z for $\gamma p \rightarrow p\pi^0$. Helicity amplitudes were generated from a multipole analysis [19], and L_Z crossed zero 3 times at 500 MeV. Then the transformation given in Eq. (16) was applied with $\phi = n\theta$. Using $\phi = \theta$ and 2θ, the number of zero crossings increased to 5 and 7, respectively. The work of Omelaenko [9] indicates that the nodal structure can also be altered by the (smaller) set of ambiguities remaining when a fixed and finite angular momentum cutoff is applied. At sufficiently low energies, a knowledge of the threshold energy dependence helps to resolve ambiguities. At higher energies, further assumptions seem necessary [20].

For the photoproduction of kaons and etas the problem is more acute. In analyzing these reactions, we have no Watson's theorem constraint and we must account for the effect of subthreshold resonances. It should also be noted that, in analyzing pion photoproduction data, the resonance positions are usually taken as known from elastic πN analyses. Given the possibility of significant contributions from ''missing resonances'' (that is, resonances very weakly coupled to πN), kaon and eta photoproduction analyses are relatively free of *a priori* constraints. Therefore they are more likely to be plagued by the kind of ambiguity discussed here.

V. SUMMARY AND CONCLUSIONS

Pion photoproduction amplitudes are not completely determined by cross-section and single-polarization measurements. This fact is exhibited by the existence of one discrete [Eq. (7)] and three continuous [Eq. (21)] transformations of the amplitudes that leave these observables invariant. The transformations, introduced in Ref. [8], are generalized in this paper. We have also shown how these transformations are related to the ambiguity found by Omelaenko [9].

In order to resolve these ambiguities, either further data or more theoretical input must be used. One theoretical constraint comes from restricting the amplitudes to contain only a certain number of partial waves. As shown in Sec. IV, this reduces the ambiguities involved. However, such a theoretical restriction seems artificial, and cannot be justified in the

case of charged-pion photoproduction (due to the t-channel pole).

Other constraints come from unitarity and the elastic πN scattering data. For energies between the πN and $\pi\pi N$ thresholds, Watson's theorem gives the phases of the photoproduction multipoles in terms of the elastic πN phase shifts. This greatly reduces the ambiguity in the photoproduction amplitudes. Above the $\pi\pi N$ threshold such a powerful constraint does not exist. However, πN data can again be used to reduce the ambiguity. We know the masses, widths, and πN couplings of the dominant resonances in the πN channel [such as the $P_{33}(1232)$, $D_{13}(1520)$, and $F_{15}(1680)$]. We can reject any transformation of the photoproduction amplitudes that significantly alters these parameters. Unfortunately, less is known about the resonances contributing to eta and kaon photoproduction.

The ambiguities discussed here are more relevant at higher energies, where there are fewer theoretical restrictions, than at lower energies, where Watson's theorem applies. This has an important consequence for the nodal trajectory method [10], since the Legendre classes it employs are not respected by the ambiguity transformations. Therefore, at energies where these transformations are allowed, the nodal trajectory method will have to account for this additional freedom.

[1] It is often said that $2N-1$ carefully chosen observables are required in order to reconstruct N independent amplitudes, up to an overall phase. That this is not generally true (due to quadrant ambiguities) is certainly "long known" if not "well known." See, for example, Ref. [11].

[2] While a great deal has been written on the identification and resolution of ambiguities, little of this has been applied in typical partial-wave analyses. The interested reader should see I. Sabba Stefanescu, Z. Phys. C **41**, 453 (1988), and references therein, as well as the book by K. Chadan and P.C. Sabitier, *Inverse Problems in Quantum Scattering Theory* (Springer-Verlag, New York, 1989).

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[15] Strictly speaking, the choice of a constant ϕ value for all values of θ is inappropriate. This is because, for example, if $\phi=\pi/2$ in Eq. (16), then the observable C_z will change sign at all angles, and yet it is constrained to equal -1 at $\theta=0°$ and $+1$ at 180°. Choices such as $\phi=\theta$ or $\phi=\epsilon\sin\theta$ leave all observables invariant at 0° and 180°. Some choices, such as $\phi=\theta$, can be eliminated if we require that the amplitudes smoothly connect to energy regions where this ambiguity is resolved.

[16] In Ref. [7] the effect of choosing $\phi=\theta$ was demonstrated using the partial-wave expansions for F and G, along with recursion relations for the Legendre polynomials. In photoproduction, a similar result follows from the recursion relations for rotation functions, using $\sin\theta=-\sqrt{2}d_{10}^1$ and $\cos\theta=d_{00}^1$ along with the expansion for $d_{\lambda\mu}^J d_{\lambda'\mu'}^{J'}$ [see, for example, A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton, New Jersey, 1957), Sec. 4.3]. However, since this choice is motivated by simplicity (and not physics), we have omitted the details.

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